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Computational approaches for estimating fault zone properties using trapped waves

Anton Kim Gulley

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**Chapter 4: The Effect of Gradational Velocities and Anisotropy on Fault Zone Trapped Waves. Submitted to Geophysical Journal International**

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<th>All writing and computations</th>
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<td>85%</td>
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<tr>
<th>Name</th>
<th>Nature of Contribution</th>
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<td>Jennifer Eccles</td>
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</tr>
<tr>
<td>Jari Kaipio</td>
<td>Computational and technical advice. Editing.</td>
</tr>
<tr>
<td>Peter Malin</td>
<td>FZTW expertise. Advice. Editing.</td>
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- that the candidate wrote all or the majority of the text.

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<tr>
<td>Jari Kaipio</td>
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<tr>
<td>Peter Malin</td>
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<td>26 Oct 2016</td>
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**Chapter 6: Inversion of Fault-Zone Trapped Waves: Gradational Velocity Models. Will be submitted to Geophysical Journal International**

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**Chapter 3: A Numerical Approach for Modelling Fault Zone Trapped Waves. Submitted to Geophysical Journal International**

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Abstract

Fault zones are regions within the Earth’s crust that accommodate the frictional sliding known as faulting. These fault-zones contain a history of past earthquake events and other tectonic processes. This means they contain valuable information on how earthquake processes work, as well as the Earth’s geologic system. The rocks of mature fault-zones are often highly altered and deformed through earthquake processes, which can result in fault-zones having lower seismic velocities than the surrounding rock. It is these lower seismic velocities that allow the fault-zone to behave as a waveguide in which fault-zone trapped waves propagate. It is thought that measurements of fault-zone trapped waves can be used for high resolution (∼ 10 m) imaging of the elastic properties and geometry of fault zones.

A novel, computationally efficient, finite element type solver for fault-zone trapped waves is developed. This solver allows for an anisotropic velocity model which varies across the fault. It is used to show how fault-zone trapped waves propagating in a, geologically realistic, gradational across-fault velocity model behave significantly differently from waves propagating in a layered across-fault velocity model. This solver is used in the forward model for a proposed inversion methodology. This methodology uses recordings of several fault-zone trapped waves as well as the Bayesian approximation error approach to estimate a two-dimensional cross-section of a fault zone. This is significant because the overall approach allows for the velocity models to be gradational in the across-fault and down-dip directions. In this thesis, preliminary investigations using fault-zone trapped waves from the Alpine Fault in the South Island of New Zealand are also carried out. These investigations suggest that the effective width and velocity contrast of the Alpine Fault is comparable to other major faults around the world.
Acknowledgements

It is very difficult to put into a few words how much the support given, by a huge number of people, has meant to me and my thesis. I hope this goes some way towards expressing my immense gratitude towards all of you.

To Jari Kaipio, you provided me so much more than just your incredible academic knowledge. Thank you for all the conversations, support, encouragement and help. Thank you for allowing me to find my own way but always finding a way to point me in the right direction when needed. To Jennifer Eccles, your effort, enthusiasm, commitment and support meant so much to me. Thank you for going the extra mile for me time and time again. I will also look back fondly on the numerous opportunities you helped provide. To Peter Malin, thank you for the support, encouragement and positive attitude. Conversations with you were always helpful and enlightening. There are many other academics that assisted me in some way, from the people at the University of Auckland, to Alpine Fault field studies as well as numerous workshops and conferences. Thank you for your time, insight, advice and knowledge.

Thank you to everyone that also helped edit my thesis; Owen Dillon, Hwan Go, Ruanui Nicholson, Rob Fullerton, Judith Fullerton, Mark Fullerton, Christina Fullerton, Kathleen Collier, Nicholas Gulley, Bruce Gulley, Marjet Pot, as well as my supervisors. Thank you for your time, attention to detail and understanding of my difficulties in writing and spelling.

To my office mates, Paul, Ru, Hwan, Owen and Attique. You made it a pleasure to turn up to work each day. The humour was unrelenting but at the same time the collaboration and academic discussions hugely influenced my thesis. Thank you also to the numerous other students and postdocs that I have met and worked with at Auckland University, on Alpine Fault field studies and on other conferences and workshops; you helped make my studies and those events informative and enjoyable. Thank you also to my other friends for the good times and the support.

To my parents, Bruce and Marjet, there are no words to describe how much your support, understanding and help has meant. Thank you for always being there and always going out of your way to help. To Judith and Rob, thank you also for the unwavering support you have given me. Thank you for taking Christina and me into your family home and helping minimize my stress during the write-up period. To my brothers, Rion and Nicholas, thank you for all your support and help. Thank you also to my extended siblings, Naomi, Lizzy, Jane and Mark and the rest of my extended family.

Finally thank you to Christina. Thank you so much for all the support you have given me through this thesis. Thank you for picking me up when I was down, helping me see the way forward and bringing so much joy to my life.

Anton
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List of Abbreviations

FZTW  Fault-Zone Trapped Wave
FZGW  Fault-Zone Guided Wave (equivalent to FZTW)
FEM   Finite Element Method
ABC   Absorbing Boundary Condition
MCMC  Markov Chain Monte Carlo
BAE   Bayesian Approximation Error
P-wave Primary/Pressure (wave)
S-wave Shear (wave)
SV    Shear waves with fault-orthogonal polarisation
SH    Shear waves with fault-parallel polarisation
DFDP  Deep Fault Drilling Project
SAFOD San Andreas Fault Observatory at Depth
GA    Genetic Algorithm
FM    Forward Modelling
3LM   Three-Layer Modelling
$M_L$  Local magnitude e.g. $M_L2.3$
3-D   Three-Dimensions
2-D   Two-Dimensions
1-D   One-Dimension
$F_L$  Love type FZTW (SH polarisation)
$F_R$  Rayleigh type FZTW (P-SV polarisation)
$F_\phi$  Leaky type FZTW (P-SV polarisation)
Vp/Vs  P- to S-wave velocity ratio
GMT   Greenwich Mean Time
List of Mathematical Symbols

- Bold upper case is used to denote matrices and tensors, e.g. $C$. Bold lower case is used to denote vectors, e.g. $c$. Non-bold upper and lower case is used to denote scalars.

- The matrix $C$ and vector $c$ have entries $C_{ij}$ and $c_i$, where $i, j$ are natural numbers. In some cases a letter is used to denote a coordinate e.g. $k_x$. Subscripts are also used to denote a region or point e.g. $k_1$ is the wave number in the 1st region.

- In the following tables, superscripts in the description indicate in which Sections the symbols are used. This is done when symbols have been used for different purposes in different Sections.

- Symbols that are used and defined in only one section have been omitted.

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<th>Modeling symbols</th>
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<tr>
<td>$y$ Measurements</td>
<td>$x$ Cartesian coordinate vector</td>
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<tr>
<td>$a(\cdot)$ Forward model</td>
<td>$x$ Coordinate from source to receiver</td>
</tr>
<tr>
<td>$\hat{a}(\cdot)$ Accurate forward model</td>
<td>$z$ Fault-orthogonal coordinate</td>
</tr>
<tr>
<td>$x$ Parameters</td>
<td>$y$ Remaining fault-parallel coordinate</td>
</tr>
<tr>
<td>$e$ Measurement error</td>
<td>$\hat{x}$ Unit vector</td>
</tr>
<tr>
<td>$\epsilon$ Modelling error$^{1,5,6}$</td>
<td>$\hat{y}$ Unit vector</td>
</tr>
<tr>
<td>$\pi(\cdot)$ Probability density function</td>
<td>$\hat{z}$ Unit vector</td>
</tr>
<tr>
<td>$\pi(a</td>
<td>b)$ Density of a given $b$</td>
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<tr>
<td>$\Gamma$ Covariance matrix$^{1,5}$</td>
<td>$\hat{\nu}_0$ Cartesian unit vector$^2$</td>
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<tr>
<td>$\delta(\cdot)$ Dirac-Delta density$^{1,5}$</td>
<td>$\hat{\theta}$ Cartesian unit vector$^2$</td>
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<tr>
<td>$|\cdot|$ Euclidean norm of a vector</td>
<td>$\hat{\phi}$ Cartesian unit vector$^2$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\nu}$ Wave propagation direction$^2$</td>
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<tr>
<td></td>
<td>$r$ Cylindrical coordinate radius$^4$</td>
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<tr>
<td></td>
<td>$\phi$ Cylindrical coordinate angle$^3$</td>
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<tr>
<td></td>
<td>$\in$ element of</td>
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Finite element symbols

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</tr>
<tr>
<td>( l )</td>
<td>( F_L ) FEM eigenvector</td>
</tr>
<tr>
<td>( r )</td>
<td>( F_R ) FEM eigenvector</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>( F_R ) radial FEM eigenvector</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>( F_R ) transverse FEM eigenvector</td>
</tr>
<tr>
<td>( N )</td>
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</tr>
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<td>( \mathbf{n} )</td>
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<tr>
<td>( \mathbf{I} )</td>
<td>The identity matrix</td>
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<td>Empty set</td>
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Wave propagation symbols

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<td>( F_R ) radial eigenvector</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>( F_R ) transverse eigenvector</td>
</tr>
<tr>
<td>( l )</td>
<td>( F_L ) eigenvector</td>
</tr>
<tr>
<td>( i )</td>
<td>Complex number</td>
</tr>
<tr>
<td>( U )</td>
<td>Group velocity</td>
</tr>
<tr>
<td>( c )</td>
<td>Phase speed</td>
</tr>
<tr>
<td>( k )</td>
<td>Wave number scalar</td>
</tr>
<tr>
<td>( \omega )</td>
<td>Angular frequency</td>
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<td>( \theta )</td>
<td>Ray angle in ( x - z ) plane</td>
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<td>( C )</td>
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<td>( s )</td>
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<td>( \alpha )</td>
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CHAPTER 1

Introduction

Earthquakes are a major hazard around the globe. For example, the magnitude 9.2 Indian Ocean Earthquake on Boxing Day 2004 killed \( \sim 230,000 \) people and the 2011 magnitude 9.0 Tohoku earthquake killed \( \sim 16,000 \) [Simons et al., 2011, Stein and Okal, 2007]. Major earthquakes such as these occur on significant weaknesses within the crust that are known as faults, the most significant of which are at tectonic plate boundaries [Marshak, 2005]. New Zealand is at risk from earthquakes, with the Alpine Fault posing the highest seismic hazard [Berryman et al., 2012b, Stirling et al., 2002]. Understanding these faults can provide insight into both past and future earthquakes in an attempt to mitigate their effects.

A mature fault-zone is a narrow (\( \sim 100 \) m) region of rock containing the principal sliding surface of the fault [Faulkner et al., 2010]. This region has been deformed by earthquake and faulting processes, therefore the fault-zone often has lower seismic (elastic wave) velocities than the surrounding more intact rock. This means that the fault-zone can behave as a waveguide and allow the propagation of fault-zone trapped waves (FZTWs) [Ben-Zion and Sammis, 2003]. These are also known as fault-zone guided waves (FZGWs). Since the FZTWs travel in the fault zone they are primarily affected by its physical properties, therefore, there is the potential for FZTWs to be used to estimate fault zone properties [Li et al., 2012].

The primary aim of this study was to develop techniques with which information on the
fault zone can be extracted from FZTW recordings. This was done by treating the problem as an inverse problem and developing the necessary forward and inverse algorithms. The definition of an inverse problem is defined in Section 1.5. This thesis focuses on creating methodology that allows for gradational velocity models in both the across-fault and vertical directions, as there is no such allowance in current inversions [e.g. Haberland et al., 2003]. The main geologic target of this study was the Alpine Fault in South Westland, New Zealand, although the majority of this work is applicable to fault zones in general.

The main contribution of this thesis is contained in Chapters 2-6. These Chapters progress from developing forward modelling methods to applying inversions and developing inverse algorithms. All of these Chapters include brief outlines of the literature and methodologies. The rest of this introduction contains a much more detailed discussion of the most important topics of this thesis.

Sections 1.1 describes the mathematics and physics of elastic (seismic) waves and some of their properties when propagating in the earth. Section 1.2 describes some of the important geologic concepts including the properties of fault zones in general and the Alpine Fault in particular. Section 1.3 describes the propagation of elastic waves in a fault zone, as fault-zone trapped waves. This section also gives an overview of the state of knowledge and methodologies used in the FZTW field, including inverse techniques and FZTW studies of different faults around the world. The numerical methods utilised in this thesis to develop models for FZTWs are described in Section 1.4. Section 1.5 describes the inverse theory and techniques used to infer fault zone properties using the numerical models and recorded FZTW data.
1.1 Elastic wave equation

The full elastic wave equation approximates the propagation of elastic waves in a medium. It describes the propagation of many types of seismic waves, including trapped waves. In this thesis, three different forms of the wave equation are considered:

1. The 2-D equation with a 1-D velocity model is used in Chapter 3 as a beginning step for the calculation of the 3-D propagation of trapped waves in a 1-D velocity model.

2. The 2-D equation with a 2-D velocity model is used in the gradational Bayesian approximation error term in Chapter 6.

3. The 3-D equation with a 2-D velocity model is used for simulation of full waveforms.

In this section, elastic waves and anisotropy are briefly described before the derivation of the elastic wave equations used in this thesis. The linear wave equation is considered and not the more complex nonlinear wave equation [McCall, 1994], or poroelastic wave equation [Yang and Zhang, 2002]. Further information on the elastic wave equation and a more detailed derivation can be seen in Aki and Richards [2009].

1.1.1 Elastic waves

Elastic waves are vector-valued waves. This is unlike the standard wave equation which is scalar valued [e.g. Salsa, 2008]. If the resistance to shearing of a isotropic medium is small (i.e. it is a fluid) then the elastic wave equation can be reduced to the pressure (sound) wave equation. The propagation of elastic waves can be broken down into two main types. The first type is the P-wave. The P-waves are compressional waves that cause displacement of the media parallel to the direction of wave propagation (longitudinal) and are essentially pressure waves. The second type is the S-wave, also known as the shear wave. These waves always travel slower than the P-waves. The S-waves cause displacement of the medium in the plane perpendicular to wave propagation direction (transverse). The first arrivals of the P-wave and S-wave can often be clearly seen in a recording of ground motion from an earthquake (seismogram). If the elastic properties of a region of rock are known, then several seismograms can be used to locate the position of the earthquake source. This technique is used all over the world to locate earthquakes [e.g. Marshak, 2005].
1.1.2 Stress and strain relationship

Stress is a force per unit area on a surface, the 3 by 3 stress tensor describes the vectored valued stress on three orthogonal surfaces. Strain is a relative deformation of a surface and is dimensionless. The 3 by 3 strain tensor describes the vector valued strain on three orthogonal surfaces, the diagonals of the strain tensor relate to compression (or extension) and the non-diagonals relate to a shearing deformation [see Aki and Richards, 2009, for more details].

Let \( \mathbf{x} \) be the spatial vector in Cartesian co-ordinates with \( x = x_1, y = x_2, z = x_3 \). Let \( s_i \) be displacement in the \( x_i \) direction. The strain tensor with entries \( \epsilon_{ij} \) is

\[
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i} \right). \tag{1.1}
\]

If the displacements \( s_i \) are small, then by Hooke’s law, the stress matrix \( \sigma_{ij} \) is linearly dependent on the different components of strain and can be written as

\[
\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \epsilon_{kl}, \tag{1.2}
\]

where \( C \) is the 3x3x3x3 elastic modulus tensor.

Because of the symmetries \( \sigma_{ij} = \sigma_{ji} \) and \( \epsilon_{ij} = \epsilon_{ji} \), \( C \) can be represented by a 6x6 symmetric matrix using Voigt notation with the relation

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{32} \\
\sigma_{31} \\
\sigma_{21}
\end{bmatrix} =
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
2\epsilon_{32} \\
2\epsilon_{31} \\
2\epsilon_{21}
\end{bmatrix}\tag{1.3}
\]

Anisotropy

Anisotropy in wave propagation means that the velocity of a wave is dependent on the direction of propagation. In general, \( C \) has 21 independent components which can be used to describe all anisotropic regimes [Carcione, 2007a]. Because of different crystal shapes and orientations of solids, there is a large range of different anisotropic regimes [e.g. Lane, 2014]. In the sub-surface of the earth, when the length scales are meters or greater, there
is at least some randomisation of individual crystals, and much of the anisotropy averages out. In seismology at these scales, there are three main regimes; orthorhombic isotropy, transverse isotropy and complete isotropy. Orthorhombic isotropy can be caused by media having perpendicular intersecting layers/fractures, and it is believed to occur in many fractured reservoirs. In this situation, $C$ has 9 independent components [Tsvankin, 1997]. This thesis does not consider trapped waves in orthorhombic media.

**Complete isotropy**

In the case of complete isotropy, $C$ only has two independent parameters that can be described by the Lamé parameters $\lambda$ and $\mu$

$$C = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda \\
\lambda & \lambda + 2\mu & \lambda \\
\lambda & \lambda & \lambda + 2\mu \\
\mu & \mu & \mu
\end{bmatrix}$$

(1.4)

here $C_{ij} = 0$ at the blank locations

These Lamé parameters, together with the density ($\rho$), can be used to derive the P-wave velocity $\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ and the S-wave velocity $\beta = \sqrt{\frac{\mu}{\rho}}$.

**Transverse isotropy**

Transverse isotropy has been used in this thesis and for the following derivation of the elastic wave equation. The term ‘transverse anisotropy’ will be used synonymously with ‘transversely isotropic’ as both mean that the velocity of waves traveling parallel to the plane of isotropy do not have directionally dependent velocities. The velocity of the wave is only dependent on the angle of propagation with respect to the normal of the plane of isotropy. Transverse anisotropy arises in many situations in geophysics. It was shown by Backus [1962] that a series of parallel isotropic layers of rock can be approximated as propagation in a homogenous transversely isotropic medium when the wavelength is much longer than the thickness of the beds. In this thesis, transverse anisotropy is considered because it is assumed that the alignment of macro and micro structures parallel with the fault, due to fault shearing, causes transverse anisotropy [Cochran et al., 2003].
In the case of transverse anisotropy, there are five independent parameters in $C$.

$$
C = \begin{bmatrix}
C_{11} & C_{11} - 2C_{66} & C_{13} \\
C_{11} - 2C_{66} & C_{11} & C_{13} \\
C_{13} & C_{13} & C_{33} \\
& & & C_{44} \\
& & & C_{44} \\
& & & C_{66}
\end{bmatrix}
$$

(1.5)

Full isotropy is achieved if $C_{44} = C_{66} = \mu$, $C_{11} = C_{33} = \lambda + 2\mu$, $C_{13} = \lambda$.

### 1.1.3 Elastic wave equation in 3-D

The force on a small region of medium comprises the change in stress across that region plus any external forcing. The full elastic wave equation can be derived by taking the limit as the size of the small region reduces to zero. This will leave us with

$$
\rho \frac{\partial^2 s_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i,
$$

$$
= \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left( \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \epsilon_{kl} \right) + f_i,
$$

$$
= \frac{1}{2} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left( \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \left( \frac{\partial s_i}{\partial x_j} + \frac{\partial s_j}{\partial x_i} \right) \right) + f_i,
$$

(1.6)

where $t$ is time, $\rho$ is density, $f$ is the forcing vector. Also, $s, f$ are dependent on $x$ and $t$ while $\rho, C$ are dependent on $x$ only.

In the case of transverse isotropy, the wave equation can be written using Eqs. (1.5-1.6) as

$$
\rho \frac{\partial^2 s_x}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{11} \frac{\partial s_x}{\partial x} + (C_{11} - 2C_{66}) \frac{\partial s_y}{\partial y} + C_{13} \frac{\partial s_z}{\partial z} \right) + \frac{\partial}{\partial y} \left( C_{66} \frac{\partial s_y}{\partial x} + C_{66} \frac{\partial s_x}{\partial y} \right) + f_x,
$$

(1.7a)

$$
\rho \frac{\partial^2 s_y}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{66} \frac{\partial s_y}{\partial x} + C_{66} \frac{\partial s_x}{\partial y} \right) + \frac{\partial}{\partial y} \left( (C_{11} - 2C_{66}) \frac{\partial s_x}{\partial x} + C_{11} \frac{\partial s_y}{\partial y} + C_{13} \frac{\partial s_z}{\partial z} \right) + f_y,
$$

(1.7b)
\[
\frac{\partial^2 s_z}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{44} \frac{\partial s_z}{\partial x} + C_{44} \frac{\partial s_x}{\partial z} \right) + \frac{\partial}{\partial y} \left( C_{44} \frac{\partial s_z}{\partial y} + C_{44} \frac{\partial s_y}{\partial z} \right) \\
+ \frac{\partial}{\partial z} \left( C_{13} \frac{\partial s_x}{\partial x} + C_{13} \frac{\partial s_y}{\partial y} + C_{33} \frac{\partial s_z}{\partial z} \right) + f_z. \quad (1.7c)
\]

Wave equation in 3-D with 2-D velocity model (2.5-D)

It is assumed here that the direction \(y\) is a horizontal direction and is parallel with the fault. An approximation for fault zones is that the elastic parameters don’t change in the horizontal fault-parallel (\(y\)) direction. This is because faults are often linear structures [Hubbert, 1972]. This is often referred to as 2.5-D [Cao and Greenhalgh, 1998]. Computationally, this means that \(\frac{\partial C_{ij}}{\partial y} = \frac{\partial \rho}{\partial y} = 0\) and, assuming infinite boundaries \(s, f\) can be separated as \(s(x, y, z, t) = s(x, z, t) \exp(ik_y y)\), \(f(x, y, z, t) = f(x, z, t) \exp(ik_y y)\) where \(i = \sqrt{-1}\) is the imaginary number and \(k_y\) is the wave number in the \(y\) direction [Salsa, 2008]. Substituting this into Eqs. (1.7) and dividing by \(\exp(ik_y y)\) leaves

\[
\frac{\rho \partial^2 s_x}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{11} \frac{\partial s_x}{\partial x} + ik_y (C_{11} - 2C_{66}) s_y + C_{13} \frac{\partial s_z}{\partial z} \right) + ik_y C_{66} \frac{\partial s_y}{\partial x} - k_y^2 C_{66} s_x \\
+ \frac{\partial}{\partial z} \left( C_{44} \frac{\partial s_z}{\partial x} + C_{44} \frac{\partial s_x}{\partial z} \right) + f_x, \quad (1.8a)
\]

\[
\frac{\rho \partial^2 s_y}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{66} \frac{\partial s_y}{\partial x} + ik_y C_{66} s_x \right) + ik_y (C_{11} - 2C_{66}) \frac{\partial s_x}{\partial x} - k_y^2 C_{11} s_y + ik_y C_{13} \frac{\partial s_z}{\partial z} \\
+ \frac{\partial}{\partial z} \left( ik_y C_{44} s_z + C_{44} \frac{\partial s_y}{\partial z} \right) + f_y, \quad (1.8b)
\]

\[
\frac{\rho \partial^2 s_z}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{44} \frac{\partial s_z}{\partial x} + C_{44} \frac{\partial s_x}{\partial z} \right) - k_y^2 C_{44} s_z + ik_y C_{44} \frac{\partial s_y}{\partial z} \\
+ \frac{\partial}{\partial z} \left( C_{13} \frac{\partial s_x}{\partial x} + ik_y C_{13} s_y + C_{33} \frac{\partial s_z}{\partial z} \right) + f_z. \quad (1.8c)
\]

1.1.4 Wave equation in 2-D

The 2-D wave equation in this thesis assumes that the elastic properties and displacement \(s\) are translationally invariant in the \(y\) direction. This can be derived directly from Eqs.
(1.7) by assuming all derivatives with respect to $y$ are 0

$$\rho \frac{\partial^2 s_x}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{11} \frac{\partial s_x}{\partial x} + C_{13} \frac{\partial s_z}{\partial z} \right) + \frac{\partial}{\partial z} \left( C_{44} \frac{\partial s_z}{\partial x} + C_{44} \frac{\partial s_x}{\partial z} \right) + f_x, \quad (1.9a)$$

$$\rho \frac{\partial^2 s_y}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{66} \frac{\partial s_y}{\partial x} \right) + \frac{\partial}{\partial z} \left( C_{44} \frac{\partial s_y}{\partial z} \right) + f_y, \quad (1.9b)$$

$$\rho \frac{\partial^2 s_z}{\partial t^2} = \frac{\partial}{\partial x} \left( C_{44} \frac{\partial s_z}{\partial x} + C_{44} \frac{\partial s_x}{\partial z} \right) + \frac{\partial}{\partial z} \left( C_{13} \frac{\partial s_x}{\partial x} \right) + f_z. \quad (1.9c)$$

For elastic waves traveling in 2-D, the $s_y$ component is decoupled from the $s_x$ and $s_z$ components meaning the equations can be solved separately. The $s_y$ component also only contains S-waves. This is partly why it is often referred to as the ‘SH’ wave propagation, where ‘H’ stands for horizontal. The coupled $s_x$, $s_z$ components contain S- and P-wave energy and is often referred to as ‘P-SV’ wave propagation, where ‘V’ is for vertical [Aki and Richards, 2009]. In this thesis, the condition that SH is horizontal and SV is vertical is relaxed and the polarisation of SH and P-SV is typically dependent on source and receiver geometries.

### 1.1.5 Anelastic attenuation

Attenuation is the process in which mechanical energy is transferred to heat. The effect on elastic waves is twofold: Firstly, it causes a decrease in the amplitude of the waves. Secondly, it causes dispersion, that is, the velocity of the waves is frequency dependent [Aki and Richards, 2009]. Attenuation is often characterised by the anelastic attenuation factor $Q$, where $2\pi Q^{-1} =$ energy loss per wavelength [Carcione, 2007b]. In general, $Q$ is dependent on the frequency of the wave $\omega$. However, in seismology, it is often approximated to be independent of $\omega$ [Carcione, 2007b]. Attenuation for general $Q$ is incorporated into the propagation of FZTWs in Chapter 3.

### 1.1.6 Boundary conditions

Conditions are required for the spatial boundaries of the partial differential equations outlined in this section. There are three main boundary conditions used in elastic waves: zero stress, zero displacement and absorbing boundary conditions.

In many cases, the physical domain has regions that have significantly different elastic properties, e.g. air and rock. In some cases, the boundary between these two regions can be approximated to a computational boundary. Let the acoustic impedance $Z$ be the
product of density and velocity. If the acoustic impedances are significantly (orders of magnitude) different between the two regions then the following two approximations can be made.

A zero stress boundary condition occurs if a wave is travelling towards a medium with a much smaller acoustic impedance [Aki and Richards, 2009]. For example, this could be a seismic wave travelling in rock (high $Z$) towards a rock-air (low $Z$) interface. In this case the approximate boundary conditions are $\sum_i \sigma_{ij} n_i = 0$ for all $j$ where $\mathbf{n}$ is the boundary normal. This is also known as a free surface boundary condition.

A zero displacement boundary condition arises when a wave is traveling towards a medium with much larger acoustic impedance. This approximate boundary condition is $\mathbf{s} = \mathbf{0}$.

In many cases the computational domain should be very large or infinite e.g. simulation of seismic waves in the earth. There is often, however, a smaller region of interest within this large domain. Absorbing boundary conditions are used to allow energy to pass out of this smaller domain without any reflections. These absorbing boundary conditions are used to allow the domain to be truncated to reduce computational time. Unfortunately, in dimensions greater than 1, there are no perfect absorbing boundary conditions and these are still a topic of research. In this thesis, two new paraxial absorbing boundary conditions are proposed. These are developed in Chapters 2 and 3.
1.2 Geology of the Alpine Fault and fault zones

Fault zone trapped waves (FZTWs) propagate within the low velocity zone associated with faults. This section provides a discussion of the geology of fault zones and their significance. In particular, attention is focused on the Alpine Fault, South Westland, New Zealand. While the majority of this thesis is spent developing methodological improvements in FZTW analysis, the primary geologic interest and case study is the Alpine Fault.

1.2.1 New Zealand geological setting

The oldest rocks of New Zealand are believed to have originated from sediment that eroded off the coast of the super continent Gondwana as long as \(~700\) million years ago [Nathan et al., 2002]. Over the next 300 million years, further sediments settled on the sea floor [Adams, 1975]. These sediments, combined with some Gondwanan Crust, now make up the western Province of the South Island of New Zealand and some parts of the North Island [Adams, 1975, Ireland, 1992] as seen in Fig. 1.1. In the period from 400 million years ago to 100 million years ago, there were further periods of erosion, volcanism and deposition on the sea floor. This is the material that makes up the rocks of the Eastern Province of the South Island [Christensen, 1984, Mortimer, 1994, Thornton, 1985], Fig. 1.1.

Approximately 85 million years ago, New Zealand separated from Gondwana [Waters and Craw, 2006]. As New Zealand moved away, upwelling magma formed the oceanic crust that underlies the Tasman sea [Thornton, 1985]. From around 25 million years ago, convergent tectonic forces began to shape New Zealand into the form seen today. A subduction zone developed off the East Coast of the North Island. This is where a dense oceanic section of the Pacific tectonic plate currently subducts under a continental portion of the Indo-Australian Plate. In Fiordland, at the south west corner of the South Island, an oceanic portion of the Australian plate is subducting under a continental portion of the Pacific plate [Walcott, 1978]. In between these two regions of subduction, the plate boundary has continental crust colliding with continental crust and both plates are too light to sink into the Earth’s interior. This continental collision results in the two plates sliding past and compressing each other as seen in Fig. 1.2. A large component of this motion is taken up by the Alpine Fault, and the uplift due to compression resulted in the formation of the Southern Alps [Coates and Cox, 2002].
Rocks of the South Island

With increased temperature and/or pressure rock will undergo metamorphism. This is where the mineralogy and mineral fabric change to form a new rock [Marshak, 2005]. Metamorphism is primarily caused by high pressure, but can also be influenced by heat and chemically active fluids. Across the eastern region of the South Island, different metamorphic rocks can be seen, from high grade metamorphic rocks that are known as schists, to low grade metamorphic rocks that are known as Greywacke [Thornton, 1985]. The rock on the Western side of the Alpine Fault is typically granitic or sedimentary material with little metamorphism seen at the surface, see Fig. 1.3. In many parts, the Alpine Fault penetrates into and is overlain by tertiary glacial deposits [Cooper and Norris, 1994].

1.2.2 The Southern Alps and Alpine Fault

The Alpine Fault is a sublinear structure that can be viewed from space, see Fig. 1.4. It represents the major interface between the Pacific plate and the Australian plate within
Figure 1.2: The motions of the Pacific Plate relative to the Indo-Australian Plate. The red arrows indicate direction of movement of the plate that is subducting under the other plate. The black lines indicate major faults. From the plate motion arrows (yellow) it can be deduced that the motion of the Alpine Fault is predominantly parallel to the fault trace but there is also a smaller perpendicular component [Norton, 2010].

Figure 1.3: Artists impression of the sub-surface of the South Island. The Torlesse terrain rock is relatively homogeneous Greywacke [Coates and Cox, 2002].
the South Island. It is $\sim 850$ km long [Barth et al., 2012] and strikes at $\sim 070^\circ$ [Norris and Cooper, 2001]. The fault trace dips $\sim 50^\circ$ to the south east beneath the Southern Alps [Davey et al., 1995]. Seismic and geologic evidence suggests that the fault penetrates to at least 25 km [Cooper, 1980, Davey et al., 1995, Grapes and Watanabe, 1992, 1994, Norris et al., 1990].

While from a distance the fault appears to be a linear feature, at higher resolution the fault is partitioned into segments. At a resolution of 1-10 km, the Alpine Fault supports smaller oblique thrust fault segments and strike slip segments [Barth et al., 2012, Norris and Cooper, 1995, 1997]. At a resolution of less than 1 km, LiDAR measurements have shown that the fault is made up of symmetric positive flower structures [Barth et al., 2012].

The Alpine Fault accommodates 70-75% of the interplate motion [Norris and Cooper, 2001]. The remaining motion is taken up by a series of faults east of the Alpine Fault [Cox and Findlay, 1995, Long et al., 2003], such as those responsible for the 2010/2011 earthquakes in Canterbury [Fry and Gerstenberger, 2011]. The central section of the Alpine Fault moves strike slip, or parallel to the plate boundary, at a rate of $27 \pm 5$ mm/year. The dip slip or vertical component varies considerably, but is generally less than 10 mm/year with the greatest rate of uplift in the vicinity of Mt Cook National Park [Norris and Cooper, 2001].

In the last 25 million years, there has been $\sim 480$ km of displacement along the Alpine Fault [Wellman, 1955]. The major uplift of the Southern Alps began as recently as 10 million years ago [Thornton, 1985], which has resulted in as much as 70 km of crustal shortening from compression orthogonal to the plate boundary [Allis, 1986, Walcott, 1979]. Much of this uplift from compression has eroded and has formed sedimentary deposits off-shore.
Large earthquakes of magnitude $\sim8$ are believed to have occurred in the past, and earthquakes of this magnitude may occur in the future on the Alpine Fault [Sutherland et al., 2007]. There is a 300-400 year cycle on large earthquakes [Berryman et al., 2012b, De Pascale and Langridge, 2012, Sutherland et al., 2007, Wells and Goff, 2007, Wells et al., 1999, Yetton, 2000]. Individual historical events have been shown to cause 6-9 m of dextral strike slip movement and up to 2 m of vertical movement [Berryman et al., 2012a, Norris and Cooper, 1995, Yetton, 2000]. It is suggested that the most recent earthquake occurring in the year 1717 ruptured 375 km of the length of the fault [Wells et al., 1999]. The probability of a ground-rupturing earthquake occurring in the next 50 years is estimated to be 30% [Biasi et al., 2015] and hence the Alpine Fault is considered to be late in its seismic cycle.

1.2.3 Alpine Fault Deep Fault Drilling Project

The Deep Fault Drilling Project (DFDP) aims to discover more about: the Alpine Fault, earthquake processes, in situ conditions of a fault zone approaching failure, earthquake hazards and increase understanding of plate tectonics. The first phase of drilling was completed in 2009 with two boreholes drilled at Gaunt Creek [Sutherland et al., 2012, Townend et al., 2009]. These boreholes, DFDP-1A and DFDP-1B and a summary of the surrounding geology are shown in Fig. 1.5. Seismometers were placed near the principal slip zone in these boreholes. These measure velocities in three directions, sampling at 200 Hz, with a corner frequency of 2 Hz. Further drilling in the Whataroa Valley (DFDP-2) was undertaken in 2014, however, the fault was not reached because of technical difficulties [Sutherland et al., 2015].

1.2.4 Fault zones

Fault rocks are derived from the surrounding host rocks. The host rocks of the Alpine Fault are discussed in Section 1.2.1. A fault zone has typically high stress, faulting and percolating hydrothermal fluids [Faulkner et al., 2010]. These processes cause fracturing, brecciation, liquid-saturation, remineralisation and high pore fluid pressure [Chester et al., 1993, Cochran et al., 2009, Mooney and Ginzburg, 1986]. Due to the fact that the temperature and the pressure increase with depth, the geologic conditions and therefore the fault rocks vary with depth. At around 12 km depth, the environment is ductile. At this depth, the process of crystal-plastic alteration causes grain size reduction. This process forms
of the rock ‘mylonite’ [Cooper and Norris, 1994, Grindley, 1963, Rattenbury, 1991, Reed, 1964, Sibson, 1977, Sibson et al., 1979, 1981]. At shallower depths, ‘cataclasite’ is formed by brittle comminution and fracturing during faulting. In the Alpine Fault, this cataclasite is overprinted on mylonite that has been tectonically uplifted [Cooper and Norris, 1994, Norris and Cooper, 1997]. Pseudotachylyte is formed by frictional partial melting during an earthquake and this has been formed on some sections of the fault [Bossière, 1991, Sibson et al., 1979]. In the near surface (few km), a 30 cm clay rich gauge has formed on the fault plane of the Alpine Fault [Cooper and Norris, 1994, Norris and Cooper, 1997]. These fault rocks are shown in Fig. 1.5.

In addition to the Alpine Fault, there are numerous other fault zones around the world that are subject to investigation and drilling projects. The most intensively studied fault is the San Andreas Fault in California, United States of America which has been investigated by the SAFOD drilling project [Zoback et al., 2011]. Other projects include the Taiwan Chelungpu Fault Drilling Project [Hung et al., 2009], Nojima Fault Scientific Drilling Program, Japan [Tanaka et al., 2001] and the Wenchuan Earthquake Fault Scientific Drilling,
China [Li et al., 2014a]. All of these projects aim to understand more about the fault zone, as the fault zone contains a geologic history of past earthquakes and tectonic process. The physical characteristics of fault zones also contain clues into the generation of future earthquakes and greater understanding may assist with earthquake prediction [Zoback et al., 2007].
1.3 Fault zone trapped waves

Fault zone trapped waves (FZTWs) are seismic energy propagating in a fault zone that have undergone multiple reflections of the higher velocity fault zone walls. FZTWs were first reported on an active fault from controlled source surface-to-borehole studies on a fault near Oroville, California [Leary et al., 1985, 1987, Li and Leary, 1990, Li et al., 1990]. These researchers studied the $F_L$ type FZTW which has fault-parallel polarisation and are analogous to Love mode surface waves. The $F_R$ type FZTWs have radial and fault orthogonal polarisation and are analogous to Rayleigh mode surface waves. The identification of $F_R$ along with leaky phases ($F_\phi$) where observed later from micro-earthquakes on the San Andreas Fault [Ellsworth and Malin, 2011, Malin and Lou, 1996]. Fig. 1.6 shows a schematic to demonstrate these different types of FZTWs.

