# Coarse Equivalences of Euclidean Buildings

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With an appendix by Jeroen Schillewaert and Koen Struyve

**Abstract**

We prove the following rigidity results. Coarse equivalences between metrically complete Euclidean buildings preserve spherical buildings at infinity. If all irreducible factors have dimension at least two, then coarsely equivalent Euclidean buildings are isometric (up to scaling factors); if in addition none of the irreducible factors is a Euclidean cone, then the isometry is unique and has finite distance from the coarse equivalence.

The appendix shows how these results can be extended to non-complete Euclidean buildings.

We prove coarse (i.e. quasi-isometric) rigidity results for leafless trees (simplicial trees and $\mathbb{R}$-trees with extensible geodesics) and, more generally, for discrete and nondiscrete Euclidean buildings. For trees, a key ingredient is a certain equivariance condition. Our main results are as follows.

**Theorem I** *Let $G$ be a group acting isometrically on two metrically complete leafless trees $T_1, T_2$. Assume that there is a coarse equivalence $f : T_1 \longrightarrow T_2$, that $T_1$ has at least 3 ends and that the induced map $\partial f : \partial T_1 \longrightarrow \partial T_2$ between the ends of the trees is $G$-equivariant. If the $G$-action on $\partial T_1$ is 2-transitive, then (after rescaling the metric on $T_2$) there is a $G$-equivariant isometry $\bar{f} : T_1 \longrightarrow T_2$ with $\partial f = \partial \bar{f}$. If $T_1$ has at least two branch points, then $\bar{f}$ is unique and has finite distance from $f$.*

The precise result is 2.18 below, where we also consider products of trees and Euclidean spaces. Without the equivariance condition, quasi-isometric rigidity fails. The letters X and H written infinitely large are, for example, trees which are coarsely equivalent without being isometric. Note, too, that any two simplicial trees of finite constant valence are coarsely equivalent without being necessarily isometric [34].

Our next results are concerned with (possibly nondiscrete) Euclidean buildings.

**Theorem II** *Let $X_1$ and $X_2$ be Euclidean buildings whose spherical buildings at infinity $\partial_{cpl}X_1$ and $\partial_{cpl}X_2$ are thick. Suppose that $f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2}$ is a coarse equivalence. Then $m_1 = m_2$ and there is a combinatorial isomorphism $f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_2$ between the spherical buildings at infinity which is characterized by the fact that for an affine apartment $A \subseteq X_1$ the $f$-image of $A \times \mathbb{R}^{m_1}$ has finite Hausdorff distance from $f_*(A) \times \mathbb{R}^{m_2}$.*

This is 5.16 below. We remark that the boundary map $f_*$ is constructed in a combinatorial way from $f$. In general, a coarse equivalence between CAT(0)-spaces will not induce a map between the respective Tits boundaries.

**Theorem III** *Let $f : X_1 \times \mathbb{R}^m \longrightarrow X_2 \times \mathbb{R}^m$ be as in Theorem II and assume in addition that $X_1$ has no tree factors and that $X_1$ and $X_2$ are metrically complete. Then there is (possibly after rescaling*

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*the metrics on the de Rham factors of $X_2$) an isometry $\bar{f} : X_1 \longrightarrow X_2$ with $(\bar{f} \times \mathrm{id}_{\mathbb{R}^m})_* = f_*$. Put $f(x \times v) = f_1(x \times v) \times f_2(x \times v)$. If none of the de Rham factors of $X_1$ is a Euclidean cone over its boundary, then $\bar{f}$ is unique and $d(f_1(x \times y), \bar{f}(x))$ is bounded as a function of $x \in X_1$.*

For a more general statement see 5.21 below.

Kleiner and Leeb proved results similar to Theorems II and III in [25, 1.1.3] and [31, 1.3] under the additional hypothesis that the Euclidean buildings are simplicial and locally finite or their buildings at infinity satisfy the Moufang condition. Their work extended Mostow-Prasad rigidity [36]. Our results offer in particular an alternative approach to these important achievements.

**Outline of the paper** In the first section we collect some basic facts about metric spaces, nonpositive curvature, ultraproducts and ultralimits. The second section is concerned with trees. We derive in particular the structure of leafless trees that admit an isometry group that is 2-transitive on the ends. Theorem I is then an application of this structural result, combined with Morse's Lemma. In the third section we collect the facts about spherical buildings and projectivities that we need. The fourth section is devoted to Euclidean buildings and their local and global structure. In the fifth section we combine all previous results. We first prove Theorem II, using the 'higher dimensional Morse Lemma'. Then we combine Theorem I and Theorem II in order to prove Theorem III. The proof relies, among other things, on Tits' rigidity result [46, Thm. 2] of 'ecological' (tree-preserving) boundary isomorphisms of Euclidean buildings.

# 1 Some metric geometry

We recall a few notions and facts about metric spaces that will be needed later. Let $(X, d)$ be a metric space. For $r > 0$ and $x \in X$ we put

$$B_r(x) = \{y \in X \mid d(x, y) < r\} \quad \text{and} \quad \bar{B}_r(x) = \{y \in X \mid d(x, y) \leq r\}.$$

For a subset $Y \subseteq X$ we put $B_r(Y) = \bigcup \{B_r(y) \mid y \in Y\}$. We call $Y \subseteq X$ *bounded* if $Y$ is contained in some sufficiently large ball, $Y \subseteq B_r(x)$ for some $x \in X$, and we call $Y$ *cobounded* if $X = B_r(Y)$ for some $r \geq 0$.

Let $f : X \longrightarrow Y$ be a map between metric spaces. If there is a constant $r > 0$ such that

$$d(f(u), f(v)) \leq r d(u, v)$$

holds for all $u, v \in X$, we call $f$ an *r-Lipschitz map*. A map $f$ which preserves distances is called an *isometric embedding*; if $f$ is in addition onto, it is called an *isometry*.

The group of all isometries of $X$ onto itself is the *isometry group* $\mathrm{Isom}(X)$. An *isometric action* of a group $G$ on $X$ is a homomorphism $G \longrightarrow \mathrm{Isom}(X)$. We call such an action (or group) *bounded* if some (or, equivalently, every) $G$-orbit is bounded in $X$. The action is called *cobounded* if there is a cobounded orbit.

**1.1 CAT($\kappa$) spaces** A *geodesic* in a metric space $X$ is an isometric embedding $\gamma : J \longrightarrow X$, where $J \subseteq \mathbb{R}$ is a closed interval. The image $\gamma(J)$ is then called a *geodesic segment*. If the domain of $\gamma$ is $J = \mathbb{R}$, then $\gamma(\mathbb{R})$ is also called a *geodesic line*, and if $J = [0, \infty)$, then $\gamma([0, \infty))$ is called a *geodesic ray*. If any two points of $X$ are contained in some geodesic segment, $X$ is called a *geodesic space*. If every geodesic $\gamma : J \longrightarrow X$ admits a geodesic extension $\bar{\gamma} : \mathbb{R} \longrightarrow X$, we say that $X$ has *extensible geodesics*.

A geodesic space $X$ is called a *CAT(0) space* if no geodesic triangle in $X$ is thicker than its comparison triangle in Euclidean space $\mathbb{R}^2$, see [7, II.1]. More generally, a geodesic space is called a CAT($\kappa$) space, for $\kappa \in \mathbb{R}$, if no geodesic triangle in $X$ is thicker than its comparison triangle in the complete simply connected Riemannian 2-manifold $M_\kappa$ of constant sectional curvature $\kappa$. For $\kappa > 0$, the space $M_\kappa$ is a sphere and the condition is only required for geodesic triangles of perimeter less than $2\frac{\pi}{\sqrt{\kappa}}$ (and the existence of geodesics is only required between points at distance less than $\frac{\pi}{\sqrt{\kappa}}$). If $(X, d)$ is CAT($\kappa$), then the same space is also CAT($\kappa'$) for all $\kappa' \geq \kappa$ [7, II.1.12]. If $r > 0$ and if $(X, d)$ is CAT($\kappa$), then the rescaled space $(X, rd)$ is CAT($\kappa/\sqrt{r}$).

If $K$ is a metrically complete convex subset in a CAT(0) space $X$, then there is a 1-Lipschitz retraction

$$\pi_K : X \longrightarrow K$$

which maps $x \in X$ to the unique closest point in $K$ [7, II.2.4].

The metric completion of a CAT($\kappa$) space is again a CAT($\kappa$) space. One important fact about complete CAT(0) spaces is the Bruhat-Tits Fixed Point Theorem: every bounded isometry group in such a space has a fixed point [7, II.2.8].

**1.2 Controlled maps** We now recall some notions from coarse geometry [37]. A map $f : X \longrightarrow Y$ between metric spaces is called *controlled* if there is a monotonic real function $\rho : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ such that

$$d_Y(f(x), f(y)) \leq \rho(d_X(x, y))$$

holds for all $x, y \in X$. If in addition the preimage of every bounded set is bounded, then $f$ is called a *coarse map*. Neither $f$ nor $\rho$ is required to be continuous. Note that the image of a bounded set under a controlled map is bounded. Two maps $g, f : X \rightrightarrows Y$ between metric spaces have *finite distance* if the set $\{d_Y(f(x), g(x)) \mid x \in X\}$ is bounded. This is an equivalence relation which leads to the *coarse metric category* whose objects are metric spaces and whose morphisms are equivalence classes of coarse maps. A *coarse equivalence* is an isomorphism in this category.

**1.3 Lemma** *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are controlled and if $g \circ f$ has bounded distance from the identity map on $X$, then $f$ is coarse. In particular, $f$ is a coarse equivalence if $f \circ g$ also has bounded distance from the identity.*

*Proof.* Suppose that $f(Z)$ is bounded. Since $g$ is controlled, $g(f(Z))$ is also bounded, $g(f(Z)) \subseteq B_r(x)$ for some $x \in X$. Now $Z$ is contained in some $s$-neighborhood of $g(f(Z))$, so $Z \subseteq B_{r+s}(x)$ is bounded. $\qquad\square$

**1.4 Quasi-isometries** A controlled map $f$ with control function $\rho(t) = t$ is the same as a 1-Lipschitz map. If $f$ admits a control function

$$\rho(t) = ct + d, \qquad \text{with } c \geq 1 \text{ and } d \geq 0,$$

then $f$ is called *large-scale Lipschitz*. A coarse equivalence is called a *quasi-isometry* if both the map and its coarse inverse are large-scale Lipschitz. More generally, we call $f$ a *quasi-isometric embedding* if $f$ is a quasi-isometry between $X$ and $f(X) \subseteq Y$. A *rough isometry* is a coarse equivalence where the control functions in both directions are of the form $\rho(t) = t + d$ (such maps are sometimes called *coarse isometries*). A map which has finite distance from an isometry is an example of a rough isometry.

In the class of geodesic spaces, controlled maps are essentially the same as large-scale Lipschitz maps.

**1.5 Lemma** *Let $f : X \longrightarrow Y$ be controlled. If $X$ is geodesic, then $f$ is large-scale Lipschitz. In particular, every coarse equivalence between geodesic spaces is automatically a quasi-isometry.*

*Proof.* For $x, y \in X$ let $d(x, y) = m + s$, with $m \in \mathbb{N}$ and $0 \leq s < 1$. Let $\gamma$ be a geodesic from $x$ to $y$. Then $d(f(\gamma(j)), f(\gamma(j + 1))) \leq \rho(1)$ for $j = 0, \ldots, m - 1$, so $d(f(\gamma(0)), f(\gamma(m + s))) \leq m\rho(1) + \rho(s) \leq (m + s)\rho(1) + \rho(1)$. $\qquad\square$

**1.6 Hausdorff distance** Two (nonempty) subsets $U, V$ of a metric space $X$ have *Hausdorff distance at most $r$* if

$$U \subseteq B_r(V) \quad \text{and} \quad V \subseteq B_r(U).$$

In this case we write $Hd(U, V) < r$. We define for $U, V, W \subseteq X$ the Hausdorff distance [19, VIII §6] as

$$Hd(U, V) = \inf\{r > 0 \mid Hd(U, V) < r\} \quad \text{and} \quad Hd_W(U, V) = Hd(U \cap W, V \cap W).$$

For example, a nonempty subset is bounded if and only if it has finite Hausdorff distance from some point. More generally, we say that $V$ *dominates* $U$ if $U \subseteq B_r(V)$ for some $r > 0$, and we write then

$$U \subseteq_{Hd} V.$$

This defines a preorder on the subsets of $X$. If $f : X \longrightarrow Y$ is controlled with control function $\rho$, and if $U \subseteq B_r(V)$, then $f(U) \subseteq B_{\rho(r)}(V)$. So if $f$ is a coarse equivalence, then $U$ and $V$ have finite Hausdorff distance if and only if $f(U)$ and $f(V)$ have finite Hausdorff distance, and $U$ dominates $V$ if and only if $f(U)$ dominates $f(V)$.

Now we turn to ultralimits and asymptotic cones of metric spaces. These constructions generalize pointed Gromov-Hausdorff convergence of metric spaces. The advantage of ultralimits is that they exist even if Gromov-Hausdorff convergence fails, and that these spaces always have good metric and functorial properties. In contrast to [24, Sec. 3.], [25, 2.4], [7, I.5], [37, 7.5], [18, 3.29], we follow the original approach of van den Dries and Wilkie [47], which is based on elementary model theory and ultraproducts. The advantage of this is viewpoint is that it keeps track of the global structure of spaces, and that it allows us in a natural way to include some further structure, such as metric balls, apartments, geodesics, or Hausdorff distance. We remark that similar techniques have been used in Banach space theory for the last 50 years.

In order to make the paper self-contained, we briefly review some elementary facts about languages, structures, and ultraproducts. They can be found in any 'old-fashioned' textbook on mathematical logic, such as [2] or [12].