Some researchers use the name Fault-Zone Guided Waves (FZGWs) [e.g. Li et al., 1997b]. In this thesis, the name fault zone trapped waves is used. This is despite the fact that the leaky phase ($F_\phi$) is not strictly trapped by an idealised waveguide. Also, real fault zones are not ‘ideal’ and the fault zone and country-rock properties change with propagation distances. This will cause some of the energy to escape the fault zone and so, in general, these waves are not strictly trapped by the fault zone, hence the word guided would be more accurate.
Before guided waves were discovered in fault zones, they were discovered in coal seams which are also low velocity waveguides [Evison, 1955, Krey, 1963]. In the related literature, the following are the most common names used: Seam waves, channel waves and Everson waves. These waves are now used routinely in the mining industry to investigate the thickness, extent and continuity of coal seams [e.g. Booer, 1982, Buchanan, 1983, Dombrowski et al., 1994]. Guided waves are also observed in subducting tectonic slabs [e.g. Hori et al., 1985, Martin et al., 2003]. At a much narrower scale, guided waves have been measured propagating within micro-fractures in laboratory experiments [Pyrak-Nolte et al., 1992].

1.3.1 Numerical and analytical models of FZTWs

The modelling of FZTWs can be broadly categorised into two types. The first method utilises existing numerical full waveform solvers. The second method uses semi-analytic solvers that are derived specifically for guided waves and approximate wave propagation in idealised waveguides.

The numerical full waveform solvers that are used in FZTW investigations are typically 2-D or 3-D finite difference approaches [e.g. Graves, 1996]. These numerical approaches generate the entire wave field produced by a source, including the FZTW. Full waveform solvers allow investigations into the properties of FZTWs as influenced by different velocity models or source types and locations [Igel et al., 1997, 2002, Jahnke et al., 2006, Li and Vidale, 1996]. The computationally intensive 3-D full waveform solvers have also been used for the estimation of fault zone parameters through forward modelling [Li and Malin, 2008, Li et al., 2014b, 2000, 2003a, Mamada et al., 2002, 2004, Mizuno et al., 2004]. The high computational time (hours) means that full waveform solvers are less practical for estimating fault zone parameters as inversions require a large number of evaluations of the forward model.

There are several different types of semi-analytic guided wave solvers. The simplest is the fully-analytic solution to $F_L$ in a homogeneous fault zone sandwiched by homogeneous higher velocity country-rock [Ben-Zion, 1998]. That paper also derives a computational model for a layered 1-D medium. Other approaches include the generalisation of surface wave methods such as propagator matrix methods [e.g. Malin and Lou, 1996]. These methods are sufficiently computationally fast to be used for inversion [Haberland et al., 2003, 2007, Wu and Hole, 2011, Wu et al., 2010]. These methods are limited to 1-D stratified velocity models which is an approximation of real world geologic complexity [Faulkner et al., 2010].
1.3.2 Numerical investigations of FZTWs

Numerous researchers have investigated the properties of FZTWs using numerical models. Numerical modelling of 3-D acoustic guided waves was first performed by Huang et al. [1995]. In that paper it was shown numerically that FZTWs can propagate in fault zones. It was also shown that FZTWs are dispersive, have high amplitude, and the duration of the FZTW coda increases with propagation distance.

A semi-analytic 2-D model for $F_L$ waves in a three-layer model was used to investigate FZTWs in Ben-Zion [1998]. It was shown that the position of the source within the fault zone significantly affects the amplitude and phase of the FZTW. An increasing velocity contrast between the fault and country-rock increases the trapping efficiency of the waveguide. They also found that the fault zone width is the primary parameter for governing which frequencies are trapped, and that attenuation with propagation distance rapidly attenuates the FZTW. In addition, they showed that if guided waves were to be used to estimate fault zone parameters, there would be significant trade-offs between fault zone width, propagation distance along the fault, velocity and density contrasts, attenuation and source location within the fault zone.

Two dimensional finite difference modelling of $F_L$ FZTWs was done by Li and Vidale [1996]. They showed that in order to observe a FZTW, the fault zone must be approximately continuous between the source and the receiver, and that the earthquake must have occurred within a few fault zone widths of the fault. They also showed that many 2-D structural variations within the fault zone such as bending, moderate changes in fault zone widths and bifurcating faults affect FZTWs but they do not prevent them from being observed. A surface sediment layer above the fault zone that is greater than a few fault zone widths thick will prevent the FZTW from begin recorded at the surface. Further 2-D finite difference modelling of both $F_L$ and $F_R$ FZTWs were simulated and compared to analytical solutions by Igel et al. [1997]. They discovered that rapid changes in the velocity model with depth that are greater than the fault zone to country-rock velocity contrast can stop the propagation of FZTWs. On the other hand, geologically expected fault zone widening at the surface has only a small effect on the waveform, while random perturbations in the velocity model that are smaller than the fault zone to country-rock velocity contrast have no obvious effect on the waveform.

Three dimensional finite difference modelling of FZTWs was followed by Igel et al. [2002] and Jahnke et al. [2006]. They showed that, in general, infinite 2-D line sources can not be used to approximate 3-D point sources for FZTW. When 2-D approximations are used to model real data (3-D), a conversion to point source solutions should be carried out.
They also showed that 3-D velocity variations, that are smaller than the fault zone to country-rock velocity contrast, overprinted on a 2-D structure do not significantly affect the recorded FZTW coda and that the observations of the above 2-D simulations still hold in the 3-D case.

Numerical modelling has also shown that guided waves can occur from sources outside the fault zone if the fault zone is shallow such that the seismic energy can enter the waveguide from below [Fohrmann et al., 2004]. Additionally, Wu et al. [2008] showed that FZTWs can tunnel into a fault zone if there is increasing velocity with depth in the fault zone.

### 1.3.3 Methods to estimate fault zone properties

Possibly the most important use of FZTWs is the potential to infer the properties of the fault zone. Here a summary is given of the different methods used by different researchers.

FZTWs are often used to infer geometrical properties of the fault. For example the connectedness at depth of fault systems has been suggested through observations of FZTWs [e.g. Calderoni et al., 2012, Li et al., 2014b]. The depth of fault zones is often inferred using travel time analysis and total dispersion of the guided waves [e.g. Ben-Zion et al., 2003b]. A discussion of these methods can be found in Li et al. [2012, Chapter 3].

FZTWs are also used to estimate the numerical values of the elastic properties of fault zones. The main four parameters being the effective width, velocity contrast between the fault zone and intact country-rock, depth of the fault zone, and attenuation (Q factor). A variety of methods are used for this, and the main ones are discussed below.

The first method of inferring fault zone parameters from FZTWs involves the assumption that the fault zone is a single, infinitely deep, homogeneous layer surrounded by homogeneous higher velocity country-rock. If this approximation is used and it is assumed that the FZTWs are of $F_L$ type, then the analytic solution of Ben-Zion and Aki [1990] can be used. This analytic model can be used for matching dispersion curves [e.g. Wu et al., 2010] or for simply matching critical frequencies [e.g. Xiaoling and Yi, 2011]. These methods are referred to here as Three-Layer Modelling (3LM).

An inversion methodology called Genetic Algorithm (GA) was first proposed by Michael and Ben-Zion [1998a]. The forward model used is the three-layer model with finite depth developed for 2-D SH waves [Ben-Zion, 1998, Ben-Zion and Aki, 1990]. Before inversion, the data is converted to 2-D then band pass filtered to remove noise. The genetic algorithm is an optimisation procedure inspired by genetic evolution (Goldberg 1989). The methods
employed involve taking a large (∼ 50) set of random parameters that describe the velocity model and earthquake. These parameter sets are then randomly mutated in a way that preferentially tends to parameter sets that fit the data, but still maintains some randomness. The major advantage of this approach is that it provides estimates on a range of waveguides that could have produced the data. Estimates of fault zone parameters using the genetic algorithm can be seen in Lewis and Ben-Zion [2010], Peng et al. [2003].

Forward Modelling (FM) is also widely used to estimate fault zone parameters. This is when the parameters are manually adjusted to find a adequate data fit. The forward model used is normally a 3-D isotropic full-elastic finite difference model [e.g. Graves, 1996].

There have been a large number of different fault zone studies using FZTWs and the above methods. These are summarised in Tables 1.1-1.2.

### 1.3.4 Shallow vs deep fault zones

There is some controversy in the literature over the depth of the low velocity zones. This debate is centred around the San Andreas Fault at Parkfield, California, USA where data quality is high due to a deep borehole seismometer from the SAFOD drilling project [e.g. Zoback et al., 2011]. Part of the difficulty in determining the depth is because of the high level of uncertainty and non-uniqueness associated with FZTWs [Michael and Ben-Zion, 1998b]. Numerous studies have suggested that the low velocity zone persists for greater than 10 km which is approximately the entire seismogenic zone [e.g. Li and Vernon, 2001, Li et al., 2000, 2004, 2002]. Other researches, utilising the same or similar data, have proposed that the low velocity zone terminates at a much shallower depth of 5 km or less [e.g. Ben-Zion et al., 2003b, Ben-Zion and Sammis, 2003, Fohrmann et al., 2004, Lewis et al., 2005, Peng et al., 2003]. More recently, Wu et al. [2008] suggested that a deep fault zone would be narrow towards the bottom, and that this narrow trapping structure would trap higher frequencies than those normally analysed. To overcome this problem, a differential dispersion methodology has been proposed [Wu et al., 2010]. These investigations used two FZTW producing microseismic events which where both vertically aligned with the seismometer to infer that the San Andreas Fault low velocity zone does go to greater than 10 km depth.

### 1.3.5 Fault healing

It has been observed through laboratory studies [e.g. Karner et al., 1997] that the velocity of a fault zone can be reduced by propagating seismic waves. This is the result of earthquake
Table 1.1: A summary of different faults studied in California, United States of America (USA)

<table>
<thead>
<tr>
<th>Fault and methodology</th>
<th>Width (m)</th>
<th>Velocity contrast (%)</th>
<th>depth (km)</th>
<th>Q</th>
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<td><strong>Studies at Parkfield, San Andreas Fault, California, USA</strong></td>
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<tr>
<td>3LM. Results at ∼6 km depth. [Li et al., 1990]</td>
<td>100-150</td>
<td>35-60</td>
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<tr>
<td>Parkfield. 3LM. [Li et al., 1997b]</td>
<td>∼170</td>
<td>∼50</td>
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<td>3-D FM. [Li et al., 2002]</td>
<td>100-200</td>
<td>25-40</td>
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<td>&gt;7</td>
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<tr>
<td>Using guided wave attenuation. [Korneev et al., 2003]</td>
<td>100-200</td>
<td>20-40</td>
<td></td>
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<tr>
<td>3-D FM. Results at surface. [Li et al., 2004]</td>
<td>∼150</td>
<td>30-40</td>
<td></td>
<td>10-50</td>
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<tr>
<td>3LM, using differential dispersion. [Wu et al., 2010]</td>
<td>∼120</td>
<td>&gt;20</td>
<td></td>
<td>&gt;10</td>
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<tr>
<td>GA. [Lewis and Ben-Zion, 2010]</td>
<td>∼150</td>
<td>30-40</td>
<td></td>
<td>∼3</td>
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<tr>
<td>Casa Loma branch. 2-D FM [Li et al., 1997a]</td>
<td>∼120</td>
<td>∼25</td>
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<tr>
<td>Hot Spring branch. Observations. [Li et al., 1997a]</td>
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<td>Waveguide not continuous</td>
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<td>25-30</td>
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<td>30-40</td>
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<td>3LM. [Li et al., 1994]</td>
<td>∼180</td>
<td>25-35</td>
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<td>∼50</td>
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<td>3-D FM. Results at &gt;1 km depth given. [Li et al., 1999]</td>
<td>∼250</td>
<td>∼45</td>
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<td>∼20</td>
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<td>Southern Landers fault zone. 3-D FM. Values at Surface given. [Li et al., 2000]</td>
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<td>Orville. 3LM. Values presented for ∼6 km depth. [Li et al., 1990]</td>
<td>∼18</td>
<td>∼65</td>
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<td>Hector Mine rupture zone. 3-D FM. Surface values stated. [Li et al., 2003a]</td>
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<td>35-45</td>
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<td>10-60</td>
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<td>Calico fault. 3-D FM. [Cochran et al., 2009]</td>
<td>∼1500</td>
<td>40-50</td>
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Table 1.2: A summary of different faults studied outside of the United States of America.

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<th>depth (km)</th>
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<td>30-60</td>
<td>40-50</td>
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<td>~35</td>
<td>~15</td>
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<td>80</td>
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<td>Mozumi-Sukenobu fault. 3-D FM. P-wave velocities quoted [Mamada et al., 2004]</td>
<td>~200</td>
<td>~20</td>
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<td>Nojima Fault. Using TOS2 borehole. 3LM - FM. [Mizuno et al., 2008]</td>
<td>~100</td>
<td>~15</td>
<td>~8</td>
<td>~60</td>
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<td>Nojima Fault. Using HRB borehole. 3LM - FM. [Mizuno et al., 2008]</td>
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<td>~15</td>
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<td>60-90</td>
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<td>Greendale fault - New Zealand. 3-D FM. Values at surface stated here. [Li et al., 2003a]</td>
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<td>35-55</td>
<td>~8</td>
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</tbody>
</table>
induced fractures causing a decrease in the strength of the rock. Over time, these fractures may close and the velocity may increase due to a process known as healing [McLaskey et al., 2012]. FZTWs have been used to investigate fault healing. These investigations have been made through repeated 3-D FZTW studies over a long period to investigate changes [Li, 2003, Li et al., 2006, Li and Vidale, 2001, Li et al., 2003b, Seeber et al., 1999].

1.3.6 Fault zone head waves

In addition to FZTWs, another waveform that is observed propagating in fault zones are fault zone head waves [Ben-Zion and Malin, 1991]. These are waves that critically refract at the faults material interface and propagate along the fault. Fault zone head waves are not considered further in this thesis.

1.3.7 Dispersion, phase velocity and group velocity

Dispersion, phase velocity and group velocity are of vital importance to FZTW propagation. A typical FZTW is a wave packet made up of many different frequencies. FZTWs are dispersive meaning that the speed of propagation along the fault zone is frequency dependant. The consequences of this are that different frequencies are observed at a receiver at different times and the time interval that the entire FZTW wave packet is recorded over typically increases with propagation distance from the source (ignoring attenuation).

Each trapped frequency ($\omega$) will also have a wave number in the fault-parallel direction of propagation of the wave packet, denoted here as $k(\omega)$. The phase speed (velocity) is defined as

$$c = \frac{\omega}{k}$$

and is the fault-parallel speed that a phase travels along the fault zone. The group velocity ($U$) describes how fast a single frequency in the wave packet moves in this fault-parallel direction along the fault zone. This is defined as

$$U = \frac{\partial \omega}{\partial k}.$$
### 1.3.8 Derivation of three-layer model

As discussed in Section 1.3.1, the three-layer model for $F_L$ (3LM) was one of the early methods used for estimation of fault zone parameters. This mathematical model was first derived by Ben-Zion and Aki [1990]. Here, a similar textbook style derivation is presented [Malin, 2013] which provides a useful step in understanding how FZTWs propagate.

![Layer 1 Country–Rock High Velocity Layer 2 Fault–Zone Low Velocity Layer 3 Country–Rock High Velocity](image)

Figure 1.7: Setup of $F_L$ three-layer system

The model of an idealised homogeneous layer of width $w$ surrounded by two homogeneous half spaces is shown in Fig. 1.7. The regions all have Lamé coefficients $\mu_j$, densities $\rho_j$ and S-wave velocities $\beta_j = \sqrt{\mu_j/\rho_j}$ for $j = 1, 2, 3$. Consider a plane wave in medium 2 with displacement in the $y$ direction travelling at angle $\theta$ to the $z$ axis as indicated by the blue arrow in Fig. 1.7. The plane wave has an angular frequency $\omega$. Reflections of the layer interfaces will produce another plane wave in medium 2 travelling at angle $\pi - \theta$ and transmissions will produce left and right propagating waves in mediums 1 and 3 respectively which are travelling at angles $\theta_j$. It can be seen from the SH part of Eq. 1.9 that the $y$ displacement field in the three different layers can be written as

\[
s_1 = b_1 \exp[i(-k_1 \cos(\theta_1)z + k_1 \sin(\theta_1)x - \omega t)],
\]

\[
s_2 = b_2^+ \exp[i(k_2 \cos(\theta)z + k_2 \sin(\theta)x - \omega t)] + b_2^- \exp[i(-k_2 \cos(\theta)z + k_2 \sin(\theta)x - \omega t)],
\]

\[
s_3 = b_3 \exp[i(k_3 \cos(\theta_3)z + k_3 \sin(\theta_3)x - \omega t)],
\]

where the subscript refers to the layer, $i = \sqrt{-1}$, $k_j = \omega/\beta_j$ and $b_1, b_2^+, b_2^-, b_3$ are complex amplitudes ($\in \mathbb{C}$). Snell’s Law states that $k_2 \sin(\theta) = k_j \sin(\theta_j)$, and therefore, the wave
numbers \( k \) can be written as:

\[
k_x = k_2 \sin(\theta) = k_j \sin(\theta_j), \tag{1.13a}
\]

\[
k_z = k_2 \cos(\theta) = \sqrt{k_2^2 - k_2^2 \sin^2(\theta)} = \sqrt{k_j^2 - k_j^2}, \tag{1.13b}
\]

\[
k_{z1} = k_1 \cos(\theta_1) = \sqrt{k_1^2 - k_1^2 \sin^2(\theta_1)} = \sqrt{k_j^2 - k_j^2} = i \sqrt{k_j^2 - k_j^2} = \hat{k}_1, \tag{1.13c}
\]

\[
k_{z3} = k_3 \cos(\theta_3) = \sqrt{k_3^2 - k_3^2 \sin^2(\theta_3)} = \sqrt{k_j^2 - k_j^2} = i \sqrt{k_j^2 - k_j^2} = \hat{k}_3, \tag{1.13d}
\]

where \( k_{z1}, k_z, k_{z3} \) refer to the wave numbers in the z direction of the layers 1, 2 and 3 respectively. Using Eqs. (1.13) in Eqs. (1.12) and letting \( \gamma = \exp[i(k_x - \omega t)] \) gives

\[
s_1 = b_1 \exp[\hat{k}_1 z] \gamma, \tag{1.14a}
\]

\[
s_2 = b_2^+ \exp[i k_z z] \gamma + b_2^- \exp[-i k_z z] \gamma, \tag{1.14b}
\]

\[
s_3 = b_3 \exp[-\hat{k}_3 z] \gamma. \tag{1.14c}
\]

As shown in Eqs. (1.13), if \( k_x > k_j, k_3 \), then \( \hat{k}_1, \hat{k}_3 \) will be real numbers and therefore \( s_1, s_3 \) in Eq. (1.14) will be exponentially decaying away from the fault zone. These are known as Evanescent waves. Evanescent waves do not have energy propagating away from the fault zone and therefore the waves are trapped.

It can be seen from Section 1.1 that the stress in the z direction of Eq. (1.14) can be calculated as \( \mu \frac{\partial u_2}{\partial z} \). Equation (1.14) must have continuity of displacement and stress at the interfaces between the layers (\( \pm w/2 \)). This gives us the following set of conditions:

Displacement at \(-w/2 \Rightarrow b_1 \exp[-\hat{k}_1 w/2] = b_2^+ \exp[-i k_z w/2] + b_2^- \exp[i k_z w/2] \) \tag{1.15a}

Displacement at \(w/2 \Rightarrow b_3 \exp[-\hat{k}_3 w/2] = b_2^+ \exp[i k_z w/2] + b_2^- \exp[-i k_z w/2] \) \tag{1.15b}

Stress at \(-w/2 \Rightarrow \mu \hat{k}_1 b_1 \exp[-\hat{k}_1 w/2] = \mu_2 k_z i (b_2^+ \exp[-i k_z w/2] - b_2^- \exp[i k_z w/2]) \) \tag{1.15c}

Stress at \(w/2 \Rightarrow -\mu_3 \hat{k}_3 b_3 \exp[-\hat{k}_3 w/2] = \mu_2 k_z i (a_2^+ \exp[i k_z w/2] - b_2^- \exp[-i k_z w/2]) \) \tag{1.15d}

Eqs. (1.15) can be written in matrix form as:

\[
\begin{bmatrix}
\exp[-i k_z w/2] & \exp[i k_z w/2] & -1 & 0 \\
\exp[i k_z w/2] & \exp[-i k_z w/2] & 0 & -1 \\
\mu_2 k_z i \exp[-i k_z w/2] & -\mu_2 k_z i \exp[i k_z w/2] & -\mu_1 \hat{k}_1 & 0 \\
\mu_2 k_z i \exp[i k_z w/2] & -\mu_2 k_z i \exp[-i k_z w/2] & 0 & \mu_3 \hat{k}_3
\end{bmatrix}
= 
\begin{bmatrix}
b_2^+ \\
b_2^- \\
b_1 \\
\hat{b}_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}, \tag{1.16}
\]
were \( \hat{b}_1 = b_1 \exp[-\hat{k}_1w/2] \) and \( \hat{b}_3 = b_3 \exp[-\hat{k}_3w/2] \).

Eq. (1.16) only has non-trivial solutions when the determinant of the matrix is zero. This requires that

\[
\exp \left[ 4i \left( k_z \frac{w}{2} \right) \right] = \frac{(\mu_1\hat{k}_1 - i\mu_2k_z)(\mu_3\hat{k}_3 - i\mu_2k_z)}{\mu_1\hat{k}_1 + i\mu_2k_z} \frac{1}{(\mu_3\hat{k}_3 + i\mu_2k_z)(\mu_3\hat{k}_3 + i\mu_2k_z)}. \tag{1.17}
\]

This can be expanded by substituting back in Eqs. (1.13) and utilising \( k_j = \omega/\beta_j \) along with \( c = \omega/k_x \), where \( c \) is the phase speed

\[
\exp \left[ 4i \frac{w}{2} \omega \sqrt{\beta_2^{-2} - c^{-2}} \right] = \frac{(\mu_1\sqrt{c^{-2} - \beta_1^{-2}} - i\mu_2\sqrt{\beta_2^{-2} - c^{-2}})}{(\mu_1\sqrt{c^{-2} - \beta_1^{-2}} + i\mu_2\sqrt{\beta_2^{-2} - c^{-2}})} \frac{(\mu_3\sqrt{c^{-2} - \beta_3^{-2}} - i\mu_2\sqrt{\beta_2^{-2} - c^{-2}})}{(\mu_3\sqrt{c^{-2} - \beta_3^{-2}} + i\mu_2\sqrt{\beta_2^{-2} - c^{-2}})}. \tag{1.18}
\]

Equation (1.18) is equivalent to equation 2.7 in Ben-Zion and Aki [1990] which they derived using the 2-D wave equation and Fourier transforms. Equation (1.18) describes the dispersion relation of the three-layer model. It can be used to compute which frequencies are trapped for a particular phase speed \( (c) \). The group velocity can also be found as a function of phase speed by differentiation

\[
U = \frac{\partial \omega}{\partial k} = \left( \frac{\partial (\omega/c)}{\partial \omega} \right)^{-1} = \left( \frac{c - \omega \frac{\partial c}{\partial \omega}}{c^2} \right)^{-1} = \frac{c^2}{c - \omega \left( \frac{\partial \omega}{\partial c} \right)^{-1}}. \tag{1.19}
\]
1.4 Numerical methods

In Section 1.1, different versions of the wave equation that are used in this thesis are derived. Except for the simplest of cases, explicit analytic solutions cannot be found [Salsa, 2008]. In most cases, this requires the use of numerical methods to get a solution. In this section numerical methods are discussed with a focus on the semi-discrete finite element method (FEM). At the end of this section, time integration methods are discussed.

1.4.1 Spatial integration methods

The finite difference method is one of the oldest spatial integration methods [Peiro and Sherwin, 2005]. It is often the method of choice for the wave equation in geophysical applications [e.g. Graves, 1996]. This is especially true in the area of FZTWs where the finite difference method has been almost exclusively used [e.g. Igel et al., 2002, Jahnke et al., 2006, Li, 2003, Mamada et al., 2004, Mizuno et al., 2004]. The finite difference method approximates a classical solution to the wave equation and relies on a local Taylor approximation to compute derivatives [Salsa, 2008]. The finite difference method is, however, difficult to adapt to non-rectangular computational grids and often restricted to constant or linear velocity models [Peiro and Sherwin, 2005]. In recent years, other methods such as the finite volume [Dormy and Tarantola, 1995] and FEM [Komatitsch and Tromp, 1999] have been gaining popularity. Both of these methods allow irregular grids. The finite volume method has the advantage that it preserves fluxes between elements and thus obeys many conservation laws [Morton and Sonar, 2007]. The finite element method has the advantage that it can have higher order basis functions allowing high polynomial order approximations and high accuracy, and it can also have arbitrary velocity models. More recently, the discontinuous Galerkin finite element method has been used [Lähivaara and Huttunen, 2011]. This method incorporates the best of both FEM and finite volume methods, however, the propagation of flux between different elements is non-trivial. In this thesis, FEM is used because of its high accuracy, ability to have irregular meshes and ability to have velocity models that change arbitrarily within one element.

1.4.2 The finite element method

Resources on such a popular method are plentiful [e.g. Pozrikidis, 2005]. The FEM is used here for problems with one and two spatial dimensions denoted by \( \text{dim}_x \) and 1-3 displacement directions denoted by \( \text{dim}_s \). Let \( \mathbf{s} \in \mathbb{R}^{\text{dim}_s} \) be a displacement vector with
entries \( s_i \) and let \( x \in \Omega \) be the spatial parameters vector with entries \( x_i \) where \( \Omega \) is the (open and bounded) computational domain. The boundary of \( \Omega \) is denoted by \( \partial \Omega \) (Lipschitz boundary) which has outwards normal defined by \( \mathbf{n} \) with entries \( n_i \). In general, the elastic wave equations in Section 1.1 can be written in the form

\[
A \frac{\partial^2 s}{\partial t^2} + \sum_{j=1}^{\dim_x} \sum_{i=1}^{\dim_x} \frac{\partial}{\partial x_i} \left( B_{ij} \frac{\partial}{\partial x_j} s \right) + \sum_{j=1}^{\dim_x} C_j \frac{\partial}{\partial x_j} s + \sum_{j=1}^{\dim_x} \frac{\partial}{\partial x_j} (D_j s) + E s + f = 0. \tag{1.20}
\]

Where, the matrices \( A, B_{ij}, C_j, D_j, E \) are \( \dim_s \times \dim_s \) matrices that, in general, depend on \( x \).

Let \( \phi \in \mathbb{R}^{\dim_s} \) be an infinitely differentiable (test) function [Salsa, 2008]. Multiplying Eq. (1.20) by \( \phi \) and integrating over \( \Omega \) yields

\[
\int_\Omega \left[ A \frac{\partial^2 s}{\partial t^2} \ominus \phi + \sum_{j=1}^{\dim_x} \sum_{i=1}^{\dim_x} \frac{\partial}{\partial x_i} \left( B_{ij} \frac{\partial}{\partial x_j} s \right) \ominus \phi + \sum_{j=1}^{\dim_x} C_j \frac{\partial}{\partial x_j} s \ominus \phi \\
+ \sum_{j=1}^{\dim_x} \frac{\partial}{\partial x_j} (D_j s) \ominus \phi + E s \ominus \phi + f \ominus \phi \right] = 0. \tag{1.21}
\]

Where \( \ominus \) implies Hadamard multiplication (term by term vector multiplication). Applying integration by parts [Salsa, 2008] to the 2nd and 4th terms results in

\[
\int_\Omega \left[ A \frac{\partial^2 s}{\partial t^2} \ominus \phi - \sum_{j=1}^{\dim_x} \sum_{i=1}^{\dim_x} \left( B_{ij} \frac{\partial}{\partial x_j} s \right) \ominus \frac{\partial \phi}{\partial x_i} + \sum_{j=1}^{\dim_x} C_j \frac{\partial s}{\partial x_j} \ominus \phi \\
- \sum_{j=1}^{\dim_x} (D_j s) \ominus \frac{\partial \phi}{\partial x_j} + E s \ominus \phi + f \ominus \phi \right] \\
+ \int_{\partial \Omega} \left[ \sum_{j=1}^{\dim_x} \sum_{i=1}^{\dim_x} n_i \left( B_{ij} \frac{\partial}{\partial x_j} s \right) \ominus \phi + \sum_{j=1}^{\dim_x} n_j (D_j s) \ominus \phi \right] = 0. \tag{1.22}
\]

In the FEM, Eq. (1.22) is used instead of Eq. (1.20). In the above derivation is has been shown that if Eq. (1.22) is solved for all infinitely differentiable \( \phi \), the solution also solves Eq. (1.20) [in the weak sense, Salsa, 2008]. In Eq. (1.22) the requirement that \( \phi \) is infinitely differentiable can in fact be relaxed, \( \phi \) along with the solution \( s \) only needs to be piecewise once differentiable \( \in H^1 \), Salsa, 2008]. The proof that Eq. (1.22) is part of the week formulation along with boundary conditions can be seen in Salsa [2008].

To proceed, the approximation to the solution \( s \) is written as a sum of \( N \) basis func-
\[ s \approx \sum_{k}^{N} s_k v_k. \] (1.23)

The Galerkin method is used, that means that the choice \( \phi = I v_k \) is made, where \( k \) is arbitrary and \( I \) is a vector of ones with dimension \( \dim s \).

A mesh is defined for the FEM by breaking the computational domain into \( N_M \) smaller domains \( \Omega_i \) such that

\[ \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \ldots \ldots \Omega_{N_M} \]
and \( \emptyset = \Omega_i \cap \Omega_j \) for all \( i \neq j \). (1.24)

These \( \Omega_i \) are referred to as elements. In one dimension, \( \Omega_i \) are lines and in this study, two dimensional elements are triangles. The entire domain \( \Omega \) is restricted to domains with sufficiently straight edges such that Eqs. (1.24) can be satisfied for our chosen sizes of \( \Omega_i \).

On this mesh, a set of \( N \) nodes are placed at locations \( x_k \). The specific location and quantity of these nodes is dependent on the choice of basis functions, \( v_k(x) \). In this thesis, the basis functions are piecewise polynomials with polynomial order \( P \). These basis functions are defined over \( \Omega \) but each basis function is only non-zero on a small number of \( \Omega_j \). These polynomials are chosen, in this thesis, to be Lagrangian interpolation polynomials [Pozrikidis, 2005] which means that \( s(x_k) = s_k \) and that these \( N \) polynomials have the property that \( v_k(x_j) = 0 \) for \( j \neq k \) and \( v_k(x_k) = 1 \). Therefore, Eq. (1.23) represents a polynomial spline interpolation of \( s \) between node locations \( x_k \). The node locations are chosen to be at the Labotto points, which means that the maximum absolute value of \( v_k \) for all \( x \in \Omega \) is \( 1 = v_k(x_k) \). This approach has been shown to be \( P^{th} \) order convergent in Pozrikidis [2005]. Examples of node locations and different elements for one and two dimensions are shown in Fig. 1.8.

In the following, only a simple form of the elastic wave equation is considered. The extension to the general elastic wave equation (Eq. 1.20) is straightforward. This simple equation is the isotropic 2-D P-SV wave equation in a fluid (\( \mu = 0 \)),

\[ \rho \frac{\partial^2 s_x}{\partial t^2} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right), \] \( (1.25a) \)

\[ \rho \frac{\partial^2 s_z}{\partial t^2} = \frac{\partial}{\partial z} \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right), \] \( (1.25b) \)
Figure 1.8: Example of finite element mesh and elements. (a) shows a 2-D example. (b) shows a 1-D example. The single basis function shown is a 4th order polynomial with its ‘1’ at an internal node. If the ‘1’ was at a shared node than the basis function would be the combination of the interpolating polynomials from the neighbouring mesh regions that are ‘1’ at that shared node.

where $s = [s_x, s_z]^T$ and $x = [x, z]^T$. Substituting Eq. (1.25) in Eq. (1.22) gives

$$
\int_{\Omega} \left[ \rho \frac{\partial^2 s_x}{\partial t^2} \phi_x + \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right) \frac{\partial \phi_x}{\partial x} - f_x \phi_x \right] = \int_{\partial \Omega} n_1 \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right) \phi_x, \tag{1.26a}
$$

$$
\int_{\Omega} \left[ \rho \frac{\partial^2 s_z}{\partial t^2} \phi_z + \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right) \frac{\partial \phi_z}{\partial z} - f_z \phi_z \right] = \int_{\partial \Omega} n_2 \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right) \phi_z. \tag{1.26b}
$$

At this point, some of the terms in the boundary integrals of Eqs. (1.26) are known. The boundary is broken up into two sub-boundaries $\partial \Omega_1$ and $\partial \Omega_2$, where $\partial \Omega_1 \cup \partial \Omega_2 = \partial \Omega$ and $\partial \Omega_1 \cap \partial \Omega_2 = \emptyset$. The sub-boundary $\partial \Omega_1$, is chosen to have outward normal $n = [1, 0]^T$ and first order paraxial absorbing boundary conditions [Engquist and Majda, 1977]. The sub-boundary $\partial \Omega_2$ is chosen to have a zero force boundary condition. These boundary conditions are

$$
\frac{\partial s_x}{\partial x} = -\sqrt{\frac{\rho}{\lambda}} \frac{\partial s_x}{\partial t} \quad \text{on} \ \partial \Omega_1, \tag{1.27a}
$$

$$
\lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} = 0 \quad \text{on} \ \partial \Omega_2. \tag{1.27b}
$$
Substituting Eqs. (1.27) into Eqs. (1.26) leaves us with

\[
\int_\Omega \left[ \rho \frac{\partial^2 s_x}{\partial t^2} \phi_x + \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right) \frac{\partial \phi_x}{\partial x} - f_x \phi_x \right] = \int_{\partial \Omega_1} \left( -\sqrt{\lambda} \rho \frac{\partial s_x}{\partial t} + \lambda \frac{\partial s_z}{\partial z} \right) \phi_x, \quad (1.28a)
\]

and

\[
\int_\Omega \left[ \rho \frac{\partial^2 s_z}{\partial t^2} \phi_z + \left( \lambda \frac{\partial s_x}{\partial x} + \lambda \frac{\partial s_z}{\partial z} \right) \frac{\partial \phi_z}{\partial z} - f_z \phi_z \right] = 0. \quad (1.28b)
\]

If the basis function approximation, Eq. (1.23) is substituted into Eq. (1.28) and \( \phi_x = \phi_z = v_j \) for arbitrary \( j \) and \( s_k = [s^x_k, s^z_k]^T \), than

\[
\int_\Omega \sum_{k=1}^N \left[ \rho \frac{\partial^2 s^x_k}{\partial t^2} v_k v_j + \left( s^x_k \lambda \frac{\partial v_k}{\partial x} + s^z_k \lambda \frac{\partial v_k}{\partial z} \right) \frac{\partial v_j}{\partial x} - f_x v_j \right] \quad (1.29a)
\]

\[
+ \int_{\partial \Omega_1} \left( v_k \sqrt{\rho \lambda} \frac{\partial s^z_k}{\partial t} - s^z_k \lambda \frac{\partial v_k}{\partial z} \right) v_j = 0, \quad (1.29b)
\]

and

\[
\int_\Omega \sum_{k=1}^N \left[ \rho \frac{\partial^2 s^z_k}{\partial t^2} v_k v_j + \left( s^x_k \lambda \frac{\partial v_k}{\partial x} + s^z_k \lambda \frac{\partial v_k}{\partial z} \right) \frac{\partial v_j}{\partial z} - f_z v_j \right] = 0. \quad (1.29c)
\]

As the Galerkin approach is being used, there are \( N \) different choices for \( \phi \). The evaluations of Eqs. (1.29) provide a (underdetermined) set of vector equations for one choice of \( \phi \). A set of (determined) vector equations is obtained using all choices of \( \phi \). To write this in matrix form, the following matrices and vectors are defined: \( M, S^{xx}, S^{zz}, S^{xz}, S^{zx}, \Gamma, \Xi, f^x, f^z \).

These have entries

\[
M_{jk} = \int_\Omega \rho v_k v_j, \quad S^{xx}_{jk} = \int_\Omega \lambda \frac{\partial v_k}{\partial x} \frac{\partial v_j}{\partial x}, \quad S^{zz}_{jk} = \int_\Omega \lambda \frac{\partial v_k}{\partial z} \frac{\partial v_j}{\partial z}, \quad S^{xz}_{jk} = \int_\Omega \lambda \frac{\partial v_k}{\partial x} \frac{\partial v_j}{\partial z}, \quad S^{zx}_{jk} = \int_\Omega \lambda \frac{\partial v_k}{\partial z} \frac{\partial v_j}{\partial x}, \quad \Gamma_{jk} = \int_{\partial \Omega_1} \lambda \frac{\partial v_k}{\partial z}, \quad \Xi_{jk} = \int_{\partial \Omega_1} \sqrt{\rho \lambda} \frac{\partial v_k}{\partial x}.
\]

Using these matrices and vectors, Eq. (1.29) can be written in matrix form

\[
M \frac{\partial^2 s_x}{\partial t^2} + \Xi \frac{\partial s_x}{\partial t} + S^{xx} s_x + S^{xz} s_z - \Gamma s_z = f^x, \quad (1.31a)
\]

and

\[
M \frac{\partial^2 s_z}{\partial t^2} + S^{zx} s_x + S^{zz} s_z = f^z, \quad (1.31b)
\]

where each row of the matrix equation represents a different value of \( j \) in Eqs. (1.29). The
following block matrices and vectors can then be defined:

\[
\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix}, \quad \hat{\mathbf{s}} = \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_z \end{bmatrix}, \\
\hat{\mathbf{C}} = \begin{bmatrix} \Xi & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{f}} = \begin{bmatrix} f^x \\ f^z \end{bmatrix}, \\
\hat{\mathbf{L}} = \begin{bmatrix} S^{xx} & S^{xz} - \Gamma \\ S^{zx} & S^{zz} \end{bmatrix}.
\] (1.32)

These definitions allow us to write Eqs. (1.31) as

\[
\hat{\mathbf{M}} \frac{\partial^2 \hat{\mathbf{s}}}{\partial t^2} + \hat{\mathbf{C}} \frac{\partial \hat{\mathbf{s}}}{\partial t} + \hat{\mathbf{L}} \hat{\mathbf{s}} = \hat{\mathbf{f}}.
\] (1.33)

The Eq. (1.33) is a linear 2nd order system of ordinary differential equations. It can be solved using numerical methods relevant to ordinary differential equations.

1.4.3 Time integration

Equation (1.33) is a 2nd order differential equation in time. Numerical integration methodology for ordinary differential equations is a huge field of research [e.g. Butcher, 2003]. Two different types of numerical integration are used here: a spectral method based on Fourier transforms [Hesthaven et al., 2007] and the Newmark’s method [Newmark, 1959].

Spectral time integration involves taking the Fourier transform of Eq. (1.33) leaving the matrix equation

\[
\left( -\omega^2 \hat{\mathbf{M}} + i \omega \hat{\mathbf{C}} + \hat{\mathbf{L}} \right) \tilde{\mathbf{s}} = \tilde{\mathbf{f}},
\] (1.34)

where the Fourier transform is defined as

\[
\tilde{g}(\omega) = \int_{-\infty}^{\infty} \hat{g}(t) \exp(-i\omega t) dt.
\] (1.35)

The numerical integration then involves solving the linear system of Eqs. (1.34) for a finite number of different \( \omega \)'s and taking the inverse Fourier transform. Spectral methods have the advantage that a comparatively low number of computations are needed to perform integration [Hesthaven et al., 2007]. Their limitations are the difficulty in incorporating initial conditions, the enforced periodic boundaries, and the fact that all computations must be done before a single time snap is available.
To perform direct time integration, Newmark’s method with a constant acceleration approximation is used [Newmark, 1959]. Let $\Delta t$ be the time step and $\hat{s}^j = \hat{s}(t = j\Delta t)$. To iterate forward in time this method solves the following system of equations:

\begin{align}
\hat{M} \frac{\partial^2 \hat{s}^j}{\partial t^2} + \hat{C} \frac{\partial \hat{s}^j}{\partial t} + \hat{L}s^j &= \hat{f}^j, \\
\hat{M} \frac{\partial^2 \hat{s}^{j+1}}{\partial t^2} + \hat{C} \frac{\partial \hat{s}^{j+1}}{\partial t} + \hat{L}s^{j+1} &= \hat{f}^{j+1}, \\
\hat{a}^{j+1/2}_{\text{mean}} &= \frac{1}{2} \left( \frac{\partial^2 \hat{s}^j}{\partial t^2} + \frac{\partial^2 \hat{s}^{j+1}}{\partial t^2} \right), \\
\frac{\partial \hat{s}^{j+1}}{\partial t} &= \frac{\partial \hat{s}^j}{\partial t} + \Delta t \hat{a}^{j+1/2}_{\text{mean}}, \\
\hat{s}^{j+1} &= \hat{s}^j + \Delta t \frac{\partial \hat{s}^j}{\partial t} + \frac{\Delta t^2}{2} \hat{a}^{j+1/2}_{\text{mean}}.
\end{align}

Note that the bottom two equations are just elementary kinematic equations. The main limitation of this method is that the acceleration in these kinematic equations is approximated as piecewise constant ($= \hat{a}^{j+1/2}_{\text{mean}}$) over the time step. This method is unconditionally stable, second order ($O(\Delta t^2)$) convergent [Newmark, 1959] and only requires the inversion of a $N \times \dim_s$ by $N \times \dim_s$ matrix as opposed to a matrix twice as big in standard Runge-Kutta methods [Butcher, 2003].

Different time steps of 3-D FZTWs computed using some of the methods outlined above are shown in Appendix A.
1.5 Inverse problems

Inverse problems involve the estimation of parameters of interest $\mathbf{x}$ based on measurements of other related parameters. They are also ill-posed in the sense of Hadamard (1902). This means that small changes in the measurements lead to large changes in the values of $\mathbf{x}$. In geophysics, the term non-unique is used instead of ill-posed [Tarantola, 2004] which is described as having a large range of values in $\mathbf{x}$ that can fit the measurements $\mathbf{y}$ within the limits of any error in the $\mathbf{y}$ values. Inverse problems occur in a large number of mathematical and physical applications such as geophysics, astronomy, medical imaging and image reconstruction.

In this thesis, the estimation of fault zone parameters based on measurements of fault-zone trapped waves (FZTWs) is treated as an inverse problem. Therefore, much of the mathematical framework used here relies on inverse problem theory. The relevant inverse problem theory and methodologies are outlined in this section.

1.5.1 The forward model

Measurement models that have additive measurement errors (noise) are considered along with an additive error term that is due to modelling errors and uncertainties [Kaipio and Somersalo, 2005]:

\[
\mathbf{y} = \hat{\mathbf{a}}(\mathbf{x}) + \hat{\mathbf{e}} + \epsilon
\]  

(1.37)

**Measurements** are the data that will be used in inversion.

**Model** is a set of equations that relates the parameters to the measurements. It is typically a set of equations or a mathematical representation that approximates the physical reality. These models are often numerical approximations to partial differential equations.

**Parameters** contain the quantities that are to be estimated, (e.g. fault zone velocity). They also often contain quantities that are not of interest but they still effect the
measurements (nuisance parameters).

**Measurement error** is any noise associated with the measurement of $y$ (and possibly any post processing).

**Modelling error** is any error associated with the model, $a(x)$. Representing the modelling error in this way is a non-standard approach [Kaipio and Somersalo, 2005, 2007] and is an important part of this thesis.

### 1.5.2 Least squares

A standard approach to estimating parameters is the least squares approach. Letting $v(x)$ be a vector of length $M$ then the least squares problem is,

$$\text{find } x \text{ such that } x \text{ minimizes } V(x) = \sum_{i=1}^{M} v_i^2(x) = ||v(x)||_2^2,$$

where $||·||_2$ is the standard Euclidean norm [Anton and Busby, 2003] and $v(x) = y - a(x)$ is the residual (discrepancy between the measurements and the model predictions).

If $v(x)$ can be written as an affine function of $x$, the solution can be found using the Moore-Penrose-Woodbury Pseudoinverse [Penrose, 1955], which is the case if the model is linear. In general, however, $v(x)$ is not an affine function. If $V(x)$ is suitably well behaved, iterative methods such as the Gauss-Newton or Levenberg-Marquardt algorithms can be used [Nocedal and Wright, 1999]. If the problem is well-posed, the least squares method can yield meaningful estimates for the unknowns. However, for ill-posed problems, the least squares solutions are meaningless [Tarantola, 2004].

### 1.5.3 Tikhonov regularisation

To cope with ill-posedness, many researches use generalised Tikhonov regularisation [Phillips, 1962, Tikhonov and Arsenin., 1977]. This method involves adding an additional regularisation term so that the minimisation functional now becomes

$$V(x) = ||y - a(x)||_2^2 + \Psi(x),$$

where $\Psi(x)$ is a convex functional. Adding in the additional regularisation term means that the estimated $x$ now also has the property that $\Psi(x)$ is small. A common choice is $\Psi(x) = \kappa||L_x x||_2^2$ where $\kappa$ is the regularisation parameter which controls the weight of the
regularisation term on the minimisation problem and $L_x$ is a regularisation operator. For example, $L_x$ is often chosen to be a discrete approximation for a differential operator, the use of which would enforce smoothness of the estimates $x$. The value of $\kappa$ used in the inversion is a non-trivial problem. The most common methods to choose the regularisation parameter are the L-curve criterion [Calvetti et al., 2000, Hansen, 1992] and the Morozov discrepancy principle [Scherzer, 1993].

1.5.4 Parameter reduction

Parameter reduction is a process where the parameters that are being estimated are simplified or reduced in number by defining a new set of parameters $x_I$, i.e.

$$x_I = f(x) \quad (1.40)$$

It is important to note that parameter reduction is almost always used. For example, when partial differential equations are involved, the elements of $x$ may belong to a function space (continuous parameters). Therefore, to make the inverse problem tractable, this parameter reduction will involve a discretisation (e.g. finite element discretisation).

Parameter reduction is normally referred to as reducing the number of parameters even further than any discretisation. Reducing the number of parameters in the inversion can make the problem less ill-posed, because fewer parameters means there are fewer combinations of parameters that can produce similar measurements. A very common technique is writing $x$ as a linear combination of the singular vectors of a model and using only some of the dominant vectors. This can also be generalised to inversion using truncated singular value decomposition [Hansen, 1987]. A very popular parameter reduction in FZTWs is to reduce the velocity model to consist of three simple homogeneous isotropic velocity models as discussed in Section 1.3.

As well as stabilising the problem, parameter reduction means there are fewer parameters to estimate and often the reduced parameter model can be computed more efficiently. It is very important to note that nearly all parameter reductions cause an increase in modelling error which may have a dramatic effect on how good the estimates are. This modelling error can also mean that the model is unable to fit the data with respect to the measurement error. This is a concept sometimes known as fidelity [Tarantola, 2004]. In this thesis the concept of fidelity is captured by our approach to modelling error which is discussed Section 1.5.7.
1.5.5 The Bayesian (probabilistic) approach

In the Bayesian approach to inverse problems, all parameters are treated as random variables, which means that a particular realisation of a parameter is assigned a probability (density) that it may occur. The Bayesian approach can therefore be used to quantify uncertainty in the estimates. A prior probability model on $\mathbf{x}$ is used in the Bayesian framework. The prior model attempts to capture all the information that is known about $\mathbf{x}$. Using priors often provides sufficient regularisation to stabilise the inverse problem.

Let $\pi_x(\mathbf{x})$ be the prior probability model and $\pi_{e+\epsilon}(\mathbf{e} + \mathbf{\epsilon})$ represent the probability density of the combined modelling and measurement errors. The posterior probability density is the probability of $\mathbf{x}$ given the measurements $\mathbf{y}$ and it is denoted $\pi(\mathbf{x}|\mathbf{y})$. Using Bayes theorem [Bayes and Price, 1763] it can be shown [Kaipio and Somersalo, 2005] that the posterior density can be represented as

$$
\pi(\mathbf{x}|\mathbf{y}) \propto \pi_x(\mathbf{x}) \pi_{e+\epsilon}(\mathbf{y} - a(\mathbf{x})|\mathbf{x}),
$$

(1.41)

where $\pi_{e+\epsilon}(\mathbf{y} - a(\mathbf{x})|\mathbf{x})$ is called the likelihood density. It describes how well a particular realisation of $\mathbf{x}$ fits the data within the error. The '$|\mathbf{x}$' is included as the measurement and modelling error may depend on $\mathbf{x}$.

1.5.6 Normal distribution

A very commonly used distribution is the normal distribution. There are two main reasons for this: Firstly, the central limit theorem says that when there are many small unrelated errors in a measurement which don’t necessarily come from a normal distribution, the sum of these errors can be approximated with a normal distribution [Rice, 1988]. Secondly, the normal distribution is computationally straightforward and many operations involving normal distributions reduce the problem to linear algebra [Kaipio and Somersalo, 2005].

A normal distribution is completely characterised by its mean and covariance matrix. Let $\mathbf{z} \in \mathbb{R}^n$ be a normally distributed column vector. Then its density can be written as

$$
\pi(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n |\Gamma_z|}} \exp \left( -\frac{1}{2} (\mathbf{z} - \mathbf{z}_0)^\top \Gamma_z^{-1} (\mathbf{z} - \mathbf{z}_0) \right)
$$

(1.42)

where $\mathbf{z}_0$ is the mean, $\Gamma_z$ is the covariance matrix, and $|\Gamma_z| > 0$ is the determinant of the covariance matrix. Also note that the symbol $\pi$ has been used in two different ways in the above equation. On the left hand side, $\pi(\mathbf{z})$ is the probability density function and on
the right hand side \( \pi \approx 3.1415 \). The meaning of the symbol \( \pi \) should be clear from the context.

### Relationship to Tikhonov regularisation

Returning to Eq. (1.41), if it is assumed that \( \pi_x \) is a normal distribution with mean \( x_0 \) and positive definite covariance \( \Gamma_x \) and it is also assumed that \( \pi_e \) is a normal distribution with mean 0 and covariance \( \sigma_e^2 I \), Eq. (1.41) can be written as

\[
\pi(x|y) \propto \exp \left( -\frac{1}{2\sigma_e^2} (y - a(x))^\top (y - a(x)) - \frac{1}{2} (x - x_0)^\top \Gamma_x^{-1} (x - x_0) \right)
\]

\[
= \exp \left( -\frac{1}{2\sigma_e^2} ||y - a(x)||_2^2 - \frac{1}{2} ||L_x(x - x_0)||_2^2 \right)
\]

(1.43a)

(1.43b)

where \( L_x^\top L_x = \Gamma_x^{-1} \). The Eq. (1.43b) reaches a most probable value when the argument in the exponential is maximised. This is equivalent to minimising

\[
||y - a(x)||_2^2 + \sigma_e^2 ||L_x(x - x_0)||_2^2.
\]

(1.44)

This is identical to Tikhonov regularisation with a specific choice for \( \Psi(x) \).

### A choice of correlated normal distribution

Let \( q \) be a vector that depends on the spatial or temporal co-ordinates \( r \). In this thesis, a particular model for the covariance of spatial variables \( q \) is used. This covariance matrix \( \Gamma \) has entries defined as

\[
\Gamma_{ij} = \sigma^2 \exp \left( -\frac{\ln(100)}{\gamma^2} ||r_i - r_j||_2^2 \right)
\]

(1.45)

where \( \sigma \) is the standard deviation of all variables \( q_i \) and \( \gamma \) is used as a measure for the correlation length, which is defined here as the distance for the correlation between two points to drop from 1 to 0.01. This normal distribution is a common choice for enforcing spatial/temporal correlations (and smoothness) of a random vector [Rasmussen and Williams, 2006]. This is used in many cases in this thesis, particularly for prior distributions to enforce smoothness.
1.5.7 Bayesian approximation error

The Bayesian Approximation Error (BAE) method is a method used to deal with modelling errors [Kaipio and Kolehmainen, 2013, Kaipio and Somersalo, 2007]. It uses samples of the difference between a highly accurate model and the model used for inversion to create an estimated probability density for the modelling error ($\epsilon$). This will now be explained in more detail.

Let $\hat{a}(x)$ be a highly accurate forward model such that it can be assumed that any related modelling errors have smaller variances than any measurement errors. Here, $x$ includes all parameters that effect $\hat{a}$. Let $a(x_I)$ be a less accurate and less computationally intensive forward model that depends on the parameters $x_I$ where $x_I$ is a reduced parameter version of $x$.

The forward model can be written as

\[
\begin{align*}
    y &= \hat{a}(x) + e, \\
    &= \hat{a}(x) + (a(x_I) - a(x_I)) + e, \\
    &= a(x_I) + (\hat{a}(x) - a(x_I)) + e, \\
    &= a(x_I) + \epsilon + e
\end{align*}
\]  

(1.46a)

(1.46b)

(1.46c)

(1.46d)

where

\[
\epsilon = \hat{a}(x) - a(x_I)
\]

(1.47)

Computing $\pi(\epsilon|\mathbf{x})$

Given Eq. (1.47) and the fact that $x_I = f(x)$ is the parameter reduction used, it can be shown that

\[
\pi(\epsilon, x_I|\mathbf{x}) = \delta(\epsilon - \hat{a}(x) + a(x_I))\delta(x_I - f(x))
\]

(1.48)

where $\delta$ is the Dirac-Delta distribution [Khuri, 2004]. Using Bayes’ theorem gives

\[
\begin{align*}
    \pi(\epsilon, x_I, \mathbf{x}) &= \pi(\epsilon, x_I|\mathbf{x})\pi(\mathbf{x}), \\
    &= \pi(\epsilon, \mathbf{x}|x_I)\pi(x_I).
\end{align*}
\]  

(1.49a)

(1.49b)
Combining Eqs. (1.48-1.49) results in

$$\pi(\epsilon, x|\mathbf{x}_I) = \delta(\epsilon - \hat{\mathbf{a}}(x) + \mathbf{a}(\mathbf{x}_I)) \delta(x_I - f(x)) \frac{\pi(x)}{\pi(\mathbf{x}_I)}.$$  \hfill (1.50)

Marginalising over $x$ reduces Eq. (1.50) to

$$\pi(\epsilon|\mathbf{x}_I) = \int \pi(\epsilon, x|\mathbf{x}_I) dx$$

$$= \frac{1}{\pi(\mathbf{x}_I)} \int \delta(\epsilon - \hat{\mathbf{a}}(x) + \mathbf{a}(\mathbf{x}_I)) \delta(x_I - f(x)) \pi(x) dx.$$  \hfill (1.51)

In the case of no parameter reduction (i.e. $\mathbf{x}_I = x$), Eq. (1.51) becomes

$$\pi(\epsilon|x_I) = \frac{1}{\pi(x_I)} \int \delta(\epsilon - \hat{\mathbf{a}}(x) + \mathbf{a}(x_I)) \delta(x_I - x) \pi(x) dx,$$

$$= \frac{1}{\pi(x_I)} \delta(\epsilon - \hat{\mathbf{a}}(x_I) + \mathbf{a}(x_I)) \pi(x_I),$$

$$= \delta(\epsilon - \hat{\mathbf{a}}(x_I) + \mathbf{a}(x_I)).$$  \hfill (1.52)

Clearly, using Eq. (1.52) directly in any inversions will result in the evaluation of $\hat{\mathbf{a}}$ which is undesirable and this formulation is unhelpful. The same is also true for Eq. (1.51). Therefore, a way of estimating $\pi(\epsilon|x_I)$ that removes the need to evaluate $\hat{\mathbf{a}}$ in the inversions is needed.

A way of approximating $\pi(\epsilon|x_I)$ involves performing pre-computations. To do this, a set of $M$ samples of $x$ is drawn from the prior distribution, and, subsequently, $x_I = f(x)$ and $\epsilon = \hat{\mathbf{a}}(x) - \mathbf{a}(x_I)$ are computed for this sample. Thus, the ensemble is

$$\left(\mathbf{x}^{(j)}, x_I^{(j)}, \epsilon^{(j)}\right) \text{ where } j = 1, 2, 3, \ldots, M.$$  \hfill (1.53)

This ensemble can then be used to approximate $\pi(\epsilon|x_I)$ by a normal distribution. The following estimates for the mean and covariance are used (The subscript $I$ is dropped in the following):

$$\mathbf{x}_0 = \frac{1}{M} \sum_{j=1}^{M} \mathbf{x}^{(j)}$$  \hfill (1.54a)

$$\epsilon_0 = \frac{1}{M} \sum_{j=1}^{M} \epsilon^{(j)}$$  \hfill (1.54b)
\[ \Gamma_x = \frac{1}{M-1} \sum_{j=1}^{M} (x^{(j)} - x_0)(x^{(j)} - x_0)^\top \] (1.54c)

\[ \Gamma_\epsilon = \frac{1}{M-1} \sum_{j=1}^{M} (\epsilon^{(j)} - \epsilon_0)(\epsilon^{(j)} - \epsilon_0)^\top \] (1.54d)

\[ \Gamma_{x\epsilon} = \frac{1}{M-1} \sum_{j=1}^{M} (x^{(j)} - x_0)(\epsilon^{(j)} - \epsilon_0)^\top \] (1.54e)

\[ \Gamma_{\epsilon x} = \Gamma_{xe}^\top \] (1.54f)

The mean and covariance of \( \pi(\epsilon|x) \) can then be derived using Schur complements [Zhang, 2006] yielding

\[ \epsilon_{0|x} = \epsilon_0 + \Gamma_{xe} \Gamma_x^{-1} (x - x_0) \] (1.55a)

\[ \Gamma_{\epsilon|x} = \Gamma_\epsilon - \Gamma_{xe} \Gamma_x^{-1} \Gamma_{xe} \] (1.55b)

In many cases, it is assumed that \( \Gamma_{xe} \Gamma_x^{-1} \approx 0 \) and therefore only the mean \( \epsilon_0 \) and covariance \( \Gamma_\epsilon \) are used. This is referred to as the enhanced error model [Kaipio and Somersalo, 2007].

In order to compute the posterior probability (Eq. 1.41), \( \pi_{e+%}(e + \epsilon|x) \) is needed. If \( e \) is normally distributed and independent of \( x \) then \( \pi_{e+%}(e + \epsilon|x) \) is also normally distributed with mean \( \epsilon_{0|x} + \epsilon_0 \) and covariance \( \Gamma_{e|x} + \Gamma_\epsilon \)

Further details can be found in [Kaipio and Kolehmainen, 2013, Kaipio and Somersalo, 2005, 2007, Lehikoinen et al., 2007]

### 1.5.8 Computational methods

The remainder of this Section on inverse problems deals with the computational algorithms that are used in this thesis to compute posterior point and uncertainty estimates for the unknowns.

**Markov chain Monte Carlo**

Equation (1.41) provides a formula for the posterior probability density. However, it is often difficult to use this formula directly to visualise or ask statistical questions about \( x \) (e.g. what is the most likely value) [Melsa and Cohn, 1978]. Markov chain Monte Carlo (MCMC) techniques provide a way of generating samples from a probability density. These
samples can be used to visualise and obtain statistical information on $x$ [Rubinstein and Kroese, 2007]: all statistical questions can be answered by computing sample means of appropriate functions of the samples.

In this thesis, the Metropolis-Hastings algorithm is used [Hastings, 1970]. This algorithm proceeds as follows [Robert and Casella, 2005]:

Let $\pi(x)$ be the probability distribution to be sampled and let $x^{(i)}$ be the most recent sample.

1. Generate a proposed $x_p$ from a proposal density $\pi_p(x_p|x^{(i)})$.

2. Evaluate

$$p = \frac{\pi(x_p) \, \pi_p(x^{(i)}|x_p)}{\pi(x^{(i)}) \, \pi_p(x_p|x^{(i)})}.$$ 

3. If $p \geq 1$,

$$x^{(i+1)} = x_p.$$ 

If $p < 1$,

$$x^{(i+1)} = \begin{cases} 
  x_p, \text{ with probability } p \\
  x^{(i)}, \text{ with probability } 1 - p.
\end{cases}$$

4. Restart at Step 1 from $x^{(i+1)}$.

Choosing a feasible proposal density $\pi_p(x_p|x^{(i)})$ is, in general, a non-trivial exercise. In this thesis, symmetric (random walk) proposal densities that have the property $\pi_p(x^{(i)}|x_p) = \pi_p(x_p|x^{(i)})$ are used. In the MCMC, the first few thousand samples are normally discarded to remove the 'burn-in'.

The main advantage of MCMC methods is that they facilitate the approximation of the full probability distribution and can therefore be used for accurate uncertainty quantification. The main disadvantage is that they are extremely computationally expensive, typically requiring thousands to millions of evaluations of $a(x)$, even for low dimensional problems ($\text{dim}_x < 10$) and the necessary number of samples increases significantly as $\text{dim}_x$ increases.
Gradient based methods

Gradient based methods can be used to find the maximum of the posterior probability distribution which is the most probable value for the unknown. The Equations (1.41) can often be rewritten to be in the form

\[ \pi(x|y) \propto \exp\left(-\frac{1}{2}||r(x)||^2_2\right). \] (1.56)

The maximum of this is when \( ||r(x)||^2_2 \) is a minimum. Gradient based methods try to find this point which is known as the maximum a-posteriori estimate \( x_{MAP} \). In this thesis, Gauss-Newton or Levenberg-Marquardt methods are used [Nocedal and Wright, 1999]. These methods necessitate the computation of the Jacobian of the mapping from \( r(x) \), denoted \( J(x) \).

Gradient based methods typically require only a few hundred evaluations of \( a(x) \) and thus are computationally inexpensive compared with MCMC. They are limited by the fact that they only provide one point estimate for the posterior probability distribution and can be difficult to tell whether the point estimate found is a local or global extremum [Nocedal and Wright, 1999].

Local approximations for the uncertainty in \( x \) can usually be defined. In this thesis a local normal approximation for the posterior covariance as a measure for the uncertainty of the estimates is used. This approximation is based on the Jacobian of the posterior functional computed at the maximum a-posteriori estimate \( x_{MAP} \) [Tarantola, 2004].
1.6 Thesis outline

Chapter 2 proposes a new absorbing boundary condition that can be used in the numerical simulation of 2-D or 3-D anisotropic elastic waves. The absorbing boundary condition developed can be used for simulation in many elastic wave settings. They are, however, developed with the intention of being used for the simulation of 2-D and 3-D elastic waves in a waveguide using 2-D and 3-D finite elements.

Chapter 3 proposes an efficient way to compute the waveform, dispersion and amplitude response from FZTWs propagating in a simplistic 1-D velocity model. This model is utilised in Chapter 4 to make synthetic observations of the properties of FZTWs. Both of these Chapters have been separately submitted to *Geophysical Journal International*.

FZTWs recorded in the fault zone of the Alpine Fault are investigated in Chapter 5. In this Chapter, forward and inverse modelling is performed using the three-layer approximation to estimate the fault zone properties.

A methodology for extracting a fully 2-D velocity profile from numerous FZTW data recordings is proposed in Chapter 6. Methodology developed in this Chapter draws heavily from the previous four Chapters. This is the primary result of the thesis and will be submitted to *Geophysical Journal International*. 
Paraxial Absorbing Boundary Conditions for 3-D Anisotropic Waves

2.1 Summary

A new approach for developing zeroth and first order paraxial absorbing boundary conditions is developed for 3-D anisotropic elastic waves. These new absorbing boundary conditions can be considered a generalisation of Stacy’s (1988) first order boundary conditions into three dimensions with an arbitrary anisotropic velocity model. These boundary conditions provide a useful alternative to the state of the art perfectly matched layer absorbing boundary conditions for solving the wave equation using finite element type methods. The methodology utilises the eigenvalue perturbation method to equate velocity variations to first order.

2.2 Introduction

In many real world waveform modelling situations there is a large area where the energy can propagate. However, we are only interested in a small region. For example, a particular simulation might only be interested in a small sedimentary basin but the wave energy
can, in principle, propagate throughout the entire earth. Absorbing boundary conditions (ABCs) are utilised in wave propagation schemes to limit the computational domains to the region of interest. They attempt to absorb all energy incident on the boundary and not cause any reflections. They are needed to dramatically reduce computational time.

The first type of ABCs developed for wave propagation were the paraxial type ABCs [Clayton and Engquist, 1977, Engquist and Majda, 1977, Reynolds, 1978, Stacey, 1988]. These paraxial type ABCs approximate the boundary condition with an advection type equation to advect energy out of the domain. Damping or sponge type ABCs were developed by Cerjan et al. [1985], Kosloff and Kosloff [1986]. These methods slowly reduce the amplitude of the energy in a damping zone placed outside of the desired boundary. Dong et al. [2005] used an eigenvalue decomposition of the propagation matrix to decompose the solution into in-going and out-going waves. More recently, ABC research has focused on the development of perfectly matched layers [Berenger, 1994, Chew and Liu, 1996, Collino and Tsogka, 2001, Hastings et al., 1996]. This method uses a damping zone around the desired boundary and has an advantage over sponge type boundary conditions in that there are significantly less reflections at the transition into the damping zone.

It is generally accepted that perfectly matched layer type ABCs are the most effective as they have an extremely high level of absorption and handle anisotropy with ease [Komatitsch and Tromp, 2003]. There are, however, several reasons why we may still be interested in using paraxial ABCs. If the computational domain has a high boundary size to domain size ratio then the addition of several damping nodes will add significant computational expense. For example, in the simulation of waves in a waveguide, the computational domain would be narrow but long. Paraxial ABCs are also significantly simpler and quicker to implement into a computational code than perfectly matched layers as the latter requires additional computational points over which a different differential equation is solved and additional memory variables are needed for this new equation [Komatitsch and Tromp, 2003]. Because their simplicity to implement, paraxial ABCs can also provide a useful first step and benchmark in the development of perfectly matched layers. Paraxial ABCs can also be used in conjunction with damping zones to reduces the required size of the damping zone [Givoli, 1991, Israeli and Orszag, 1981].

In this paper, we develop zeroth and first order paraxial ABCs for 3D elastic waves using the eigenvalue perturbation method. The methodology allows for arbitrary anisotropy and we demonstrate the case for transverse anisotropy.
2.3 Zeroth order ABC

In the following, we derive the zeroth order boundary conditions in the standard way [e.g. Clayton and Engquist, 1977]. Let us assume that a plane wave is travelling in the direction of \( \hat{\nu} = [\hat{\nu}_1, \hat{\nu}_2, \hat{\nu}_3] \) in the cartesian coordinate system \( x = [x, y, z] \). A plane wave solution can be written in the form of

\[
s = v \exp(i(k\hat{\nu} \cdot x - \omega t)), \tag{2.1}\]

where \( \omega/k = \gamma \) is the wave speed, \( s \) is the displacement, \( v \) is the displacement direction vector and \( i = \sqrt{-1} \). Substituting this in the full elastic wave equation with arbitrary anisotropy [Aki and Richards, 2009] leaves us with the matrix equation

\[
\gamma^2 v = K v. \tag{2.2}\]

In the case of transverse anisotropy

\[
K = \frac{1}{\rho} \begin{bmatrix}
C_{11}\hat{\nu}_1^2 + C_{66}\hat{\nu}_2^2 + C_{44}\hat{\nu}_3^2 & (C_{11} - C_{66})\hat{\nu}_1\hat{\nu}_2 & (C_{13} + C_{44})\hat{\nu}_1\hat{\nu}_3 \\
(C_{11} - C_{66})\hat{\nu}_1\hat{\nu}_2 & C_{66}\hat{\nu}_1^2 + C_{11}\hat{\nu}_2^2 + C_{44}\hat{\nu}_3^2 & (C_{13} + C_{44})\hat{\nu}_2\hat{\nu}_3 \\
(C_{13} + C_{44})\hat{\nu}_1\hat{\nu}_3 & (C_{13} + C_{44})\hat{\nu}_2\hat{\nu}_3 & C_{44}\hat{\nu}_1^2 + C_{44}\hat{\nu}_2^2 + C_{33}\hat{\nu}_3^2
\end{bmatrix}. \tag{2.3}\]

Note that for the isotropic case, we have \( C_{11} = C_{33} = \lambda + 2\mu, \ C_{13} = \lambda, \ C_{44} = C_{66} = \mu \) where \( \lambda, \mu \) are Lamé parameters.

Equation (2.2) is an eigenvalue equation of a symmetric positive definite matrix. Let the diagonal matrix of eigenvalues (\( \gamma_i^2 \)) be \( \Lambda^2 \) and the corresponding orthonormal eigenvector matrix be \( V \). Then any plane wave traveling in direction \( \hat{\nu} \) can now be written in the form

\[
s = V \begin{bmatrix}
b_1 \exp(i\omega(\gamma_1^{-1}\hat{\nu} \cdot x - t)) \\
b_2 \exp(i\omega(\gamma_2^{-1}\hat{\nu} \cdot x - t)) \\
b_3 \exp(i\omega(\gamma_3^{-1}\hat{\nu} \cdot x - t))
\end{bmatrix}, \tag{2.4}\]

where \( b = [b_1, b_2, b_3] \) is the arbitrary weights of the different eigenmodes. Computing the derivatives of Eq. 2.4 gives us:

\[
\frac{\partial s}{\partial \hat{\nu}} = \hat{\nu}_1 \frac{\partial s}{\partial x} + \hat{\nu}_2 \frac{\partial s}{\partial y} + \hat{\nu}_3 \frac{\partial s}{\partial z} = \omega V \Lambda^{-1} b, \tag{2.5a}\]

\[
\frac{\partial s}{\partial t} = -\omega V b. \tag{2.5b}\]
Utilising the fact that $V^\top = V^{-1}$, we can write

$$\frac{\partial \mathbf{s}}{\partial \mathbf{\nu}} = -V\Lambda\mathbf{s}^{-1}V^\top \frac{\partial \mathbf{s}}{\partial t}.$$  \hspace{1cm} (2.6)

Therefore, $\mathbf{s}$ in Eq. (2.4) is also the solution to Eq. (2.6) and so Eq. (2.6) is an advection type equation that can be used as an ABC. This can be applied by setting Eq. (2.6) as the boundary condition at the computational domain boundary.

### 2.4 First order ABC

While Eq. (2.6) is an ABC it only absorbs energy coming from the $\mathbf{\nu}$ direction and any other plane waves will incur at least some reflection. We derive an improvement that increases absorption from all directions close to the $\mathbf{\nu}$ direction.

Let us define an orthonormal co-ordinate system with unit vectors $\mathbf{\nu}_0, \mathbf{\theta}, \mathbf{\phi}$. Letting $\delta_\theta$ and $\delta_\phi$ be small numbers, we define the direction of travel of a plane wave as $\mathbf{\hat{\nu}}$, where

$$\mathbf{\hat{\nu}} = \cos(\delta_\theta) \cos(\delta_\phi) \mathbf{\nu}_0 + \sin(\delta_\theta) \mathbf{\hat{\theta}} + \sin(\delta_\phi) \cos(\delta_\theta) \mathbf{\hat{\phi}}$$

$$\approx \mathbf{\nu}_0 + \delta_\theta \mathbf{\hat{\theta}} + \delta_\phi \mathbf{\hat{\phi}} \text{ to first order.} \hspace{1cm} (2.7)$$

Substituting the plane wave solution ($\mathbf{s} = \mathbf{v} \exp(ik\mathbf{\nu} \cdot \mathbf{x} - i\omega t)$) in to the full elastic wave equation whilst keeping terms up to first order leaves us with the eigenvalue problem

$$\gamma^2 \mathbf{v} = \mathbf{K} \mathbf{v} \approx (\mathbf{K}_0 + \delta_\theta \mathbf{K}_\theta + \delta_\phi \mathbf{K}_\phi) \mathbf{v}. \hspace{1cm} (2.8)$$

In the case of transverse anisotropy we have:

$$\mathbf{K}_0 = \frac{1}{\rho} \begin{bmatrix}
C_{11}\hat{\nu}_1^2 + C_{66}\hat{\nu}_2^2 + C_{44}\hat{\nu}_3^2 & (C_{11} - C_{66})\hat{\nu}_1\hat{\nu}_2 & (C_{11} + C_{44})\hat{\nu}_1\hat{\nu}_3 \\
(C_{11} - C_{66})\hat{\nu}_1\hat{\nu}_2 & C_{66}\hat{\nu}_1^2 + C_{11}\hat{\nu}_2^2 + C_{44}\hat{\nu}_3^2 & (C_{11} + C_{44})\hat{\nu}_2\hat{\nu}_3 \\
(C_{11} + C_{44})\hat{\nu}_1\hat{\nu}_3 & (C_{11} + C_{44})\hat{\nu}_2\hat{\nu}_3 & C_{44}\hat{\nu}_1^2 + C_{44}\hat{\nu}_2^2 + C_{33}\hat{\nu}_3^2
\end{bmatrix} \hspace{1cm} (2.9a)$$

$$\mathbf{K}_\theta =$$

$$\frac{2}{\rho} \begin{bmatrix}
\frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_1\hat{\nu}_1 + \theta_1\hat{\nu}_3) & \frac{1}{2}(C_{11} - C_{66})(\hat{\theta}_1\hat{\nu}_2 + \theta_2\hat{\nu}_1) & \frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_1\hat{\nu}_3 + \theta_3\hat{\nu}_1) \\
\frac{1}{2}(C_{11} - C_{66})(\hat{\theta}_1\hat{\nu}_2 + \theta_2\hat{\nu}_1) & \frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_1\hat{\nu}_2 + \theta_2\hat{\nu}_2) & \frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_1\hat{\nu}_3 + \theta_3\hat{\nu}_2) \\
\frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_1\hat{\nu}_3 + \theta_3\hat{\nu}_1) & \frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_2\hat{\nu}_2 + \theta_2\hat{\nu}_2) & \frac{1}{2}(C_{11} + C_{44})(\hat{\theta}_2\hat{\nu}_3 + \theta_3\hat{\nu}_3)
\end{bmatrix} \hspace{1cm} (2.9b)$$
Computing the different directional differentials of Eq. (2.13) at $x$

Later, we will need four terms in Eq. 2.10 and the four different matrices of vectors $\tilde{V}$

Any wave traveling in direction $\hat{\nu}$ equal to the corresponding terms in Eq. 2.10b.