**1.7 Languages and structures** Suppose that $\mathcal{L}$ is a first-order language. Such a language is a set consisting of symbols for finitary functions, relations and constants. The language is allowed to be infinite. A specific example which will be important here is the language $\mathcal{L}_{\mathrm{of}} = \{+, \cdot, \leq, 0, 1\}$ of ordered fields. An $\mathcal{L}$-*structure* is a tuple $\mathfrak{A} = (A, F, R, C)$ consisting of a nonempty set $A$ (the *universe* of the structure), a set $F$ of finitary functions, a set $R$ of finitary relations, and a set $C \subseteq A$ of constants. In addition, we have an *interpretation* which assigns to every function/relation/constant symbol in $\mathcal{L}$ a function/relation/constant in $\mathfrak{A}$. For example, the reals form an $\mathcal{L}_{\mathrm{of}}$-structure $\mathfrak{R}$ if we interpret the symbols $+, \cdot, 0, 1, \leq$ in their usual way. Abusing notation, we denote symbols and their interpretations

by the same letters—this should cause no confusion, as the interpretation will always be the natural one.

Metric spaces fit well into this concept. The relevant language is $\mathcal{L}_{\mathrm{ms}} = \mathcal{L}_{\mathrm{of}} \cup \{d, X, R\}$. If $(X, d)$ is a metric space, then we consider the $\mathcal{L}_{\mathrm{ms}}$-structure $\mathfrak{X}$ with universe $X \,\dot\cup\, \mathbb{R}$, functions $+, \cdot, d$, relations $R, X, \leq$, and constants $0, 1$. The unary relation $X(p)$ says that $p$ is a point in $X$ and the unary relation $R(t)$ says that $t$ is a real number. The binary function $d$ gives the distance on points in $X$, while the binary functions $+, \cdot$ give the usual arithmetic operations in the real numbers.

**1.8 Formulas** From a language $\mathcal{L}$ we build formulas by the standard rules of first-order logic, using the logical symbols $\exists, \forall, \dot=, \neg, \rightarrow$ and an infinite set of symbols for variables. Variables in a formula which are not bound by a quantifier are called *free*. We write $\varphi(x)$ for such a formula with a free variable $x$. Abusing notation, $x$ may also denote a finite tuple of free variables (and we also allow tacitly that the formula $\varphi(x)$ contains no free variables at all). If we want to interpret a formula in a given $\mathcal{L}$-structure, we first have to substitute the free variables by elements of the universe, that is, we plug in specific values $a$ for the free variables $x$. For this substitution, we write $\varphi(a)$. A formula without free variables is called a *sentence*. If a sentence $\varphi$ holds in the $\mathcal{L}$-structure $\mathfrak{A}$, one writes $\mathfrak{A} \models \varphi$. For example, we have in the metric space $\mathfrak{X}$ that

$$\mathfrak{X} \models \forall u, v, w \left( X(u) \wedge X(v) \wedge X(w) \rightarrow d(x, z) \leq d(x, y) + d(y, z) \right)$$

that is, the triangle inequality holds for all triples of points in $X$. The formula

$$\varphi(x_1, x_2) = \left( X(x_1) \wedge X(x_2) \wedge (d(x_1, x_2) \leq 1) \right)$$

with free variables $x_1, x_2$ holds in the metric space $X$ if we substitute points $u, v$ for $x_1$ and $x_2$ whose distance is at most 1.

**1.9 Ultraproducts** Suppose that $K$ is a countably infinite set and that $D$ is a nonprincipal ultrafilter on $K$. This means that $D$ is a collection of subsets of $K$, containing all cofinite subsets of $K$, which is closed under finite intersections, closed under going up (if $a \in D$ and $K \supseteq b \supseteq a$, then $b \in D$), and which contains for every subset $a \subseteq K$ either $a$ or its complement $K - a$, but not both. Using Zorn's Lemma, it is easy to see that nonprincipal ultrafilters exist [12, 4.1.3]. Suppose that $(\mathfrak{A}_k)_{k \in K}$ is a family of $\mathcal{L}$-structures $\mathfrak{A}_k = (A_k, F_k, R_k, C_k)$. We introduce an equivalence relation on $K$-sequences $(a_k), (b_k) \in \prod_k A_k$ by putting

$$(a_k) \sim_D (b_k) \quad \text{if} \quad \{j \in K \mid a_j = b_j\} \in D.$$

The resulting set of equivalence classes is the *ultraproduct* $A_D = \prod_k A_k / D$. On the functions, relations and constants in $\prod_k F_k$, $\prod_k R_k$ and $\prod_k C_k$ we introduce the same equivalence relation. The resulting $\mathcal{L}$-structure

$$\mathfrak{A}_D = \left( A_D, F_D, R_D, C_D \right)$$

is the *ultraproduct* of the family $(\mathfrak{A}_k)_{k \in K}$. If all structures in the family are equal, one calls $\mathfrak{A}_D$ also an *ultrapower*.

Ultraproducts are useful because of the following two facts. Firstly, Łos' Theorem says that $\mathfrak{A}_D$ has the same first-order properties as 'the majority of the $\mathfrak{A}_k$'. The second fact is that the ultraproduct is 'huge' and contains 'strange' elements: it is $\omega_1$-saturated (this is essentially the same as the 'overspill' mentioned in [47, Sec. 3]). For the next two theorems we assume that $(\mathfrak{A}_k)_{k \in K}$ is a countable family of $\mathcal{L}$-structures and that $D$ is a nonprincipal ultrafilter on $K$.

**1.10 Theorem (Łos' Theorem)** *Let $\varphi(x)$ be an $\mathcal{L}$-formula (possibly with a free variable $x$). Let $a = (a_l) \in \prod_k A_k$. Then*

$$\mathfrak{A}_D \models \varphi(a) \quad \text{if and only if} \quad \{j \in K \mid \mathfrak{A}_j \models \varphi(a_j)\} \in D.$$

*(If there is no free variable in $\varphi$ this means that $\varphi$ holds in $\mathfrak{A}_D$ if and only if the set of indices $k$ for which $\varphi$ holds in $\mathfrak{A}_k$ is in $D$.)*

*Proof.* The proof is an easy induction on the complexity of formulas in [12, 4.1.9] [2, §5 2.1] [22, 9.5.1]. □

The following fact is an elementary consequence.

**1.11 Lemma** *If $\mathfrak{A}$ is a finite structure, then the ultrapower $\mathfrak{A}_D$ is isomorphic to $\mathfrak{A}$.* □

If we put $\mathfrak{A}_k = \mathfrak{R} = (\mathbb{R}, +, \cdot, \leq, 0, 1)$, we end up with the field of *nonstandard reals*

$$\mathfrak{R}_D = ({}^*\mathbb{R}, +, \cdot, \leq, 0, 1).$$

By Łos' Theorem, this is a real closed field. If each $\mathfrak{X}_k$ is a metric space, then $\mathfrak{X}_D$ is a generalized metric space, where the distance function takes it values in the nonstandard reals ${}^*\mathbb{R}$. We will come back to these generalized metric spaces below. Variants of the following result are folklore. For the sake of completeness, we include a proof.

**1.12 Theorem ($\omega_1$-saturation of ultraproducts)** *Let $\Phi$ be countable set of $\mathcal{L}$-formulas in the free variable $x$. Suppose that for every finite subset $\Psi \subseteq \Phi$ there exists an $K$-sequence $(a_k)$ such that $\mathfrak{A}_D \models \psi(a)$ holds for all $\psi(x) \in \Psi$. Then there exists an $K$-sequence $(a_k)$ such that $\mathfrak{A}_D \models \varphi(a)$ holds for all $\varphi(x) \in \Phi$.*

*Proof.* We may assume that $K = \mathbb{N}$ and that $\Phi$ is countably infinite. Accordingly, let $\Phi = \{\varphi_k(x) \mid k \in \mathbb{N}\}$ be an enumeration of $\Phi$. We have by Łos' Theorem 1.10 that

$$K_n = \{k \in \mathbb{N} \mid k \geq n \text{ and } \mathfrak{A}_k \models \exists x\, \varphi_0(x) \wedge \cdots \wedge \varphi_n(x)\} \in D.$$

Since $\bigcap_{n \geq 0} K_n = \varnothing$, there exists for each $i \in K_0$ a largest $n(i)$ with $i \in K_{n(i)}$. For $k \in \mathbb{N} - K_0$ we choose $a_k \in A_k$ arbitrarily. For $k \in K_0$ we choose $a_k \in A_k$ in such a way that $\mathfrak{A}_k \models \varphi_0(a_k) \wedge \cdots \wedge \varphi_{n(k)}(a_k)$ holds. We claim that the $K$-sequence $a = (a_k)$ has the required property. Let $n \in \mathbb{N}$. For all $k \in K_n$ we have that $n \leq n(k)$, whence $\mathfrak{A}_k \models \varphi_n(a_k)$. Thus $\mathfrak{A}_D \models \varphi_n(a)$ holds. □

As an illustration, consider the countable set of $\mathcal{L}_{\text{of}}$-formulas

$$\varphi_n(x) = \big(x > 0 + \underbrace{1 + 1 + \cdots + 1}_{n \text{ summands}}\big).$$

By 1.12, there are elements in ${}^*\mathbb{R}$ which satisfy these formulas simultaneously. This means that there are elements in ${}^*\mathbb{R}$ which are bigger than every natural number. In other words, ${}^*\mathbb{R}$ is a *non-archimedean real closed field*. We call an element $r \in {}^*\mathbb{R}$ *finite* if $|r| \leq n$ holds for some $n \in \mathbb{N}$. It is readily verified that the finite elements form a valuation ring ${}^*\mathbb{R}_{\text{fin}} \subseteq {}^*\mathbb{R}$. Its maximal ideal is the set ${}^*\mathbb{R}_{\text{inf}} = \{r \in {}^*\mathbb{R} \mid |r| \leq \frac{1}{n+1} \text{ for all } n \in \mathbb{N}\}$ of *infinitesimal elements*. One can show that the quotient field ${}^*\mathbb{R}_{\text{fin}}/{}^*\mathbb{R}_{\text{inf}}$ is order-isomorphic to $\mathbb{R}$ [41, 23.8]. The corresponding ring epimorphism

$$std : {}^*\mathbb{R}_{\text{fin}} \longrightarrow \mathbb{R}$$

assigns to every finite nonstandard real $x \in \mathbb{R}_{\mathrm{fin}}$ its so-called standard part $std(x) \in \mathbb{R}$. We note also that $^*\mathbb{R}$ contains canonically a copy of $\mathbb{R}$ (represented by constant $K$-sequences of reals).

$$0 \longrightarrow {}^*\mathbb{R}_{\mathrm{inf}} \longrightarrow {}^*\mathbb{R}_{\mathrm{fin}} \longleftrightarrows \mathbb{R} \longrightarrow 0$$

**1.13 Ultralimits and asymptotic cones** For the remainder of this section, $D$ will denote a nonprincipal ultrafilter on the countably infinite set $K$. We noted above that the ultraproduct $\mathfrak{X}_D$ of a family of metric spaces is a generalized metric space, whose distance function takes its values in the nonstandard reals[1]. Let $p$ be a point in the ultraproduct $X_D$ and let $r$ be a positive nonstandard real. Let

$$X_D^{(p,r)} = \left\{ q \in X_D \mid \tfrac{1}{r} d(p,q) \in {}^*\mathbb{R}_{\mathrm{fin}} \right\}.$$

Then $\tilde{d} = std \circ \tfrac{1}{r} d$ is a real-valued pseudometric

$$X_D^{(p,r)} \times X_D^{(p,r)} \xrightarrow{\ std \circ \frac{1}{r} d\ } \mathbb{R},$$

where points with infinitesimal $\tfrac{1}{r} d$-distance have $\tilde{d}$-distance 0. Identifying points at $\tilde{d}$-distance 0, we obtain a metric space which we denote by $C(X_D, p, r)$,

$$C(X_D, p, r) \times C(X_D, p, r) \xrightarrow{\ \tilde{d}\ } \mathbb{R}.$$

This is the *ultralimit* of the family $(X_k)_{k \in K}$ with respect $p$ and $r$. If we represent the nonstandard real $r$ and the point $p$ by $K$-sequences $(r_k)_{k \in K}$ and $(p_k)_{k \in K}$, respectively, then these are the sequences of scaling factors and basepoints as in [25, 2.4.2] and [37, 7.5]. In the notation of these authors, we have

$$C(X_D, p, r) = D\text{-}\lim \tfrac{1}{r_k} X_k.$$

If $r$ is infinite and all $X_k$ are equal to one fixed metric space $X$, then $C(X_D, p, r)$ is the *asymptotic cone* of $X$. It is immediate that ultralimits commute with metric product decompositions. From the surjection $std : {}^*\mathbb{R}_{\mathrm{fin}} \longrightarrow \mathbb{R}$ we have in particular an isometry

$$C({}^*\mathbb{R}^n, p, r) \cong \mathbb{R}^n$$

for any choice of $p$ and $r$.

Ultralimits have many good properties. For example, they are spherically complete (and in particular complete, [15, §32]). Spherical completeness means that every nested sequence $\bar{B}_{t_0}(x_0) \supseteq \bar{B}_{t_1}(x_1) \supseteq \bar{B}_{t_2}(x_2) \supseteq \bar{B}_{t_3}(x_3) \supseteq \cdots$ of closed balls has nonempty intersection.

**1.14 Lemma** *The ultralimit $C(X_D, p, s)$ of any countably infinite family of metric spaces $(X_k)_{k \in K}$ is spherically complete.*

 *Proof.* Let $\bar{B}_{t_0}(x_0) \supseteq \bar{B}_{t_1}(x_1) \supseteq \bar{B}_{t_2}(x_2) \supseteq \bar{B}_{t_3}(x_3) \supseteq \cdots$ be a nested sequence of closed balls in $C(X_D, p, r)$. We consider the canonical surjection

$$X_D \supseteq X_D^{(p,r)} \longrightarrow C(X_D, p, r)$$

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[1]Generalized metric spaces, where the metrics take values in some ordered abelian group appear already in Hausdorff [19, VIII §5 III]. See also [14, Ch. 1.2] [33, II.1].

which identifies points whose $\frac{1}{r}d$-distance is infinitesimal. For each $x_n$ we choose a preimage $z_n \in X_D^{(p,r)}$. For $k, \ell \in \mathbb{N}$ we put

$$E_{k,\ell} = \left\{ z \in X_D \mid \tfrac{1}{r}d(z, z_k) \leq t_k + \tfrac{1}{\ell+1} \right\}.$$

This family has the finite intersection property. Indeed, we have $z_m \in E_{k,\ell}$ for all $m \geq k$. Adding constant symbols $z_n$ and $t_n$ to our language $\mathcal{L}_{\mathrm{ms}}$, we may apply Theorem 1.12. The countable set of formulas

$$\varphi_{k,\ell}(x) = \left( \tfrac{1}{r}d(x, z_k) \leq t_k + \tfrac{1}{\ell+1} \right)$$

can be satisfied by a single element $z \in X_D$, that is, $z \in \bigcap_{k,\ell} E_{k,\ell}$. The image of this element in $C(X_D, p, r)$ is contained in $\bigcap_{n \geq 0} \bar{B}_{r_n}(x_n)$. $\qquad \square$

Now we suppose that each space $X_k$ is $\mathrm{CAT}(\kappa_k)$. Let $\kappa$ denote the nonstandard real corresponding to the $K$-sequence $(\kappa_k)$. We also fix a positive nonstandard real $r$. The next result is well-known [24, 3.6] [25, 2.2.4]; from the viewpoint of ultraproducts and Łos' Theorem, it is almost trivial.

**1.15 Lemma** *For every real number $\lambda$ with $\lambda \geq \sqrt{r}\kappa$, the space $C(X_D, p, r)$ is $CAT(\lambda)$. In particular, $C(X_D, p, r)$ is $CAT(0)$ if each $X_k$ is $CAT(0)$.*

*Proof.* This is obvious from Łos' Theorem 1.10 and the fact that the $\mathrm{CAT}(\kappa)$ condition can be stated in the language $\mathcal{L}_{\mathrm{ms}}$. The details are as follows. As in the standard definition of the $\mathrm{CAT}(\kappa)$ condition [7, II.1], we put $D_\kappa = \infty$ for $\kappa \leq 0$ and $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$. For nonstandard reals $s < t$ we define the 'nonstandard interval'

$$[s, t] = \{x \in {}^*\mathbb{R} \mid s \leq x \leq t\},$$

and we define 'nonstandard geodesics' in $X_D$ in the obvious way as isometric injections of such nonstandard intervals. It is clear from the fact that the $X_k$ are $\mathrm{CAT}(\kappa_k)$ spaces and from Łos' Theorem that any two points $u, v \in X_D$ with $d(u, v) < D_\kappa$ can be joined by a nonstandard geodesic. We put $\varepsilon = 0$ if $\kappa = 0$ and $\varepsilon = \kappa/|\kappa|$ else. Let $M^\varepsilon$ denote the simply connected complete Riemannian surface of constant sectional curvature $\varepsilon$, with its metric $d^\varepsilon$, and let $M_D^\varepsilon$ denote its ultrapower. For $\kappa \neq 0$ we define a metric $d^\kappa$ on $M_D^\varepsilon$ by $d^\kappa = \frac{1}{\sqrt{|\kappa|}}d^\varepsilon$. Again, we see from Łos' Theorem 1.10 that triangles of perimeter less than $2D_\kappa$ in $(X_D, d)$ are not thicker than triangles in $(M_D^\varepsilon, d^\kappa)$. Thus $X_D$ is a 'nonstandard $\mathrm{CAT}(\kappa)$ space': it satisfies the usual $\mathrm{CAT}(\kappa)$ comparison triangle condition, except that the model space $M^\varepsilon$ is replaced by its nonstandard version $M_D^\varepsilon$. If we rescale the metric on $X_D$ by the factor $\frac{1}{r}$, we obtain a nonstandard $\mathrm{CAT}(\sqrt{r}\kappa)$ space. If we identify in this space $(X_D, \frac{1}{r}d)$ points at infinitesimal distance, nonstandard geodesics become ordinary geodesics. The $\mathrm{CAT}(\lambda)$ inequality also remains valid for all real $\lambda \geq \sqrt{r}\kappa$. Thus $C(X_D, p, r)$ is a $\mathrm{CAT}(\lambda)$ space. $\qquad \square$

Asymptotic cones may be used to 'make large-scale Lipschitz maps continuous' as follows.

**1.16 Lemma** *Suppose that we have a countably infinite family of large-scale Lipschitz maps*

$$f_k : X_k \longrightarrow Y_k, \quad \text{with} \quad d(f_k(u), f_k(v)) \leq s_k d(u, v) + t_k.$$

Let $s$ and $t$ denote the nonstandard reals represented by the $K$-sequences $(s_k)$ and $(t_k)$ in $^*\mathbb{R}$. Assume that $s$ is finite with standard part $std(s) = s' \in \mathbb{R}$. Suppose that $r$ is a positive nonstandard real. If $\frac{t}{r}$ is infinitesimal, then $f$ induces an $s'$-Lipschitz map

$$C(f) : C(X_D, p, r) \longrightarrow C(Y_D, f_D(p), r).$$

This holds in particular if the $s_k$ and $t_k$ are constant and $r$ is any positive infinite nonstandard real.

*Proof.* We have $\frac{1}{r}d(f_D(u), f_D(v)) \leq \frac{1}{r}d(u,v) + \frac{t}{r}$ by Łoś' Theorem. Identifying points at infinitesimal distance, we obtain a well-defined Lipschitz map between the ultralimits. $\square$

**1.17 Corollary** *There is no quasi-isometric embedding $f : [0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.*

*Proof.* Otherwise, we could choose an infinite positive nonstandard real $r$ to obtain a continuous injection $C(f)$

$$
\begin{array}{ccc}
C(([0,\infty) \times \mathbb{R}^n)_D, 0, r) & \xhookrightarrow{\quad C(f) \quad} & C(\mathbb{R}_D^n, 0, r) \\
\cong \downarrow & & \cong \downarrow \\
[0, \infty) \times \mathbb{R}^n & & \mathbb{R}^n.
\end{array}
$$

Such a map would embed a closed $n+1$-cube homeomorphically into $\mathbb{R}^n$, which is impossible [17, 7.1.1, 7.3.20]. $\square$

## 2 Coarse equivalences of trees

In this section we prove Theorem I from the introduction. We first investigate the structure of group actions on trees that are 2-transitive on the ends.

**2.1 Definition** A metric space $T$ is called a *tree* (or $\mathbb{R}$-tree) if it has the following two properties.

**(T1)** For any two points $x, y \in T$, there is a unique geodesic $\gamma : [0, d(x,y)] \longrightarrow T$ with $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. We put $[x, y] = \gamma([0, d(x,y)])$.

**(T2)** If $0 < r < s$ and if $\gamma : [0, s] \longrightarrow T$ is an injection such that $\gamma|_{[0,r]}$ and $\gamma|_{[r,s]}$ are geodesics, then $\gamma$ is a geodesic.

Trees are CAT($\kappa$) for every $\kappa \in \mathbb{R}$, and in particular CAT(0) [7, II.1.15]. Basic references for trees are [1], [14] and [33]. An $\mathbb{R}$-tree with extensible geodesics is called a *leafless tree*. An *apartment* in $T$ is an isometric image of $\mathbb{R}$ (a geodesic line).

If $z$ is in the geodesic segment $[x, y]$ but different from $x$ and $y$, we say that $z$ is *between* $x$ and $y$. Given two geodesic segments $[x, y]$ and $[x, y']$ in a tree, there is a unique point $z$ with $[x, y] \cap [x, y'] = [x, z]$ [14, p. 30]. A point $z$ in the tree is called a *branch point* if there are three points $u, v, w$ distinct from $z$ such that $[u, v] \cap [v, w] \cap [w, u] = \{z\}$.

**2.2 Ends** A *ray* in a tree $T$ is an isometric image of $[0, \infty)$ (a geodesic ray). Two rays are called equivalent (or parallel) if their intersection is again a ray; the resulting equivalence classes are called the *ends* of $T$. The set of all ends of $T$ is denoted by $\partial T$. Given $x \in T$ and $u \in \partial T$, there is a unique geodesic $\gamma : [0, \infty) \longrightarrow T$ with $\gamma(0) = x$ whose image is in the class of $u$ [14, p. 60]; we put $\gamma([0, \infty)) = [x, u)$ and $(x, u) = [x, u) - \{x\}$.

Every apartment in $T$ determines two ends. Conversely, if $u, v$ are distinct ends, then there is a unique apartment whose ends are $u$ and $v$ [14, p. 61] and which we denote $(u, v) \subseteq T$. If $A \subseteq T$ is an apartment and $z \in T$, then there exists a unique point $\pi_A(z)$ in $A$ which has minimal distance from $z$ and every geodesic segment $[z, x]$ with $x \in A$ contains $\pi_A(z)$ [14, p. 61]. This definition is compatible with 1.1.

We now consider the following condition.

$(2\text{-}\partial)$ The group $G$ acts isometrically on the leafless metrically complete tree $T$, and this action is 2-transitive on $\partial T$: given ends $u \neq v$ and $u' \neq v'$, there is an element $g \in G$ with

$$g(u) = u' \quad \text{and} \quad g(v) = v'.$$

We note that such a group acts transitively on the set of apartments of $T$. We now derive the structure of trees $T$ satisfying the condition $(2\text{-}\partial)$. We call a point $x$ in the tree $G$-isolated if it is the unique fixed point of its stabilizer,

$$T^{G_x} = \{x\}.$$

For such a $G$-isolated point $x$, the stabilizer $G_x$ is by the Bruhat-Tits Fixed Point Theorem [7, II.2.8] a maximal bounded subgroup (i.e., $G_x$ is not properly contained in a bounded subgroup of $G$).

**2.3 Proposition** *Let $u, v, w$ be three distinct ends of a tree $T$ satisfying $(2\text{-}\partial)$ and let $x$ be the branch point determined by these three ends, $(u, v) \cap (u, w) = (u, x]$. Then there exists an element $g \in G_x$ which fixes $u$ and maps $v$ to $w$.*

*Proof.* Since $G$ is 2-transitive on $\partial T$, we find $h \in G_u$ such that $h(v) = w$.
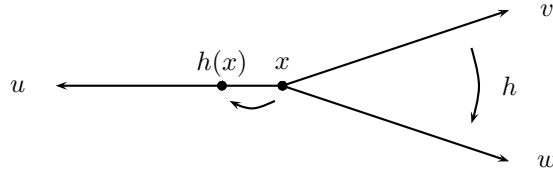


Since $h(x) \in h(u, v) = (u, w)$, one of the following three cases must hold:
(1) If $h(x) = x$ we are done, with $g = h$.
(2) If $h(x) \in (u, x)$, then $A = \bigcup \{h^{-n}(u, x] \mid n \in \mathbb{N}\}$ is an $h$-stable apartment, on which $h$ induces a translation of length $d(x, h(x))$. As all apartments are conjugate under $G$, we find an isometry $h'$ fixing $u$ and $w$ which maps $h(x)$ to $x$. Thus $g = h'h$ fixes $x$ and $u$ and maps $v$ to $w$.
(3) If $x \in (u, h(x))$, we put $A = \bigcup \{h^n(u, x] \mid n \in \mathbb{N}\}$ and argue similarly as in (2). $\square$

**2.4 Corollary** *In a tree satisfying $(2\text{-}\partial)$, all branch points are $G$-isolated.* $\square$

We have the following a topological dichotomy.

**2.5 Proposition** *In a tree satisfying $(2\text{-}\partial)$, the set of branch points is either closed and discrete, or it is dense in every apartment.*

*Proof.* Let $A = (u, v)$ be an apartment and assume that the set of branch points is not dense in $A$. If $A$ contains no branch point, then $A = T$, and if $A$ contains only one branch point, then every apartment contains only one branch point. In either case, the claim follows.

Assume that $x \in A$ is a branch point and that $A$ contains another branch point $y$. By 2.3 we find an element $g \in G_x$ which interchanges $(u, x]$ and $(v, x]$. Then $g(y)$ is also a branch point. In this

way, we get infinitely many branch points in $A$ at uniform distance $d(x,y)$. Because the set of branch points in $A$ was assumed not to be dense, there has to be a minimal distance $t$ between these, and they are distributed uniformly at this distance in $A$.

Since all apartments are conjugate, the set of branch points is either dense or discrete with uniform distance $t$ in every apartment. In the discrete case, the minimal distance between two branch points is $t$, so the set is closed and discrete in $T$. □

**2.6 Corollary** *Assume that the tree $T$ satisfies (2-∂) and that the set of branch points is discrete. There are three possibilities for the structure of $T$.*

*Type (0). There are no branch points, $T \cong \mathbb{R}$.*

*Type (I). There is a single branch point. Then $T$ is the Euclidean cone over its set of ends $\partial T$, i.e. $T$ is a quotient of $[0,\infty) \times \partial T$, where $0 \times v$ is identified with $0 \times u$, for all $u, v \in \partial T$.*

*Type (II). There is an infinite discrete set of branch points. Then $T$ is a simplicial metric tree, every vertex has valence at least 3, and all edges have the same length $t$.* □

This classification can be refined in terms of the $G$-action. For type (0), $G$ induces a group $\{\pm 1\} \ltimes R$ on $T$, where $R$ is a subgroup of $(\mathbb{R},+)$. If $R \neq 0$, there are infinitely many $G$-isolated points. For type (I), there is a unique $G$-isolated point. For type (II), $G_z$ acts transitively on the set of edges containing $z$, for each branch point $z$. Hence there are two subcases: either $G$ acts transitively on the branch points (vertices), or $T$ is bipartite and $G$ has two orbits on the vertices. In the first case, the mid-points of the edges are $G$-isolated (and $G$ acts with inversion), in the second case, only the branch points are $G$-isolated. We note that for the types (I), (II), $T$ admits the structure of a simplicial metric tree with a simplicial $G$-action.