Writing these as equations in matrix form we have

$$K =$$

Let $\gamma_{\nu}$ be the square root of the eigenvalues of $K_0$ and the corresponding eigenvector be $v_{0\nu}$.

The eigenvalue perturbation is a method of approximating the eigenvalues and eigenvectors of a matrix that has been perturbed slightly and the eigenvalues and vectors of the original matrix are already known [Golub and Van Loan, 2013]. We can therefore use the eigenvalue perturbation matrix to estimate how the eigenvectors and eigenvalues change when the matrix $K$ is perturbed by the matrices $\delta_\theta K_\theta$ and $\delta_\phi K_\phi$. These formulas are

$$\gamma_i^2 = \gamma_{0\nu}^2 + \delta_\theta v_{0\nu}^T K_\theta v_{0\nu} + \delta_\phi v_{0\nu}^T K_\phi v_{0\nu}, \quad (2.10a)$$

$$v_i = v_{0\nu} + \delta_\theta \sum_{j: \gamma_{0\nu} \neq \gamma_j} \frac{v_{0j}^T K_\theta v_{0\nu}}{\gamma_{0\nu}^2 - \gamma_j^2} v_{0j}^T + \delta_\phi \sum_{j: \gamma_{0\nu} \neq \gamma_j} \frac{v_{0j}^T K_\phi v_{0\nu}}{\gamma_{0\nu}^2 - \gamma_j^2} v_{0j}^T. \quad (2.10b)$$

Later, we will need $\gamma_i^{-1}$, this can be written to first order as

$$\gamma_i^{-1} = \gamma_{0\nu}^{-1} - \frac{\delta_\theta}{2\gamma_{0\nu}^3} v_{0\nu}^T K_\theta v_{0\nu} - \frac{\delta_\phi}{2\gamma_{0\nu}^3} v_{0\nu}^T K_\phi v_{0\nu}. \quad (2.11)$$

Writing these as equations in matrix form we have

$$\Lambda^{-1} = \Lambda_0^{-1} + \delta_\theta \Lambda_\theta^{-1} + \delta_\phi \Lambda_\phi^{-1}, \quad (2.12a)$$

$$V = V_0 + \delta_\theta V_\theta + \delta_\phi V_\phi, \quad (2.12b)$$

where the four different diagonal matrices ‘$\Lambda$’ have their entries equal to the corresponding four terms in Eq. 2.10 and the four different matrices of vectors ‘$V$’ have their columns equal to the corresponding terms in Eq. 2.10b.

Any wave traveling in direction $\hat{\nu}$ can now be written, to first order, as

$$s = V \begin{bmatrix} b_1 \exp(i\omega(\gamma_1^{-1}\hat{\nu} \cdot x - t)) \\ b_2 \exp(i\omega(\gamma_2^{-1}\hat{\nu} \cdot x - t)) \\ b_3 \exp(i\omega(\gamma_3^{-1}\hat{\nu} \cdot x - t)) \end{bmatrix}. \quad (2.13)$$

Computing the different directional differentials of Eq. (2.13) at $x = 0$ leaves us with:
\[
\frac{\partial s}{\partial \hat{\nu}} = i\omega V \Lambda^{-1} b
\]
\[
\approx i\omega \left[ V_0 \Lambda_0^{-1} + \delta_\theta \left( V_0 \Lambda_0^{-1} + V_0 \Lambda_\theta^{-1} \right) + \delta_\phi \left( V_\phi \Lambda_0^{-1} + V_0 \Lambda_\phi^{-1} \right) \right] b,
\] (2.14a)

\[
\frac{\partial s}{\partial \hat{\theta}} = i\omega \delta_\theta V \Lambda^{-1} b
\]
\[
\approx i\omega \delta_\theta V_0 \Lambda_0^{-1} b,
\] (2.14b)

\[
\frac{\partial s}{\partial \hat{\phi}} = i\omega \delta_\phi V \Lambda^{-1} b
\]
\[
\approx i\omega \delta_\phi V_0 \Lambda_0^{-1} b,
\] (2.14c)

\[
\frac{\partial s}{\partial t} = -i\omega V b
\]
\[
\approx -i\omega \left( V_0 + \delta_\theta V_\theta + \delta_\phi V_\phi \right) b.
\] (2.14d)

We can use derivatives with respect to \( \hat{\nu} \) and \( t \) in the above equations to cancel out the unknown vector \( b \) to zeroth order (in \( \delta_\theta, \delta_\phi \)).

\[
V_0 \Lambda_0^{-1} V_0^T \frac{\partial s}{\partial t} + \frac{\partial s}{\partial \hat{\nu}} = i\omega \delta_\theta \left( V_\theta \Lambda_0^{-1} + V_0 \Lambda_\theta^{-1} - V_0 \Lambda_0^{-1} V_0^T V_\theta \right) b
\]
\[
+ i\omega \delta_\phi \left( V_\phi \Lambda_0^{-1} + V_0 \Lambda_\phi^{-1} - V_0 \Lambda_0^{-1} V_0^T V_\phi \right) b.
\] (2.15)

This is exactly our zeroth order ABC derived earlier. Using Eqs. (2.14d-2.15), it is straightforward to see that the first order ABCs are

\[
0 = \frac{\partial s}{\partial \hat{\nu}_0} + B_\theta \frac{\partial s}{\partial \hat{\theta}} + B_\phi \frac{\partial s}{\partial \hat{\phi}} + B_t \frac{\partial s}{\partial t},
\] (2.16)

where

\[
B_\theta = - \left( V_\theta \Lambda_0^{-1} + V_0 \Lambda_\theta^{-1} - V_0 \Lambda_0^{-1} V_0^T V_\theta \right) \Lambda_0 V_0^T,
\] (2.17a)

\[
B_\phi = - \left( V_\phi \Lambda_0^{-1} + V_0 \Lambda_\phi^{-1} - V_0 \Lambda_0^{-1} V_0^T V_\phi \right) \Lambda_0 V_0^T,
\] (2.17b)

\[
B_t = V_0 \Lambda_0^{-1} V_0^T.
\] (2.17c)

Equation 2.16 provides an equation that can be applied at a computational boundary to absorb outwards propagating waves so that they do not reflect. The Equation is dependent on the unit vector \( \hat{\nu}_0 \), which represents the direction that incoming waves will achieve maximum absorption and can be set by the user of the boundary condition depending on the problem. The unit vectors \( \hat{\theta}, \hat{\phi} \) must be chosen to complete the right hand coordinate system.
2.5 Isotropic case

In the isotropic case, the eigenvectors are independent of propagation direction. We can assume, without loss of generality, that $\hat{\nu}_0, \hat{\theta}, \hat{\phi}$ are unit vectors in the $x, y, z$ direction respectively. This leaves us with a number of simplifications:

$$[\gamma_1, \gamma_2, \gamma_3] = [\alpha, \beta, \beta],$$

$$\Lambda_\theta^{-1} = \Lambda_\phi^{-1} = \mathbf{0},$$

$$V_0 = \mathbf{I},$$

$$V_\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V_\phi = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

where $\alpha$ and $\beta$ are the P- and S-wave velocities and $\mathbf{I}$ is the identity matrix. This reduces Eqs. (2.17) to:

$$B_\theta = \Lambda_0^{-1} V_\theta \Lambda_0 - V_\theta = \begin{bmatrix} 0 & 1 - \beta \alpha^{-1} & 0 \\ \alpha \beta^{-1} - 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_\phi = \Lambda_0^{-1} V_\phi \Lambda_0 - V_\phi = \begin{bmatrix} 0 & 0 & 1 - \beta \alpha^{-1} \\ 0 & 0 & 0 \\ \alpha \beta^{-1} - 1 & 0 & 0 \end{bmatrix},$$

$$B_t = \Lambda_0^{-1}.$$  

Substituting (2.19) in to (2.16) leaves us with the first order isotropic paraxial ABCs

$$\frac{\partial s_x}{\partial x} = -\frac{1}{\alpha} \frac{\partial s_x}{\partial t} + \frac{\beta - \alpha}{\alpha} \frac{\partial s_y}{\partial y} + \frac{\beta - \alpha}{\alpha} \frac{\partial s_z}{\partial z},$$

$$\frac{\partial s_y}{\partial x} = -\frac{1}{\beta} \frac{\partial s_y}{\partial t} + \frac{\beta - \alpha}{\beta} \frac{\partial s_x}{\partial y},$$

$$\frac{\partial s_z}{\partial x} = -\frac{1}{\beta} \frac{\partial s_z}{\partial t} + \frac{\beta - \alpha}{\beta} \frac{\partial s_y}{\partial z}. $$
If we take the 2D approximation that all derivatives with respect to $y$ are zero, then we obtain Stacy’s (1988) first order ABCs

\[
\begin{align*}
\frac{\partial s_x}{\partial x} &= -\frac{1}{\alpha} \frac{\partial s_x}{\partial t} + \frac{\beta - \alpha}{\alpha} \frac{\partial s_z}{\partial z}, \\
\frac{\partial s_y}{\partial x} &= -\frac{1}{\beta} \frac{\partial s_y}{\partial t}, \\
\frac{\partial s_z}{\partial x} &= -\frac{1}{\beta} \frac{\partial s_z}{\partial t} + \frac{\beta - \alpha}{\beta} \frac{\partial s_x}{\partial z}.
\end{align*}
\tag{2.21}
\]

2.6 Numerical experiments and discussion

In order to test these new ABCs, we perform four different numerical experiments. These experiments are shown in Figure 2.1. It is clear from the Figure that the first order ABCs are significantly better than zeroth order and work equally well for isotropic and anisotropic velocity models. The slight discrepancy between first order anisotropic and isotropic simulations is because of the anisotropy causing the rays to travel at a greater angle to the boundary. The last column in the simulation shows an ABC where the direction of maximum absorption is ‘angled’ towards the source. This shows that if the approximate propagation direction of a wave incident on a boundary is known, then the absorbing boundary can be significantly improved. This angled ABC has a maximum reflection amplitude of 2% of the incident wave amplitude. This is compared with 12% for the 1st order ABC simulation with maximum absorption normal to the boundary.

From Figure 2.1, these ABCs are clearly not as useful as perfectly matched layer type boundary conditions which can be tuned to absorb nearly all energy from all angles [Komatitsch and Tromp, 2003]. However, these ABCs do still have their place as they are computationally more efficient and much quicker to conceptualise and implement into a code than perfectly matched layer type boundary conditions. There is also scope for them to be used in conjunction with PML ABCs in order to reduce the necessary thickness of the perfectly matched layer.
Figure 2.1: Numerical experiments of absorbing boundary conditions (ABCs). The columns show four different numerical experiments of 2-D elastic wave propagation with 3 displacement directions. All experiments have had a point source located in the bottom left hand corner and have an absorbing boundary on the right hand side. Only the magnitude (||s||) of the displacement vector is plotted and displacement values less than 5% of the maximum are not plotted (white area). The first column is for a homogeneous isotropic velocity model with first order absorbing boundaries (Stacey) and the P-wave velocity 1.5 times faster than the S-wave. The other three columns have homogeneous transversely isotropic velocity models with the P-wave 20% faster in the direction across the page and the SH wave is 25% faster than the SV wave in the direction across the page. The first three columns all have ABCs with maximum absorption for rays traveling normal to the boundary. The fourth column has maximum absorption from rays travelling directly from the source. The first row shows the simulation. The second row shows the waves reflected from the absorbing boundary. These have been amplified by 6 and plotted on the same colour scale as the simulation.
A Numerical Approach for Modelling Fault-Zone Trapped Waves

3.1 Summary

We develop a computationally efficient approach to compute the waveforms and the dispersion curves for fault-zone trapped waves guided by arbitrary transversely-isotropic across-fault velocity models. The approach is based on a Green’s function type representation for $F_L$ and $F_R$ type fault-zone trapped waves. The model can be used for simulation of the waveforms generated by both infinite line sources (2-D) and point sources (3-D). The numerical scheme is based on a high order finite element approximation and, to increase computational efficiency, we make use of absorbing boundary conditions and mass lumping of finite element matrices.

3.2 Introduction

A mature fault zone represents a significant zone of weakness within the earth’s crust that accommodates focused crustal strain as active earthquake rupture, slow slip or creep [Peng and Gomberg, 2010]. Characteristics such as fracturing, brecciation, remineralisation, and
high pore-fluid pressure in the fault often result in lower bulk seismic velocities than in the surrounding country-rock [Chester et al., 1993, Cochran et al., 2009, Mooney and Ginzburg, 1986].

Fault zone trapped (or guided) waves (FZTWs) are seismic waves that travel in the low velocity region of a fault zone [Li et al., 2012]. These waves are characterised by being at least partially trapped or guided in the fault zone through internal reflections off the higher velocity country-rock. The FZTWs have many similar properties to surface waves and there are FZTWs that correspond to the Love, Rayleigh and leaky surface wave types. The $F_L$ FZTWs are analogous to Love mode surface waves and have fault-parallel polarisation [fault SH motion, Li et al., 2012]. The observation of $F_L$ phases on an active fault were first reported from controlled source surface-to-borehole studies at Oroville, California [Leary et al., 1985, 1987, Li and Leary, 1990, Li et al., 1990]. The $F_R$ FZTWs are analogous to Rayleigh mode surface waves and have radial and fault orthogonal polarisation (fault PSV motion). The $F_\phi$ phases are a leaky FZTW type that arrive between the P- and S-wave first arrivals and are only partially trapped by the waveguide. The $F_R$ and leaky ($F_\phi$) FZTWs were first identified in passive source experiments on the San Andreas Fault [Ellsworth and Malin, 2011, Malin and Lou, 1996].

FZTWs are dispersive body waves. The $F_R$ and $F_L$ waves arrive after the S-wave and typically have a higher amplitude after significant propagation distances [Huang et al., 1995]. The amplitude and frequency content of FZTWs depend strongly on the physical properties and geometry of the fault zone and the surrounding rock [Ben-Zion, 1998, Igel et al., 2002, Li and Vidale, 1996]. Modelling of FZTWs has therefore been used to estimate the elastic properties and geometry of a fault zone at a resolution of 10s to 100s of metres [Ben-Zion et al., 1992, 2003b, Eccles et al., 2015, Ellsworth and Malin, 2011, Haberland et al., 2003, 2007, Hough et al., 1994, Lewis et al., 2005, 2007, Li et al., 1999, 1998, 1997b, Li and Malin, 2008, Li et al., 2014b, Li and Vernon, 2001, Mizuno et al., 2004, Wu et al., 2010].

Modelling of FZTWs can be broadly categorized into two types. These two methodologies are numerical full waveform solvers and semi-analytic trapped wave solvers.

The numerical full waveform solvers that are used in FZTW investigations are typically 2-D or 3-D finite difference approaches [e.g. Graves, 1996]. These numerical approaches output the entire wave field produced by a source including the FZTW. Full waveform solvers allow investigations into the properties of FZTWs as influenced by different velocity models or source types and locations [Igel et al., 1997, 2002, Jahnke et al., 2006, Li and Vidale, 1996]. The computationally intensive 3-D full waveform solvers have also been used for the estimation of fault zone parameters through forward modelling [Li and Malin,
The high computational time (hours) means that full waveform solvers are often impractical for estimating fault zone parameters through inversion which requires many evaluations of the forward model.

There are several different types of semi-analytic trapped wave solvers. The simplest is the fully analytic solution to $F_L$ in a homogenous fault zone sandwiched by homogenous higher velocity country-rock [Ben-Zion, 1998]. This Chapter also derives a computational model for a layered 1-D medium. Other approaches include the generalisation of surface wave methods such as propagator matrix methods [e.g. Malin and Lou, 1996]. These methods are sufficiently computationally fast for inversion and have been previously used in this way [Haberland et al., 2003, 2007, Wu and Hole, 2011, Wu et al., 2010]. These methods are limited to 1-D stratified velocity models which may poorly represent real world geologic complexity [Faulkner et al., 2010].

The described limitation of these semi-analytic trapped wave solvers means that many of the inversions performed to date assume that the fault zone is a single homogenous layer surrounded by homogenous country-rock [Malin and Lou, 1996, Wu and Hole, 2011]. Many fault zones, however, have gradational boundaries [Leary et al., 1987], which is shown to significantly affect the dispersion characteristics of the trapped wave in Gulley et al. [2017a].

It is expected that many fault zones have bulk anisotropic elastic properties [Cochran and Vidale, 2001, Leary et al., 1987, Li et al., 1987]. This anisotropy could be caused by parallel layering, preferential alignment of fractures or other intrinsic rock properties such as crystal orientation and type. In some cases this anisotropic elastic velocity profile can be well approximated by a transversely-isotropic velocity model [Leary et al., 1987, Li et al., 1987]. In this Chapter transverse-isotropy in fault zones means that any wave traveling parallel to the central fault plane has the same velocity, but the velocities change if the wave direction has a component normal to the fault plane. Anisotropy in coal seam waveguides has been well explored [Buchanan et al., 1983]. A methodology for modelling guided waves with arbitrary fracture orientation was developed by Lou and Crampin [1991]. This methodology assumes the waveguide is made up of several homogeneous anisotropic layers and uses layer matching techniques. It is shown in Lou and Crampin [1991] that guided waves in anisotropic waveguides have similar properties to those in isotropic waveguides however anisotropy does cause variations in group velocities. These variations may be important if guided waves are being used to image waveguides.

In this Chapter, we derive a semi-analytical computational model for $F_L$ and $F_R$ type FZTWs that propagate in three dimensions. The velocity model is restricted to wave vari-
ations in the across-fault direction only (1-D) and it has fault-parallel transverse-isotropy. This model has two aims. The first is to develop a model that attempts to allow for a velocity structure with gradational boundaries and anisotropy but is sufficiently efficient such that it could be employed in the inversion. The second is to compute Green’s function type representations (amplitude response) for FZTW from a point source with an arbitrary 1-D velocity model. Such a representation of the amplitude response and dispersion curves allows for further analysis of FZTW. This analysis could include understanding which properties of the waveguide or earthquake have the most significant effect on the amplitude or dispersion of particular frequencies, and such an analysis would be difficult if only modelling the coda in the time domain.

3.3 A numerical approach to compute FZTWs generated by an arbitrary 1-D velocity model

3.3.1 Overview of the approach

We developed a numerical and computational scheme to compute the trapped frequencies and waveforms of the $F_L$ and $F_R$ FZTWs with a transversely-isotropic velocity model that allowed for arbitrary velocity variations in the direction across-fault. The numerical approach was based on a high order finite element method because of its high computational efficiency [Wiggins, 1976]. The finite element method also has an advantage over methods such as matrix propagator methods [e.g. Haskell, 1953, Thomson, 1950] or layer matching methods [Ben-Zion, 1998] as it allows for the elastic properties to be varied over an element, thus reducing the number of points needed to approximate complex velocity models. On the truncation boundary of the computational domain, we employed absorbing boundary conditions. Further computational speed was gained by the use of a mass lumping approximation [Cohen et al., 1993], which reduces the complexity of the eigenvalue problem.

Once the trapped modes and dispersion curves of the FZTWs have been calculated, we use a generalisation of the equations in Aki and Richards [2009] for transversely-isotropic trapped waves to calculate the Greens function (amplitude response). This allows the FZTW waveform to be computed efficiently for multiple different source types. These source types include infinite line sources (2-D wave propagation) and moment tensor point sources (3-D wave propagation from an earthquake).

The capabilities of this approach are shown in Fig. 3.1. Figure 3.1a shows an example
velocity model with gradational boundaries consistent with decreasing fracture density with distance from the fault core [Faulkner et al., 2010] provides the waveguide. The FZTWs are generated from a strike slip earthquake occurring at 3km depth and 3km along-strike from the receiver. Dispersion curves, Green’s function (amplitude response) and the associated FZTW waveform are shown.

Figure 3.1: Example outputs from proposed finite element fault-zone trapped wave modelling methodology (a) S-wave velocity model. The P-wave velocity and the density have a similar shape to the S-wave velocity but with ranges of 5 km/s-5.3 km/s and 2.6 g/cm³-2.7 g/cm³ respectively. Isotropy is assumed. The P-wave (attenuation) Q factor has the same profile with a range of 20-100 and the S-wave Q factor has a range of 10-50. (b) Dispersion curves of the fundamental and 1st harmonics. (c) FZTW waveforms produced from a purely strike slip earthquake in a vertically oriented fault that is located at 3 km depth and 3 km along the strike from the receiver. The source and receiver are located 25 m from the central fault plane, the source spectrum used is 1/frequency and only the fundamental and 1st harmonic is used. The numbers on the axis show the relative amplitudes of the waveforms. (d) The Green’s function (amplitude response) that, together with the dispersion curve produced the waveforms seen in (c). The numbers on the axis show the relative amplitudes of the waveforms.

3.3.2 Anisotropic propagation in a waveguide

We derive the equations and solutions for a FZTW with an arbitrary transversely-isotropic velocity model in which the velocity varies only in the direction (denoted by z below)
Figure 3.2: The coordinate system used. An individual wave ray is traveling in the direction described by the ‘ray angle’ but due to the waveguide the FZTW net energy is traveling in the $x$ direction. P-SV waves (i.e. $F_R$) have displacement in the $x$-$z$ directions and the SH waves ($F_L$) have displacement in the $y$ direction.

across the fault. The notations and the general approach largely follows the treatment of surface waves in Aki and Richards [2009], which we generalise here for transversely-isotropic trapped waves. We assume that the FZTW propagates in the $x$ direction. The plane wave solution to the full elastic wave equation can be written as

$$
\begin{align*}
   u(x, z, \omega, k, t) &= r_1(k, z, \omega) \exp(i(kx - \omega t))\hat{x} + \\
   &+ ir_2(k, z, \omega) \exp(i(kx - \omega t))\hat{z} + \\
   &+ l(k, z, \omega) \exp(i(kx - \omega t))\hat{y},
\end{align*}
$$

where $i = \sqrt{-1}$ is the imaginary unit and is introduced in the $\hat{z}$ term to make the $r_1$ and $r_2$ components in-phase with each other. The unit vector in the direction across the fault is $\hat{z}$ and $\hat{x}, \hat{y}$ are the two remaining unit vectors in the right hand coordinate system. The angular frequency is $\omega$, $k$ is the wave number in the $x$ direction and the phase speed is $c = \omega/k$.

We take the transverse-isotropy planes of symmetry to be the $z = $ constant (fault-parallel) planes and thus, without loss of generality, we will only consider the $x$ and $z$ directions. We define the propagation angle $\theta$ that describes wave propagation direction in the $x$-$z$ plane with $\theta = 0^\circ$, meaning propagation in the fault-perpendicular $z$ direction, and $\theta = 90^\circ$ referring to propagation in the fault-parallel $x$ direction as seen in Fig. 3.2. The elastic tensor coefficients $C_{ij}$ and the density $\rho$ are functions of $z$ only. A transversely-isotropic material has five independent elastic tensor coefficients $C_{11}, C_{33}, C_{44}, C_{66}$ and $C_{13}$, which determine the velocities of different propagation angles $\theta$ as described in Thomsen [1986].

Routine substitution of the above in the full elastic wave equation [Aki and Richards, 2009]
Chapter 3. A Numerical Approach for Modelling FZTWs

This gives us two decoupled systems of generalised quadratic eigenvalue problems:

\[ \omega^2 \left( \frac{C_{66}}{c^2} - \rho \right) l = \frac{\partial}{\partial z} \left( C_{44} \frac{\partial l}{\partial z} \right), \quad (3.2a) \]
\[ kC_{13} \frac{\partial r_2}{\partial z} + k^2 (C_{11} - c^2 \rho) r_1 = \frac{\partial}{\partial z} \left( C_{44} \frac{\partial r_1}{\partial z} - kC_{44} r_2 \right), \quad (3.2b) \]
\[ -kC_{44} \frac{\partial r_1}{\partial z} + k^2 (C_{44} - c^2 \rho) r_2 = \frac{\partial}{\partial z} \left( C_{33} \frac{\partial r_2}{\partial z} + kC_{13} r_1 \right), \quad (3.2c) \]

where (3.2b-3.2c) are coupled.

The solution to Eq. (3.2a) yields the \( F_L \) mode FZTW which is analogous to Love type surface waves and has particle motion only in the \( y \) direction. The related eigenvalue is \( \omega \) which corresponds to trapped frequencies for each particular phase speed \( (c) \) and the related eigenvector is the trapped mode \( l(z) \). The solution to the coupled Eqs. (3.2b) and (3.2c) yields the \( F_R \) type FZTWs which are analogous to Rayleigh type surface waves, and these have motion in the \( x \) and \( z \) directions. The eigenvalue in these two coupled equations is the wave number \( k \) with eigenvectors \( \{ r_1(z), r_2(z) \} \). The lowest eigenvalue for each \( c \) is called the fundamental and the higher eigenvalues are referred to as harmonics.

For \( \omega \) to be real-valued in Eq. (3.2a) and hence for the \( F_L \) trapped mode to exist, we must have \( c^{H}_{\text{min}} = \min(\beta^H_x(z)) < c < \min(\beta^H_x(\pm\infty)) = c^{H}_{\text{max}} \). For \( F_R \) to exist, we must have \( c^{V}_{\text{min}} = \min(\beta^V_x(z)) < c < \min(\beta^V_x(\pm\infty)) = c^{V}_{\text{max}} \). Note that the superscript \( H \) and \( V \) refer to SH and SV polarisation respectively.

We use the anisotropy parameters that are defined by Thomsen [1986], these are:

\[ \alpha_0 = \sqrt{\frac{C_{33}}{\rho}} = \alpha(0^\circ), \quad (3.3a) \]
\[ \beta_0 = \sqrt{\frac{C_{44}}{\rho}} = \beta^V(90^\circ) = \beta^V(0^\circ) = \beta^H(0^\circ), \quad (3.3b) \]
\[ \epsilon = \frac{C_{11} - C_{33}}{2C_{33}} \Rightarrow \alpha(90^\circ) = \alpha_0 \sqrt{1 + 2\epsilon}, \quad (3.3c) \]
\[ \gamma = \frac{C_{66} - C_{44}}{2C_{44}} \Rightarrow \beta^H(90^\circ) = \beta_0 \sqrt{1 + 2\gamma}, \quad (3.3d) \]
\[ \delta = \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})} \Rightarrow, \quad (3.3e) \]

For a fully isotropic media, we would have \( \epsilon = \gamma = \delta = 0 \). To visualise this parametrisation of anisotropy, consider briefly the case of weak elastic anisotropy \((|\epsilon|, |\gamma|, |\delta| \ll 1)\). In this
case, it was shown in Thomsen [1986] that the velocities are approximately given by

\[
\alpha(\theta) = \alpha_0 \left[ 1 + \delta \sin^2(\theta) \cos^2(\theta) + \epsilon \sin^4(\theta) \right], \\
\beta^V(\theta) = \beta_0 \left[ 1 + \frac{\alpha_0^2}{\beta_0^2} (\epsilon - \delta) \sin^2(\theta) \cos^2(\theta) \right], \\
\beta^H(\theta) = \beta_0 \left[ 1 + \gamma \sin^2(\theta) \right].
\] (3.4a, 3.4b, 3.4c)

### 3.3.3 Absorbing boundary conditions

We need to pose a boundary model on Eqs. (3.2). It is known that \(l, r_1, r_2 \to 0\) as \(z \to \pm\infty\) however significant computational improvements can be obtained by employing absorbing boundary conditions [Engquist and Majda, 1977]. To this end, we choose the computational domain to be bounded by \(z_{\text{min}}\) and \(z_{\text{max}}\). We approximate \(C_{ij}\) and \(\rho\) as constants in the regions outside the computational domain, that is, \(z \leq z_{\text{min}}\) and \(z \geq z_{\text{max}}\). To compute the absorbing boundary condition we start with the analytic solutions to Eqs. (3.2) for the regions outside of the truncation boundaries. We then apply the conditions that \(l, r_1, r_2 \to 0\) as \(z \to \pm\infty\). This leaves us with the following equations:

\[
l(z) = A\exp(\omega_k z), \\
r_1(z) = A\nu S_1 \exp(\omega_k \nu z) + A_p P_1 \exp(\omega_k p z), \\
r_2(z) = A\nu S_2 \exp(\omega_k \nu z) + A_p P_2 \exp(\omega_k p z),
\] (3.5a, 3.5b, 3.5c)

where,

\[
n = \begin{cases} 
-1, & \text{if } z = z_{\text{min}}, \\
1, & \text{if } z = z_{\text{max}},
\end{cases}
\] (3.6a)

\[
Q_0 = ((2\epsilon + 1)c^{-2} - \alpha_0^{-2})(c^{-2} - \beta_0^{-2}), \\
Q_1 = \alpha_0^{-2} + \beta_0^{-2} + 2c^{-2}((\alpha_0^2\beta_0^{-2} - 1)(\delta - \epsilon) - 1 - \epsilon), \\
k_h = -n \sqrt{(2\gamma + 1)c^{-2} - \beta_0^{-2}}, \\
k_\nu = -n \sqrt{\frac{1}{2}(-Q_1 - \sqrt{Q_1^2 - 4Q_0})}, \\
k_p = -n \sqrt{\frac{1}{2}(-Q_1 + \sqrt{Q_1^2 - 4Q_0})}, \\
S_1 = ((2\epsilon + 1)\alpha_0^2c^{-2} - 1 - k_\nu \beta_0^2)^{-1}, \\
S_2 = \left(-k_\nu c^{-1}\sqrt{(\alpha_0^2 - \beta_0^2)((2\delta + 1)\alpha_0^2 - \beta_0^2)}\right)^{-1}.
\] (3.6b, 3.6c, 3.6d, 3.6e, 3.6f, 3.6g, 3.6h)
\[ P_1 = \left( (2\epsilon + 1)\alpha_0^2 c^{-2} - 1 - k_p^2 \beta_0^2 \right)^{-1}, \]  
\[ P_2 = \left( -k_p c^{-1} \sqrt{\alpha_0^2 - \beta_0^2} \right) \left( (2\delta + 1)\alpha_0^2 - \beta_0^2 \right)^{-1}, \]

and \( A_h, A_v, A_p \) are arbitrary constants.

It can be shown that Eqs. (3.5) are the unique solution to the first order differential equations

\[ \frac{\partial l}{\partial z} = \omega k h l, \]  
\[ \frac{d r_1}{d z} = k \xi [(S_1 P_2 k_v - P_1 S_2 k_p) r_1 + (k_p - k_v) S_1 P_1 r_2], \]  
\[ \frac{d r_2}{d z} = k \xi [(k_v - k_p) S_2 P_2 r_1 + (S_1 P_2 k_p - P_1 S_2 k_v) r_2], \]

where \( \xi = c(S_1 P_2 - P_1 S_2)^{-1} \).

If Eqs. (3.7) are applied at the truncation boundaries then the interior solution will behave as though the solution domain goes to \( \pm \infty \) and therefore Eqs. (3.7) are the absorbing boundary conditions.

### 3.3.4 Finite element approximation

We use the finite element method to solve Eqs. (3.2) with boundary conditions (3.7). In the finite element method, the solution is approximated by a weighted sum of \( N \) basis functions. For example, we write \( l(z) \approx \sum_{k=1}^{N} l(z_k) v_k(z) \), where the basis functions have the property that \( v_k(z_k) = 1 \) and \( v_k(z_j) = 0 \) for \( k \neq j \), and where \( z_k \), for \( k = 1, 2, 3, \ldots, N \) are a set of ordered locations (nodes) of the finite element mesh with \( z_1 = z_{\text{min}} \) and \( z_N = z_{\text{max}} \).

Another property of the basis functions is that they are non-zero only on a region close to the corresponding nodes and in this non-zero region they can be represented as Lagrange interpolation polynomials of order \( P \), which is referred to as the order of the approximation. Here, we choose the node locations specifically so that these polynomials have zeros at the so-called Lobatto points, leading to increased computational accuracy. For further details on the finite element scheme employed in this Chapter see Pozrikidis [2005].

Let us define the vector \( l \) with entries \( l_k = l(z_k) \) and the vectors \( r_1 \) and \( r_2 \) defined in the same way. We define the following elements of finite element matrices:

\[ M_{ji}^f = \int_{z_1}^{z_N} f(z) v_i(z) v_j(z) dz, \]  
\[ \text{(3.8a)} \]
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\[ S^f_{ji} = \int_{z_1}^{z_N} f(z) \frac{\partial v_i(z)}{\partial z} \frac{\partial v_j(z)}{\partial z} dz, \quad (3.8b) \]

\[ D^f_{ji} = \int_{z_1}^{z_N} f(z) \frac{\partial v_i(z)}{\partial z} v_j(z) dz, \quad (3.8c) \]

\[ H^f_{ji} = \int_{z_1}^{z_N} f(z) v_i(z) \frac{\partial v_j(z)}{\partial z} dz, \quad (3.8d) \]

\[ \Gamma^f_{ji} = \begin{cases} -f(z_1)v_i(z_1) \frac{dv_j}{dz} \bigg|_{z_1}, & \text{if } i = j = 1, \\ f(z_N)v_i(z_N) \frac{dv_j}{dz} \bigg|_{z_N}, & \text{if } i = j = N, \\ 0, & \text{otherwise}, \end{cases} \quad (3.8e) \]

\[ \Xi^f_{ji} = \begin{cases} -f(z_1)v_i^2(z_1), & \text{if } i = j = 1, \\ f(z_N)v_i^2(z_N), & \text{if } i = j = N, \\ 0, & \text{otherwise}, \end{cases} \quad (3.8f) \]

where the subscripts refer to the \( j \)th row and \( i \)th column of the matrices and \( f \) is a related function of \( z \) which depends on which particular coefficient the matrix is related to. We compute the integrals using Gaussian quadrature.

Thus, following the standard finite element procedure, Eqs. (3.2) are approximated with the quadratic eigenvalue (matrix) problems

\[ 0 = (S^{C^{44}} - \Gamma^{C^{44}})l - \omega^2(M^\rho - c^{-2}M^{C^{66}})l, \quad (3.9a) \]

\[ 0 = (S^{C^{44}} - \Gamma^{C^{44}})r_1 + k(D^{C^{13}} - H^{C^{44}} + \Xi^{C^{44}})r_2 
+ k^2(M^{C^{11}} - c^2M^\rho)r_1, \quad (3.9b) \]

\[ 0 = (S^{C^{33}} - \Gamma^{C^{33}})r_2 + k(H^{C^{13}} - \Xi^{C^{13}} - D^{C^{44}})r_1 
+ k^2(M^{C^{44}} - c^2M^\rho)r_2. \quad (3.9c) \]

Adopting the absorbing boundary model (Eqs. 3.7) causes the \( \Gamma \) matrices to be substituted by \( \Xi \) matrices. The finite element approximation, Eqs. (3.9), can now be written in the form

\[ 0 = S^{C^{44}}l - \omega\Xi^{g_0}l - \omega^2(M^\rho - c^{-2}M^{C^{66}})l, \quad (3.10a) \]

\[ 0 = S^{C^{44}}r_1 + k(D^{C^{13}} - H^{C^{44}} - \Xi^{g_2-C^{44}})r_2, 
- k\Xi^{g_1}r_1 + k^2(M^{C^{11}} - c^2M^\rho)r_1, \quad (3.10b) \]

\[ 0 = S^{C^{33}}r_2 + k(H^{C^{13}} - \Xi^{C^{13}+g_3} - D^{C^{44}})r_1, 
- k\Xi^{g_4}r_2 + k^2(M^{C^{44}} - c^2M^\rho)r_2. \quad (3.10c) \]
where,

\[ g_0(z_b) = C_{44} k_b, \]  
(3.11a)

\[ g_1(z_b) = C_{44} \xi (S_1 P_2 k_v - P_1 S_2 k_v), \]  
(3.11b)

\[ g_2(z_b) = C_{44} \xi (k_p - k_v) S_1 P_1, \]  
(3.11c)

\[ g_3(z_b) = C_{33} \xi (k_v - k_p) S_1 P_2, \]  
(3.11d)

\[ g_4(z_b) = C_{33} \xi (S_1 P_2 k_p - P_1 S_2 k_s), \]  
(3.11e)

and \( z_b = z_1 \) or \( z_N \).