We say that $T$ is of *type (III)* if the set of branch points is dense. By 2.5, the branch points are then dense in every apartment.

**2.7 Proposition** *Assume that the tree $T$ satisfies (2-∂) and that the set of branch points is dense. Then every point $x \in T$ is $G$-isolated.*

*Proof.* Let $x \in T$. If $G_x$ fixes another point $y \neq x$, then $G_x$ fixes the geodesic segment $[x,y]$. There is a branch point $z$ between $x$ and $y$. By 2.3 $y$ is not a fixed point of $G_{x,z}$. But $G_{x,y} = G_x \supseteq G_{x,z}$, a contradiction. □

Every tree satisfying (2-∂) corresponds to one of the four types (0)-(III). Now we clarify the role of the maximal bounded subgroups. We noted already that by the Bruhat-Tits Fixed Point Theorem, the stabilizers of $G$-isolated points are maximal bounded subgroups.

**2.8 Proposition** *Assume (2-∂) and that $P \subseteq G$ is a maximal bounded subgroup. Then $P$ is the stabilizer of a $G$-isolated point.*

*Proof.* Assume this is false, so the fixed point set $T^P$ contains a geodesic segment $[x,y]$ with $x \neq y$. If $[x,y]$ contains a branch point $z$, then $G_z = P$ (by maximality), whence $T^P = T^{G_z} = \{z\}$, a contradiction. From 2.5 we see that the set of branch points is closed and discrete in $T$. By 2.6 we can assume that $T$ is a simplicial tree. Then $P$ fixes some simplicial edge of $T$ elementwise. Therefore it fixes also some branch point $z$, which is a contradiction to 2.3. □

In a tree $T$ satisfying (2-∂), let $i_G(T)$ denote the set of $G$-isolated points of $T$. By the results above, the set $i_G(T)$ corresponds bijectively to the set of maximal bounded subgroups of $G$. With respect to

the conjugation action of $G$ on subgroups, this correspondence is $G$-equivariant. Our aim is to show that $T$ can be recovered from the $G$-actions on $i_G(T)$ and $\partial T$. We let

$$b(T) \subseteq i_G(T)$$

denote the set of branch points and consider the different types (0)-(III) of trees. By the *combinatorial structure* of a tree we mean the underlying set $T$ together with the collection of all apartments in $T$ (without any metric). First, we dispose of the two degenerate types (0) and (I).

**2.9**  Type (0) is characterized by $\#\partial T = 2$ (and $b(T) = \varnothing$). The tree $T \cong \mathbb{R}$ is unique up to isometry and the group induced by $G$ splits as a semidirect product $\mathbb{Z}/2 \ltimes R$, where $R$ is a subgroup of $(\mathbb{R}, +)$. Since $\mathbb{R}$ contains subgroups which are abstractly, but not topologically, isomorphic, the $G$-action on $T$ can in general not be recovered from the $G$-action on $i_G(T)$ and $\partial T$.

Type (I) is characterized by $\#\partial T \geq 3$ and $\#i_G(T) = 1$. This determines both the combinatorial structure of the tree $T$ and the $G$-action, but not the metric.

In the remaining cases, both $\partial T$ and $i_G(T)$ are infinite. This situation is much more rigid.

**2.10**  Suppose that $T$ is of type (II). Then $x \in i_G(T)$ is a branch point if and only if $G_x$ has no orbit of length 2 in $i_G(T)$. So we can recover the set $b(T)$ of branch points in $i_G(T)$. By 2.3, two branch points $x, y$ are adjacent if and only if the only branch points fixed by $G_{x,y}$ are $x$ and $y$, hence the simplicial structure of $T$ can be recovered from the $G$-action on $i_G(T)$. We note that then $G_{x,y}$ has at most 3 fixed points in $i_G(T)$. Since all edges of $T$ have the same length, $T$ is determined as a metric space up to a scaling factor.

The following result shows how for type (III) apartments can be described in terms of the $G$-action. Recall that $\pi_A : T \longrightarrow A$ denotes the retraction that sends $x$ to the nearest point $\pi_A(x) \in A$.

**2.11 Lemma**  *Assume that $T$ is of type (III). Let $A = (u, v)$ be an apartment. Then $x$ is contained in $A$ if and only if $x$ is the unique fixed point of $\langle G_{x,u} \cup G_{x,v} \rangle$.*

*Proof.*  Suppose that $x \notin A$. Then both $G_{x,u}$ and $G_{x,v}$ fix $\pi_A(x) \neq x$. Now assume that $x \in A$ and that $z \neq x$. If $\pi_A(z) = x$ then $x$ is a branch point and $G_{u,x}$ moves $z$ by 2.3. If $\pi_A(z)$ is between $x$ and $u$, then there is a branch point between $x$ and $\pi_A(z)$ and therefore $G_{v,x}$ moves $z$ by 2.3. Similarly, $G_{u,x}$ moves $z$ if $\pi_A(z)$ is between $x$ and $v$. $\square$

**2.12**  If $T$ is of type (III), then $i_G(T) = T$ by 2.7. The stabilizer $G_{x,y}$ of two distinct points $x, y$ fixes the infinite set $[x, y] \subseteq i_G(T)$, so type (III) can be distinguished from type (II). By 2.11 we can recover the apartments in $T$ from the group action. Let $A = (u, v)$ be an apartment and let $z \in A$. Then $(u, z] = A \cap T^{G_{u,z}}$, so we can also recover the rays in $A$. Therefore $G$ determines the topology of $A$. Since the branch points are dense $A$, the group $G_{u,v}$ induces a group $H$ of translations on $A$ which has a dense orbit. Up to a scaling factor, there is just one $H$-invariant metric on $A$ which satisfies $d(h^n(x), y) = n d(x, y)$ for all $h \in H$ and all $n \in \mathbb{N}$. So the metric on $A$ is determined up to scaling. Since all apartments are conjugate, the metric on $T$ is unique up to scaling.

Summarizing these facts we have the following result.

**2.13 Proposition**  *Assume that $T$ is a tree with $\#\partial T \geq 3$ and that $G$ acts on $T$ in such a way that (2-$\partial$) holds. Given the $G$-actions on $\partial T$ and $i_G(T)$, the combinatorial structure of the tree $T$ is*

*uniquely determined. If $T$ is of type (II) or type (III), the metric is determined up to a scaling factor.*
$\square$

Now we get to our main rigidity result for trees. Suppose that

$$f : T_1 \longrightarrow T_2$$

is a coarse equivalence between two trees. We recall a result about coarse equivalences between $\delta$-hyperbolic spaces which goes back to M. Morse. It says that the coarse image of a geodesic is Hausdorff close to a geodesic [7, III.H.1.7] [10, 1.3.2]. Since trees are $\delta$-hyperbolic for every $\delta > 0$, it follows that there is a positive constant $r_f > 0$ such that the image $f(A)$ of an apartment $A \subseteq T_1$ has Hausdorff distance at most $r_f$ from a unique apartment $A' \subseteq T_2$. We put $f_* A = A'$. Note that $f_*$ just maps apartments to apartments, it is not a map between the trees.

**2.14** For our application to Euclidean buildings we have to consider the more general situation of a coarse equivalence

$$f : T_1 \times \mathbb{R}^{m_1} \longrightarrow T_2 \times \mathbb{R}^{m_2}.$$

A leafless $\mathbb{R}$-tree is a 1-dimensional Euclidean building, so we may apply the higher dimensional Morse Lemma 5.9. It follows that there is a constant $r_f > 0$ such that for every apartment $A \subseteq T_1$ there is a unique apartment $A' \subseteq T_2$ such that $f(A \times \mathbb{R}^{m_1})$ has Hausdorff distance at most $r_f$ from $A' \times \mathbb{R}^{m_2}$. Moreover,

$$m_1 = m_2$$

by 5.13. We define a map $f_*$ from the set of apartments in $T_1$ to the set of apartments in $T_2$ by putting

$$f_* A = A'.$$

If $g$ is a coarse inverse for $f$, then $g_*$ is an inverse for $f_*$. We also need the following auxiliary result on trees.

**2.15 Lemma** *Let $\mathcal{F}$ be a collection of apartments in a tree $T$ and let $r > 0$. If the subset $X = \bigcap \{B_r(A) \mid A \in \mathcal{F}\} \subseteq T$ is nonempty and unbounded, then the apartments in $\mathcal{F}$ have a common end.*

*Proof.* The result is a special case of 5.15 below, but we give a direct proof. Let $(u, v) \in \mathcal{F}$ and choose a sequence $x_n \in X$ such that $\pi_{(u,v)}(x_n)$ converges to one end of $(u, v)$, say $u$. This is possible since $X \subseteq B_r((u,v))$ is unbounded. Let $A \in \mathcal{F}$. Then $d(\pi_A(x_n), \pi_{(u,v)}(x_n)) \leq 2r$ and the unbounded sequence $\pi_A(x_n)$ subconverges to an end $w$ of $A$. If $w \neq u$, then the set $\{d(\pi_A(x_n), \pi_{(u,v)}(x_n)) \mid n \in \mathbb{N}\}$ would be unbounded. Thus $w = u$. $\square$

In the previous lemma the end is unique, unless $\mathcal{F}$ consists of a single apartment.

**2.16 Proposition** *If $T_1$ has at least one branch point and if $f : T_1 \times \mathbb{R}^m \longrightarrow T_2 \times \mathbb{R}^m$ is a coarse equivalence, then the map $f_*$ between the sets of apartments of the trees induces a bijection $f_* : \partial T_1 \longrightarrow \partial T_2$ between the ends of the trees, in such a way that $f_*(u, v) = (f_* u, f_* v)$.*

*Proof.* Let $r_f > 0$ be as in 2.14 and let $\mathcal{F}$ be a finite collection of apartments in $T_1$ having an end $u$ in common. Put $(u, x] = \bigcap \{A \mid A \in \mathcal{F}\}$ and $Y = \bigcap \{B_{r_f}(f_* A) \mid A \in \mathcal{F}\}$. Then $f((u, x] \times \mathbb{R}^m) \subseteq Y \times \mathbb{R}^m$. If the set $f_* \mathcal{F}$ has no common end, then $Y$ is bounded by 2.15, so $Y \times \mathbb{R}^m$ is quasi-isometric to $\mathbb{R}^m$. We obtain then a quasi-isometric embedding of $[0, \infty) \times \mathbb{R}^m$ into $\mathbb{R}^m$, which is impossible

by 1.17. Therefore $Y$ is unbounded and $f_*\mathcal{F}$ has a common end. Applying the same argument to a coarse inverse $g$ of $f$, we see that a finite set $\mathcal{F}$ has an end in common if and only if $f_*\mathcal{F}$ has an end in common.

For $i = 1, 2$, consider the simplicial complex $AC(T_i)$ whose simplices are the finite collections of apartments in $T_1$ having a common end. The ends of $T_i$ correspond to the maximal complete subcomplexes of $AC(T_i)$ (a simplicial complex is called complete if any two simplices are contained in some simplex). We have shown that $f_*$ induces an isomorphism $f_* : AC(T_1) \longrightarrow AC(T_2)$. It follows that $f_*$ induces a bijection $\partial T_1 \longrightarrow \partial T_2$. $\qquad\square$

The complex $AC(T_i)$ is a special case of the apartment complex of a spherical building; see 3.9. It is the nerve of the covering of $\partial T_i$ given by all two-element subsets. The last paragraph of the proof is a special case of 3.10 below. The next result is a special case of [25, 8.3.11].

**2.17 Proposition** *Let $G$ be a group acting isometrically on two metrically complete leafless trees $T_1, T_2$. Assume that $f : T_1 \times \mathbb{R}^m \longrightarrow T_2 \times \mathbb{R}^m$ is a coarse equivalence and that the induced map $f_*$ on the apartments is $G$-equivariant. If $T_1$ has at least three ends, then a subgroup $P \subseteq G$ has a bounded orbit in $T_1$ if and only if it has a bounded orbit in $T_2$.*

*Proof.* Suppose that $P \subseteq G$ has a bounded orbit in $T_1$. Let $x$ be a branch point in $T_1$ and consider the bounded orbit $P(x)$. We put

$$f(x \times 0) = y \times q$$

and we show that $y$ has a bounded orbit $P(y)$ in $T_2$.

Let $\mathcal{F}$ denote the set of all apartments in $T_1$ which intersect the orbit $P(x)$ nontrivially. This set $\mathcal{F}$ is obviously $P$-invariant and has no common end (because $x$ is a branch point). Let $\mathcal{F}' = f_*\mathcal{F}$ denote the corresponding set of apartments in $T_2$. Since we assume that $f_*$ is $G$-equivariant, $\mathcal{F}'$ is also $P$-invariant, and the apartments in $\mathcal{F}'$ have no common end by 2.16. For $s > 0$ consider the $P$-invariant set

$$X_s = \bigcap\{B_s(A') \mid A' \in \mathcal{F}'\} \subseteq T_2.$$

By 2.15, the set $X_s$ is bounded (or empty). Let $r' = \sup\{d(x, p(x)) \mid p \in P\}$. For every $A \in \mathcal{F}$ we have $d(x, \pi_A(x)) < r'$. Put $f(\pi_A(x) \times 0) = y' \times p'$. Then $d(y', \pi_{f_*A}(y')) \le r_f$ by 2.14. If $\rho$ denotes the control function for $f$, then $d(y, y') \le \rho(r')$. This holds for all $A \in \mathcal{F}$, whence

$$y \in X_{\rho(r')+r_f}.$$

As this set is bounded and $P$-invariant, $P(y)$ is bounded. If $g$ is a coarse inverse of $f$, then $g_*$ is $G$-equivariant (because it is the inverse of the equivariant map $f_*$), so we obtain the converse implication by the same arguments. $\qquad\square$

Now we prove our main result on trees, which implies Theorem I in the introduction. The map $f_* : \partial T_1 \longrightarrow \partial T_2$ is defined as in 2.16.

**2.18 Theorem** *Let $G$ be a group acting isometrically on two metrically complete leafless trees $T_1, T_2$, with $\#\partial T_1 \ge 3$ and assume that the action of $G$ on $\partial T_1$ is 2-transitive. Suppose that there is a coarse equivalence*

$$f : T_1 \times \mathbb{R}^m \longrightarrow T_2 \times \mathbb{R}^m$$

*and that the induced map $f_* : \partial T_1 \longrightarrow \partial T_2$ is $G$-equivariant. Then we have the following.*

*(i) After rescaling the metric on $T_2$ by a constant $r > 0$, there is a $G$-equivariant isometry*

$$\bar{f} : T_1 \longrightarrow T_2.$$

*For every apartment $A \subseteq T_1$ we have $\bar{f}(A) = f_* A$. If $T_1$ has at least 2 branch points, then both $\bar{f}$ and $r$ are unique.*

*(ii) Put $f(x \times p) = f_1(x \times p) \times f_2(x \times p)$. If $T_1$ has at least 2 branch points, then there is a constant $s > 0$ such that $d(f_1(x \times p), \bar{f}(x)) \leq s$ holds for all $x \times p \in T_1 \times \mathbb{R}^m$. The constant $s$ depends only on $T_1$, the control function $\rho$, $a$ and the constant $r_f$ from 2.14. In particular, $f$ and $\bar{f}$ have finite distance if $m = 0$.*

*(iii) If $f$ is a rough isometry (see 1.4), we may put $r = 1$.*

*Proof.* (i) By 2.17 both $G$-actions have the same set of maximal bounded subgroups. These subgroups correspond by 2.8 to the $G$-isolated points. Therefore we have an equivariant bijection

$$\bar{f} : i_G(T_1) \longrightarrow i_G(T_2).$$

The types (I), (II), and (III) can be distinguished by the $G$-action on $i_G(T_1)$; see 2.9, 2.10 and 2.12 (our assumptions exclude trees of type (0)). The combinatorial structure is also encoded in the $G$-action, as we noted in 2.13. Also, we can rescale the metric on $T_2$ by a constant $r > 0$ in such a way that $\bar{f} : i_G(T_1) \longrightarrow i_G(T_2)$ extends to an equivariant isometry $T_1 \longrightarrow T_2$ which we also denote by $\bar{f}$. From the construction it is clear that $\bar{f}(A) = f_* A$. For trees of type (II) and (III), $\bar{f}$ and $r$ are unique by 2.13.

(ii) Let $z \in b(T_1)$ and put $r_1 = 1 + \inf\{d(x, y) \mid x, y \in b(T_1), x \neq y\}$. Then $T_1$ is covered by the $G$-translates of $B_{r_1}(z)$. Let $\mathcal{F}$ be the collection of all apartments of $T_1$ containing $z$. By 2.14 there is a constant $r_f > 0$ such that $f_1(z \times p) \in \bigcap\{B_{r_f}(f_* A) \mid A \in \mathcal{F}\} = \bigcap\{B_{r_f}(\bar{f}(A)) \mid A \in \mathcal{F}\} = B_{r_f}(\bar{f}(z))$. It follows that $d(f_1(x \times p), \bar{f}(x)) \leq r r_1 + r_f + \rho(r_1)$ for general $x \in T_1$.

(iii) For trees of type (I) it is clear that no rescaling is necessary in order to find $\bar{f}$. Suppose that $f$ is a rough isometry, with control function $\rho(t) = t + b$. For trees of type (II) and (III) we have by (ii) $d(\bar{f}(x), \bar{f}(x')) \leq d(x, x') + b + 2(r r_1 + r_f + b)$. On the other hand, $d(\bar{f}(x), \bar{f}(x')) = r d(x, x')$. Since $T_1$ is unbounded, $r \leq 1$. Applying the same argument to $\bar{f}^{-1}$, we see that $r = 1$. $\square$

For the special case of a coarse equivalence between locally finite simplicial trees, a similar result is proved in [31, 4.3.1].

# 3   Spherical buildings

In this section we record some basic notions and facts about spherical buildings. Everything we need can be found in [8], [38], [45] and [48]. For our present purposes it is convenient to view buildings as simplicial complexes. This is essentially Tits' approach in [46]; see also [16].

**3.1   Simplicial complexes**   Let $V$ be a set and $S$ a collection of finite subsets of $V$. If $\bigcup S = V$ and if $S$ is closed under going down (i.e. $a \subseteq b \in S$ implies $a \in S$), then the poset $(S, \subseteq)$ is called a *simplicial complex*. The $k + 1$-element subsets $a \in S$ are called *$k$-simplices*. More generally, any poset isomorphic to such a poset $(S, \subseteq)$ will be called a simplicial complex. The *join* $S * T$ of two simplicial complexes $S, T$ is the product poset; it is again a simplicial complex. Homomorphisms between simplicial complexes are defined in the obvious way as order-preserving maps which do not

raise the dimension of simplices [7, 7A.1]. A homomorphism which maps $k$-simplices to $k$-simplices is called *non-degenerate*. The *geometric realization* $|S|$ of the simplicial complex $S$ is the set

$$|S| = \left\{ p : V \longrightarrow [0,1] \mid p^{-1}(0,1] \in S \text{ and } \textstyle\sum_{v \in V} p(v) = 1 \right\}$$

Endowed with the weak topology, $|S|$ is then a CW complex. Moreover,

$$|S * T| \cong |S| * |T|,$$

where the right-hand side is the topological join. In the case of spherical buildings, $|S|$ is often endowed with a stronger, metric topology [7, II.10A]. A result due to Dowker says that the identity map is a homotopy equivalence between these two topologies [7, I.7].

**3.2 Coxeter groups and buildings** Let $(W, I)$ be a Coxeter system [6, IV.3] [8, II.4]. Thus $W$ is a group with a (finite) generating set $I$ consisting of involutions and a presentation of the form

$$W = \left\langle I \ \middle| \ (ij)^{ord(ij)} = 1 \text{ for all } i, j \in I \right\rangle.$$

For a subset $J \subseteq I$ we put $W_J = \langle J \rangle$. If $J$ is nonempty, then $(W_J, J)$ is again a Coxeter system [6, IV.8]. The poset

$$\Sigma = \Sigma(W, I) = \bigcup \{ W/W_J \mid J \subseteq I \},$$

ordered by reversed inclusion, is a simplicial complex, the *Coxeter complex* [8, III.1]. The *type* of a simplex $wW_J$ is $t(wW_J) = I - J$. The type function may be viewed as a non-degenerate simplicial epimorphism from $\Sigma$ to the power set $2^I$ of $I$ (viewed as a simplicial complex).

A *building* $B$ is a simplicial complex together with a collection $Apt(B)$ of subcomplexes, called *apartments*, which are isomorphic to a fixed Coxeter complex $\Sigma$. The apartments have to satisfy the following two compatibility conditions.

**(B1)** For any two simplices $a, b \in B$, there is an apartment $A \in Apt(B)$ containing $a, b$.

**(B2)** If $A, A' \subseteq B$ are apartments containing the simplices $a, b$, then there is a (type preserving) simplicial isomorphism $A \longrightarrow A'$ fixing $a$ and $b$.

The type functions of the apartments are therefore compatible and extend to a non-degenerate surjective simplicial map $t : B \longrightarrow 2^I$, the type function of the building. The cardinality of $I$ is the *rank* of the building,

$$\#I = \text{rank}(B) = \dim(B) + 1.$$

A building of rank 1 is just a set (of cardinality at least 2), the apartments are the two-element subsets.

The maximal simplices in a building are called *chambers*. We denote by $Cham(B)$ the set of all chambers. Every simplex of a building is contained in some chamber (so buildings are *pure* simplicial complexes). Recall that the *dual graph* of a pure complex is the graph whose vertices are the maximal simplices and whose edges are the simplices of codimension 1; this is the *chamber graph* of $\Delta$. A *gallery* is a simplicial path in the chamber graph, and a nonstammering gallery is a path where consecutive chambers are distinct. The chamber graph of a building is always connected. A *minimal gallery* is a shortest path in the chamber graph.

A building is called *thick* if every non-maximal simplex is contained in at least 3 distinct chambers (it is always contained in at least 2 distinct chambers). If every simplex of codimension 1 is contained in exactly two chambers, the building is called *thin*. Thin buildings are Coxeter complexes. We allow non-thick buildings (in [45], all buildings are assumed to be thick, but the results from [45] which we collect in this section hold for non-thick buildings as well).

**3.3 Residues and panels** Let $a \in B$ be simplex of type $J$. The *residue* of $a$ is the poset

$$\text{Res}(a) = \{b \in B \mid b \supseteq a\}.$$

If $a$ is not a chamber, then this poset is again a building, whose Coxeter complex is modeled on $W_{I-J}$ [45, 3.12]. If $a$ is a simplex of codimension 1 and type $I - \{j\}$, then $\text{Res}(a)$ is called a *j-panel*.

There is an order-reversing poset isomorphism between the simplicial complex $B$ and the set of all residues in $B$. Residues can also be defined in terms of the chamber graph, viewed as an edge-colored graph. Basically, this is a dictionary which allows the passage from buildings, viewed as simplicial complexes (as in [45]) to buildings viewed as edge colored graphs (or chamber systems) (as in [48]). In view of this correspondence, we call a simplex of type $I - \{j\}$ also a $j$-panel.

The join of two buildings is again a building. Conversely, a building decomposes as a join if its Coxeter group is decomposable (i.e. if $I \subseteq W$ decomposes into two subsets which centralize each other).

**3.4 Spherical buildings** A Coxeter complex $\Sigma$ is called *spherical* if it is finite. Then the geometric realization $|\Sigma|$ is a combinatorial sphere of dimension $\#I - 1$. A building is called *spherical* if its apartments are finite (the building may nevertheless be infinite). Two simplices $a, b$ in a spherical Coxeter complex $\Sigma$ are called *opposite* if they are interchanged by the antipodal map (the opposition involution) of the sphere $|\Sigma|$ [45, 3.22]. In a spherical building, two simplices are called opposite if they are opposite in some (hence every) apartment containing them.

**3.5 Thick reductions** A non-thick spherical building $B$ can always be 'reduced' to a thick building as follows. There exists a thick spherical building $B_0$ such that $B$ is a simplicial refinement of a join $\mathbb{S}^0 * \cdots * \mathbb{S}^0 * B_0$ (we view $\mathbb{S}^0$ as a thin spherical building of rank 1) [11] [13] [25, 3.7] [27, 3.8] [42]. For the geometric realization, we have then

$$|B| = \mathbb{S}^k * |B_0|,$$

where $k$ is the number of $\mathbb{S}^0$-factors in the join. So non-thick spherical buildings are suspensions of thick spherical buildings. The geometric realization of a thin spherical building is a sphere.

The following lemma will be used later.

**3.6 Lemma** *Let $B$ be a spherical building and let $A \subseteq B$ be an apartment. If every panel $a \in A$ is contained in at least three different chambers of $B$, then $B$ is thick.*

*Proof.* Let $a$ be an arbitrary panel in $B$, and let $c_0, \cdots, c_k$ be a shortest gallery with the property that the first chamber $c_0$ contains $a$ and the last chamber $c_k$ is in $A$. This gallery can be continued inside $A$ as a minimal gallery until it reaches a panel $b \in A$ which is opposite $a$. Then there is a bijection $[a; b]$ between $\text{Res}(a)$ and $\text{Res}(b)$; see 3.12 below. It follows that $a$ is contained in at least three different chambers. □

We will see that Euclidean buildings give rise to a family of building epimorphisms. The following fact is useful; see [5, 2.8].

**3.7 Lemma** *Let $B, B'$ be spherical buildings of the same type and let $\varphi : B \longrightarrow B'$ be a type-preserving simplicial map. Then $\varphi$ is surjective if and only if its restriction to every panel is surjective.*

*Proof.* Since the chamber graph of $B'$ is connected, it is clear that the local surjectivity condition implies global surjectivity. Conversely, suppose that $\varphi$ is surjective. Let $a', b' \in B'$ be $i$-adjacent chambers and let $a$ be a preimage of $a'$. We have to find a preimage $b$ of $b'$ which is $i$-adjacent to $a$. Let $c'$ be opposite $a'$ and let

$$a' \underset{i}{\quad\rule{1cm}{0.4pt}\quad} b' \quad\rule{1cm}{0.4pt}\quad \cdots \quad\rule{1cm}{0.4pt}\quad c'$$

be a minimal gallery which we denote by $\gamma'$. Let $c$ be a preimage of $c'$. Since $\varphi$ does not increase distances in the chamber graph, $c$ is opposite $a$. Thus there is a gallery

$$a \underset{i}{\quad\rule{1cm}{0.4pt}\quad} b \quad\rule{1cm}{0.4pt}\quad \cdots \quad\rule{1cm}{0.4pt}\quad c$$

in $B$ of the same type as $\gamma'$, which we denote by $\gamma$. Since $\gamma'$ is the unique gallery of its type in $B'$ from $a'$ to $c'$, it follows that $\varphi(\gamma) = \gamma'$. In particular, $\varphi(b) = b'$. $\qquad\square$

**3.8 Corollary** *Let $B, B'$ be spherical buildings of the same type and let $\varphi : B \longrightarrow B'$ be a type-preserving simplicial surjective map. If $B'$ is thick, then $B$ is also thick.* $\qquad\square$

A thick spherical building is determined by its apartment complex which we introduce now. This complex appeared already in 2.16 for the special case of rank 1 buildings. We will see later that the apartment complex of the spherical building at infinity is a coarse invariant of a Euclidean building.

**3.9 The apartment complex** Let $B$ be a building and $Apt(B)$ its set of apartments. The *apartment complex* $AC(B)$ is the simplicial complex whose simplices are finite subsets $A_1, \ldots, A_k$ of apartments, with $A_1 \cap \cdots \cap A_k \neq \varnothing$ (in other words, $AC(B)$ is the *nerve* of the covering $Apt(B)$ of $B$).