### 3.3.5 Derivation of the related linear eigenvalue equations

Next, we transform the generalised quadratic eigenvalue problems (Eq. 3.10) to the generalised (nonquadratic, linear) eigenvalue problems. To this end, we define first the following (block) vector and matrices:

\[ \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \]  
(3.12a)

\[ \mathbf{A}_S = \begin{bmatrix} S^{44} & 0 \\ 0 & S^{33} \end{bmatrix}, \]  
(3.12b)

\[ \mathbf{A}_D = \begin{bmatrix} 0 & D^{13} - H^{44} \\ H^{13} - D^{44} & 0 \end{bmatrix}, \]  
(3.12c)

\[ \mathbf{A}_F = \begin{bmatrix} M^{11} & 0 \\ 0 & M^{44} \end{bmatrix}, \]  
(3.12d)

\[ \mathbf{A}_M = \begin{bmatrix} M^\rho & 0 \\ 0 & M^\rho \end{bmatrix}, \]  
(3.12e)

\[ \mathbf{A}_\Xi = \begin{bmatrix} \Xi^{g_1} & \Xi^{g_2 - C^{44}} \\ \Xi^{g_1 + C^{13}} & \Xi^{g_4} \end{bmatrix}, \]  
(3.12f)

where \( \mathbf{0} \) is a zero matrix of appropriate dimensions. The Equations (3.10) can now be written as

\[ \mathbf{0} = [S^{44} - \omega \Xi^{g_1} - \omega^2 (M^\rho - c^{-2} M^{66})] \mathbf{l}, \]  
(3.13a)

\[ \mathbf{0} = [\mathbf{A}_S + k (\mathbf{A}_D - \mathbf{A}_\Xi) + k^2 (\mathbf{A}_F - c^2 \mathbf{A}_M)] \mathbf{r}. \]  
(3.13b)
The Equations (3.13) specify two uncoupled quadratic eigenvalue problems. To reduce these to generalised (linear, non-quadratic) eigenvalue problems, we define further:

\[
\begin{align*}
I^* &= \begin{bmatrix} I & 0 \\ \omega I & 0 \end{bmatrix}, \\
J^* &= \begin{bmatrix} J & 0 \\ kJ & 0 \end{bmatrix}, \\
\tilde{S}^L &= \begin{bmatrix} 0 & I \\ S_{C44} & -\Xi_{90} \end{bmatrix}, \\
\tilde{M}^L &= \begin{bmatrix} I & 0 \\ 0 & M^\rho - c^{-2}M^C_{66} \end{bmatrix}, \\
\tilde{S}^R &= \begin{bmatrix} 0 & I \\ -A_S & A_\Xi - A_D \end{bmatrix}, \\
\tilde{M}^R &= \begin{bmatrix} I & 0 \\ 0 & A_F - c^2A_M \end{bmatrix},
\end{align*}
\]

where \( I \) are identity matrices of appropriate dimensions. Thus, the Eqs. (3.13) can be written in the form of

\[
\begin{align*}
\tilde{S}^L I^* &= \omega \tilde{M}^L J^*, \\
\tilde{S}^R J^* &= k \tilde{M}^R J^*.
\end{align*}
\]

We note that the above representation is not unique. There are also representations in which the matrices \( \tilde{S}, \tilde{M} \) are symmetric which would yield certain computational advantages. In our case, the choice we have made yield diagonally dominant matrices, which we exploit in the mass lumping below.

### 3.3.6 Mass lumping

The Equations (3.15) define generalised eigenvalue problems. In the chosen form, the matrices on the right hand side (\( \tilde{M} \)) are made up of matrices on the diagonal blocks which are either identity matrices or of the mass matrix type \( M^f \). For these, we employ mass lumping [Cohen et al., 1993] in which the matrices \( \tilde{M} \) are approximated by diagonal
matrices $\tilde{M}$ as follows:

$$
\tilde{M}_{ji} = \begin{cases} 
\sum_{i=1}^{N} \tilde{M}_{ji}, & \text{if } i = j, \\
0, & \text{if } i \neq j. 
\end{cases} \quad (3.16)
$$

Mass lumping is equivalent to calculating the $M_{ji}^f$ integrals in Eqs. (3.8) using inexact quadrature. Let $P$ be the polynomial order of the basis functions used, then the integrands of the $M_{ji}^f$ integrals are at least of order $2P$ (equality when $f = \text{constant}$). The mass lumping approximation is equivalent to evaluating these integrals with Lobatto quadrature that is exact when the integrands have polynomial order up to $2P-1$ [Chapter 3, Pozrikidis, 2005]. As $2P-1 < 2P$, mass lumping is always an approximation but it is often a feasible one.

Since the approximate $\tilde{M}$ is diagonal, we define $\tilde{S} = \tilde{M}^{-1}\tilde{S}$ to reduce the Eqs. (3.15) to the ordinary eigenvalue problems

$$
\tilde{S}^k(c)l^* = \omega l^*, \quad (3.17a) \\
\tilde{S}^R(c)r^* = kr^*, \quad (3.17b)
$$

where we have indicated $c$ to stress that these matrices depend on the phase speed. The eigenvalues of the problems (3.17) can be solved with various methods such as the pre-built eigenvalue solvers in MATLAB® [Chapter 7, Golub and Van Loan, 2013] which we use here. An example of a velocity model and eigenvectors $(l(z), r_1(z), r_2(z))$ from three different eigenvalues (trapped frequencies) are shown in Fig. 3.3.

Reducing errors induced by mass lumping

The use of mass lumping allows us to solve the much less computationally intensive simple eigenvalue matrix problem. Numerical evaluation of the mass lumping approximation indicates that the loss of accuracy associated with the mass lumping is small but tends to increase at higher frequencies ($\omega$) and elastic models with high gradients within an element. In the following, we make use of a correction that can be applied to improve the eigenvalue estimates.

Multiplying Eqs. (3.2) by $l$, $r_1$ and $r_2$, respectively, and carrying out integration by parts
Figure 3.3: Trapped modes of a velocity model. (a) S-wave velocity model used. The P-wave velocity and the density have a similar shape to the S-wave velocity but with ranges of 5 km/s-5.3 km/s and 2.6 g/cm$^3$-2.7 g/cm$^3$ respectively. There is no anisotropy assumed here. (b-d) Examples of the eigenvectors (trapped modes) for three different eigenvalues (frequencies). The eigenvectors are normalised by kinetic energy.

before summing the $F_R$ equations, yields the following equations:

$$0 = \omega^2(c^{-2}I_2^l - I_1^l) + I_3^l,$$

$$0 = k^2(I_2^r - c^2I_1^r) + kI_3^r + I_4^r,$$

where the $I^r,l_k$ are the energy integrals,

$$I_1^l = \frac{1}{2} \int_{-\infty}^{\infty} \rho l^2 dz,$$

$$I_2^l = \frac{1}{2} \int_{-\infty}^{\infty} C_{66} l^2 dz,$$

$$I_3^l = \frac{1}{2} \int_{-\infty}^{\infty} C_{44} \left( \frac{dl}{dz} \right)^2 dz,$$

$$I_1^r = \frac{1}{2} \int_{-\infty}^{\infty} \rho (r_1^2 + r_2^2) dz,$$
The above equations are given in Aki and Richards [2009] for isotropic surface waves, but they are presented here for transversely isotropic trapped waves. These integrals are computed from $-\infty$ to $+\infty$. To compute these integrals we must account for the fact that we have only computed the eigenvectors between $z_{\min} = z_1$ and $z_{\max} = z_N$ and the energy that is absorbed by the absorbing boundaries. To do this we make use of the fact that we already know the form of the wave outside of the boundaries, these are given in Eq. 3.5.

The computational procedure is defined in the following derivation:

$$I_1^r = \frac{1}{2} \int_{-\infty}^{\infty} C_{11} r_1^2 + C_{44} r_2^2 dz,$$  

$$I_2^r = \int_{-\infty}^{\infty} C_{13} r_1 \frac{dr_2}{dz} - C_{44} r_2 \frac{dr_1}{dz} dz,$$  

$$I_3^r = \frac{1}{2} \int_{-\infty}^{\infty} C_{33} \left(\frac{dr_2}{dz}\right)^2 + C_{44} \left(\frac{dr_1}{dz}\right)^2 dz.$$  

where we have shown the derivation only for $I_1^r$ and $\sum_{\pm} f(z) = f(z_1) + f(z_N)$ and $(\cdot)^\top$ denotes transpose.
We have also used the following matrices:

\[
\hat{r} = \frac{\xi}{c} \begin{bmatrix} P_2 & -P_1 \\ -S_2 & S_1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix},
\]

\[
\mathbf{I}_{r1} = -\frac{n}{2\omega} \begin{bmatrix} S_1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} \frac{1}{k_v} & \frac{2}{k_v+k_p} \\ \frac{2}{k_v+k_p} & \frac{1}{k_p} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & P_1 \end{bmatrix},
\]

\[
\mathbf{I}_{r2} = -\frac{n}{2\omega} \begin{bmatrix} S_2 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \frac{1}{k_v} & \frac{2}{k_v+k_p} \\ \frac{2}{k_v+k_p} & \frac{1}{k_p} \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & P_2 \end{bmatrix},
\]

\[
\mathbf{I}_{dr1} = \frac{-n\omega}{2} \begin{bmatrix} S_1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} k_v & \frac{2k_v k_p}{k_v+k_p} \\ \frac{2k_v k_p}{k_v+k_p} & k_p \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & P_1 \end{bmatrix},
\]

\[
\mathbf{I}_{dr2} = \frac{-n\omega}{2} \begin{bmatrix} S_2 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} k_v & \frac{2k_v k_p}{k_v+k_p} \\ \frac{2k_v k_p}{k_v+k_p} & k_p \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & P_2 \end{bmatrix},
\]

\[
\mathbf{I}_3 = \frac{n}{2} \begin{bmatrix} (C_{44} - C_{13})S_1 S_2 & g \\ g & (C_{44} - C_{13})P_1 P_2 \end{bmatrix},
\]

where

\[
g = \frac{C_{44}(S_1 P_2 k_v + S_2 P_1 k_p) - C_{13}(S_1 P_2 k_p + S_2 P_1 k_v)}{k_v + k_p}.
\]

As has been mentioned previously, an initial estimate for the eigenvectors and eigenvalues can be obtained from the mass lumped Eqs. (3.17). We can then use these estimates to compute the energy \(I\) using Eqs. (3.20). Once this has been done we can re-estimate the eigenvalues using Eqs. (3.18). It has been shown in Aki and Richards [2009] that estimating \(\omega\) using Eq. (3.18a) is accurate even if there are first order errors in the eigenvectors and eigenvalues used to compute the energy integrals. Whilst there is no such proof shown for Eq. (3.18b) we note the similarity between Eqs. (3.18) and the eigenvalue perturbation method [Golub and Van Loan, 2013], which is used to update eigenvalues when small changes are made to matrices (in our case this small change is mass lumping). Our numerical experiments show that this correction causes a significant reduction in error, and that this reduction is greater than that obtained using eigenvalue perturbation methods. An example of the relative errors of the mass lumped solution with and without the correction are shown in Fig. 3.4. We assess that the errors induced by the mass lumping with the correction are small enough to be dominated by the overall approximate character of the velocity model. Numerical experiments suggest that using mass lumping together with the correction is up to 50% faster than using the un-lumped equations however precise computational speeds and speed ups are determined by the polynomial order and number of elements used.
3.3.7 Computation of group velocities

The solution of (3.18) can then be used to find the frequency $\omega$ as a function of phase speed $c$ where $k = \omega/c$ for the $F_L$ and $F_R$ modes. In Aki and Richards [2009, chapter 7], it was shown that the group velocities $U_l$ and $U_r$ for $F_L$ and $F_R$, respectively, are given by

$$U_l(k, \omega) = \frac{I^L_2}{c I^L_1}, \quad (3.22a)$$
$$U_r(k, \omega) = \frac{I^R_5 + \frac{r^R}{2\pi}}{c I^R_1}. \quad (3.22b)$$

With the above, we can compute $U$ and $\omega$ for any $c$. We want, however, to compute $U$ as a function of $\omega$. In order to explain our approach and to justify our choice of method, we will start by briefly discounting two standard approaches. The first standard way to compute $U$ as a function of $\omega$ would be to iteratively (using the Gauss-Newton method, for example) find the $c$ corresponding to a desired $\omega$ and subsequently compute the group velocity $U$. This would, however, be computationally expensive, since this procedure would need to be carried out for all required $\omega$ and each $\omega$ would require many eigenvalue computations in the iterative process. The second standard method would be
to use a spline interpolation between a set of pre-computed \( \{ U, \omega \} \) points. For a linear spline to be sufficiently accurate, we would require many points to be pre-computed, which would be computationally expensive. To improve on this, we could use a quadratic spline; however, due to the unavoidably irregular spacing between known \( \omega \) points, a quadratic spline that uses three points suffers heavily from Runge’s phenomenon and is inaccurate as pointed out by Pozrikidis [2005].

Our method for computing \( U \) as a function of \( \omega \) uses a quadratic spline that has even \( \omega \) point spacing over one quadratic interpolation and so does not suffer from Runge’s phenomenon. To derive our method, we start from the definition of group velocity \( U = \frac{d\omega}{dk} \).

Assuming that \( M \) pre-computed pairs of \( \{ U, \omega \} \) are ordered, we can consider the \( j \)th value \((j < M)\) and we can write

\[
U = \frac{d\omega}{dk} \Rightarrow k_{j+1} - k_j = \int_{\omega_j}^{\omega_{j+1}} \frac{d\omega}{U(\omega)}, \quad (3.23)
\]

Provided \( M \) is large enough (e.g. \( M \sim 20 \) with evenly spaced \( c \) values), the Simpson’s rule will be accurate enough to compute the above integral, thus

\[
k_{j+1} - k_j = \int_{\omega_j}^{\omega_{j+1}} \frac{d\omega}{U(\omega)}, \quad (3.24a)
\]

\[
\approx \frac{\omega_{j+1} - \omega_j}{6} \left( \frac{1}{U_j} + \frac{4}{U_{j+\frac{1}{2}}} + \frac{1}{U_{j+1}} \right), \quad (3.24b)
\]

where \( U_{j+\frac{1}{2}} = U\left(\frac{\omega_{j+1} + \omega_j}{2}\right) \). Rearranging leaves us with

\[
U_{j+\frac{1}{2}} \approx 4 \left( 6 \frac{k_{j+1} - k_j}{\omega_{j+1} - \omega_j} - \frac{1}{U_j} - \frac{1}{U_{j+1}} \right)^{-1}. \quad (3.25)
\]

This gives us an extra point in the middle of \( \omega_j \) and \( \omega_{j+1} \) that can be used for the quadratic spline between these two points. The accuracy of this method is shown in Fig. 3.5. We can now find \( U \) for any \( \omega \) by only pre computing \( M \sim 20 \) eigenvalues. Thus, we now have an efficient way of computing the dispersion curves for any arbitrary transversely-isotropic velocity model.

### 3.3.8 Calculation of FZTW waveforms

Green’s (amplitude) function representations can be used to compute the FZTW waveform induced by different source types, including a moment tensor point source, as seen in
Aki and Richards [2009]. These representations are based both on the trapped modes (eigenvectors) and the associated resonant frequencies (eigenvalues) which can be computed using the approach described above.

The Green’s representation for the waveform (three components of the displacement) \( s \) from a single type of guided wave is of the form

\[
s(r, \phi, z, t) = \sum_{H=1}^{\infty} \int_{\omega_H}^{\infty} g(\omega, k_H, r, \phi, z) \exp(i(k_H(\omega)r - \omega_H t)) d\omega
\]

(3.26)

where \( r \) is the distance from the source to the receiver along-fault, \( z \) is the distance from the centre of the fault, \( \phi \) is the angle on the \( x\)-\( y \) plane, and \( t \) is time. The waveform \( s \) is the (vector-valued) displacement in the three directions and \( g \) is the (vector-valued) Green’s function. The Green’s function \( g \) depends on the type and properties of the source, the elastic model and the (possible) attenuation. The index \( H \) is the harmonic number, and typically only the first few harmonics are needed since the higher harmonics are strongly attenuated or are not generated in the first place. Furthermore, each harmonic has a minimum frequency \( \omega_H^* = \omega_H(c_{\text{max}}) \).
For the computation of the integrals in Eq. (3.26) by any quadrature scheme, we need to compute the values \( g(\omega_H, k_H, r, \phi, z) \) at the quadrature points (frequencies). To compute each of these, we need to solve the respective eigenvalue problems, which is computationally the most complex task. To minimise the number of solutions to eigenvalue problems, we aim first to change the integration with respect to \( \omega \) into an integral with respect to the phase speed \( c \). Let us define the following functions:

\[
\frac{d\omega_H}{dc} = \left( \frac{dc}{d\omega_H} \right)^{-1} = \left( \frac{d(\omega_H/k_H)}{d\omega_H} \right)^{-1} = \frac{k^2_H}{k_H - \omega_H/U_H}, \tag{3.27a}
\]

\[
\frac{dk_H}{dc} = \left( \frac{dc}{dk_H} \right)^{-1} = \left( \frac{d(\omega_H/k_H)}{dk_H} \right)^{-1} = \frac{k^2_H}{k_HU_H - \omega_H}, \tag{3.27b}
\]

\[
\hat{g}_H(c, r, \phi, z) = -g(\omega_H, k_H, r, \phi, z)\frac{d\omega_H}{dc}, \tag{3.27c}
\]

\[
\phi_H(c, x, t) = k_H(c)r - \omega_H(c)t, \tag{3.27d}
\]

where \( U_H = d\omega_H/dk_H \) is the group velocity calculated from the Eqs. (3.22). As a result changing the integration variable, the definite integrals are now over a finite interval,

\[
s(r, \phi, z, t) = \sum_{H=1}^{\infty} \int_{c_{\min}}^{c_{\max}} \hat{g}_H(c, r, \phi, z) \exp(i\phi_H(c, x, t)) dc, \tag{3.28}
\]

where \( c_{\min} \) and \( c_{\max} \) are the corresponding integration limits over \( c \). For the quadrature with respect to \( c \), we can choose the quadrature points to be evenly spaced. The negative sign in Eq. (3.27c) is due to having \( \omega_H(c_{\max}) = \omega^*_{H} \) and \( \omega_H(c_{\min}) = \infty \) and the related exchange of integration limits.

To interpolate between values of \( \omega \) and \( k \) we utilise the fact that \( \frac{d\omega}{dc} \) and \( \frac{dk}{dc} \) are known at each point (Eqs. 3.27a-3.27b). This allows us to interpolate between the known \( k \) and \( \omega \) points with the third order polynomial spline that has the correct derivative and correct value at each known point. This cubic interpolation loses accuracy near \( c_{\min} \) and \( c_{\max} \) because the magnitude of \( \frac{d\omega}{dc} \) or \( \frac{dk}{dc} \) tends to \( \infty \) near these boundaries. In practice, this is not an issue since \( \hat{g} \approx 0 \) near \( c_{\min} \) and \( c_{\max} \). Once we have increased the number of known points using this cubic spline we can then numerically integrate by treating \( \hat{g}, k, \omega \) as linear functions between known points [e.g. Aki and Richards, 2009].

An example of a computed waveform generated by an infinite line source using this interpolation method is shown in Fig. 3.6. It can be seen that using this interpolation method reduces the error by around an order of magnitude.
Figure 3.6: Effect of interpolating extra frequency and wavenumber points before the Green’s functions are integrated to produce a waveform. (a) True $F_L$ waveform (displacement amplitude) for an infinite line source (2D) using the velocity profile shown in Fig. 3.3a. The source and receiver are centred in the fault zone 5 km apart and only the fundamental harmonic is shown. This true waveform is computed using 10,000 eigenvalues. (b) The error in the waveform when it is calculated using 20 eigenvalues only and when an additional 20 points are interpolated between each eigenvalue and amplitude point. It can be seen that using the interpolation method increases accuracy by around an order of magnitude.

3.4 Comparison with layered velocity model

The strength of this finite element method is its ability to deal with gradational velocity models efficiently. Gradational velocity may also be approximated using many homogenous layers. This allows one to use methods such as the propagator matrix [Haskell, 1953, Thomson, 1950]. To compare these two methods, we compute the eigenvectors with one seven point polynomial basis vector and compare the result to the solution from a seven layer approximation of the velocity model. Such a model is shown in Fig. 3.7. These two methods both have the same number of unknowns and therefore similar computational complexity and computational time. Figure 3.7 shows that the finite element method used here is significantly more accurate when dealing with gradational velocity models. Further computations suggest that, typically, $\sim 40$ homogenous layers are needed to approximate the velocity model seen in Fig. 3.7 in order to obtain accuracy similar to that derived from the finite element method with one seven point polynomial basis function. This ultimately means that, when simulating trapped waves in a gradational velocity model, the layer matching methods are going to have $\sim 5$ times larger matrices than the methods proposed in this paper and therefore will be at least 5 times slower.

If the velocity model of interest can be well approximated by a small number of homogenous layers, then it is likely that the methods in this Chapter are less efficient than other methods
Figure 3.7: Shows the effectiveness of the proposed finite element method of dealing with gradational velocity models. (a) The true velocity model and a seven layer approximation. (b) The true eigenvalues (frequencies) of the $F_L$ fundamental harmonic for the true velocity model that is calculated using 50 7th order basis functions. (c) The errors induced by using one 6th order polynomial (7 unknowns) as a finite element basis or using a seven homogeneous layer approximation. It can be seen that the finite element method is several orders of magnitude more accurate.

[e.g. Ben-Zion, 1998, Haskell, 1953, Thomson, 1950]. Lithologically bounded waveguides such as coal seams can often be thought of as homogenous layers, and while the methodology in this Chapter will work in these situations, previously published methods are more efficient. Many fault zones, however, can be thought of as having gradational boundaries [Leary et al., 1987], therefore our methodology is primarily aimed at FZTWs.

### 3.5 Discussion

We proposed a computational scheme to compute the dispersion curves and observed waveforms corresponding to arbitrary transversely-isotropic velocity models. The motivation for a computationally efficient approach is to allow the use of such a model in inversions. While the forward model may need to be computed a large number of times when employing, for example, a Gauss-Newton type algorithm, we note that the exploration of the (posterior) uncertainty using a Markov chain Monte Carlo algorithm typically calls for forward computations in the order of millions of times. In such cases, even moderate
speed-ups are welcome.

The main limitation of this model is that it assumes no variation in structure in the direction of FZTW propagation. While this model is an improvement on previously employed semi-analytic models, even this model is unrealistic since the elastic properties of the fault zone and country-rock typically change with depth due to increasing pressures and temperatures. To try to reduce the above limitation, an approximation can be made that represents this 2-D variability with 1-D slices, as was done in Wu et al. [2010]. However, in general, a fault zone that changes in width will have different dispersion and amplitude information due to different reflection angles than a model that is based on 1-D slices [Li and Vidale, 1996]. Such a model also does not allow for variation in the third (transverse to propagation) direction.

We have assumed that transverse-isotropy is the dominant form of anisotropy. It is possible, however, that the anisotropic structure of fault zones is more general [Li et al., 1987]. The proposed approach could in principle be generalised to an arbitrary anisotropic model. This would, however, incur a significant additional numerical cost since the displacements in all three directions would be coupled where currently the SH motion is decoupled.
The Effect of Gradational Velocities and Anisotropy on Fault-Zone Trapped Waves

4.1 Summary

Synthetic Fault-Zone Trapped Wave (FZTW) dispersion curves and amplitude responses for $F_L$ (Love) and $F_R$ (Rayleigh) type phases are analysed in transversely-isotropic 1-D elastic models. We explore the effects of velocity gradients, anisotropy, source location and mechanism. These experiments suggest: (i) A smooth exponentially decaying velocity model produces a significantly different dispersion curve to that of a three-layer model, with the main difference being that Airy phases are not produced. (ii) The FZTW dispersion and amplitude information of a waveguide with transverse-isotropy depends mostly on the shear wave velocities in the direction parallel with the fault, particularly if the fault zone to country-rock velocity contrast is small. In this low velocity contrast situation, fully isotropic approximations to a transversely-isotropic velocity model can be made. (iii) Fault-aligned fractures and/or thin layers of different rock composition in the fault zone that cause transverse-isotropy enhance the amplitude and wave-train length of the $F_R$ type FZTW. (iv) Moving the source and/or receiver away from the fault zone removes the higher frequencies first, similar to attenuation. (v) In most physically realistic cases, the radial component of the $F_R$ type FZTW is significantly smaller in amplitude than the
transverse.

4.2 Introduction

Fault-Zone Trapped Waves (FZTWs) are seismic energy that is guided by the low velocity zone of a mature fault through total internal reflection. Three different modes of FZTWs have been observed. The $F_L$ FZTWs are analogous to Love mode surface waves. These are made up entirely of shear waves (S-waves) and have fault-parallel polarisation that is orthogonal to the direction of travel (radial direction). This is also known as SH motion on a fault. The first reported observations of $F_L$ in a fault zone were in controlled source surface-to-borehole studies at Oroville, California [Leary et al., 1985, 1987, Li and Leary, 1990, Li et al., 1990]. The $F_R$ FZTWs are analogous to Rayleigh mode surface waves [Malin and Lou, 1996]. Most of the $F_R$ energy propagates as S-waves with a smaller proportion of the energy propagating as Primary waves (P-waves). They have radial and fault orthogonal polarisation which is also known as P-SV motion on a fault. While the $F_L$ and $F_R$ FZTWs are observed as dispersive coda arriving after the S-wave, $F_\phi$ waves arrive between the P- and S-wave direct arrivals. The $F_\phi$ are a leaky wave type FZTW that are only partially trapped by the waveguide. Like the $F_R$, $F_\phi$ have P-SV polarisation. However, the $F_\phi$ energy is dominated by P-waves and energy leaks from the waveguide in the form of S-waves. These were first observed in passive source experiments on the San Andreas Fault [Ellsworth and Malin, 2011].

FZTWs are primarily used for the imaging and understanding of fault zones. They have been used to estimate the across-fault properties of a low velocity zone at a resolution of tens of metres [Ben-Zion et al., 1992, 2003b, Eccles et al., 2015, Ellsworth and Malin, 2011, Haberland et al., 2003, 2007, Hough et al., 1994, Lewis et al., 2005, 2007, Li et al., 1999, 1998, 1997b, Li and Malin, 2008, Li et al., 2014b, Li and Vernon, 2001, Mizuno et al., 2004, Wu et al., 2010]. Several different analytical, numerical and computational approaches have been proposed and employed to investigate the effects of 2-D and 3-D variations in rock properties, source type and location. In the following we give a brief review of these approaches and findings.

Two-dimensional finite difference modelling of $F_L$ was done in Li and Vidale [1996]. They showed that in order to observe a FZTW, the fault zone must be approximately continuous between the source and the receiver and that the earthquake must have occurred within a few fault zone widths of the fault. They also showed that many 2-D structural variations within the fault zone such as bending, moderate changes in fault zone widths and bifurcating faults cause a reduction in FZTW energy but do not prevent them from being
The analytic solution for two dimensional $F_L$ waves for a fault zone made of homogenous layers was derived by Ben-Zion [1990] and Ben-Zion [1998]. They showed that increasing velocity contrast between the fault and country-rock increases the trapping efficiency of the waveguide. There are significant trade-offs between fault zone width, propagation distance along the fault and velocity contrast which means that a variety of different fault zone models can produce the same or similar FZTWs.

Two-dimensional finite difference modelling of both $F_L$ and $F_R$ FZTWs were compared to analytical solutions by Igel et al. [1997]. They discovered that rapid changes in the S-wave velocity model with depth that is greater than the fault zone to country-rock velocity contrast can destroy FZTWs. On the other hand, a geologically expected fault zone widening at the surface has only a small effect on the waveform, while random perturbations in the velocity model that are smaller than the velocity contrast have no significant effect on the waveform.

The depth extent of low velocity zones has proven controversial. Numerous studies have suggested that the low velocity zone persists for greater than 10 km, which is approximately the entire seismogenic zone [e.g. Li and Vernon, 2001, Li et al., 2000, 2004, 2002]. Other researchers, utilising some of the same or similar FZTW data, have proposed that the low velocity zone terminates at a much shallower 5 km depth or less [e.g. Ben-Zion et al., 2003b, Ben-Zion and Sammis, 2003, Fohrmann et al., 2004, Lewis et al., 2005, Peng et al., 2003]. Part of the difficulty in determining the depth is because of the high level of uncertainty and non-uniqueness associated with FZTWs [Michael and Ben-Zion, 1998b]. This means that different fault zone models can produce the same or very similar waveforms. Another issue is that it is difficult to determine if seismic waves are entering the fault zone from beneath or tunnelling in higher up in a fault zone [Wu, 2008].

Three-dimensional finite difference modelling of FZTWs was followed by Igel et al. [2002] and Jahnke et al. [2006]. They showed that, in general, infinite 2-D line sources can not be used to approximate 3-D point sources for FZTW. However, applying an approximate 3-D to 2-D conversion to the data can be used to make this approximation feasible in some cases. They also showed that 3-D velocity variations that are smaller than the velocity contrast between the fault and country-rocks overprinting on a 2-D structure do not significantly affect the recorded FZTW coda and that the other observed 2-D effects mentioned above still hold in the 3-D case.

A widely used approximation is that a fault zone can be thought of as a single layer of low velocity rock surrounded by intact rock [Ben-Zion and Aki, 1990, Calderoni et al., 2012,
Eccles et al., 2015, Lewis and Ben-Zion, 2010, Li et al., 1998, 1990, Mizuno and Nishigami, 2006, Wu et al., 2008]. Such a three-layer model is computationally straightforward to implement [Ben-Zion and Aki, 1990] and allows for a simple comparison between different fault zones. A five-layer extension for the three-layer model was used with finite difference modelling in Li and Malin [2008]. While the fault core (e.g. fine grained fault gouge) may create a sharp velocity contrast, the geological fault core is observed to be far narrower (0.01 m-10 m) than the modelled waveguide (40-1000 m) [Kolyukhin and Torabi, 2012]. The waveguide is more commonly attributed to the fracture damage zone of the fault and hence velocities across the fault zone, like fractures, could be predicted to behave in exponential or power law fashion across the fault [Johri et al., 2014, Leary et al., 1987, Li and Vidale, 1996, Mitchell and Faulkner, 2009, Savage and Brodsky, 2011].

Due to preferential alignment of fractures and parallel layering, fault zones are generally considered to be anisotropic [Cochran and Vidale, 2001, Leary et al., 1987, Li et al., 1987]. In many cases, this anisotropy can be approximated by assuming transverse-isotropy, which means that rays travelling parallel to the principal fault plane have the same velocity [Leary et al., 1987, Li et al., 1987]. Anisotropy in wave-guides has been widely explored in the context of coal seam seismics [Buchanan et al., 1983, Lou and Crampin, 1991]. They showed that the dispersion curve of an anisotropic waveguide is similar to an isotropic waveguide but the precise numerical values of group velocity and phase velocity differ. Furthermore, Nakamura and Takenaka [2006] used a matrix propagator method to simulate FZTWs in a 2-D homogeneous transversely-isotropic waveguide surrounded by isotropic country-rock. These synthetic earthquakes are located beneath an array of seismometers. They performed a shear wave splitting analysis to the synthetic waveforms and concluded that anisotropy can be detected when the focal mechanisms of the earthquake are dip-slip but not when they are strike-slip.

In this Chapter, we study the differences in the FZTWs that are generated by different types of 1-D fault-perpendicular velocity profiles, source/detector locations and anisotropy; the first and last of these have never previously been considered via synthetic modelling. In particular, we focus on the dispersion curves and amplitude responses. We also study the implications of the focal mechanism to the modelled FZTWs.

4.3 Computational Approach Overview

For the computation of the FZTW responses, we used a finite element type solver that computes dispersion, amplitude response and FZTW waveforms for a velocity model that varies in the across-fault direction only [Gulley et al., 2017b]. This solver can efficiently
Figure 4.1: Setup of the fault zone and coordinate system used. The direction $x$ is pointing directly from source to receiver, parallel the the fault. The direction $z$ is normal to the fault and $y$ is the remaining direction in the right hand coordinate system. All FZTWs have energy traveling in the $x$ direction. The P-SV displacement is in the $x$-$z$ plane. This corresponds to the $F_R$ FZTW with the $F_R$ radial displacement being in the $x$ direction and the $F_R$ transverse displacement in the $z$ direction. The $F_L$ FZTWs have displacement in the $y$ direction. The ray angle defined here is used in Section 4.5. Note that this Figure is not to scale and FZTW producing sources will typically be several fault widths from the receiver.

handle arbitrary gradational transversely-isotropic velocity models allowing them to be easily investigated. It also allows for separate computation of dispersion curves and can compute the FZTW waveform from different source types including a moment tensor point source.

4.4 Isotropic models: gradational velocities, source locations and FZTW partitioning

1-D isotropic velocity models are used to investigate the effects on modelled FZTWs of gradational versus layered velocity models, the dependance on source and receiver locations, the relative strengths of the $F_L$ or $F_R$ components and the dependence of these components on the direction of the earthquake slip vector. The coordinate system and displacement directions are shown in Fig. 4.1.

4.4.1 Gradational versus layered velocity profiles in the fault zone

In order to investigate the effect of gradational velocity models we compute the dispersion and amplitude response of three different velocity models. The velocity models are a three-layer model, a five-layer model and an exponentially decaying ($1/cosh$) model.
These velocity models, amplitude responses and the group velocities of the two lowest harmonics of the $F_L$ and $F_R$ FZTWs are shown in Fig. 4.2. It can be seen that the three-layer and the five-layer velocity models have significantly different dispersion curves to the exponentially decaying velocity model. The group velocities of the three- and five-layer models have local minima below the minimum velocity of the waveguide (the Airy phase) and even lower group velocities are seen in higher harmonics. On the other hand, the exponentially decaying velocity profile does not exhibit an Airy phase. The exponentially decaying velocity model has a minimum group velocity which is close to the minimum velocity of the waveguide for all harmonics. Lower minimum group velocities lead to increased wave-train/coda length. The five-layer model gives a reasonable approximation of the exponentially decaying model only at low frequencies, albeit with lower amplitude responses.

The velocity profile also affects the distribution of energy over the fault. The amplitude responses shown Fig. 4.2 represent the amplitude of each frequency component and shows how well the waveguide propagates each frequency from the source to the receiver location. As seen in Fig. 4.2 the three-layer waveguide propagates around 20% more seismic energy to the receiver located near the middle of the waveguide than the exponentially decaying (1/cosh) waveguide. Although not shown in any figures, the amplitude response of the exponentially decaying waveguide is greater than the three-layer waveguide on a receiver located many fault widths away from the fault zone. This insures that both waveguides trap the same amount of energy as is physically expected (ignoring attenuation). It is also important to note that while the amplitude responses shown Fig. 4.2 have a maximum, this may not correspond to the frequency with the maximum amplitude seen on the recorded waveform. This is because the recorded waveform depends on both the amplitude response and the dispersion. For example, a recorded waveform may have a maximum recorded frequency at the Airy frequency due the effect of dispersion causing many similar frequencies to arrive at the same time.