If $B$ is thick, then every simplex $a \in B$ can be written as an intersection of finitely many apartments.

**3.10 Proposition** *Let $B_1, B_2$ be thick spherical buildings and let $\varphi : AC(B_1) \longrightarrow AC(B_2)$ be a simplicial isomorphism. Then there is a unique simplicial isomorphism $\Phi : B_1 \longrightarrow B_2$ such that $\varphi(A) = \Phi(A)$ for all $A \in Apt(B_1)$.*

*Proof.* The map $\varphi$ sends sets of apartments with the finite intersection property to sets with the finite intersection property. Because $B_1$ is thick and has finite apartments, the maximal subsets with the finite intersection property in $Apt(B_1)$ are precisely the sets $\mathcal{S}_v = \{A \in Apt(B_1) \mid v \in A\}$ for all vertices $v$ of $B$. Therefore $\varphi$ induces a bijection $\Phi$ between the vertices of the buildings. If $v \in A$, then $\Phi(v) \in \varphi(A)$. Since $B_1$ is thick, two vertices $u, v$ in $B_1$ are adjacent if and only if the following holds: no other vertex $w$ is in the intersection of all apartments containing $u$ and $v$. This can be expressed in $AC(B_1)$ as follows: if $\mathcal{S}_w \supseteq \mathcal{S}_u \cap \mathcal{S}_v$, then $\mathcal{S}_w = \mathcal{S}_u$ or $\mathcal{S}_w = \mathcal{S}_v$. So $\Phi$ preserves the 1-skeleton of $B_1$. But every building is the flag complex of its 1-skeleton [45, 3.16], therefore $\Phi$ is simplicial. $\qquad\square$

In the previous proof, thickness is essential. It is clear that the thick reduction (see 3.5) of a spherical building has the same apartment complex as the building itself. We remark that a simplicial isomorphism between two spherical buildings maps apartments to apartments, even if it is not type-preserving.

**3.11 Projections in buildings** Let $c$ be a chamber and $a$ a simplex in a building. Then there is a unique chamber $d$ in $\mathrm{Res}(a)$ which has minimal distance from $c$ (with respect to the distance in the chamber graph), and which is denoted $d = \mathrm{proj}_a c$. If $b$ is a simplex then $\mathrm{proj}_a b$ is defined to be the simplicial intersection of the chambers $\mathrm{proj}_a c$, where $c$ ranges over all chambers containing $b$ [45, 3.19].

If $a, b$ are opposite simplices in a spherical building, then $\mathrm{proj}_a : \mathrm{Res}(b) \longrightarrow \mathrm{Res}(a)$ is a simplicial isomorphism [45, 3.28]. The following observations are due to Knarr [26] and Tits [46]. They were rediscovered by Leeb [31, Ch. 3].

**3.12 Perspectivities** Let $B$ be a spherical building. We noted already that if $a, b$ are opposite simplices in $B$, then $\mathrm{proj}_b : \mathrm{Res}(a) \longrightarrow \mathrm{Res}(b)$ is a building isomorphism (not necessarily type preserving) between $\mathrm{Res}(a)$ and $\mathrm{Res}(b)$. We denote this isomorphism by

$$[b; a] : \mathrm{Res}(a) \longrightarrow \mathrm{Res}(b)$$

and call it a *perspectivity*. A concatenation of perspectivities is called a *projectivity*; we write

$$[c; b] \circ [b; a] = [c; b; a] : \mathrm{Res}(a) \longrightarrow \mathrm{Res}(c)$$

etc. The inverse of $[b; a]$ is $[a; b]$. A projectivity is called *even* if it can be written as a composition of an even number of perspectivities.

**3.13 Projectivities** Recall that a *groupoid* is a small category where every arrow is an isomorphism. The *projectivity groupoid* $\Pi_B$ of a spherical building $B$ is the category whose objects are the simplices of $B$, and whose morphisms are projectivities. It is closely related to the *opposition graph* $\mathrm{Opp}(B)$ whose vertices are the simplices of $B$ and whose edges are unordered pairs of opposite simplices. Every simplicial path in $\mathrm{Opp}(B)$ induces a projectivity. We denote by

$$\Pi_B(a) = \mathrm{Hom}_{\Pi_B}(a, a)$$

the group of all automorphisms of $\mathrm{Res}(a)$ induced by maps in $\Pi_B$. The subgroup $\Pi_B(a)^+$ consisting of all even projectivities is a normal subgroup of index 1 or 2 in $\Pi_B(a)$. If $f : B_1 \longrightarrow B_2$ is an isomorphism of spherical buildings, then $f$ induces an isomorphism between $\Pi_{B_1}$ and $\Pi_{B_2}$ in the obvious way.

The following result is essentially Knarr's [26, 1.2]. We use several facts about galleries and distances which can all be found in [48].

**3.14 Theorem** *Suppose that $B$ is a thick spherical building and that $r$ is an $i$-panel. If $i$ is not an isolated node in the Coxeter diagram of $B$, then $\Pi_B(r)^+$ is a 2-transitive permutation group on $R = \mathrm{Res}(r)$.*

*Proof.* Let $a, b, b'$ be three distinct chambers in $R$. We construct a projectivity which fixes $a$ and maps $b$ to $b'$. Let $j$ denote the type of a neighboring node of $i$, i.e. $ij \neq ji$ in $W$. We choose a nonstammering gallery $b \overset{i}{\underline{\quad}} a \overset{j}{\underline{\quad}} c \overset{i}{\underline{\quad}} d$ in $B$ (the superscripts indicate the types of panels in the gallery). Since $ij \neq ji$, this gallery is minimal. Therefore it is contained in some apartment $A \subseteq B$. Let $s$ be the panel in $A$ opposite to the $j$-panel $a \cap c$. Let $e$ be a chamber in $S = \mathrm{Res}(s)$ which is not in $A$ (here we use that $B$ is thick). Then $e$ is opposite to $a$ and $c$. There is a unique panel $t \subseteq e$

which is opposite both to $r$ and to the panel $q = c \cap d$. Since $b$ is not opposite $e$, $\mathrm{proj}_t b = e$. Similarly $\mathrm{proj}_t d = e$, whence $[q; t; r](b) = d$. We claim that $\mathrm{proj}_t(a) = \mathrm{proj}_t(c)$. Assuming for the moment that this is true, we have $[t; r](a) = [t; q](c)$, whence $[q; t; r](a) = c$. Applying the same construction to $b' \xrightarrow{\ i\ } a \xrightarrow{\ j\ } c \xrightarrow{\ i\ } d$, with a second apartment $A'$ and a panel $t'$, we obtain the projectivity $[r; t'; q; t; r]$ with the required properties.

It remains to show that $\mathrm{proj}_t(a) = \mathrm{proj}_t(c)$. Let $A''$ denote the apartment spanned by the opposite chambers $a$ and $e$. Let $f$ denote the chamber in $\mathrm{Res}(t) \cap A''$ different from $e$ and let $k$ denote the gallery distance between $a$ and $e$. Then $a$ and $f$ have gallery distance $k - 1$. Since $t$ and $p$ are not opposite, $g = \mathrm{proj}_p f$ has gallery distance $k - 2$ from $f$. So there is a gallery $(f, \ldots, g, a)$ of length $k - 1$, which is therefore minimal. It follows that $(f, \ldots, g, c)$ is also a minimal gallery, and $f$ has gallery distance $k - 1$ from $c$. Since $t$ is opposite to $r$ and $q$, we have $\mathrm{proj}_t(a) = f = \mathrm{proj}_t(c)$. $\qquad \square$

Let $a, b$ be opposite panels in $B$ and let $B(a, b)$ denote the union of all apartments containing $a$ and $b$. For a chamber $c \in B$, let $c_a = \mathrm{proj}_a c$ and $c_b = \mathrm{proj}_b c$. If $A$ is an apartment containing $a$ and $b$ and $c \in A$ is a chamber, then $c_a, c_b \in A$ and $\mathrm{proj}_a c_b = c_a$. In particular, each pair of distinct chambers $c, d \in \mathrm{Res}(a)$ determines a unique apartment in $B(a, b)$ containing $c$ and $d$ and the chambers in $\mathrm{Res}(a)$ correspond bijectively to the half apartments of $B(a, b)$ having $a$ and $b$ as boundary panels.

**3.15 Lemma** *The subcomplex $B(a, b) \subseteq B$ is a weak building of the same type as $B$ and every apartment of this weak building contains $a$ and $b$. If $b'$ is another panel opposite $a$, then there is a unique simplicial isomorphism $B(a, b) \longrightarrow B(a, b')$ which fixes $B(a, b) \cap B(a, b')$.*

*Proof.* Let $c, d$ be chambers in $B(a, b)$. Then there exists an apartment $A$ containing $a, b, c_a$ and $d_a$. It follows that $A$ contains $c_b$ and $d_b$ and therefore $c$ and $d$. This shows that $B(a, b)$ is a building and that every apartment of this building contains $a$ and $b$.

Suppose now that $b'$ is also opposite $a$. For every apartment $A$ containing $a, b$, there is a unique apartment $A'$ containing $a, b'$, such that $A \cap \mathrm{Res}(a) = A' \cap \mathrm{Res}(a)$. Since $A \cap A'$ contains chambers, there is a unique isomorphism $A \longrightarrow A'$ fixing $A \cap A'$. For $c \in A$, the image $c' \in A'$ can be described as follows. It is the unique chamber which has the same $W$-valued distances from the two chambers $a_c$ and $\mathrm{proj}_{b'} a_c$ as $c$ has from $a_c$ and $\mathrm{proj}_b a_c = b_c$. This description is independent of $A$ and $A'$ and shows that we obtain a well-defined map on the chambers. This map is adjacency-preserving on each apartment and hence a building isomorphism. $\qquad \square$

**3.16** Suppose that $a = a_0$ is a panel and $a_0 \text{ ——— } a_1 \text{ ——— } \cdots \text{ ——— } a_k$ is a path in the opposition graph. By the previous lemma we obtain a sequence of canonical isomorphisms $B(a_0, a_1) \cong B(a_1, a_2) \cong \ldots \cong B(a_{k-1}, a_k)$ fixing the intersections of consecutive buildings. If $a_0 = a_k$, then the composition of these isomorphisms yields an automorphism of $B(a_0, a_1)$. If $k$ is even, then this automorphism fixes $a$ and its restriction to $\mathrm{Res}(a)$ coincides with the projectivity $[a_k; \ldots; a_0]$.

# 4   Euclidean buildings

The notion of a (nondiscrete) Euclidean building is due to Tits [46]. Prior to their axiomatization in [46], the nondiscrete Euclidean buildings that arise from reductive groups over valued fields were studied in [9]. We rely on Parreau's work [35] which contains many important structural results for Euclidean buildings. She showed in particular that the axioms given by Kleiner-Leeb [25] are equivalent to Tits' original axioms plus metric completeness.

**4.1 The affine Weyl group** We fix a spherical Coxeter group $(W, I)$ in its standard representation on $\mathbb{R}^n$ (where $n = \#I$). A *Weyl chamber* or *sector* in $\mathbb{R}^n$ is the closure of a connected component in $\mathbb{R}^n - (H_1 \cup H_2 \cup \cdots \cup H_r)$, where the $H_k$ are the reflection hyperplanes of $W$ corresponding to the conjugates of the generators $i \in I$. The closure of a connected component of $\mathbb{R}^n - H_k$ is called a *half space*. A *Weyl simplex* in $\mathbb{R}^n$ is an intersection of Weyl chambers. A *wall* is a reflection hyperplane.



| Weyl chamber | half space | Weyl simplex | wall |
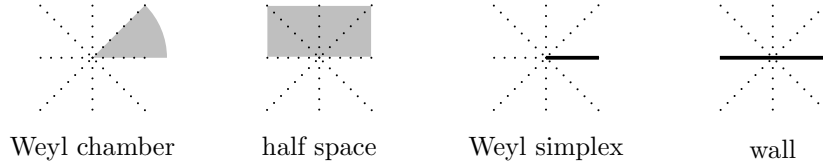
Note that we do not require that $W$ be irreducible. The Weyl simplices in $\mathbb{R}^n$, ordered by inclusion, form a simplicial complex which is isomorphic to the Coxeter complex $\Sigma$ of $W$.

We also fix a $W$-invariant inner product on $\mathbb{R}^n$. The corresponding norm will be denoted by $||\cdot||$. Up to scaling factors on the irreducible $W$-submodules of $\mathbb{R}^n$, such an inner product is unique. The group $W$ normalizes the translation group $(\mathbb{R}^n, +)$ and the semidirect product $W\mathbb{R}^n$ acts isometrically on $\mathbb{R}^n$. We call this group $W\mathbb{R}^n$ the *affine Weyl group*.

**4.2 Euclidean buildings** Let $W$ be a spherical Coxeter group and $W\mathbb{R}^n$ the corresponding affine Weyl group. Let $X$ be a metric space. A *chart* is an isometric embedding $\varphi : \mathbb{R}^n \longrightarrow X$, and its image is called an *affine apartment*. We call two charts $\varphi, \psi$ *W-compatible* if $Y = \varphi^{-1}\psi(\mathbb{R}^n)$ is convex (in the Euclidean sense) and if there is an element $w \in W\mathbb{R}^n$ such that $\psi \circ w|_Y = \varphi|_Y$ (this condition is void if $Y = \varnothing$). We call a metric space $X$ together with a collection $\mathcal{A}$ of charts a *Euclidean building* if it has the following properties.

**(EB1)** For all $\varphi \in \mathcal{A}$ and $w \in W\mathbb{R}^n$, the composition $\varphi \circ w$ is in $\mathcal{A}$.