### 4.4.2 Varying source and receiver locations across the fault zone

Different source/receiver locations can affect the amplitude response of FZTWs. In particular, we are interested in how close to the fault zone a source/receiver needs to be to generate/observe FZTWs. We note that keeping the receiver in a constant location and moving the source is approximately equivalent to keeping the source constant and moving the receiver which can be seen through analysis of the surface wave functions in Aki and Richards [2009]. The topic of source and receiver locations was first investigated using an analytic solution for FZTWs in a layered medium by Ben-Zion and Aki [1990]. Later, 2-D
Figure 4.2: (a) The three isotropic velocity models. The exponential decay model is a 1/cosh function. The elastic values and boundaries of the three-layer and five-layer stratified fault zone models are chosen so that they are least squares approximations to the exponential velocity model. The effective width of all three models equals 100 m. Only the S-wave velocities are shown; the P-wave velocities and density profiles have the same shape with minimum and maximum values of 2.85-3.2 km/s and 2600-2700 kg/m$^3$. (b-c) The $F_L$ and $F_R$ dispersion curves for the fundamental and the lowest two harmonics, respectively. The exponential velocity model has qualitatively different dispersion to the layered velocity models. In particular, the exponentially decaying model does not exhibit an Airy phase (a local minimum). (d-f) The amplitude responses of the fundamental generated by an isotropic infinite line source that is an impulse function in time. The source and receiver are both offset 25 m from the fault centre and the radial distance from source to receiver is 10 km. The radial direction refers to the direction oriented along-fault between the source and the receiver. The $F_R$ transverse component has displacement perpendicular to the fault plane and the $F_L$ component has displacement parallel with the fault plane. A very low spatial attenuation of $Q_\alpha = Q_\beta = 1000$ is assumed.
finite difference modelling was done by Li and Vidale [1996]. These researchers suggested that FZTWs can be observed from sources and receivers that are within three fault widths of the receiver. Their study considers only the waveforms rather than the amplitude responses. Here, we compute the amplitude responses for the fundamental and the first harmonic and their dependence on the source location.

The amplitude responses and the corresponding waveforms associated with three infinite line sources that are located at increasing distances away from the fault zone are shown in Figs. 4.3 and 4.4, respectively. These Figures indicate that moving the source away from the fault zone causes a reduction in trapped amplitude at all frequencies but the higher frequencies are removed fastest. The corresponding eigenmodes are shown in Fig. 4.5. This removal of the higher frequencies occurs because the lower frequency trapped modes penetrate further into the country-rock as indicated by Fig. 4.5. Anelastic attenuation also causes the reduction of higher frequencies faster than low frequencies. This means that, when imaging fault zones using FZTWs, there is a significant trade-off between the anelastic attenuation parameter ‘Q’ and the location of the source across the fault zone. Both of these parameters significantly affect the amplitude of the FZTW and have only a minor effect on the dispersion of the wave. This means that the influence of these two parameters can be significantly reduced when modelling FZTWs by considering the dispersion.

4.4.3 Relative amplitudes of FZTW components

FZTWs can cause displacement in all three directions (see Fig. 4.1). These components often contain different amounts of energy. This can be understood by looking at the resonant waveforms of the waveguide (eigenvectors). The resonant waveforms correspond to free oscillations in the waveguide at a fixed frequency. Examples of different resonant waveforms are shown in Fig. 4.5. In what follows, “radial” refers to particle motion that is oriented along the line between source and receiver while “transverse” refers to motion that is perpendicular to this direction, as shown in Fig. 4.1. The results show that the radial component of the \( F_R \) resonant waveform is significantly smaller than the transverse component. This is because the \( F_R \) resonant waveform is dominated by S-waves while the P-waves propagate in an evanescent form. For example, there is \( \sim 8 \) times the amount of kinetic energy in the shearing motion (S-wave) than in the compressional motion (P-wave) of the 15 Hz trapped \( F_R \) fundamental resonant waveforms as seen in Fig. 4.5. As the FZTW propagates predominantly in the radial direction, the transverse component contains most of the S-wave energy while the radial component contains the majority of the P-wave energy. The result is that the radial component of the full \( F_R \) waveform can be
expected to be significantly lower in amplitude than the transverse component as shown in Figs. 4.3 and 4.4. This can be observed in FZTWs produced on the Alpine Fault [Eccles et al., 2015].

The exception to this rule is if the P-wave velocity of the fault zone is less than the S-wave velocity of the country-rock. In such a case, the P component would not propagate as an evanescent wave and the guided wave would therefore no longer be dominated by the S-wave energy. However, such a situation is unlikely as the maximum country-rock to fault rock S-wave velocity ratio that has been estimated is 1.6 [Haberland et al., 2003] which is less than the fault rock P-wave to S-wave velocity ratios recorded in fault zones around the world [Hung et al., 2009, Li et al., 2004].

Figure 4.3: Amplitude responses for the velocity models shown in Fig. 4.2a for different source offsets from the fault centre. The sources are located 10 km along-fault from the receiver and the offsets of the five sources from the fault are 0, 1/2 and 2 fault widths (100 m) from the fault centre. These offsets are indicated in the top middle subplot. All sources are isotropic infinite line sources (2-D) with a temporal impulse and the receiver is offset from the fault centre by 1/4 of the fault zone width (25 m). Here, we have assumed very low spatial attenuation of $Q_\alpha = Q_\beta = 1000$. It can be seen that the effect of moving the source away from the fault zone removes high frequencies. It is also seen that the radial component has complex behaviour and is generally much lower in amplitude (also see Fig. 4.5).
Figure 4.4: The observed waveforms generated by the amplitude responses and source types shown in Fig. 4.3. The corresponding velocity models and dispersion curves are shown in Fig. 4.2. The three source positions used are when the source is 0 (top), 1/2 (middle) and 2 (bottom) fault widths (100 m) from the fault centre as indicated in the first column of subplots. The exponential decay waveguide exhibits a shorter wave train with frequency increasing monotonously with time. The fundamental and all harmonics of the exponentially decaying waveguide finish at a single point in time that corresponds to propagation distance divided by minimum velocity of the waveguide (6.7 s = 10000 m /1500 m/s). The three and five-layer velocity models have some differences in their FZTW behaviour. There is a maximum amplitude peak which corresponds to the Airy phase after which the lower amplitude higher harmonics can be observed. At approximately 6.7 s for the three and five-layer wave-guides multiple frequencies from the same harmonic are recorded. This does not occur in the exponential decay waveguide.
Figure 4.5: Three different resonant waveforms (waveguide eigenvectors) from the three different velocity models in Fig. 4.2. The waveforms are normalised by the respective kinetic energies and plotted on the same scale. The radial and transverse $F_R$ resonant waveforms are coupled together. The $F_R$ radial resonant waveform is significantly smaller in amplitude than the $F_R$ transverse component. As the FZTW waveform is made up of all resonant waveforms, this implies that the radial part of the $F_R$ guided wave should be significantly lower in amplitude than the transverse part. It can also be seen that, for the same harmonic, the lower frequencies penetrate further in the country-rock.

The shape of the trapped modes is similar for the $F_L$ mode and the transverse component of the $F_R$ mode as shown in Fig. 4.5. This is because the $F_L$ mode is made up of shearing motion only and the $F_R$ mode’s transverse component is shearing dominated, as discussed above.

The trapped modes shown in Fig. 4.5 also indicate that the $F_R$ radial components of the fundamental have a zero near the middle of the fault, which is where the $F_R$ transverse component and $F_L$ exhibit their highest amplitudes. Thus, recording FZTWs in the centre of the fault zone will maximise the signal to noise ratio of the transverse components of the fundamental FZTWs but will result in the radial component having a smaller amplitude.
Figure 4.6: The amplitude response at 15 Hz (fundamental) of the 100 m wide three-layer model waveguide corresponding to Fig. 4.2a. This amplitude response was generated from a moment tensor point source. The source and receiver are located at the edge of the fault zone (50 m). The earthquake slip vector is assumed to be on the same plane as the fault zone. The figure shows that when the earthquake slip vector is in the direction towards the receiver, only the $F_R$ mode is generated. Correspondingly, when the slip vector is perpendicular to the direction towards the receiver only the $F_L$ mode is generated. Similar results are produced by higher harmonics, different frequencies and different elastic models.

4.4.4 Orientation of the slip vector to the receiver

This section looks at the source receiver orientation for an earthquake with a moment tensor point source. We assume here that the earthquake slip vector lies on the plane at the centre of the fault zone. When the earthquake slip vector is aligned with the source-receiver direction, only the $F_R$ mode is generated in those directions, i.e. $F_L$ is not seen (Fig. 4.6). This is because with this slip vector there is little energy generated in the direction in which $F_L$ waves are polarised. Correspondingly, when the slip vector is perpendicular to the source-receiver direction, only the $F_L$ mode is recorded at the receiver. The implications are that in some cases, focal mechanisms may need to be taken in to account when modelling FZTWs. This also means that FZTWs could possibly be used in the future to assist in the estimation of focal mechanisms, provided there are multiple seismometers that are spaced sufficiently far apart along the fault zone to capture a large range of different angles to the source.

4.5 Anisotropy

Fault-zones are expected to be anisotropic due to preferential alignment of fractures and fault-parallel layering of rock beds [Cochran and Vidale, 2001, Leary et al., 1987, Li et al., 1987]. Many fault zones can be assumed to be transversely-isotropic if the fractures and/or rock layers are aligned parallel to the fault plane [Leary et al., 1987]. Here we investigate
the effects of transverse-isotropy on FZTWs.

### 4.5.1 Parameterisation of anisotropy

We assume that the fault plane is the \(x-y\) plane and that this is the plane of transverse-isotropy. Without loss of generality, we can therefore take wave propagation directions to be only in the \(x-z\) plane. Figure 4.1 shows the propagation angle \(\theta\) that describes a ray angle in the \(x-z\) plane with \(\theta = 0^\circ\) meaning propagation in the \(z\) direction and \(\theta = 90^\circ\) being propagation in the \(x\) direction. In this situation, the P-waves have particle displacement in the \(x-z\) plane only and we denote the respective velocity by \(\alpha\). The S-waves with particle displacement in the \(x-z\) plane have velocities denoted \(\beta^V\) and S-waves with particle displacement in the \(y\) direction only have velocity \(\beta^H\).

To investigate the effects of anisotropy on FZTWs we make use of the parametrization of transverse-isotropy as defined in Thomsen [1986]. These parameters are \(\alpha_0, \beta_0, \epsilon, \gamma, \delta\). The \(\alpha_0\) and \(\beta_0\) are the P- and S-wave velocities in the \(z\)-direction. The parameters \(\epsilon\) and \(\gamma\) describe the anisotropy of the P- and SH-velocities, respectively, and \(\delta\) is a further parameter related to P- and SV-wave velocities. The velocity of a particle \(\alpha(\theta)\) propagating at an angle \(\theta\) and its relationship to the other parameters were derived in Thomsen [1986] as:

\[
\alpha(\theta) = \alpha_0 \left[ 1 + \epsilon \sin^2(\theta) + D^*(\theta) \right]^{1/2}, \tag{4.1a}
\]

\[
\beta^V(\theta) = \beta_0 \left[ 1 + \alpha_0^2 \beta_0^{-2} \epsilon \sin^2(\theta) - \alpha_0^2 \beta_0^{-2} D^*(\theta) \right]^{1/2}, \tag{4.1b}
\]

\[
\beta^H(\theta) = \beta_0 \left[ 1 + 2 \gamma \sin^2(\theta) \right]^{1/2}, \tag{4.1c}
\]

where

\[
D^*(\theta) = \frac{1 - \beta_0^2 \alpha_0^{-2}}{2} \left( 1 + \frac{4(2\delta - \epsilon)}{1 - \beta_0^2 \alpha_0^{-2}} \sin^2(\theta) \cos^2(\theta) + \frac{4\epsilon(1 - \beta_0^2 \alpha_0^{-2} + \epsilon)}{(1 - \beta_0^2 \alpha_0^{-2})^2} \sin^4(\theta) \right)^{1/2} - 1. \tag{4.1d}
\]
From equations (4.1), we have:

\[
\begin{align*}
\alpha(0) &= \alpha_0, \\
\alpha(90^\circ) &= \alpha_0(1 + 2\epsilon)^{1/2}, \\
\beta^V(90^\circ) &= \beta^V(0) = \beta^H(0) = \beta_0, \\
\beta^H(90^\circ) &= \beta_0(1 + 2\gamma)^{1/2}.
\end{align*}
\]

### 4.5.2 Anisotropy in wave-guides

As fault zones are known to be anisotropic, we investigate how several geologically plausible fault zones will effect FZTWs. One situation is when the fault zone has fault-aligned fractures, foliation or layers of different rock composition [e.g. Leary et al., 1987]. In this case, the \( F_R \) FZTW will have a longer wave-train and higher amplitude than the \( F_L \). This is because the fault zone will have lower SV velocities than SH velocities, which increases the ability of the waveguide to trap SV waves and corresponds to lower minimum group velocities (longer wave-trains) and higher amplitude responses. This result can be seen in the anisotropic fault zone model in Fig. 4.7. In the extreme case, in which the waveguide is almost entirely caused by anisotropy, there would be very little energy in the \( F_L \) as seen in Fig. 4.7. An alternative geologic model is an isotropic fault zone (e.g. unfoliated cataclasites) within a transversely-isotropic country-rock (e.g. schist) which could be imagined from a structure such as New Zealand’s Alpine Fault [Toy et al., 2015]. In this case the SV velocities in the country-rock are lower than the SH velocities, resulting in reduced trapping for the \( F_R \) FZTW. The \( F_R \) therefore have a shorter wave-train and lower amplitude as seen in Fig. 4.7.

### 4.5.3 Anisotropy approximations

In many cases in modelling we need to make assumptions or approximations due to insufficient knowledge or recourses. Here, we derive and investigate several high angle approximations to Eqs. (4.1). The aim of these approximations is to provide insight into the most important parameters for modelling FZTWs and to provide possible parameter reductions for modelling and inversions. We note that these approximations apply only to the fundamental (trapped) frequency. Guided waves are trapped when a propagating waves undergoes total internal reflection at each boundary [Ben-Zion, 1998]. The Snell’s
Figure 4.7: Investigates the effect of several different anisotropic fault zone models. The first row gives the anisotropic velocity model. $\theta = 90^\circ$ is for a ray traveling in the waveguide parallel direction and $\theta = 0^\circ$ is in the waveguide normal direction although we only show the range $45^\circ \leq \theta \leq 90^\circ$. The second row shows the $F_L$ and $F_R$ dispersion curves of the fundamental. The third row shows the amplitude response from the fundamental of $F_R$ transverse component and the $F_L$ wave. An infinite line source is used and the source and receiver are both located 25 m from the centre of the fault. The scale of velocity amplitude is arbitrary. The first column is an isotropic velocity model to be used for comparison. The second column is a situation that could arise from fault-aligned fractures or thin layers of different rock composition in the fault zone. The third column shows a situation where the waveguide is mostly caused by anisotropy from fault-aligned fractures or thin layers of different rock composition. The fourth column shows the effect of fault-aligned fractures or thin layers of different rock composition in the country-rock but not in the fault zone. Only the three-layer velocity model and the fundamental harmonic is shown. These results, however, are similar to other velocity models and harmonics. It can be seen that anisotropy in the fault zone causes $F_R$ to have a greater amplitude response and lower minimum group velocity than $F_L$. This leads to greater amplitude $F_R$ with longer wave trains. The country-rock having anisotropy has the opposite effect.
law for critical reflection angles shows that for guided waves to exist we must have

$$\theta \geq \theta_{\text{min}} = \sin^{-1}\left(\frac{\min(\beta)}{\min(\beta(\pm\infty))}\right) = \sin^{-1}(1 - \text{velocity contrast})$$  \hspace{1cm} (4.3)

where $\theta_{\text{min}}$ is the angle of propagation of the critically reflected ray at the location where $\beta$ is smallest. This is the smallest possible angle of propagation for the FZTW.

Fault-zones around the world have typically been modelled to have fault zone to country-rock velocity contrasts ranging between 5-50% [Ben-Zion et al., 2003b, Eccles et al., 2015, Li et al., 1994, Li and Malin, 2008, Mizuno and Nishigami, 2006]. For small velocity contrasts, the inequality (4.3) suggests that the propagation angles on the $x-z$ plane will be close to 90° (radial direction), and therefore taking a high angle approximation to Eqs. (4.1) is justified.

For larger velocity contrasts, the inequality (4.3) still suggests that the high angle velocities are significantly more important than the velocities in the direction orthogonal to the fault, since orthogonal rays are not trapped at all. In addition, a high degree of additional dependence on the velocities parallel to the fault is suggested by the fact that the range of possible trapped phase speeds depends only on the these velocities (parallel to the fault) as shown by Gulley et al. [2017b]. Note that while the S-waves are traveling in the fault-parallel direction their associated particle motion is in the fault-perpendicular direction.

The upcoming approximations are made with only a few parameter sets of country-rock and fault zone $\alpha_0$, $\beta_0$, $\epsilon$, $\delta$ and $\gamma$. However, our testing has shown that these results generalise to other choices of these parameters.

**High angle approximation for the anisotropy parameter $\delta$**

The anisotropy parameter $\delta$ influences $\beta^V(\theta)$ and $\alpha(\theta)$ when $0^\circ < \theta < 90^\circ$ as is evident from Eqs. (4.1) and (4.2). The parameter can typically only be measured accurately with very specific apparatus such as the laser-based apparatus [Blum et al., 2013]. In many cases sufficient information on $\delta$ may not be available and so $\delta$ needs to be approximated in some way.

For this outcome, we look for a high angle approximation to Eqs. (4.1). From these equations, it can be seen that as $\theta \to 90^\circ$ the term with $\cos^2(\theta)$ tends to zero suggesting that $\delta$ is negligible at high angles. If the anisotropy is due to thin layering of isotropic materials, we have $\delta < \epsilon$ [Berryman, 1979]. In order to fulfil the requirement that the elastic parameter $C_{13} > 0$, $\delta$ needs to be greater than a small negative number [Thomsen,
Thus, choosing $\delta = \epsilon/2$ seems to be a feasible central choice for $\delta$. Furthermore, setting $\delta = \epsilon/2$ makes the $\cos^2(\theta)$ term vanish in Eqs. (4.1).

As can be seen from Eq. (4.1c), $\delta$ does not affect the SH velocities so we only need to test this approximation on the $F_R$ guided waves. In order to do this, we compute the dispersion curves for $\epsilon = 0.15$ in the country-rock, $\epsilon = 0.3$ is the maximum value in the fault zone and $\delta = \epsilon/4$. This simulation is compared with the approximation $\delta = \epsilon/2$. These values for $\epsilon$ correspond to $\sim 25\%$ P-wave anisotropy in the fault zone and are realistic upper bounds [Leary et al., 1987, Thomsen, 1986]. The values $\alpha_0$, $\beta_0$ and $\rho$ are the same for all simulations in this section.

The first row of Fig. 4.8 shows different dispersion curves that are produced by this simulation for three different choices of the minimum value of $\beta_0$. It can be expected that this approximation is better at higher propagation angles. For a single dispersion curve, higher propagation angles correspond to higher frequencies and, as is seen in Fig. 4.8, the error is smaller at higher frequencies. The error is also less for smaller velocity contrast. This can be seen in the inequality (4.3) where a smaller velocity contrast leads to a minimum angle closer to $90^\circ$, hence the high angle approximation is better.

The errors are the greatest for the three-layer model and the smallest for the exponentially decaying model. This is because the exponentially decaying model and five-layer model have narrower regions where the velocity is equal to the minimum velocity. Using Snell’s law, it can be seen that a ray moving to a faster region will have an increasing ray angle. This means that waves in the exponentially decaying model and five-layer model will spend more time propagating at higher angles which increases the validity of the approximation and therefore reduces the error.

**Isotropic approximations for anisotropic propagation**

In many cases, computational FZTW models are based on isotropic wave propagation. Here, we investigate the errors that are induced by approximating an anisotropic velocity model with an isotropic approximation. A velocity model with transverse-isotropy is defined by the five parameters $(\alpha_0, \beta_0, \epsilon, \delta, \gamma)$ as described in Section 4.5.1, whereas an isotropic velocity model has only two parameters $(\alpha, \beta)$.

The high angle approximations of Eqs. (4.1) used for the $F_R$ mode are $\beta = \beta_0$ and $\alpha = \alpha_0\sqrt{1 + 2\epsilon}$. The percentual errors in the group velocity with this approximation are shown in Fig. 4.8. As expected, when comparing the second and third rows of Fig. 4.8, it can be seen that the isotropic approximation is worse than the $\delta = \epsilon/2$ approximation.
Figure 4.8: The top row shows the dispersion curve produced by setting the anisotropy parameters $\epsilon = 0.15$ in the country-rock, $\epsilon = 0.3$ is the maximum value in the fault zone and $\delta = \epsilon/4$. The three different curve sets all have maximum SV-wave velocities in the waveguide parallel direction of 2 km/s, the top curve set has a minimum S-wave velocity in the waveguide parallel direction of 1.67 km/s and the middle and bottom curve sets have 1.33 km/s and 1 km/s respectively. More information on the velocity models can be seen in Fig. 4.2. The middle row shows the percentual errors in the group velocity when approximating the anisotropy parameter $\delta$ with $\delta = \epsilon/2$. The bottom row shows the percentual errors in the group velocity when approximating the anisotropic velocity model with an isotropic model that has the same velocities in the fault-parallel direction. It can be seen that the errors increase for greater velocity contrasts and for velocity models that have a larger cross section of the fault zone at the minimum velocity.
Figure 4.9: The top row shows the dispersion curve produced by setting the anisotropy parameters $\gamma = 0.15$ in the country-rock while $\gamma = 0.3$ is the maximum value in the fault zone. The three different curve sets all have maximum SH-wave velocities in the waveguide parallel direction of 2 km/s, the top curve set has a minimum S-wave velocity in the waveguide parallel direction of 1.67 km/s and the middle and bottom curve sets have 1.33 km/s and 1 km/s respectively. More information on the velocity models can be seen in Fig. 4.2. The bottom row shows the percentual errors in the group velocity when approximating the anisotropic velocity model with an isotropic model that has the same velocities in the fault-parallel direction.

The high angle approximation for the $F_L$ mode is $\beta = \beta_0 \sqrt{1 + 2\gamma}$. To test the feasibility of this approximation, we compute a simulation with $\gamma = 0.15$ in the country-rock, while $\gamma = 0.3$ is the maximum value in the fault zone. We find that, as with $F_R$, these approximations are worse for greater velocity contrasts and for velocity models that have a larger cross section of the fault zone at the minimum velocity. The errors of these approximations can be seen in Fig. 4.9. This shows that in many cases, depending on required accuracy, the isotropic approximation can be used to simplify models.

### 4.6 Implications for modeling FZTWs

Imaging using FZTWs often produces non-unique solutions [Ben-Zion, 1998]. This means that the imaging process is sensitive to measurement and modelling errors [Tarantola,
2004]. FZTWs produced from a three-layer model have a lower minimum group velocity than a fault zone with a smoothly decaying velocity model. Therefore, making a three-layer approximation for the purposes of estimating fault zone parameters could lead to underestimation of the velocity contrast, underestimation of fault zone depth, or overestimation of fault zone width. This means that, in order to obtain more accurate constraints on variations with depth [e.g. Ben-Zion et al., 2003b], the across-fault profile should be considered carefully. Improved assumptions about the nature of the fault zone could be constrained using evidence from boreholes, geological evidence, numerical fault mechanical studies or FZTW inversions with a rich supply of data from across-fault seismometer arrays.

Our results suggest that the amplitude information of FZTWs is significantly affected by attenuation, source receiver orientation and focal mechanisms, as well as the elastic properties of the fault zone and country-rock. The dispersion information is, however, primarily affected only by the rock velocities and densities. Therefore, when modelling fault zone properties, the number of parameters can be reduced by using dispersion information only.

The anisotropic investigations have shown that the $F_L$ and $F_R$ FZTW properties are dominated by the fault-parallel S-wave velocities. These fault-parallel velocities are different for $F_L$ and $F_R$. In many cases when modelling FZTWs, isotropic approximations may be made despite the knowledge that there is actually transverse-isotropy. When this is done the S-wave velocity for $F_L$ and $F_R$ should be considered to be independent.

### 4.7 Conclusions

Analysis of synthetic dispersion curves and amplitude responses for $F_L$ and $F_R$ type Fault-Zone Trapped Waves (FZTW) in different transversally isotropic 1-D elastic models revealed:

- A smooth exponentially decaying velocity model has a qualitatively different dispersion curve to a three-layer model, the main differences being that there is no Airy phase and that the FZTW train is shorter with smaller amplitude.

- The radial component of the $F_R$ is smaller than the transverse component for most realistic fault zone velocity models.

- Moving the source or receiver away from the fault has an effect similar to attenuation, in that higher frequencies are lost first.

- A FZTW producing earthquake that has slip vector aligned towards the receiver will
cause $F_R$ to be the dominant FZTW recorded. A perpendicular slip vector will see $F_L$ being dominant.

- Fault-aligned fractures and thin layers of different rock composition causing transverse-isotropy in the fault zone will cause an increase in the $F_R$ amplitude and wave-train length compared to $F_L$. Transverse-isotropy in the country-rock reduces the $F_R$ amplitude and wave-train length compared to $F_L$.

- The transverse-isotropy parameter $\delta$ only has a small effect on the amplitude and dispersion of FZTWs when the velocity contrast is small. In this case, we suggest that fixing $\delta = \epsilon/2$ is a reasonable choice.

- Also in the case of a low velocity contrast, a transversally isotropic fault zone can be further approximated by an isotropic velocity model where the isotropic velocities are equal to the anisotropic velocities in the direction parallel to the fault. We note that the isotropic approximation is different for $F_L$ and $F_R$ mode guided waves in the same transversely-isotropic waveguide.
Investigation of the Alpine Fault’s low velocity zone using fault-zone trapped waves

5.1 Preliminary

The style of this Chapter is different from the rest of the chapters in that it is not written in the form of a scientific paper, the reason being that part of this Chapter shows the modelling that has already been published by Eccles et al. [2015]. While the author of this thesis is only a co-author of that paper, he did perform a significant proportion of the modelling. That modelling has been reproduced here and is clearly referenced.

5.2 Introduction

Recordings of FZTWs are used in this Chapter to estimate the effective width and velocity contrast of the Alpine Fault. This is achieved through the use of forward modelling of dispersion, forward modelling of the full waveform, and inverse modelling of dispersion. The three-layer model is used for all inversions, which means that the fault zone is approximated to be a simple homogeneous layer surrounded by homogeneous country-rock. An example of this is depicted in Fig. 5.1. In many cases, this three-layer model is not a good
approximation as it produces significantly different FZTWs when compared with a more geologically relevant gradational velocity model (cf. Chapter 4). The three-layer model is used here because it is simple and computationally efficient. The modelling is intended as an initial inversion on Alpine Fault data in order to provide understanding of working with real FZTW data. The results serve as initial estimates of fault zone properties upon which more advanced methodology can be built. The three-layer model has also been used elsewhere [Hough et al., 1994, Li et al., 1990, Mizuno et al., 2008, Wu et al., 2010] and therefore the present results can be compared with these previous studies.

5.3 Data

FZTW events were recorded on the 2 Hz DFDP-1 seismometers (see Section 1.2.3) over a 125 day period starting February 2011. The earthquakes were located using the methodology described in [Boese et al., 2012] and had local magnitude ($M_L$) calculated where possible using the methodology defined in [Boese et al., 2012]. The events of highest quality FZTW coda were kept for modelling and their locations are shown in Fig. 5.2. An example of an earthquake coda containing a FZTW is shown in Fig. 5.3.

5.4 Dispersion measurements

Dispersion measurements for the micro-seismic events are made by fitting the function $f(t) = A(t) \sin(\phi(t))$ to the $F_R$ signal, where $t$ is the time, $A$ is the amplitude, $\phi$ is the phase and $\omega(t) = \frac{d\phi}{dt}$ is the angular frequency. The curve fitting is done using a Bayesian
Figure 5.2: Location map of FZTW producing earthquakes. M refers to magnitude of the FZTW event. These events are recorded at the DFDP-1 boreholes at Gaunt Creek. The GEONET [Petersen et al., 2011] and SAMBA [Boese et al., 2012] stations were used for the location of the FZTW events. Events A, B, and C are referred to later in the text. Figure modified from [Eccles et al., 2015].
Figure 5.3: Micro-earthquake recorded at DFDP-1A for $M_L 1.9$ event on 16 February 2011. Located as Event A in Fig. 5.2. Data are rotated into: (a) Radial component. Particle motion in the direction of the source-receiver path. (b) Transverse component. Particle motion is perpendicular to the fault plane and perpendicular to source-receiver path. (c) Transverse component. Particle motion is parallel to the fault and perpendicular to source-receiver path. The partitioning of energy in the dispersive post-S-wave phase indicates a dominant $F_R$ (Rayleigh-type) FZTW. The vertical axis is relative velocity.

The following information is used:

- **Error model** It is assumed that the noise predominantly comes from the S-wave coda overlapping the $F_R$ signal. The form of the noise was therefore established by calculating the autocorrelation function of the S-wave coda and assuming the error model is Gaussian.

- **Prior model** The amplitude ($A(t)$) and phase ($\phi(t)$) functions have Gaussian smoothness priors. Their covariances are created using the methodology outlined in Section 1.5.6. The prior mean for $A(t)$ was calculated by a smooth interpolation between local extrema of the absolute value of the $F_R$ signal and the prior mean for $\omega(t)$ was calculated using the analysis of zeros technique [Mazeika and Drauviliene, 2010].

- **Post processing** Once the phase $\phi(t)$ has been calculated, numerical differentiation and error propagation methods [Kaipio and Somersalo, 2005] are used to obtain our estimated mean and covariance for the dispersion curve $\omega(t)$.

An example of a dispersion measurement is shown in Fig. 5.4.
5.5 The forward model

The three-layer model for \( F_R \) is used. This is where there is a central fault zone of width \( w \) surrounded by two layers of infinitely thick rock. These regions each have a P-wave velocity \( \alpha_j \), S-wave velocity \( \beta_j \), Vp/Vs ratios \( \alpha_j/\beta_j \) and density \( \rho_j \). The central fault zone has the subscript \( j = 2 \) and the country-rock regions have the subscripts \( j = 1, 3 \) for the hanging-wall and footwall respectively. Dispersion curves of group velocity (\( U \)) and frequency (\( \omega \)) as functions of the phase speed (\( c \)) can be computed using the methodology developed in Chapter 3. The frequency (\( \omega \)) can then be found for any \( U \) by utilising a Gauss-Newton iteration. This model is written as

\[
\omega = a(U; w, \alpha, \beta, \rho) + e
\]  

(5.1)

where \( e \) is measurement error and \( \alpha, \beta, \rho \) are vectors of \( \alpha_j, \beta_j, \rho_j \) respectively. The group velocity \( U \) can be written as \( U = D/(t - t_0) \) where \( t \) is arrival time, \( t_0 \) is the source time of the earthquake and \( D \) is the distance to the earthquake. Using this, the model can be re written as

\[
\omega = a(t; D, t_0, w, \alpha, \beta, \rho) + e = a(t; x) + e
\]

(5.2)

where \( x = \{D, t_0, w, \alpha, \beta, \rho\} \) represents the parameters of interest. This can be written in vector format \( a(x) \) with \( a_i(x) = a(t_i; x) \) and \( t_i \) is the different time values in the seismogram. The vectors \( \omega \) and \( e \) are also defined in a similar way.

In general Eq. (5.2) can be used to calculate all the harmonics in a FZTW signal. Only the first harmonic is used because the observed wave train is short and the higher harmonics are likely to have been attenuated [Lou et al., 1997].

In this Chapter, dispersion is both computed and simulated with \( \omega(t) \) where \( t \) is the independent variable. In most geophysical texts time/velocity are the dependent variables and \( \omega \) is the independent. We have done it with \( t \) as the independent variable because we...
have treated time points as fixed in the inversion and therefore time needs to be treated as the independent variable. An issue with our approach is that, physically, it is possible to have more than one frequency arrive at the same time as seen in Fig. 4.2. It is, however, observed in our data that frequency is only increasing with time and therefore the data suggests that we don’t have two different frequencies that arrive at the same time. Based on this observation of the data we restrict our model to only consider frequencies increasing with time i.e. the model does not produce frequencies greater than the critical frequency.

5.6 Dispersion forward modelling results

In this Section, forward modelling of dispersion is done to estimate effective width and velocity contrast of the Alpine Fault’s low velocity zone. These results have been published in Eccles et al. [2015]. The forward modelling is done by matching the dispersion curves through manual tuning of the parameters. The uncertainty estimates are obtained by finding the range of modelled dispersion curves that fit the measured data within their ±2 standard deviation estimates. Only 9 of the 28 analysed FZTW events had high enough signal to noise ratio to perform the dispersion measurements. The range of values produced by the forward modelling for fault zone width is 60-200 m and reduction in S-wave velocity between fault zone and country-rock ranged between 15-40%. While those are the range of values produced, the range of values that are contained within all of the individual earthquake forward modelling error estimates suggest that the fault zone is approximately 100-150 m wide and the velocity contrast is 15-25%. These results are shown in Table 5.1.

5.7 Waveform forward modelling results

To complement the dispersion forward modelling, waveform forward modelling of three earthquakes was also performed. The results of this section have been published in Eccles et al. [2015]. The waveforms are calculated with the computational methodology derived in Chapter 3 using moment tensor point sources. The forward modelling is done on events A,B,C shown in Fig. 5.2. The estimated parameters and waveform fits are shown in Fig. 5.5.
Table 5.1: Dispersion forward modelling results. Country-rock values used are S-wave velocity of 3450 m/s [Eberhart-Phillips and Bannister, 2002, Van Avendonk et al., 2004] and Vp/Vs ratio of 1.64 [Hung et al., 2009, Li et al., 2004, Townend et al., 2013]. Events A,B,C that are indicated in the date column are used for further analysis in this Chapter. The table is modified from the supplementary material of Eccles et al. [2015].

<table>
<thead>
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<th>Date (GMT)</th>
<th>Time</th>
<th>Latitude</th>
<th>Longitude</th>
<th>Depth (km)</th>
<th>ML</th>
<th>Fault zone width (m)</th>
<th>Fault zone velocity contrast</th>
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<td>170.37</td>
<td>5.3</td>
<td>1.8</td>
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<td>2.2</td>
<td>60-140</td>
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<td>60-140</td>
<td>20-35%</td>
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<td>5.5</td>
<td>1.3</td>
<td>60-130</td>
<td>15-40%</td>
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<td>170.3</td>
<td>4.9</td>
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<td></td>
</tr>
<tr>
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<td>-43.37</td>
<td>170.32</td>
<td>5.2</td>
<td>1.5</td>
<td></td>
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<td>11/04/2011</td>
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<tr>
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<td>-43.27</td>
<td>170.51</td>
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</table>
Figure 5.5: Forward modelling of three different FZTW producing earthquakes. These earthquakes are seen in Fig. 5.2. The country-rock parameters used are an S-wave velocity of 3450 m/s and a Vp/Vs ratio of 1.64. The velocity contrast refers to the country-rock to fault zone S-wave velocity contrast and the fault zone has a Vp/Vs ratio of 1.81. Figure is modified from Eccles et al. [2015]

5.8 Dispersion inversion results

In this section, a Bayesian inversion of a single earthquake on the Alpine Fault using a simple three-layer model is performed. The purpose of this is to explore the quality of information contained by FZTWs on fault zone parameters and the trade-offs that exist among the parameters. This is done through the use of probability densities. These inversions also aim to provide some validation of the forward modelling.

5.8.1 Data for inversion

The earthquake denoted A in Fig. 5.2 is used in the inversion. This data and $F_R$ dispersion curve can be seen in Figs. 5.3 and 5.4 respectively. To perform the inversion, the Bayesian
approach is used and the posterior probability distribution is written as

\[ \pi(x|\omega) \propto \pi_x(x)\pi_e(\omega - a(x)) \]  \hspace{1cm} (5.3)\

where \( \pi_x \) is a prior model for the unknowns \( x \). This prior model is discussed in the next section. The probability distribution of the measurement error is \( \pi_e \) which is assumed to be normally distributed zero mean with the covariance that was derived in the dispersion measurements (Section 5.4). The forward model is \( a(x) \) defined in Section 5.5, for a given \( x \). All parameters 'x' are treated as variables in the inversion.

### 5.8.2 Prior distribution

The prior model contains any information already available on the model variables from other measurements and observations. Values taken from the literature provide a guide but as information is limited and is often only collected from the near surface, ranges are generally broadened to ensure that the prior model is not unnecessarily restrictive. In particular, almost no prior information is included on the fault zone width and fault zone velocity contrast as little information is available on these parameters. The notation of mean ± standard deviation is used. All prior densities are normal densities except when indicated. The actual numerical values that take in to account all prior information are presented in Fig. 5.6 and Table 5.2 as part of the inversion results.

- Positivity constraints are enforced where appropriate. The constraints \( \beta_2 < \beta_1 = \beta_3 \) and \( \alpha_2 < \alpha_1 = \alpha_3 \) are used to ensure the fault zone has a lower velocity than the country-rock.