**(EB2)** Any two points $x, y \in X$ are contained in some affine apartment.

**(EB3)** The charts are $W$-compatible.

The charts allow us to map Weyl chambers, walls and half spaces into $X$; their images are also called Weyl chambers, walls and half spaces. The first three axioms guarantee that these notions are coordinate independent. We call $\mathcal{A}$ an *atlas* or *apartment system* for $X$.

**(EB4)** If $a, b \subseteq X$ are Weyl chambers, then there is an affine apartment $A$ such that the intersections $A \cap a$ and $A \cap b$ contain Weyl chambers.

**(EB5)** If $A_1, A_2, A_3$ are affine apartments which intersect pairwise in half spaces, then $A_1 \cap A_2 \cap A_3 \neq \varnothing$.

The last axiom may be replaced by the following axiom [35, Thm. 1.21].

**(EB5')** If $A \subseteq X$ is an affine apartment and $x \in A$ a point, then there is a 1-Lipschitz retraction $\rho : X \longrightarrow A$ with $\rho^{-1}(x) = \{x\}$. The point $x$ is called the *center* of the retraction.

In fact, the retractions can be chosen in such a way that they have the following slightly stronger property.

**(EB5$^+$)** If $A \subseteq X$ is an affine apartment and $x \in A$ a point, then there is a 1-Lipschitz retraction $\rho : X \longrightarrow A$ with $d(x, y) = d(x, \rho(y))$ for all $y \in X$.

The number $n$ is called the *dimension* of the Euclidean building $X$. It coincides with the topological (covering) dimension of $X$ [28, 7.1] [30, 3.3]. In [46], [25] or [35], the translation group of the affine Weyl group may be some $W$-invariant subgroup of $(\mathbb{R}^n, +)$. Since we are in this paper only concerned with metric properties of Euclidean buildings, and since the affine Weyl group can always be enlarged to the full translation group without changing the underlying metric space and the set of affine apartments [35, 1.2], there is no loss in generality here. From our viewpoint, every point $p \in X$ is a *special point* [9, 1.3.7]. We remark that the Coxeter group $W$ of a Euclidean building need not be crystallographic [21] [4].

**4.3 Example: Euclidean cones over spherical buildings** Let $B$ be a spherical building and let $\mathrm{EB}(B)$ denote the quotient of $|B| \times [0, \infty)$ where $|B| \times 0$ is identified to a point. Let $d_{|B|}$ denote the spherical metric on $|B|$, and put $d(x \times s, y \times t)^2 = s^2 + t^2 - 2st \cos(d_{|B|}(x, y))$; see [7, I.5.6]. With this metric, $\mathrm{EB}(B)$ is the infinite Euclidean cone over $|B|$. It is not difficult to see that $\mathrm{EB}(B)$ is a Euclidean building. The affine apartments in $\mathrm{EB}(B)$ correspond bijectively to the apartments of $B$. These buildings are generalizations of the trees of type (I), and we call them *Euclidean buildings of type (I)*. One can view $\mathrm{EB}(B)$ as the affine building with respect to the trivial valuation on the spherical building $B$. We note that this construction is functorial: every automorphism of $B$ extends to an isometry of $\mathrm{EB}(B)$. In this way, every spherical building can be viewed as a Euclidean building. In [39], $\mathrm{EB}(B)$ is called an *immeuble vectoriel*.

**4.4 Example: leafless trees** A leafless tree is a 1-dimensional Euclidean building. In particular $\mathbb{R}$ is a Euclidean building. The affine Weyl group is $W\mathbb{R} = \{x \longmapsto \pm x + c \mid c \in \mathbb{R}\}$.

**4.5 Example: simplicial Euclidean buildings** The geometric realization of an affine simplicial building is a Euclidean building [8] [46].

The images of the Weyl simplices in a Euclidean building $X$ under the charts are also called Weyl simplices. The image of the origin in $\mathbb{R}^n$ is called the *base point* or *tip* of the Weyl simplex.

**4.6 The vector distance** Let $a_0 \subseteq \mathbb{R}^n$ be a fixed Weyl chamber. Given two points $p, v$ in the Euclidean building $X$, there exists a chart $\varphi : \mathbb{R}^n \longrightarrow X$ that maps $a_0$ to a $p$-based Weyl chamber containing $v$. Let $\Theta(p, v) \in a_0 \subseteq \mathbb{R}^n$ denote the vector that is mapped to $v$. By (EB3), the vector $\Theta(p, v)$ is independent of $\varphi$. We thus have a well-defined map

$$X \times X \longrightarrow a_0, \qquad (p, v) \longmapsto \Theta(p, v)$$

which we call the *vector distance* [35, 1.3.1]. We remark that the map $\Theta(p, -) : X \longrightarrow a_0$ is 1-Lipschitz. We also note the following. The involution $x \longmapsto -x$ on $\mathbb{R}^n$ induces an involution $j : x \longmapsto w_0(-x)$ on the Weyl chamber $a_0$, where $w_0$ is the unique longest element in the spherical Coxeter group $W$. We have the symmetry relation

$$\Theta(x, y) = j(\Theta(y, x)),$$

for all $x, y \in X$.

The affine apartment $X = \mathbb{R}^n$ with $\mathcal{A} = W\mathbb{R}^n$ is an example of a Euclidean building and we have the following small fact which will be needed later. A related result is proved in [35, 2.15].

**4.7 Lemma** *Let $W$ be a spherical Coxeter group, acting in the standard representation on $\mathbb{R}^n$. Let $\Theta$ denote the corresponding vector distance on $\mathbb{R}^n$. Suppose that $K \subseteq \mathbb{R}^n$ is a nonempty convex set, that $g : K \longrightarrow K$ is a bijection, and that*

$$\Theta(p, v) = \Theta(g(p), g(v))$$

*holds for all $p, v \in K$. Then there exists an element $w$ in the affine Weyl group $W\mathbb{R}^n$ whose restriction to $K$ coincides with $g$.*

*Proof.* First we note that $g$ is an isometry of $K$, because $||\Theta(p,v)|| = d(p,v)$. We choose the pair $p, v \in K$ in such a way that the smallest Weyl simplex $b \subseteq a_0$ that contains $\Theta(p, v)$ has maximal dimension. Applying an element of the affine Weyl group $W\mathbb{R}^n$ to $K$, we may assume that $p = 0$ and that $v \in b \subseteq a_0$. We choose $w \in W\mathbb{R}^n$ in such a way that $w(0) = g(0)$ and $w(v) = g(v)$. We claim that $g(x) = w(x)$ for all $x \in K$.

In general, if $\Theta(0, u) = \Theta(0, u')$, then $u$ and $u'$ lie in the same finite $W$-orbit in $\mathbb{R}^n$. Since $b$ is maximal, $v$ is an interior point of the Weyl simplex $b$ and there is an $\varepsilon > 0$ such that $g(u) \in w(b)$ holds for all $u \in b \cap K$ with $d(u,v) \le \varepsilon$. This in turn implies that $g(u) = w(u)$ for all $u \in b \cap K$ with $d(u,v) \le \varepsilon$. Since both $g$ and $w$ are isometries, we conclude that $g$ and $w$ agree on the convex set $b \cap K$. Let now $u \in K$ be arbitrarily. If $t > 0$ is small we have necessarily $z = ut + v(1 - t) \in b$ by the maximality of $b$. Since both $g$ and $w$ are isometries and $g(v) = w(v)$ and $g(z) = w(z)$, we have $g(u) = w(u)$. $\qquad\square$

The Weyl simplices lead to two spherical buildings. One of them captures the asymptotic geometry of $X$, while the other is an 'infinitesimal' version of the Euclidean building, similar to the tangent space of a Riemannian manifold. The first one, the spherical building at infinity, will be considered now and the second in 4.14.

**4.8   The spherical building at infinity** We call two Weyl simplices $a, a' \subseteq X$ *Hausdorff equivalent* if they have finite Hausdorff distance. The equivalence class of $a$ is denoted $\partial a$. The preorder $\subseteq_{Hd}$ defined in 1.6 induces a partial order on these equivalence classes. Let $\partial_\mathcal{A} X$ denote the set of all equivalence classes of Weyl simplices, partially ordered by domination $\subseteq_{Hd}$. For every affine apartment $A$, the poset $\partial A$ consisting of the Weyl simplices contained in $A$ may be viewed as a sub-poset of $\partial_\mathcal{A} X$.

**4.9 Proposition** *The poset $\partial_\mathcal{A} X$ is a spherical building. The map $A \longmapsto \partial A$ is a one-to-one correspondence between the affine apartments in $X$ and the apartments of the spherical building $\partial_\mathcal{A} X$.*

*Proof.* See Parreau [35, 1.5]. $\qquad\square$

We have also the following fact.

**4.10 Lemma** *Suppose that $a$ and $a'$ are Weyl chambers with tips $x, x'$. If $\partial a = \partial a'$, then*

$$Hd(a, a') \le d(x, x').$$

*Proof.* Let $\psi : a \longrightarrow a'$ denote the unique $\Theta$-invariant isometry. Given $z \in a$, there is a unique geodesic $\gamma : [0, \infty) \longrightarrow a$ with $\gamma(0) = x$ and $\gamma(s) = z$, for some $s \ge 0$. The geodesic $\gamma' = \psi \circ \gamma$ is at finite distance from $\gamma$, because $\partial a = \partial a'$. Since the metric of a CAT(0) space is convex, we have $d(\gamma(s), \gamma'(s)) \le d(x, x')$ [7, II.2.2]. $\qquad\square$

**4.11 The maximal atlas** The spherical building at infinity depends very much on the chosen set of charts $\mathcal{A}$. Similarly to a differentiable manifold, a Euclidean building admits a unique maximal atlas $\hat{\mathcal{A}}$ [35, 2.6]. The set $\hat{\mathcal{A}}$ is called the *complete apartment system*. We denote the spherical building at infinity which corresponds to the complete apartment system by $\partial_{cpl}X$. Its metric realization coincides with the Tits boundary of $X$ [35, Cor. 2.19].

**4.12 Product decompositions and reductions** If the Coxeter group $W$ is reducible, then $X$ decomposes as a metric product $X = X_1 \times X_2$ of Euclidean buildings, with $\partial_{\mathcal{A}_1}X_1 * \partial_{\mathcal{A}_2}X_2 = \partial_{\mathcal{A}}X$ [35, 2.1]. If $\mathbb{S}^k * B_0 = \partial_{\mathcal{A}}X$ is the thick reduction of $\partial_{\mathcal{A}}X$ and if $W_0$ is the Coxeter group of $B_0$, then there is a Euclidean building $X_0$ with affine Weyl group $W_0\mathbb{R}^m$ and atlas $\mathcal{A}_0$ and an isometry $X \cong \mathbb{R}^{n-m} \times X_0$, with $\partial_{\mathcal{A}_0}X_0 = B_0$ and $k = n - m - 1$ [25, 4.9].

We record some more results about Euclidean buildings which will be needed later.

**4.13 Proposition** *Let $X$ be an $n$-dimensional Euclidean building. If $A \subseteq X$ is isometric to $\mathbb{R}^n$, then $A$ is an apartment in the complete apartment system of $X$.*

    *Proof.* See [35, Prop. 2.25]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We remark that the previous proposition is in the approach by Kleiner and Leeb essentially an axiom.

**4.14 The residue at a point** Let $p$ be a point in $X$. Two $p$-based Weyl simplices $a, b$ have *equal germs near $p$* if

$$B_r(p) \cap a = B_r(p) \cap b$$

holds for some $r > 0$. The corresponding equivalence classes form a spherical building $X_p$ which we call the *residue* of $X$ at $p$; see [35, 1.6]. The Coxeter group of $X_p$ is $W$, hence $X_p$ has the same type as the spherical building at infinity $\partial_{\mathcal{A}}X$. We call the point $p$ *thick* if the residue $X_p$ is thick. If $a \subseteq X$ is a Weyl simplex, then $\partial a$ has a unique representative which is a $p$-based Weyl simplex [35, Cor. 1.9]. The apartments of $X_p$ arise from the affine apartments containing $p$ (but this correspondence is in general not 1-to-1). We thus obtain a type-preserving surjective simplicial map
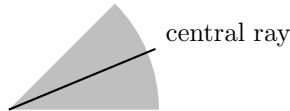
$$\partial_{\mathcal{A}}X \longrightarrow X_p,$$

and also a surjective canonical 1-Lipschitz map $\mathrm{EB}(\partial_{\mathcal{A}}X) \longrightarrow X$ (which depends on $p$).

**4.15** Given a $p$-based Weyl chamber $a$, let $\zeta_a$ denote the spherical barycenter of the Weyl chamber $\partial a$. We call the geodesic ray

$$[p, \zeta_a) \subseteq a$$

the *central ray* of $a$.



central ray

The possible angles between all centrals rays starting at $x$ form a finite subset of $[0, \pi]$, since the spherical Coxeter complex of $W$ is finite. If $[p, \zeta_b)$ and $[p, \zeta_c)$ are the central rays of two $p$-based Weyl chambers $b, c$, then the Alexandrov angle between these rays is 0 if and only if $b$ and $c$ have the same germ near $p$.

If two sufficiently long geodesic segments in a tree have small Hausdorff distance, then they intersect in a long geodesic segment. The next proposition is the higher dimensional analog of this fact.

**4.16 Proposition** *Let $X$ be a Euclidean building and let $s > 0$ be a positive real number. There exists a positive real number $c_s > 0$ such that the following holds for all $t > 0$. If $A, A' \subseteq X$ are affine apartments and $y$ is a point in $X$ such that $Hd_{B_{t+c_s}(y)}(A, A') \leq s$, then*

$$B_t(y) \cap A = B_t(y) \cap A'.$$

*Proof.* Let $a_0 \subseteq \mathbb{R}^n$ denote the standard Weyl chamber. There exists a number $c_s > 0$ such that for the point $z_a \in [0, \zeta_a)$ with $d(0, z_a) = c_s$, the closed ball $\bar{B}_{2s}(z_a) \cap a$ contains only interior points of $a$.