- The earthquake (EQ A) was located using the method outlined in Boese et al. [2012] but also including data from the DFDP-1 seismometer. The source-receiver distance is \( D = 20.5 \pm 1.5 \) km. The error in the origin time was estimated using the root mean squared value from the location algorithm, \( t_0 = 0 \pm 0.11 \) s.

- It is assumed that velocities in the hanging-wall and footwall are the same. The DFDP-1 drilling and geophysical well logging resolved the shallow hanging-wall and fault properties in detail [Townend et al., 2013] and previous active source seismic experiments [e.g. Davey, 2010, Van Avendonk et al., 2004] have provided some velocity constraint. These studies are however, not deemed to be representative of the elastic properties at depth. Hence, the prior velocity model is based on joint active source and earthquake seismology inversions [Eberhart-Phillips and Bannister, 2002], yielding S-wave velocities \( \beta_{1,3} = 3.45 \pm 0.15 \) km/s, and \( Vp/Vs \) ratios are
\[ \frac{\alpha_{1,3}}{\beta_{1,3}} = 1.64 \pm 0.03. \]

- Given the limitations in the assumed, homogeneous with depth, country-rock velocity model and earthquake location, an additional constraint is required for the model to be faithful to the observed P- and S-wave arrival times. The difference between the direct body wave arrival times was simulated using this homogeneous velocity model and those from a range of 1-D velocity gradients suggested by the 3-D tomographic models of Eberhart-Phillips and Bannister [2002]. These simulations provide the constraints

\[ t_0 + D/\alpha_{1,3} - t_p = 0.15 \pm 0.15 \text{ s} \quad \text{and} \quad t_0 + D/\beta_{1,3} - t_p = 0.2 \pm 0.2 \text{ s} \]

where \( t_p \) and \( t_s \) are the arrival times of the P- and S-wave respectively.

- The absolute values of the densities have no influence on the model, only the density contrasts. The prior model for density is based on geophysical well logs from the Alpine Fault [Townend et al., 2013] as well as the Taiwan Chelungpu Fault [Hung et al., 2009] and the San Andreas Fault [Springer et al., 2009], giving

\[ \rho_c = 1 - \frac{\rho_2}{\rho_1}, 3 = 0.1 \pm 0.05. \]

- No prior information other than positivity is included because there is no information known about the fault zone width.

- The recorded FZTW shows that frequency increases with time. Therefore, parameters in the forward model (Eq. 5.2) are restricted so that frequency cannot decrease with time.

### 5.8.3 Inversion results

The MCMC was iterated 5,000,000 times and \( \sim 250,000 \) independent samples were produced. The posterior probability densities generated are all approximately normal densities and so can be well characterised by means, standard deviations and correlation coefficients. Figure 5.6 shows the values for the individual means, standard deviations and probability densities for each parameter and compares them with the prior densities. Most significantly, the Alpine Fault is estimated to have an effective width between 210-270 m. The S-wave velocity of the Alpine Fault is estimated to be between 10-15% slower than the country-rock velocity.

Many of the parameters are correlated in the posterior probability densities. The joint probability densities of the five most significant correlations are shown in Fig. 5.7. The Pearson's correlation coefficients [Pearson, 1895] between all parameters from both the prior and posterior densities are given in Table 5.2.
Figure 5.6: Posterior and prior probability densities of modelled parameters. Higher values represent values that are more likely to have caused the observed FZTW coda. The solid black lines represent the posterior densities and the blue dashed lines represent the prior densities. The black text on the left of the distribution represents the mean±standard deviation of the posterior densities. The prior densities are included in blue on the right of the distribution if these differ from the posterior. There are three parameters where the posterior is significantly different from prior. These are the only parameters that the inversion provides further constraints than the prior model. Note that the densities are scaled so that they have the same maximum amplitude.

Table 5.2: Pearson's correlation coefficient between modelled parameters [Pearson, 1895]. A coefficient of 1 implies perfectly correlated, 0 is uncorrelated and -1 is perfectly negatively correlated. The values in brackets are the correlation coefficients calculated from the prior densities. The values in bold imply a difference to the prior of greater than 0.3 and these are the major trade-offs when imaging using this FZTW model and data. See Fig. 5.7 for images of these major trade-offs.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\beta_{1,3}$</th>
<th>$\beta_c$</th>
<th>$\rho_c$</th>
<th>$w$</th>
<th>$\alpha_{1,3}/\beta_{1,3}$</th>
<th>$\alpha_2/\beta_2$</th>
<th>$D$</th>
<th>$\Delta t_0$</th>
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<td>-0.1</td>
<td><strong>0.8</strong></td>
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</tr>
<tr>
<td>Country-rock Vp/Vs- $\alpha_{1,3}/\beta_{1,3}$</td>
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<td>0</td>
<td>0.3</td>
<td>0</td>
<td>-0.1</td>
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<tr>
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<td>-0.1</td>
<td>0</td>
<td>(0.3)</td>
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</tr>
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</table>

5.9 Discussion of modelling results

For the same earthquake the forward modelling and inverse modelling has produced different results. The forward modelling produced a fault zone width of between 100-150
Figure 5.7: Joint posterior probability densities. Lighter colours represent values that are more likely to have caused the observed FZTW codas. See Table 5.2 for the Pearsons correlation coefficient and further explanation. Note that the densities are normalised so that they have the same maximum amplitude.

m and velocity contrast of 15-25% whereas the inverse modelling has estimated the fault zone width as 210-270 m and velocity contrast as 10-15%. This difference is not acceptable and part of the reason for this difference is that in the inversion all parameters including country-rock parameters are estimated in the inversion, which allows for the error in them to be accounted for. This difference between forward and inverse modelling is often the case for ill-posed problems and has been observed in synthetic FZTW data in Gulley et al. [2015].

It is not possible to compare these results to DFDP-1 geophysical well logs across the Alpine Fault as this drilling did not span the entire deformation zone [Townend et al., 2013]. These width estimates of the forward and inverse modelling are comparable to many other major fault-zones imaged using FZTWs, such as the San Andreas fault at Parkfield California [100-200 m; Li and Malin, 2008], the San Andreas fault at Landers California [~180 m; Li et al., 1994], the Nojima Fault in southwest Japan [150-290 m; Mizuno and Nishigami, 2006] and the Karadere-Duzce branch of the North Anatolian Fault, western Turkey [~100 m; Ben-Zion et al., 2003b].
These velocity contrast estimates of the forward and inverse modelling are comparable to the Najoma Fault (5-25%) and the San Andreas Fault at Parkfield (25%). However they are notably less than the San Andreas Fault at Landers (≈33%) and the North Anatolian Fault (≈50%). A possible explanation for this difference is that the measurements of the San Andreas Fault at Landers and the North Anatolian Fault were made within a year of a major earthquake rupture and that these faults had been significantly weakened by the earthquakes [Li et al., 2004]. The Alpine Fault is, in contrast, considered to be late in its seismic cycle [Biasi et al., 2015]. Another explanation for this smaller velocity contrast could be that the Alpine Fault low velocity zone does not persist to the depth of the earthquake as described in [Ben-Zion et al., 2003b]. This would result in a longer path through intact rock before entering the fault zone. As the three-layer model calculates an average fault zone velocity, a shallow rooted fault low velocity zone would mean a faster apparent S-wave velocity and hence a smaller velocity contrast. This scenario could be the result of the high uplift rates and elevated isotherms in the vicinity of the Alpine Fault [Allis and Shi, 1995].

The data and three-layer model employed primarily contain information on fault zone width, S-wave velocity contrast and the fault zone Vp/Vs ratio. This can be seen in Fig. 5.6 where, for all other parameters, the posterior distribution is essentially the same as the prior distribution. This is partly because these three parameters have very little prior constraints. In the case that more is known about the fault zone width and velocities, it is undetermined whether the improved priors on these three parameters would cause the posterior densities to contain improved estimates on any of the remaining parameters.

The country-rock velocity, the density contrast and earthquake distance also all have a significant effect on the width and/or velocity contrast even though the data and method used here is unable to provide further constraints on these parameters other than the priors. This can be seen in Fig. 5.7 and Table 5.2 where these parameters have a high correlation with width and/or velocity contrast. A high correlation coefficient means that there is a trade-off between these parameters. These correlations mean that any widening or narrowing of one the priors of the country-rock velocity, density contrast or earthquake distance will cause a widening or narrowing of the posterior densities of the width and/or velocity contrast. Most notably, earthquake distance has a big influence on both width and velocity contrast. Therefore, in order to obtain reliable estimates from FZTW it is important that feasible prior models for the country-rock velocity, the density contrast and, in particular, earthquake location are used. The accurate determination of all these properties is limited from the surface but can be greatly improved through drilling projects. A useful but logistically challenging experiment for determining shallow structure would be a deep borehole active source to near surface receiver FZTW experiment. The near surface
to deep borehole case would also be useful provided the source can be located to cause sufficient trapped energy [Leary et al., 1985, 1987, Li and Leary, 1990, Li et al., 1990]. These experiments would allow much greater control on the source/receiver(s) geometry. The associated geophysical well logging downhole will provide further information that can be incorporated into priors. While the borehole would provide a 1-D localised profile of the fault zone, the additional constraint would enable improved estimates of 2- or 3-D geometries.

The velocity contrast and the fault zone width are strongly negatively correlated in the posterior distribution with a correlation coefficient of -0.5. This means that an increase/decrease in velocity contrast can be matched by a decrease/increase in fault zone width and the resulting FZTW dispersion curve would be similar. While this has been noted by other authors [e.g. Ben-Zion et al., 1992, Michael and Ben-Zion, 1998a], Fig. 5.7 provides new insight by showing exactly which combination of parameters are more or less probable. There are also strong correlations between some of fault zone width, velocity contrast, country-rock velocity, density contrast and distance to the earthquake.

## 5.10 Inverting FZTWs

Probability densities have also been utilised in the inversion of FZTWs in [Ben-Zion et al., 2003b], which utilises the genetic algorithm of [Michael and Ben-Zion, 1998b]. The main advantages of their approach is the use of 2-D analytical solutions of Ben-Zion [1990, 1998] which compute the component that is transverse to propagation direction but parallel with the fault of the entire elastic wave coda from a line (2-D) source in a vertically and horizontally stratified media. This method therefore has the advantages that it incorporates an amplitude response, attenuation, a 2-D source location as well as allowing for the fault zone to have a variable depth. These improvements, however, can also be incorporated in to a MCMC inversion. The Bayesian inversion used in this Chapter finds the range of parameters that fit the data with respect to the noise in the FZTW signal. Thus, it provides a better representation of the actual uncertainty in the model with respect to the parameters. The present approach is also able to investigate the effect of densities, location errors and P-wave velocities.

The results of this Chapter show that the use of Bayesian statistics and MCMC can estimate the uncertainties of modelling FZTWs and the trade-off between fault zone properties. This method also provides useful insight into the information that can be extracted from FZTWs. This approach can be transferred to models that contain greater complexity as well as to simultaneously include multiple FZTW coda. The MCMC approach does carry a
computational cost. The three-layer model used here is computationally efficient with the 5,000,000 iterations run on a local desktop in only a few hours. For more complex models the MCMC approach can be parallelised for use on computation clusters but application to full wave solvers [e.g. Jahnke et al., 2006] is not yet feasible.

The three-layer model used in this Chapter is a one-dimensional approximation. In reality, the Alpine Fault is two dimensional with small scale variations in the third dimension [Eberhart-Phillips and Bannister, 2002]. These small scale variations have minimal effect on the observed FZTW coda [Jahnke et al., 2006]. There is still much debate as to whether two dimensional fault zone cross-sections can be estimated from FZTW signals [Ben-Zion et al., 2003b]. Therefore, this initial study using the three-layer model combined with the Bayesian inversion provides a valuable first step in investigating the properties of the Alpine Fault. Further limitations of our simple three-layer model are that the Alpine Fault country-rock is anisotropic [Okaya et al., 1995] and the footwall and hanging-wall most likely have different elastic properties [Townend et al., 2013]. Fault-zones are also highly attenuating [Li et al., 2012]. As these have some effect on the recorded coda, including the dispersion, these limitations must be considered when interpreting these results.
Inversion of Fault-Zone Trapped Waves: Gradational Velocity Models

6.1 Summary

Fault-zone trapped waves are seismic body waves that are guided by the low velocity zone of a mature fault. Inversions of fault zone trapped wave data face the challenge of many parameters of variable importance, i.e. the presence of nuisance parameters. While fixing these parameters can lead to poor estimates of fault zone properties, we demonstrate that feasible estimates for the fault zone properties can be obtained by treating the uncertainty in the nuisance parameters as a modelling error. The Bayesian approximation error method can be used to account for this modelling error. It is also geologically expected that many fault-zones will have gradational velocity variations in both the across-fault and the down-dip directions. Approximating a gradational velocity model with a layered model for the purpose of inversion can lead to misleading estimates of the fault zone properties. This discrepancy between gradational and non-gradational models may also be treated as modelling error, and the Bayesian approximation error approach can again be used to retain computational efficiency. These results are demonstrated using synthetic 3-D waveform data to invert for 2-D fault zone structure.
6.2 Introduction

The characterisation of low velocity zones of mature faults can provide important information for the understanding of earthquake physics and fault properties [Ben-Zion and Sammis, 2003]. Traditional direct Primary (P) and Shear (S) wave tomography in the vicinity of the fault does not have sufficient resolution to image the narrow (10s-100s of metres) fault zone [e.g. Eberhart-Phillips and Michael, 1993, Thurber et al., 2004]. This is because these body waves only spend a very small percentage of their travel time in these narrow fault zones. Fault-Zone Trapped Waves (FZTWs) propagate along fault zones, internally reflecting off the higher velocity country-rock walls. FZTWs, therefore, spend a significant percentage of time propagating in the fault zone and so have the potential to be utilised for high resolution imaging of fault zone structure [∼10 m, Li et al., 1990].

FZTWs are a dispersive body waves and three different motion types have been observed: $F_L$, $F_R$ and $F_φ$. The $F_L$ waves are analogous to Love mode surface waves and have fault-parallel polarisation (fault SH motion). These were first reported in an active fault in controlled source surface-to-borehole studies at Oroville, California [Leary et al., 1985, 1987, Li and Leary, 1990, Li et al., 1990]. The $F_R$ waves are analogous to Rayleigh mode surface waves and have radial and fault orthogonal polarisations [fault P-SV motion, Malin and Lou, 1996]. The $F_φ$ are leaky wave type FZTWs that arrive between the P- and S-direct arrivals, and are only partially trapped by the waveguide [Ellsworth and Malin, 2011].

For approximately the past 25 years, researchers have been using FZTWs to investigate fault zone properties. They have observed that fault zone widths range from a few tens to several hundreds of metres and that the S-wave velocity contrast between country-rock and fault zones may be up to 60% [Haberland et al., 2003]. There have been numerous methods used to obtain these estimates; two of the most prominent being the genetic inversion algorithm [Michael and Ben-Zion, 1998a] and 3-D finite difference forward modelling [e.g. Graves, 1996].

The genetic inversion algorithm was first proposed by Michael and Ben-Zion [1998a]. Before inversion, the data is converted to 2-D by multiplying by a correction term and band pass filtered to remove high frequency noise. The fault zone is then approximated to be a single homogenous block with unknown width, depth and velocity contrast using the 2-D SH model developed by Ben-Zion and Aki [1990] and Ben-Zion [1998]. The genetic algorithm is an optimisation procedure based around genetic evolution [Goldberg, 1989]. The major advantage of this approach is that it provides uncertainty estimates for the full range of wave-guides that could have produced the data. Examples of the genetic algorithm were
produced by Lewis and Ben-Zion [2010] and Peng et al. [2003].

Another common method of estimating fault zone parameters is 3-D forward modelling using an isotropic full elastic finite difference model [e.g. Cochran et al., 2009, Li et al., 2002, Mamada et al., 2002]. The process involves trial and error to find physical parameters that match the data. Forward modelling has the advantage that the velocity model can be arbitrary and allows for a wide range of geologically plausible situations. The 3-D full waveform solvers tend not to be used in inversions of FZTWs as the computational time is typically prohibitive.

Due to the approximately linear increase in pressures with depth in the absence of major lithological changes, it could be expected that fault zone properties would change smoothly with depth [Marshak, 2005]. It might also be expected that the velocity profile would be smoothly varying in the across-fault direction. This could be caused by gradationally increasing fracture densities and gradational lithological changes. A range of geological and numerical studies have proposed that gradational across-fault velocity models could be expected [Johri et al., 2014, Leary et al., 1987, Li and Vidale, 1996, Mitchell and Faulkner, 2009, Savage and Brodsky, 2011]. Despite this, in inversions of FZTWs, approximations employing layers are typically used, as these have been computationally fast enough.

The depth extent of a low velocity fault zone remains a topic of debate. Some researchers have suggested that low velocity zones persist for the entire seismogenic zone (>10 km) [e.g. Li and Vernon, 2001, Li et al., 2000, 2004, 2002, Wu et al., 2010]. Others have suggested that the low velocity zones only penetrate more shallowly to a depth of <5 km [e.g. Ben-Zion et al., 2003b, Fohrmann et al., 2004, Lewis et al., 2005, Peng et al., 2003]. Part of the difficulty in resolving this debate is the significant non-uniqueness and uncertainty associated with estimating fault zone parameters using FZTW forward and inversion models [Michael and Ben-Zion, 1998b].

The inversion of FZTW data is inherently ill-posed and non-unique [Ben-Zion, 1998, Michael and Ben-Zion, 1998b]. This means that there are a large range of solutions that can fit the data within the bounds of the measurement error. Consequently, it is desirable to find the entire range of solutions that fit the data rather than a single point estimate [Scales and Snieder, 2000]. The Bayesian approach provides a framework for the inverse problem to be treated statistically. This allows the estimation of an ensemble of possible solutions that fit the data [Tarantola, 2004]. A notable inclusion within the Bayesian framework is the use of prior information in the form of a prior probability density. This prior density attempts to capture everything that is already known about the parameters. A methodology that has recently been developed within the Bayesian framework is the Bayesian Approximation Error (BAE) method [Arridge et al., 2006, Kaipio and Kolehmainen, 2013, Kaipio
and Somersalo, 2007]. The BAE is used to account for modelling errors and uncertainties. Some recent examples of modelling errors accounted for with the BAE include model reduction in full wave ultrasound tomography [Koponen et al., 2014], uncertain boundaries in electrical impedance tomography [Calvetti et al., 2015] and approximating the poroelastic wave equation with the elastic wave equation in aquifer parameter estimation [Lähivaara et al., 2015, 2014].

A difficulty that arises in many inverse problems, including inverting FZTWs, is that there are large numbers of nuisance parameters. These parameters are typically less geologically interesting than the others. In this Chapter, we demonstrate that miss-modelling these parameters can lead to poor estimates of fault zone properties. Estimating these parameters as part of the inversion process can produce feasible results, but incurs increased computational difficulties because of the greater number of variables that must be estimated. Ideally, we would like to be able to account for the uncertainty associated with errors in the nuisance parameters without estimating them in the inversion (a pre-marginalisation step). We demonstrate below how the BAE can be used to perform this pre-marginalisation.

Performing inversions on FZTW data using gradational velocity models, as outlined above, remains an ongoing problem. Part of the difficulty in inverting for increasingly realistic geological models is that computational models fast enough to be used for inversion tend to require layered velocity models. We demonstrate how the BAE method can be used to allow for gradational velocity models in an inversion despite the use of one of these computationally efficient FZTW forward solvers. This is achieved by treating the difference between a gradational and layered velocity profile as a modelling error. An approximate numerical gradational correction algorithm is derived, which significantly reduces the time required to compute the BAE probability density.

### 6.3 Inverting for gradational across-fault velocity profiles

#### 6.3.1 Data generation

The aim of any inversion methodology is to invert real world data to obtain reliable estimates. Inverting real world data is inherently more difficult than inverting synthetic data because of the fact that computational models are only ever approximations of real world physics. The related discrepancy in the accuracy of the estimates was analysed quantitatively by Kaipio and Somersalo [2007]. This means that if inversions performed on synthetic
data do not produce feasible results, it is highly unlikely that a similar inversion on real world data will do so. Performing inversions on synthetic data is therefore an important step in the development of inversion methodologies. This is particularly true when it is difficult to validate real world inversion findings, as is the usual case in geophysics.

Synthetic earthquake data is generated using a 2.5-D full waveform solver [Gulley, 2017], where 2.5-D implies that the waves propagate in 3-D but the velocity model can be described by only two spatial parameters. This solver uses Fourier transforms in the dimension perpendicular to the strike of the fault and 2-D finite elements [e.g. Komatitsch and Tromp, 1999] in the remaining two dimensions. The velocity model used to produce the synthetic dataset is shown in Fig. 6.1. The fault zone consists of two different regions: the first begins at the surface and finishes at the same depth as the first horizontal layer, the second region is below the first, and finishes somewhere in the second layer. Both fault zone regions have an across-fault velocity profile that decreases in the shape of a Gaussian. The horizontal layers consist of three isotropic units and the entire velocity model is translationally invariant in the along-strike direction. Eight different sources from a point force are simulated, as seen in Fig. 6.1. An example of a simulated earthquake with its FZTWs can be seen in Fig. 6.2. The simulated data is rotated in to radial, fault-orthogonal and fault-perpendicular components and the $F_L$ FZTW is identified.

### 6.3.2 Data spatial resolution

The earthquake locations in Section 6.3.1 are specifically chosen to cause the inversion for velocity model to have variable resolution. How well the data can resolve a layer is dependent on having multiple earthquakes with ray paths that spend different amounts
Figure 6.2: Simulated earthquake seismograph example. This data is from the earthquake shown in Fig. 6.1 at \( \sim 2 \) km depth and \( \sim 1 \) km along-strike. The dashed lines show the \( F_L \) component that will be used to obtain the dispersion measurements.

of time in the layer. For example, we might expect the top layer (Fig. 6.1) to be poorly resolved by inversion. This is because all of the earthquakes will have ray paths that are refracting upwards and so they all will have a near vertical ray path through that layer. In addition, this layer is thin and so all rays will spend a small amount of time in this layer meaning it has a small effect on the data.

6.3.3 Measuring dispersion

We chose to use dispersion information, but not the amplitude information, in the inversion. This is because amplitude information is affected by source type, source orientation, source frequency and source offset from the fault, whereas dispersion information is not affected by these parameters [Gulley et al., 2017a]. Therefore, use of only the dispersion information only simplifies the inversion. We will now outline the method for obtaining estimates of dispersion from a waveform.

Let \( \mathbf{p} \) be the position with respect to the receiver of a FZTW generating source. For a guided wave propagating in a laterally homogeneous fault zone, Aki and Richards [2009] showed that the waveform from a single harmonic can be written as

\[
    u(\mathbf{p}, t) = \int_0^\infty \tilde{A}(\omega) \exp \left[ i \omega \left( \frac{||\mathbf{p}||}{c(\omega)} - t \right) \right] d\omega, \tag{6.1}
\]

where \( \omega \) is the angular frequency, \( A(\omega) \) is an amplitude function and \( c(\omega) \) is the phase speed in this fault zone, \( ||\mathbf{p}|| \) is the source-receiver distance and \( i = \sqrt{-1} \). The above equation also assumes that the source is located in the centre of the fault zone. If the fault
zone consists of $M$ different regions then the waveform can now be written as

$$u(p, t) = \int_0^\infty A(\omega) \exp \left[ i \omega \left( \sum_{j=1}^M q_j(\omega) c_j(\omega) \right) - t \right] d\omega,$$

(6.2)

where $A(\omega)$ is a different amplitude function, $q_j(\omega)$ is the distance travelled in each fault zone region and $c_j(\omega)$ is the phase speed in each region. This allows us to define the phase time as

$$t_p(\omega) = \sum_{j=1}^M \frac{q_j(\omega)}{c_j(\omega)}.$$  

(6.3)

The group velocity is defined as $U = \frac{d\omega}{dk}$, where $k = \frac{\omega}{c}$. The group time can thus be written as

$$t_g(\omega) = \frac{d(t_p(\omega))}{d\omega} = \sum_{j=1}^M \frac{d(k_j q_j)}{d\omega} = \sum_{j=1}^M \frac{q_j}{U_j} + \sum_{j=1}^M k_j \frac{d q_j}{d\omega}.$$  

(6.4a)

(6.4b)

(6.4c)

Re-writing Eq (6.2) using the phase time gives us,

$$u(p, t) = \int_0^\infty A(\omega) \exp \left[ i \omega (t_p(\omega) - t) \right] d\omega,$$

(6.5)

where it is important to note that $t_p$ is dependent on the earthquake position and the fault zone properties.

If we assume that only one harmonic is present, any trapped wave can be written in the form of Eq. (6.5). We can therefore fit this equation to trapped wave data to estimate an amplitude function ($A(\omega)$), and a dispersion curve ($t_p(\omega)$). To do this we use a standard Bayesian curve fitting routine [e.g. Tarantola, 2004]. We use exponential smoothness priors [Rasmussen and Williams, 2006] to restrict values of $A$ and $t_p$ to a range of sensible values. The noise in the data is approximated by estimating the autocorrelation of the S-wave data. This autocorrelation information is then used as rows/columns in the Toeplitz covariance matrix. A Gauss Markov estimate [Melsa and Cohn, 1978] for the posterior mean and covariance is made. This covariance information is needed in order to estimate the range of possible dispersion curves that could have produced that data. An example of a dispersion curve computed in this way is shown in Fig. 6.3.
Figure 6.3: Example of dispersion measurements. The top figure shows the $F_L$ waveform from between the two dashed lines in Fig. 6.2. Also in the top figure are the credibility intervals of the best fit reconstructed waveform computed in the dispersion fitting using equation Eq. (6.5). The bottom figure shows the estimated group time with credibility intervals. The theoretical dispersion curve is calculated using the forward model (see section 6.3.4). The credibility intervals widen at the low and high frequencies where there is very little energy. The credibility intervals show the 68%, 95% and 99.7 % likelihood regions ($\pm 1, 2, 3$ standard deviations).

6.3.4 The forward model

Whilst our synthetic data was created with a full waveform solver, trying to use that same solver to infer fault zone parameters would be impractical without supercomputing power [Hong and Sen, 2009]. We wish to perform these inversions on a local desktop. A more computationally efficient mode is, therefore, needed. However, a more computationally efficient model will induce modelling errors, so we need to choose a model that compromises computational time with accuracy.

The computational forward solver that we use assumes that the velocity model can be broken up into several simple (1-D) fault zone regions. In each of these layers the group velocity, $U_j(\omega)$, and wave number curves, $k_j(\omega)$, are computed using the methodology developed by Gulley et al. [2017b]. We can then make use of ray-tracing and the conservation of boundary-parallel wave numbers (Snell’s law) to compute the refraction of each frequency. A seismic wave from an earthquake that is below the fault zone must first travel to the bottom of the low velocity zone. This is incorporated into the ray tracing algorithm by preserving fault-parallel wavenumbers. Once the ray tracing has been completed, the group time can be calculated using Eq. (6.5). We ignore the second term in Eq. (6.5) because our numerical experiments have shown that it is small ($< 10^{-3}s$) for these simulated events. This is because an increase in the distance travelled in one layer ($q_j$) is likely to cause a similar decrease in distance travelled in another layer to ensure that the
ray still arrives at the receiver; therefore $\sum dq_j d\omega \approx 0$. This approximation is most valid for earthquakes that are directly down-dip of the receiver. An example of the ray tracing for a single earthquake can be seen in Fig. 6.4. The time taken to compute the forward model locally is in the order of $\sim 0.1$s.

6.3.5 Inversion methodology

The Bayesian framework is used to perform the inversion. Let $y$ be the measurements which, in this case, represent the dispersion measurements from all earthquakes. Let $x$ be the parameters that we wish to estimate and $a(x)$ be the forward model. Using Bayes’ theorem, we can write the posterior probability density [Tarantola, 2004] as:

$$\pi(x|y) \propto \pi_e(y - a(x)|x)\pi_x(x), \quad (6.6)$$

where $\pi_e$ is the probability density of any errors (measurement or modelling) and $\pi_x$ is the prior probability density of $x$. This prior model attempts to contain all information that we already know about $x$. The prior model stabilises the inversion by reducing the probability that unrealistic solutions are deemed suitable by the posterior density. To estimate the posterior probability density (Eq. 6.6) we use a Markov chain Monte Carlo because of its ability to produce an ensemble of suitable solutions that fit the data and therefore estimate uncertainty. The Metropolis-Hastings algorithm is employed [Hastings, 1970].

This presented inversion algorithm is used to perform a joint inversion using data from all of the recorded earthquakes. Whilst the data used in this Chapter is only from a single receiver, the methodology dose not restrict this and data from different receivers can be included if available.
6.3.6 Parameters

The forward model contains a number of parameters (52), which are denoted by $x$. We are, however, only interested in a small number of them (10) denoted, $x_I$. These parameters of interest are those that influence the S-wave velocities of the fault zone. Specifically, these are the widths, depths and velocity contrasts of the fault zone regions, the country-rock S-wave velocities and the layer depths. The other parameters (those that we are not interested in), are termed ‘nuisance parameters’ ($x_N$). These parameters describe the location and time of each earthquake as well as the rock densities and P-wave velocities of the country-rock and fault zone. These parameters are referred to as nuisance parameters because the $F_L$ FZTWs are not affected by P-wave velocities and the densities will only have a small effect on the dispersion [Gulley et al., 2017b]. This means that we are unlikely to be able to use FZTWs to improve the estimates of P-wave velocities and densities beyond our prior model. It is also unlikely that the earthquake locations will have improved estimates beyond the prior after inversion of FZTW data. However, the earthquake locations, P-wave velocities and densities may still affect the inversion results of the parameters of interest [Gulley, 2017]. Therefore, these values are ‘nuisance parameters’.

Probability densities are depicted graphically in this Chapter as credibility intervals. Credibility intervals show the region that the probability density indicates to be the most plausible. In this Chapter, a curve is deemed to be feasible with respect to a (posterior) probability density if it lies within the 99.7% credibility intervals.

6.3.7 The prior model

The prior models used in Bayesian inversions are chosen in an attempt to make any geologically plausible solution probable, and any geologically implausible solutions are attempted to be made impossible or improbable. In this case, the prior model relies on other results (e.g. tomography) to constrain P- and S-wave velocities in the country-rock, and these are expected to be under reasonable constraint. In the prior model, we constrain the effective width and velocity contrast of the fault zone to be getting smaller with depth. We also employ numerous positivity constraints.

Also contained within the prior model is a ray tracing earthquake location algorithm. This is included to ensure that the body S- and P-waves arrive at the receiver at the approximate time that they arrive in the simulated earthquake recording (see data in Fig. 6.2). It is likely, for real world application of this methodology, that we will already have a good prior model for the earthquake locations through other means. However, this earthquake
location algorithm is included here as it is important that the trade-off between earthquake location and country-rock velocity is well characterised. Note that this is mathematically equivalent to doing a joint inversion with earthquake location [Michael and Ben-Zion, 1997]. It is included in the prior model here, so that the posterior estimates of fault zone parameters due to the FZTW data can be easily isolated.

6.3.8 Consequences of assuming the across-fault velocity profile form

In the first inversions shown here, we look at the effect of using different across-fault velocity profiles in the forward model. The velocity profiles in question are a three-layer model, a Gaussian model and an exponentially decaying model (using \(1/\cosh\)), as seen in Fig. 6.5. As described in Section 6.3.1 the data being inverted was generated with a Gaussian across-fault velocity model. In these inversions the nuisance parameters are fixed to their true values in order to isolate the effects of velocity gradient assumptions. In this Chapter, we consider an inversion to be feasible if the true solution is within the 99.7% credibility intervals.

When a Gaussian or exponentially decaying across-fault profile is used for the inversion, the true solution is contained within the 99.7% credibility intervals of the inversion as seen in Fig. 6.6. When a three-layer model is assumed to represent the across-fault profile, the true solution is not captured by the 99.7% credibility intervals of the inversion, as can also be seen in Fig. 6.6.

The three-layer profile fails because, for the same value of velocity contrasts and effective widths, it has a qualitatively different dispersion curve from the Gaussian model. This means that for the same parameters \(\mathbf{x}_f\) these two across-fault profiles will produce dif-
Figure 6.6: Three different inversions comparing the effect of using different across-fault velocity profiles in the (forward) model used to estimate the parameters. The different across-fault profiles are shown in Fig. 6.5. The posterior credibility intervals show the 68%, 95% and 99.7% regions. A posterior probability density is deemed feasible if the 99.7% credibility intervals contain the true solution. Note that these posterior densities represent an ensemble of different layered solutions.
ferent dispersion curves in the forward model. In Gulley et al. [2017a], it was shown that the three-layer model dispersion curve produces smaller group velocities than an exponentially decaying or Gaussian type forward model. The inversion shown in Fig. 6.6 has thus compensated for the smaller group velocities by increasing the estimates of fault zone velocities (smaller velocity contrast). On the other hand, the exponential decay and Gaussian across-fault profiles have similar dispersion curves, so the inversion with the exponential decay also produced feasible results.

6.4 Accounting for errors in nuisance parameters

In the next set of inversions, we look at the effect caused by (realistic) errors in the fixed value of the nuisance parameters. To achieve this, we perturb the nuisance parameters away from the ‘true’ in such a way that they are still contained within the prior distribution. In particular, the earthquakes are moved by up to 500 m in space and 0.1 s in time; the two shallowest earthquakes in Fig. 6.1 are perturbed so that they are still on the fault plane. These inversions use the previous data set from Fig. 6.1. The across-fault profile for the forward model used here is the Gaussian.

The inversion where the nuisance parameters are fixed at incorrect values is shown in the top row of Fig. 6.7. As can be seen, this inversion overestimates the fault zone width in the top layer and underestimates the fault zone velocity contrast. This failure is attributed to the errors in the nuisance variables, as this inversion worked well with accurate nuisance variables (as seen in the top row of Fig. 6.6).

The standard approach to address this issue is to allow the nuisance parameters to be variables in the inversion. This inversion is shown in the second row of Fig. 6.7. As can be seen, this inversion worked well, although there is now significant uncertainty in the top fault zone width. This is, in part, because this region is small; therefore, the seismic waves only spend a small amount of time in this region, reducing the effect that this region has on the data. Although this inversion works well it is of limited practicality because the number of parameters has increased from 10 to 52. The number of parameters would increase by four for each additional earthquake included. Many inversion algorithms, including the Markov chain Monte Carlo algorithms, become intractable if the number of parameters gets too large [Kaipio and Somersalo, 2005]. To avoid this problem, we employ a parameter map and the Bayesian Approximation Error (BAE) method.
Figure 6.7: Demonstration of how errors in the nuisance parameters can be efficiently dealt with using the BAE. The top row shows an inversion where there are errors in the fixed values of the nuisance parameters used (e.g., the earthquake location). The second row shows an inversion where the nuisance parameters are also treated as variables in the inversion. The third row shows an inversion where there are errors in the nuisance parameters, but the Bayesian Approximation Error (BAE) method has been used to account for these errors. The posterior credibility intervals show the 68%, 95% and 99.7% regions.
6.4.1 Parameter map

As mentioned above, the prior model also contains the uncertainty that is inherited from the earthquake location algorithm. Therefore, the prior model enforces that the direct P- and S-wave arrive when the earthquake data indicate that they should. This is observed in the prior distribution as correlations between parameters such as the earthquake location and the country-rock velocity. In the inversion, however, the nuisance parameters are fixed. The nuisance parameters include the earthquake location parameters but not the country-rock S-wave velocity. This means that in the inversion, the country-rock S-wave velocity will be limited to a very small range in order for the modelled P- and S-waves arrival times to be consistent with the observed P- and S-waves arrival times.

To account for this problem, we make the nuisance parameters linearly dependent on the parameters of interest using what we have called a ‘parameter map’. This parameter map is defined as

\[ x_N = v + Vx_I, \]

where \( x_I \) and \( x_N \) are the parameters of interest and the nuisance parameters respectively. The vector \( v \) and matrix \( V \) are obtained from least squares estimates of prior distribution samples. Note that the use of the parameter map means that the nuisance parameters are no longer fixed in the inversion but instead are modelled as affine maps of the primary unknowns.

6.4.2 Bayesian approximation error method

To further account for the errors in the nuisance parameters, we make use of the BAE method. Up until this point we have assumed that there are measurement errors, but no modelling errors. The errors induced by fixing the nuisance parameters to incorrect values can be interpreted as modelling error. The BAE method gives us a way of estimating this modelling error. In the following, we give a brief overview of how the BAE method works. For more details see Arridge et al. [2006], Kaipio and Somersalo [2007] and Kaipio and Kolehmainen [2013].

Let \( \hat{a}(x) \) represent an accurate forward model and \( a(x_I) \) be a less accurate model that only has the parameters of interest as variables. The measurement error is \( e \). We can now write
where $\epsilon(x) = \hat{a}(x) - a(x_f)$ is the BAE additive error term. The idea behind the BAE is to find a way of estimating a probability density function for $\epsilon$ offline so that no evaluations of the computationally intensive accurate forward model are needed during the inversion. The standard approach is a sample-based method that incorporates the following steps:

1. Randomly generate an ensemble of $x^{(i)}$ from the prior distribution.
2. Compute an ensemble of $\epsilon^{(i)} = \hat{a}(x^{(i)}) - a(x_f^{(i)})$.
3. Using the statistics of $\{\epsilon^{(i)}, x^{(i)}\}$ a probability density for $\pi_{\epsilon,x}$ can be formed. This is done by approximating $\pi_{\epsilon,x}$ as a multivariate Gaussian and computing the joint mean and covariance of the ensemble.
4. Combining $\pi_{\epsilon,x}$ with $\pi_\epsilon$ to form an estimate of $\pi_{\epsilon+\epsilon}(\epsilon + \epsilon|x)$.

The posterior probability distribution Eq. (6.6) can be re written as:

$$
\pi(x|y) \propto \pi_{\epsilon+\epsilon|x}(y - a(x)|x) \pi_x(x). 
$$

(6.9)

It has been shown that the accurate model does not need to be an exact representation of reality; it just needs to be sufficiently accurate for the statistics of $\pi_{\epsilon+\epsilon|x}$ to be approximately correct [Arridge et al., 2006, Koponen et al., 2014].

### 6.4.3 Inversion using the Bayesian approximation error method

To perform the inversion while taking into account errors in the nuisance parameters, we use the following models. The accurate model $\hat{a}(x)$ uses a Gaussian across-fault velocity model and the numerical discretisation has 25 sixth order basis functions [see Gulley et al., 2017b]. The less accurate model $a(x_f)$ utilises the parameter map (Section 6.4.1). It also assumes that the velocity model is Gaussian but, for numerical efficiency, only 13 sixth order polynomials are used. The BAE is calculated using these models.

The inversion results are shown in the bottom row of Fig. 6.7. As can be seen, the posterior credibility intervals capture the true solution. This result is comparable to the full inversion
using all parameters which, as seen in the middle row of Fig. 6.7. The main difference between these two inversions is that the credibility intervals of the BAE inversion are wider.

We define the ‘desired dispersion’ to be the dispersion produced by the inaccurate model when the parameters of interest are at their true value, but the nuisance parameters are at their fixed values used in the inversion. This is \( \mathbf{a}(\mathbf{x}_{\text{true}}) \), and represents the dispersion that we want the model to produce. Note that this ‘desired dispersion’ will be different when the parameter map is used.

A demonstration of the effect of the parameter map and BAE is shown in Fig. 6.8. It shows that the BAE method causes the region of more probable dispersion curves to widen and the centre of this region to shift due to the non-zero mean of \( \pi_{\epsilon \mid \mathbf{x}} \). The parameter map also shifts the desired dispersion closer to the measurements. The BAE method and the parameter map cause this desired dispersion to be within the 95% credibility interval of the allowable error in the measurements, thus making the true velocity model a plausible solution to the inversion result. Note that these 95% credibility intervals do not demonstrate any of the covariance information contained within the measurement + modelling error densities. This covariance information has a significant effect on which modelled dispersion curves are consistent with the data + modelling error.

### 6.5 Inverting for gradational across-fault and down-dip velocity profiles

Up until this point, we have assumed that we have a gradational velocity model in the across-fault direction (Gaussian). In the down-dip direction we have assumed that the fault zone is made up of different vertically homogeneous, horizontal layers as shown in Fig. 6.1. FZTWs are caused by constructive interference of rays reflecting off the higher velocity fault zone walls. A fault zone that is fully gradational would cause the waves to reflect off the country-rock at different angles from those observed in a layered (down-dip) model. Therefore, these two models would have different dispersion curves [Li and Vidale, 1996]. Additionally, if the layered approximation only used a few layers, there would likely be insufficient flexibility in the parameters to adequately fit the data [Sen and Stoffa, 1991]. To demonstrate this point, Fig. 6.9 compares earthquakes simulated from a gradational velocity model and an approximating layered velocity model. As can be seen in Fig. 6.9, earthquakes 3 and 4 have almost identical waveforms; thus, for these earthquakes, the layered velocity model is a good approximation. Earthquakes 1 and 2,
Figure 6.8: The effect of the parameter map and Bayesian Approximation Error (BAE). This data is for the same earthquake shown in Fig. 6.3. The desired dispersion is produced by the forward model when the parameters of interest are set to their true values but the nuisance parameters are set to the incorrect values used in the inversion. The change in the desired dispersion from A to B is due to the parameter map. The measurements’ 95% credibility interval represents the region of more probable dispersion curves that fit the data. The downward shift and widening of these credibility intervals from A to B is due to the BAE. The true dispersion can be seen in Fig. 6.3. In the inversion method, the BAE density is dependent on $x_l$. The credibility intervals shown here are evaluated at $x_l = x_l^{true}$. Note that the best estimate for the true dispersion is the measured dispersion in Fig. A.

however, have different travel times, so the layered velocity model is a poor approximation in this case. This is because the average slowness (1/velocity) between source and receiver is different for earthquakes 1 and 2 whereas for earthquakes 3 and 4, the average slowness is similar.

The obvious way to deal with this poor approximation is to include more layers. The difficulty with doing this in an inverse problem is that it will require the computation of dispersion for additional layers and introduce several more parameters, both of which make the inverse problem more difficult. Instead, we chose to continue using only a few layers in the inversion. The BAE method is used to account for the modelling errors caused by this poor approximation.
Figure 6.9: Demonstration of the difficulty in approximating a gradational velocity model with a small number of layers. The velocity models are each used for the simulation of two different 2-D SH earthquakes (EQs). The seismograms show that earthquake 2 travels faster than earthquake 1 but earthquakes 3 and 4 have similar waveforms. The source is a 7 Hz Ricker wavelet, and the dashed lines show the width (half height width) of the Gaussian waveguide.

Figure 6.10: Cross section of the S-wave velocity model and earthquake locations used for generating synthetic measurement data to test gradational inversions. The P-wave velocities and the densities have similar profiles to this S-wave profile, and all are isotropic in the along-strike direction (2-D velocity model). The across-fault velocity profile is a Gaussian. The earthquake along-strike distance represents the distance from the receiver.

### 6.5.1 Data generation

Data for gradational inversions are simulated using the time domain 3-D full waveform solver outlined in Section 6.3.1. The velocity model and earthquake locations used are shown in Fig. 6.10. Dispersion measurements are made in the same way as in Section 6.3.3.
6.5.2 Bayesian approximation error for gradational down-dip velocity model

As the use of 3-D full waveform solvers for inversion is computationally unrealistic, we investigate whether the BAE can be used to account for the discrepancy between a fully 3-D gradational velocity structure and the layered approach already considered. To do this we first need to define a transformation that converts a gradational model to a layered model and vice versa. We chose to use a simple spline type model. An example of its behaviour can be seen in Fig. 6.11.

The forward model described in Section 6.3.4 is used. A new accurate model is denoted as $\hat{b}(x)$, which allows for gradational velocities. This model has the input parameters of the layered model and has three stages: firstly it converts the layered velocity model to gradational; secondly, it performs a 3-D full waveform computation; and thirdly, the dispersion measurement routine is used (Section 6.3.3).

We can now redefine our BAE additive error term from Eq. (6.8d) as

$$\epsilon(x) = \hat{b}(x) - a(x_I).$$  \hspace{1cm} (6.10)

Computing the above BAE additive error term would require hundreds of 3-D wave propagation simulations for each earthquake. This would require a large number of computational hours on a super computer. We would like to be able to perform these computations on a local machine, so we need to find a way of approximating $\hat{b}(x) - a(x_I)$.

To make this approximation, we assume that the error induced by having gradational velocity models in a 3-D wave field is similar to that induced in a 2-D wave field. The 2-D surface that we use is the plane that cuts orthogonally through the fault zone and captures the source and receiver as indicated in the second image in Fig. 6.12. For this approximation, we need to define two new models that utilise the velocity model and
this 2-D surface. The model $\hat{b}_{2D}(x)$ is a 2-D full wave form solver that operates on the gradational velocity model and $\hat{a}_{2D}(x)$ is the 2-D version of our accurate forward model $\hat{a}(x)$ (Section 6.4.3). We can now write this approximation as

$$\hat{b}(x) = \hat{a}(x) + \hat{b}(x) - \hat{a}(x)$$  \hspace{1cm} (6.11a)
$$\approx \hat{a}(x) + [\hat{b}_{2D}(x) - \hat{a}_{2D}(x)],$$  \hspace{1cm} (6.11b)
$$\Rightarrow \epsilon(x) \approx [\hat{a}(x) - a(x_I)] + [\hat{b}_{2D}(x) - \hat{a}_{2D}(x)].$$  \hspace{1cm} (6.11c)

The first term in Eq. (6.11c) is identical to the BAE term in Section 6.4.3. The second term is the gradational correction term. The main error in this approach is that the dispersion calculated in $[\hat{b}_{2D}(x) - \hat{a}_{2D}(x)]$ does not take into account the influence 3-D refraction has on the dispersion curve. A schematic of how one sample is calculated is shown in Fig. 6.12. The probability distribution $\pi_{e+\epsilon|x}(e + \epsilon|x)$ is estimated from samples using the same recipe outlined in Section 6.4.3.
Figure 6.13: Inversions demonstrating the effect of using the down-dip gradational correction term in the Bayesian Approximation Error (BAE). This is the main result of the Chapter. The top row shows an inversion without using the BAE. The second row shows the inversion using the BAE to account for errors in the nuisance parameters. The third row shows the inversion when using the BAE to account for the errors in the nuisance parameters and to account for the layered-velocity model approximation. The posterior credibility intervals show the 68%, 95% and 99.7% regions.

6.5.3 Results

We perform three different inversions to test the effectiveness of the gradational BAE. The first inversion does not use the BAE method, and therefore assumes a layered velocity model. This is shown in the top row of Fig. 6.13. This inversion performs poorly as the posterior credibility intervals do not capture the true solution.
The second inversion makes a layered approximation and utilises the BAE that accounts for nuisance parameters. These results are shown in the second row of Fig. 6.13. This inversion does a good job of approximating the average velocity over each layer, even though we have not included the gradational correction.

The third inversion uses the BAE with the gradational correction. As can be seen in the third row of Fig. 6.13, this inversion performs best. In particular, it provides consistent estimates of the velocity contrast and the width profile with depth, and allows good estimates of the point at which the fault zone terminates. We have not presented any correlation information or example solutions to the inversion shown in the third row of Fig. 6.13. However, many solutions do follow a similar trajectory to the true solution and thus the inversion results are feasible. The inversion result in the third row of Fig. 6.13 suggests that it is likely that the change in fault zone width with depth has a significantly steeper gradient than the true solution. This is due to the fact that the fault zone width in the top layer is poorly constrained by the data as outlined in Section 6.3.2.

A demonstration of how the gradational correction term affects the error models is shown in Fig. 6.14. The left hand subfigure shows only the dispersion measurement and estimated credibility intervals of the measurement error. The desired dispersion is not contained within these credibility intervals. Adding the BAE with the gradational correction term causes the credibility intervals to shift and widen. This shift is caused by the non-zero mean of the gradational BAE probability density. Adding the gradational BAE causes the desired dispersion to be contained within the credibility intervals.
Chapter 6. Inversion of Fault-Zone Trapped Waves: Gradational Velocity Models

Figure 6.14: The effect of the gradational correction to the BAE. These data are for the earthquake seen in Fig. 6.10 at 2 km depth and 1 km along-strike. The desired dispersion is produced by the forward model when the parameters of interest are set to their values that best approximate the gradational velocity model but the nuisance parameters are set to the incorrect values used in the inversion. Note that the parameter map has caused the change in the desired dispersion from A to B. The measurements with 95% credibility intervals represent the region of more probable dispersion curves that fit the data. In the inversion method, the BAE density is dependent on \( x_I \). The credibility intervals shown here are evaluated at \( x_I = x_I^\text{true} \). Note that the best estimate for the true dispersion is the measured dispersion in Fig. A.

6.6 Discussion

The approaches developed here have allowed us to perform feasible estimates of fault zone parameters using fully gradational 2.5-D synthetic data. In particular, the estimated credibility intervals for how the fault zone parameters change with depth contain the predetermined model values. This Chapter proposes a method to partially account for real world complexity in the inversion of fault-zone trapped waves (FZTWs).

Given unlimited computational recourses, this inversion would be best done using a 3-D full waveform solver with all parameters treated as variables. The trade-off involved in using a lesser forward model in conjunction with the BAE is the widening of the posterior credibility intervals. This occurs because the BAE increases the allowable misfit in the data leading to a wider range of acceptable parameters. There are several approaches that can be investigated to reduce the uncertainty in the posterior estimates. The first is the use of a more accurate forward model, which would lead to the BAE error term being smaller. We have employed this method to a small degree by using the parameter map. The second approach is to better approximate the density of the BAE error samples. The Gaussian fit
used here is the typical choice used for BAE [Kaipio and Kolehmainen, 2013]. However, it has been shown by Lipponen et al. [2013] that improved approximations of the BAE density narrows the posterior credibility intervals. Case study specific geological information may aid in the construction of the the prior model, which may in turn result in the decrease of the prior variances. This, in turn, would result in the narrowing of the posterior credibility intervals. Other ways to reduce uncertainty include the use of data with lower measurement error variances; this might be achieved by having a deeper borehole. In many cases, more data or an additional receiver at another depth may help.

This widening of the posterior credibility intervals is most prevalent in the fault zone width estimates in the top layer. There are two reasons for this: Firstly, the region is relatively small in comparison to the second fault zone layer, meaning that variations in the parameters will have less of effect on the data. Secondly the trapped waves used here have a spectrum of around 2-15 Hz, as the fault zone gets wider, these frequencies will essentially travel as body waves through the low velocity fault zone [see dispersion curves in Gulley et al., 2017b] so further widening does not affect the velocity of these body waves. If this happens, the posterior density of the top fault zone width will tend to the prior density.

The pre-marginalisation of the earthquake location error could be translated or generalised to other methodologies. Some imaging techniques eliminate the effect of earthquake location error. For example, the two station method [e.g. Woods and Okal, 1996] eliminates the effect of source location error by only considering the waves traveling between two stations (with known locations). The method proposed here would still allow for the estimation of velocities between these two receivers along with the estimation properties between a source and a receiver. In this case, the credibility intervals between the two receivers would be much narrower than those between the source and receiver due to uncertainty in earthquake location.

A parameter of interest that is not estimated by this method is the attenuating parameter ‘Q’. It is not estimated because Q is contained within the amplitude part of Eq. 6.5 as shown in Aki and Richards [2009]. Therefore, the effect of Q has been pre-marginalised and the results allow for all sensible values of Q. If estimating Q were of critical importance, either the method developed here would need to be adapted to include amplitude information in the inversion, or a completely different approach would be needed.

Using the wrong across-fault velocity profile can have a significant negative effect on the feasibility of the estimated fault zone properties. If the across-fault profile were unknown, then it could potentially be estimated in the inversion. However, we believe that this would be highly non-unique. If there is reasonable geological evidence for a particular across-fault
profile, then it is possible that any uncertainty in this could be accounted for using the BAE.

The methods proposed in this Chapter could potentially be used to contribute significant information to the debate on the depth extent of low velocity zones associated with faults. The use of Markov chain Monte Carlo means that the depth extent of the fault could be presented with credibility intervals. Not only would this provide an estimate, but it could also be used to demonstrate how much information on the depth of the fault was contained by a particular FZTW dataset.

There are a number of real world phenomena that the model does not capture. Most significantly, earthquakes must be allocated as either below the fault or in it before the inversion is carried out. The seismic waves from a source below the fault zone tunnel into it at its bottom. However, real FZTWs could potentially tunnel in mid fault zone [Wu et al., 2008], particularly if the fault has a non-vertical dip. If a full waveform solver were used for the inversion, this would not be a concern. Further complexities that are not included are anisotropy, fault asymmetry and multiple receivers. Adding these factors to the methods developed in this Chapter is likely to prove straightforward.

6.7 Conclusions

The inversion method proposed in this Chapter was shown to produce feasible estimates for 2-D gradational fault zone models using synthetic data. The Bayesian approach used has demonstrated the non-uniqueness involved in inverting FZTWs. The inversion method developed within this Chapter could be used to inform (with uncertainty) the debate on the depth extent of FZTWs. This would likely require many well-located FZTW events that were approximately down-dip of the receiver to ensure that 2-D assumptions were valid. There is also the potential that the gradational BAE could be adapted to other inversion methods such as genetic algorithms.
This study aimed to investigate how a number of geologically realistic situations, such as anisotropy or gradational across-fault velocity profiles, effect the propagation of Fault-Zone Trapped Waves (FZTWs). This information was to be utilised for an alternative inversion methodology that allows these to be taken into account when deemed geologically necessary. A second goal was to investigate and utilise the FZTWs that have been observed on New Zealand’s Alpine Fault in order to estimate the properties of the Alpine Fault’s fault zone.

7.1 Overview of progress

7.1.1 Paraxial absorbing boundary condition

In Chapter 2 a method was proposed for computing Absorbing Boundary Conditions (ABCs) for numerical simulation of anisotropic elastic waves. This was developed because a computationally efficient ABC was needed for the Bayesian Approximation Error (BAE) method where hundreds of 2-D elastic waves were computed. The work done in this Chapter is somewhat distinct from both FZTW studies and Alpine Fault studies but was used in the numerical wave propagation performed in this thesis. These boundary conditions were
derived by assuming that waves arriving at the boundary are arriving from a particular chosen direction. An advection type equation can then be used to describe these waves, and this advection equation is a zeroth-order paraxial absorbing boundary condition. The eigenvalue perturbation method was used to increase the absorption for waves arriving at the boundary from directions close to the chosen direction. These boundary conditions are very computationally efficient but not as effective at absorbing outgoing waves as the current ‘state of the art’, perfectly matched layer absorbing boundary conditions [Komatitsch and Tromp, 2003]. The comparatively poor absorption is not a major problem when used to simulate FZTWs. Because of the trapped nature of FZTWs, they have high amplitude compared with outwards radiating waves that arrive at the numerical boundaries. Therefore, small superfluous boundary reflections that arrive at the receiver will have a very small amplitude compared with the FZTWs and so the comparatively poor boundary absorption of the proposed method is less important when simulating FZTWs.

7.1.2 Numerical model for FZTWs

In order to investigate the influence of geologically realistic situations on FZTWs a method for computing FZTWs was developed in Chapter 3. This method assumes the velocity model is transversely isotropic and has variations in the across-fault velocity profile only (1-D velocity model). The model uses Helmholtz decompositions and the finite element method to re-write the equations of motion as an eigenvalue problem. The eigenvalue is a trapped frequency and the eigenvector corresponds to the trapped waveform.

A key feature of this model is that it allows for any arbitrary 1-D velocity model which could be gradational or layered. Therefore, this model can be used to investigate the effect of gradational velocities or estimate fault zone parameters allowing for gradational velocities. The flexibility over the velocity model is made possible by use of the finite element method which does not impose any restrictions of the velocity model.

This computational model was also developed with the intention of using it for inversions using Bayesian statistics and a Markov chain Monte Carlo. This method of inverting allows the uncertainty in the estimates of fault zone parameters to be accurately characterised. However, the disadvantage is that it requires many (> 10⁶) evaluations of the forward model. Therefore, a number of techniques were utilised to make this numerical FZTW model computationally more efficient. High order polynomials were used as basis functions for the finite element method. These high order approximations reduce the number of basis functions needed to maintain a specific accuracy. Absorbing boundary conditions where used to remove the need to include the country-rock in the computational domain,
significantly reducing the number of basis functions needed. The third technique used was mass lumping, a technique which reduces the generalised eigenvalue problem to a standard eigenvalue problem, significantly reducing the computational complexity. A correction was then applied which accounts for most of the errors induced by mass lumping.

The final significant feature of the computational scheme developed in Chapter 3 was the use of a transversely isotropic velocity model. This is important because many fault zones have been shown to be anisotropic [Cochran et al., 2003, Hung et al., 2009, Leary et al., 1987]. While anisotropy is widely considered in the analogous context of coal seam waveguides where the low velocity zone has strong lithologic control [e.g. Buchanan et al., 1983], it has not been considered when estimating fault zone parameters from fault trapped waves in previous studies.

This computational methodology was also used in the development of another dispersion forward model in Chapter 6. This dispersion forward model assumed that the velocity model can be made up of horizontal 1-D layers effectively making this a 2.5-D dispersion forward model. Snell’s law was utilised in order to compute the ray paths in the different layers. This 2.5-D dispersion forward model was used for inversions.

7.1.3 Observations of the properties of FZTWs for modelling and inversion

Computational experiments using the developed forward models showed that a gradational across-fault velocity model produces a qualitatively different dispersion curve from a layered across-fault velocity model. In particular, a three-layer velocity model has an Airy phase while a gradational ‘Gaussian’ type model does not. This means it is important to consider the across-fault velocity profile when using FZTWs to estimate fault zone properties.

It was shown that there are a number of parameters that effect the amplitude of the FZTWs but not their dispersion. The most notable parameters are the earthquake source frequency spectrum and source tensor, as well as the sources fault-perpendicular offset from the centre of the fault zone. It may be impossible to obtain good estimates of these parameters. For example, earthquakes often have location errors of hundreds of metres which is larger than the effective widths of many low velocity fault zones. Therefore, modelling fault zones with dispersion only can reduce the complexity of the imaging process.

The effect of transverse isotropy was also investigated. It was shown that while anisotropy is important in the propagation of FZTWs, approximations can be made either to reduce the number of parameters or, if insufficient information is available, to account for
anisotropy. The geologically plausible situation of fault-aligned fractures in the fault zone would increase the wave train length and amplitude of $F_R$ FZTWs compared to the $F_L$. In contrast, if the fractures were located in the country-rock and not in the fault zone, the $F_L$ FZTWs would have a longer wave train and amplitude response.

Preliminary inversions of Alpine Fault data also provided valuable information on the properties of FZTWs. This inverse modelling provided a demonstration of the significant uncertainty and trade-offs that exist when inverting FZTW data. The largest trade off was between fault zone width and distance to the earthquake but significant trade-offs also exist between other parameters. The country-rock parameters and earthquake location parameters were shown to have a posterior distribution that are almost equivalent to the prior distributions. This means that FZTWs do not contain sufficient information on these parameters to improve estimates further than the prior distribution. This significantly influenced the decision to treat many of these parameters as nuisance parameters and pre marginalise over them using the Bayesian approximation error method in Chapter 6.

Preliminary Bayesian inversions on synthetic data were carried out. These inversions re-confirmed that using an incorrect across-fault velocity profile will lead to poor estimates of fault zone widths and fault zone to country-rock velocity contrasts. It was also shown that using incorrect fixed values of the nuisance parameters (e.g. earthquake location) would also lead to poor estimates of the fault zone parameters.

### 7.1.4 Preliminary Alpine Fault investigations

FZTWs recorded in the fault zone of the Alpine Fault, New Zealand were investigated in Chapter 5. Forward and inverse modelling was performed on the waveform to estimate fault zone proprieties. This was done by approximating the fault zone as a homogeneous dipping layer surrounded by homogeneous country-rock. Forward modelling suggested that the fault has an effective width of 100-150 m and velocity contrast of 15-25% whereas inverse modelling of a single earthquake suggested that the effective width is 210-270 m and velocity contrast is 10-15%. A likely cause of this discrepancy is the errors in the nuisance parameters which were shown in Chapter 6 to significantly affect results. Although the forward and inverse modelling estimates produce different estimates for width, the values are still in the range of other major fault zones, whereas the velocity contrasts produced by forward and inverse modelling are less than fault zones that recently had major earthquakes. The discrepancy between the Alpine Fault and recently active faults could potentially be attributed to fault healing [Li and Vidale, 2001].
7.1.5 Inversion methodology

The major contribution of this thesis is the inversion methodology that was developed in Chapter 6. The inversion methodology is focused on developing an inverse algorithm that takes into account the observations about FZTWs made in this study and elsewhere (see Section 1.3.2). In particular, the inversion methodology was developed to allow for 2-D gradational velocity models. The forward model used was the 2.5-D dispersion forward model and only dispersion information was used in inversions. A methodology for performing inversions with feasible estimates despite incorrect fixed values of the nuisance parameters was first developed. This methodology utilised the ‘parameter map’ developed in this thesis and the Bayesian approximation error method to pre-marginalise over the errors induced by the uncertainty in the nuisance parameters. The inversions were extended to be gradational in the down-dip direction by further use of the Bayesian approximation error. In order to compute the Bayesian approximation error of the gradational term, an approximation for 3-D wave propagation in a gradational velocity model was developed using 2-D slices. This correction induced by the BAE resulted in the inversion estimates being feasible.

7.2 Discussion

7.2.1 Forward model

The forward model developed in this thesis allows the computation of the dispersion curves of a point source (3-D) in a 2-D velocity model (fault zone), making this a 2.5-D forward solver. The velocity model is restricted to being made up of layers in the down-dip direction but can be fully gradational in the across-fault direction. There are two other types of prominent forward models currently in use, a 3-D finite difference full waveform solver and the semi-analytic guided wave solver. These models have both strengths and weaknesses compared with the proposed forward model and the best choice of model would depend on the modelling situation and requirements.

The 3-D finite difference full waveform solver has an advantage over the proposed forward model in that it can compute the full-waveform and can allow for 3-D velocity models, although presently only 2-D velocity models have been used in 3-D finite difference solvers [e.g. Cochran et al., 2009, Li et al., 2003a, Mamada et al., 2004]. The major weakness of these 3-D full waveform solvers is that they are computationally very slow, requiring many hours of computational time. In contrast the forward model developed in this thesis takes
micro seconds to compute. This is important because it allows the proposed model to be used in inversions which typically require hundreds of evaluations of the forward model [Kaipio and Somersalo, 2005].

A semi-analytic guided wave solver was developed by Ben-Zion and Aki [1990] and Ben-Zion [1998]. This model computes the full waveform from a 2-D SH line source and the velocity model is a low-velocity well (three-layer model to finite depth). This forward model is also very computationally efficient. The limitations of this model are that it can only compute the \( F_L \) FZTW and is also only 2-D. Furthermore, the velocity model is very simplistic and does not accurately represent real world geology.

### 7.2.2 Estimating fault zone parameters

The forward model developed in Chapters 3 and 6 was used in an inverse algorithm. This inverse algorithm uses Bayesian statistics and Markov chain Monte Carlo (MCMC) to estimate uncertainty. It also uses the Bayesian Approximation Error (BAE) method to account for errors in parameters that we do not wish to estimate (nuisance parameters) and to account for the errors induced by the layered down-dip velocity model. The prominent inversion algorithm currently in use is the genetic algorithm proposed by Michael and Ben-Zion [1998a]. This genetic algorithm uses a semi-analytic guided wave solver [Ben-Zion, 1998, Ben-Zion and Aki, 1990] and uses data that has been multiplied by a scaling factor to make it appear 2-D. The algorithm works by evolving an ensemble of solutions until it adequately fits the data. This ensemble of solutions provides an estimate of the uncertainty in the inversion. This distribution is based on the fitness value used [Chatterjee et al., 1996]. These genetic algorithm inversions also inherit the limitations of the forward model used. However, there is potential that the BAE method could be used to overcome some of these limitations in a similar way that the limitations in the forward model proposed in this thesis where overcome (e.g. accounting for the forward model not being gradational in the down-dip direction). The main step that this would require (apart from the BAE) is for the fitness value used in the genetic algorithm to be improved into a likelihood function that accounts for the error in the measurements [Kaipio and Somersalo, 2005].

Another potential inversion methodology would be the use of full waveform inversions and gradational methods. This has not been investigated so far. It would mostly likely require the use of adjoint methods [Tromp et al., 2005]. If this is successful it would be the better inversion method for accounting for real-world geologic complexity. A disadvantage of this method is that the quantification of uncertainty would be less reliable [Melsa and Cohn, 1978].
Forward modelling is also used to investigate fault zone parameters. However, it is difficult to quantify uncertainty in forward modelling, or to determine whether it has converged to the best fit solution. Also, forward modelling does not take into account the nuisance parameters (cf. Chapter 6). Therefore, in this thesis it is proposed that the forward modelling of FZTWs has potentially not produced results consistent with the real world. It is also proposed that it is possible that the inversions using genetic algorithm have not produced results that are close to the real world geologic situation. This is because the limitations of the velocity model used and also because the genetic algorithm does not account for all nuisance parameters. While the inversion algorithm proposed in this thesis takes into account errors in nuisance parameters and allows for 2-D velocity models, it has not been tested with real world data. Methods to test our methodology are proposed in the Section on further work (7.3.3).

### 7.2.3 Geological advances that could be obtained from FZTW studies

An algorithm that produces feasible estimates of the faults low velocity zone would contribute significantly to the knowledge on the mechanics of earthquakes and plate tectonics. It is well known that fault zones are highly complex internally [e.g. Sibson, 1986]. However, FZTWs may be able to capture the bulk properties of the fault zone. These bulk properties include the effective width, velocity contrast, anelastic attenuation factor and depth of the fault.

Fluids are a vitally important component of the fault. For example, they are essential in the development in fault gauge and their frictional properties [Collettini et al., 2009]. The frictional properties of the fault are highly important because it effects the style and type of earthquake [Bar-Sinai et al., 2015]. A wider fault with greater velocity contrast may be attributed to greater permeability [Micarelli et al., 2006]. Attenuation is often attributed to fluids therefore this parameter also provides insight into fluids in the fault [Worthington and Hudson, 2000]. Reliable estimates of the depth extent of the low velocity zone would help establish the depth the high permeability zone goes and its role in transporting fluids from depth [Faulkner et al., 2010]. The permeability of the fault zone is also highly important in the extraction of hydrocarbons [Manzocchi et al., 1999].

Accurate characterisation of the faults properties over time will assist in understanding fault healing. This is a processes where fractures that where opened by seismic activity are closed [e.g. McLaskey et al., 2012]. Understanding fault healing is important because it establishes control on the earthquake cycle between major events and may aid understanding
of when a fault is approaching failure [Ben-Zion et al., 2003a].

Understanding the width of fault zone’s provides insight into previous earthquakes and the style of earthquakes. For example, a wider fault zone typically implies that the fault has accumulated more displacement [Shipton et al., 2013]. Large mature faults however typically have widths on the scale of 100s of meters but there is large variation between different faults [Savage and Brodsky, 2011]. A wider fault might imply that the faulting is less localised on a single plane which has implications for fault roughness and the style of earthquake [Faulkner et al., 2010]. Most notably, a single localised slip surface that persists over large areas would produce larger earthquakes because an earthquake can propagate further without being ‘obstructed’ [Candela et al., 2011].

7.3 Future Work

7.3.1 Alpine Fault inversions

An essential next step in this work is to apply the inversion methodology developed here to FZTWs recorded in the fault zone of the Alpine Fault. There are still a number of issues that would need to be resolved first.

Tunnelling into the fault is a significant obstacle to obtaining feasible estimates. This is particularly a problem for a dipping fault, as vertically refracting waves are more likely to be incident upon the fault zone increasing the possibility for tunnelling in to the fault zone midway along it. It is plausible that this could be resolved through trial and error involving 2-D simulations of potential fault zone models. It is also possible that the Bayesian approximation error gradational correction method already accounts for this but this could only be confirmed through simulations of fault zone models.

An interesting property of FZTWs on the Alpine Fault is that there is a significant delay between the first S-waves and the FZTW (see Fig. 5.3). This is not seen on other vertically dipping faults [e.g. Li et al., 2014b]. Part of this travel time difference is undoubtedly caused because the Alpine Fault is dipping at an angle and, therefore, the refracted S-wave will arrive before FZTWs that are travelling in the direct path along the fault. It is possible that this is exacerbated by fault-parallel anisotropy which may cause the FZTW travelling along the fault to travel slower (see Fig. 4.7). A method to overcome this problem would be to have a broad prior distribution allowing for both situations to be the cause. However, this may result in the posterior estimates being very wide and essentially non-informative because of the non-uniqueness of the problem. Alternatively, further synthetic and geologic
investigations may help form a more informative prior or, the best model could be identified using model selection techniques [Chipman et al., 2001]. An understanding of anisotropy on the Alpine Fault will also be greatly enhanced by further recordings of $F_L$ FZTWs. It is unclear whether any $F_L$ have been identified on the Alpine Fault and the majority of FZTWs identified have been $F_R$ dominant. Evidence suggesting the presence or absence of $F_L$ would aid understanding of anisotropy due to the different polarisation between $F_L$ and $F_R$.

### 7.3.2 Fully 3-D inversions

While most FZTW investigations assume 2-D velocity models, there is evidence that 3-D effects may be important for FZTWs that are generated a significant distance from the receiver along-strike [Mizuno et al., 2008]. In the current inversions, the velocity model is made up of 1-D layers and the Bayesian approximation error method is used in the inversion to make the velocity model appear fully 2-D. It would be straightforward to transition the methodology of Chapter 6 into 3-D by assuming the velocity model is made up of a grid of 1-D polygons (instead of layers). The Bayesian approximation error method could potentially still be used to transition this 1-D polygon grid into a fully 3-D velocity model.

Whilst the transition of the forward model would be seamless, other computation issues would arise. As shown in Chapter 6, the dispersion properties of each layer must be calculated and therefore, if using many 1-D polygons instead of a few layers, the number of rock dispersion properties calculated would significantly increase. These dispersion calculations are already one of the most computationally intensive parts of the forward model, thus the computational time of the forward model would increase significantly. As each 1-D section would have its own acoustic properties, the number of parameters needing to be estimated would greatly increase. This will, therefore, mean that the number of earthquakes needed to constrain these parameters will also increase (100 s). Which would further increase computational time as ray paths would need to be computed for each earthquake and Bayesian approximation error computations need to be carried out for each earthquake. However, the above three computational time increases would all be trivial to parallelise and, therefore, this 3-D approach would be best to carried out on a super computer. Whilst a super computer can resolve the above computational problems, the increased number of parameters that need to be inverted would make the inversion significantly more difficult or even intractable for MCMC techniques. This is because additional parameters mean that more samples need to be collected and tuning the MCMC is more difficult [Robert and Casella, 2005].
7.3.3 Validation of model

It is invariably the case that inversions carried out with real data do not perform as well as with synthetic data [Kaipio and Somersalo, 2005, Tarantola, 2004]. This is because models are always approximations to real world phenomena and inverse problems are highly sensitive to errors. It is, therefore, always desirable to test the methodology with real data where the true model is known. This is particularly difficult in geophysics where it is a financial and technological challenge to access deep rock [Sutherland et al., 2015]. Initially, the methods developed in this thesis were going to utilise the data and be validated against results of the DFDP-2 drilling project. However, technical difficulties meant that the drilling did not reach the fault zone [Sutherland et al., 2015]. An alternative way to validate the computational model could be to use data from laboratory scale fault zones such as the data observed in [Pyrak-Nolte et al., 1992].

7.4 Conclusion

In this study it was demonstrated that gradational velocity models, anisotropy and nuisance parameters are important considerations when performing fault zone investigations using FZTWs. A methodology for inverting FZTW data for a fully gradational 2-D cross section of a fault zone was developed. This is a significant contribution to fault zone studies as current inversion algorithms do not allow for gradational velocity models. Whilst this methodology has not been tested on real world data, it has the potential to provide more reliable estimates of the elastic properties of real fault zones. This could significantly increase our understanding of faulting processes at depths below the reaches of surfaces studies and drilling.
References


References


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References


This appendix shows an example of fault-zone trapped wave (FZTW) propagation. The images were generated using the 3-D finite element wave propagation solver outlined in Section 1.4. The forcing term is a small point-like source in space and a Ricker wavelet in time. The legend and caption for all the simulation images are given in Fig. A.1 and the following figures show time snaps of the simulation.

Figure A.1: Legends and setup of the following simulation images. The wave propagation is shown on three different planes that are ‘cut-out’ from a section of ‘earth’. On these planes, the S-wave velocity is shown in grey scale. The dark band is the fault zone. Each column of the following simulation images show ground displacement in one of the three displacement directions indicated by the arrows at the top of each column. The colours represent ground displacement. Each row has a different maximum displacement, along with the time, shown on the label of each row.
Appendix A. FZTW simulation images

Fault-Perp. Displacement

Fault-Parallel Displacement

Vertical Displacement

Time #50
Max Displacement=178

Time #100
Max Displacement=48

Time #150
Max Displacement=37
Appendix A. FZTW simulation images

Fault-Perp. Displacement

Fault-Parallel Displacement

Vertical Displacement

Time #350
Max Displacement=21

Time #400
Max Displacement=19

Time #450
Max Displacement=18