Suppose now that $t > 0$, that $A, A' \subseteq X$ are as in the claim of the proposition and that $p \in B_t(y) \cap A$. We claim that $p \in B_t(y) \cap A'$. We choose a $p$-based Weyl chamber $b \subseteq A$ and $z_b \in [p, \zeta_p)$ with $d(p, z_b) = c_s$. The condition on the Hausdorff distance ensures that we can find a point $v' \in A'$ with $d(z_b, v') \leq 2s$. From the comparison triangle in Euclidean space we see that the angle between the geodesic segments $[p, v']$ and $[p, z_b]$ is so small that $[p, v']$ intersects the Weyl chamber $b$ in an interior point.
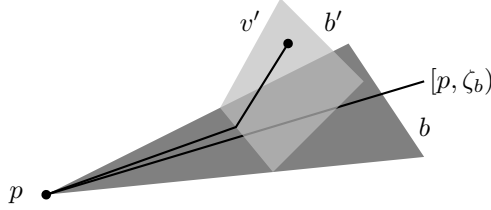


We now choose a geodesic ray $[v', \eta) \subseteq A'$ which extends $[p, v']$. Thus $[p, v'] \cup [v', \eta)$ is a geodesic ray. This ray is contained in a unique $p$-based Weyl chamber $b'$ whose germ near $p$ is contained in $A$.

We now consider the $p$-based Weyl chamber $c \subseteq A$ opposite $b$, and we choose $z_c \in [p, \zeta_c)$ similarly as before, with $d(p, \zeta_c) = c_s$. Then we find a $p$-based Weyl chamber $c'$ whose germ near $p$ is opposite to $b'$. These two Weyl chambers are contained in a unique affine apartment [35, Prop. 1.12], which is therefore $A'$. It follows that $p$ is contained in $A'$. Thus $B_r(y) \cap A \subseteq B_r(y) \cap A'$. The other inclusion follows by symmetry. □

We call an affine wall $M \subseteq X$ *thick* if $M$ is the intersection of three affine apartments. We have the following local thickness criterion for walls.

**4.17 Lemma** *Let $A \subseteq X$ be an affine apartment, let $p \in A$ and let $M \subseteq A$ be a wall containing $p$. Let $r \in X_p$ be a panel in the wall determined by $M$ in $X_p$. If $r$ is contained in three distinct chambers of $X_p$, then $M$ is thick.*

*Proof.* Let $A_p$ denote the apartment induced by $A$ in $X_p$, let $s \in A_p$ be the panel opposite $r$ and let $a, b \in A_p$ be the two chambers containing $s$. We represent $a, b, r, s$ by $p$-based Weyl simplices in $A$. By assumption, there is a chamber containing $r$ which is opposite both to $a$ and to $b$. By 3.8 we can find a $p$-based Weyl chamber $c$ representing this chamber, such that $\partial c$ is adjacent to $\partial A$. Since the panel $\partial c \cap \partial A$ has a unique $p$-based representative, this representative is contained in $M$. It follows that $c \cap A \subseteq M$. Let $A'$ denote the affine apartment spanned by $c$ and $a$; see [35, 1.12].
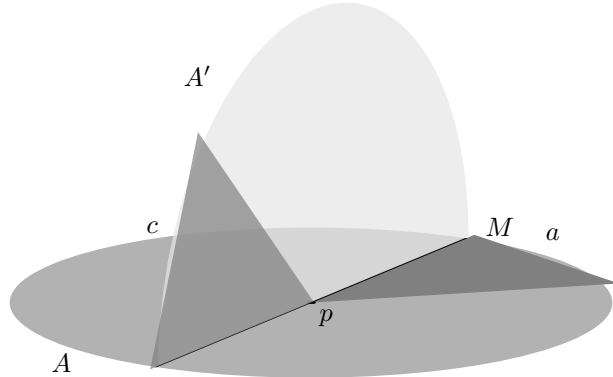
Then $A \cap A'$ is a half space whose boundary is $M$ and therefore $M$ is thick. $\qquad\square$

**4.18 Lemma** *Let $p$ be a point of an affine apartment $A \subseteq X$. Then $p$ is thick if and only if every wall of $A$ containing $p$ is thick.*

*Proof.* If $p$ is thick, then every wall through $p$ is thick by 4.17. Conversely, if every wall of $A$ containing $p$ is thick, then all panels in $A_p$ are contained in at least 3 chambers. Thus, $p$ is thick by 3.6. $\qquad\square$

**4.19 Lemma** *Let $A$ be an affine apartment and assume that $L, M \subseteq A$ are non parallel thick walls. Then the reflection of $L$ along $M$ in $A$ is also thick.*

*Proof.* Let $A_\pm \subseteq A$ denote the half spaces bounded by $M$ and let $A_0 \subseteq X$ be a third half space with $A \cap A_0 = M$. There is a unique affine wall $L'$ in the affine apartment $A' = A_+ \cup A_0$ extending $L \cap A_+$. By 4.17, $L'$ is thick. Now let $L''$ be the wall in $A_0 \cup A_-$ which extends $L' \cap A_0$. Again by 4.17, $L''$ is thick. Finally, let $L'''$ be the wall in $A$ which extends $L'' \cap A_-$. A third application of 4.17 yields that $L'''$ is thick.
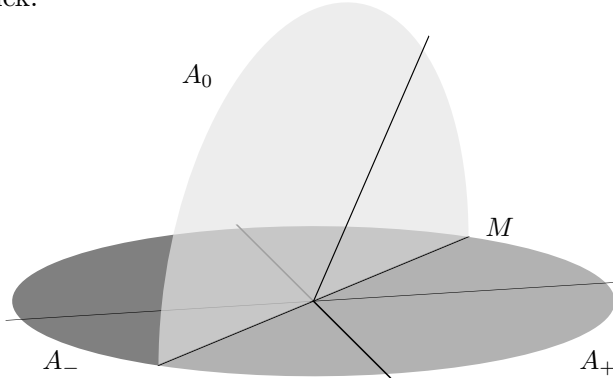


But $L'''$ is precisely the reflection of $L$ along $M$. $\qquad\square$

**4.20 Lemma** *The Euclidean building $X$ contains a thick point if and only if $\partial_{\mathcal{A}} X$ is thick.*

*Proof.* By 3.8 thickness of $X_p$ implies thickness of $\partial_{\mathcal{A}} X$. Now suppose that $\partial_{\mathcal{A}} X$ is thick, let $A$ be an affine apartment, let $c$ be a Weyl chamber of $A$ and let $M_1, ..., M_n$ be the walls of $A$ bounding $c$. Since $\partial_{\mathcal{A}} X$ is thick, we can choose thick walls $M_1', \ldots, M_n'$ in $A$ such that $M_i'$ is parallel to $M_i$ for each $i$. The intersection of the thick walls $M_1', \ldots, M_n'$ contains a point $p$. By 4.18 and 4.19, $p$ is thick. $\qquad\square$

The following trichotomy is analogous to the case of trees in 2.6. In higher dimension, however, we need no assumptions on group actions.

**4.21 Proposition** *Suppose that $X$ is a Euclidean building of dimension $n \geq 2$ and that $\partial_A X$ is irreducible and thick. Let $th(X) \subseteq X$ denote the set of thick points. There are the following three possibilities.*

*(I) There is a unique thick point which is contained in every affine apartment of $X$.*

*(II) The set of thick points is a closed, discrete and cobounded subset in $X$ and in every apartment of $X$.*

*(III) The set of thick points is dense in $X$ and in every apartment of $X$.*

*Proof.* We noted in 4.20 that $th(X) \neq \varnothing$. For an affine apartment $A \subseteq X$ we denote by $R(A) \subseteq \mathrm{Isom}(A)$ the group generated by reflections along the thick walls in $A$. If $p$ is a thick point, then the $R(A)$-stabilizer of $p$ is $R(A)_p \cong W$. We start with two observations.

(i) Suppose that $p$ is a thick point in an affine apartment $A$, and that $M \subseteq A$ is a thick wall not containing $p$. Such an apartment $A$ exists if $X$ contains at least two thick points. Since the Weyl group $W$ is irreducible and $\dim(A) \geq 2$, the point $p$ is the intersection of thick walls in $A$ which are not parallel to $M$. It follows from 4.19 that the reflection of $p$ along $M$ in $A$ is again a thick point, so the $R(A)$-orbit of $p$ consists of thick points.

(ii) For any two affine apartments $A, A'$, there is a sequence of affine apartments $A = A_0, \ldots, A_k = A'$ such that $A_j \cap A_{j+1}$ is a half space. (Using galleries, this is easily seen to be true for apartments in the spherical building at infinity.) It is clear that the sequence of walls determined by these half spaces is thick.

Suppose first that $th(X) = \{p\}$ consists of a single point and that $A$ is an affine apartment containing $p$. Then (i) shows that the thick walls in $A$ are precisely the ones passing through $p$. If $A'$ is any other affine apartment, then (ii) and a simple induction on $k$ show that all apartments $A_0, \ldots, A_k$ in the sequence connecting $A$ and $A'$ contain $p$. This is case (I).

Suppose now that $th(X)$ contains at least two points $p, q$ and that $A$ is an apartment containing $p$ and $q$. Then the $R(A)$-orbit of $q$ consists of thick points and is cobounded in $A$, because the Weyl group $W$ is irreducible, and because $R(A)$ contains translations. Let $T \subseteq R(A)$ denote the translational part (i.e. the kernel of the action of $R(A)$ on $\partial A$). Any closed subgroup of the vector group $(\mathbb{R}^n, +)$ is a product of a free abelian group of finite rank and a vector group. If $T$ is discrete, then $R(A)$ is an affine reflection group. If $T$ is not discrete, then the closure of $T$ in $\mathrm{Isom}(A)$ consists of all translations in $A$ (because $W$ acts irreducibly on the set of all translations), so $th(X) \cap A$ is dense.

If $A'$ is any other affine apartment having a half space in common with $A$, then the thick walls in $A$ propagate to thick walls in $A'$. The isometry $A \longrightarrow A'$ also preserves thick points. We obtain a canonical isomorphism $R(A) \cong R(A')$. From (ii) we conclude that the isometry groups $R(A)$ and $R(A'')$ are isomorphic for any affine apartment $A'' \subseteq X$, and that this isomorphism maps $th(X) \cap A$ onto $th(X) \cap A''$. In the discrete case, we have (II), whereas the nondiscrete case is (III). $\qquad \square$

**4.22 Corollary** *Let $X$ be as in 4.21 (I). Then $X \cong \mathrm{EB}(\partial_A X)$ is a Euclidean cone.*

*Proof.* Let $p \in X$ be the unique thick point and let $A$ be an affine apartment. Since we are in case (I), the point $p$ is contained in $A$. The canonical surjective 1-Lipschitz map $\mathrm{EB}(\partial_A X) \longrightarrow X$ sends therefore $\mathrm{EB}(\partial A)$ isometrically onto $A$. Since any two points in $\mathrm{EB}(\partial_A X)$ are in some apartment, $\mathrm{EB}(\partial_A X) \longrightarrow X$ is an isometry. $\qquad \square$

The following observation will not be used, but it illustrates how simplicial affine buildings fit into the picture. In the remaining case (III), $X$ is a nondiscrete Euclidean building.

**4.23 Corollary** *Let $X$ be as in 4.21 (II). Then $X$ is the metric realization of an affine simplicial building.*

*Proof.* The groups $R(A)$ are affine Coxeter groups; see [8, Ch.VI]. In this way, every apartment has a canonical simplicial structure as a Coxeter complex. The axioms of an affine simplicial building follow from 4.2. $\square$

In general, a Euclidean building is not determined by its spherical building at infinity. For example,

$$\partial_{\mathcal{A}}\mathrm{EB}(\partial_{\mathcal{A}}X) = \partial_{\mathcal{A}}X.$$

Some additional data are needed, which are encoded in the panel trees. The following material is due to Tits [46]. A proof of 4.28 with all details filled in can be found in [20].

**4.24  Wall trees and panel trees**  Let $(X, \mathcal{A})$ be a Euclidean building and let $a, b$ be a pair of opposite panels in the spherical building at infinity. Let $X(a, b)$ denote the union over all affine apartments in $\mathcal{A}$ whose boundary contains $a$ and $b$,

$$X(a, b) = \bigcup\{A \mid A \subseteq X \text{ affine apartment in } \mathcal{A} \text{ and } a, b \in \partial A\}.$$

Then $X(a, b)$ is a Euclidean building, which factors metrically as

$$X(a, b) = T \times \mathbb{R}^{n-1},$$

where $T$ is a leafless tree; see [25, 4.8.1], [40, 3.9] [46]. We call this tree $T$ the *wall tree* associated to $(a, b)$, because it depends only on the unique wall of $\partial_{\mathcal{A}}X$ containing $a$ and $b$. In the notation of 3.15, the spherical building at infinity of $X(a, b)$ is

$$\partial(X(a, b)) = (\partial_{\mathcal{A}}X)(a, b).$$

If $X$ is metrically complete, then $T$ and $X(a, b)$ are also metrically complete [44].

**4.25 Lemma** *Let $X$ be a Euclidean building and let $a, b, b'$ be panels in $\partial_{\mathcal{A}}X$. Suppose that $b$ and $b'$ are opposite $a$. Then there is a unique isometry $X(a, b) \longrightarrow X(a, b')$ which fixes $X(a, b) \cap X(a, b')$ pointwise.*

*Proof.* Let $A$ be an affine apartment whose boundary contains $a, b$, and let $A'$ be the corresponding affine apartment whose boundary contains $\mathrm{Res}(a) \cap \partial A$ and $b'$; see 3.15. Since $A \cap A'$ contains Weyl chambers, there is a unique isometry $A \longrightarrow A'$ fixing $A \cap A'$. This proves the uniqueness of the isometry. For the existence, we show that these isometries $A \longrightarrow A'$ of the individual affine apartments fit together.

Let $A_1, A_2 \subseteq X(a, b)$ be two affine apartments containing a point $x \in A_1 \cap A_2$. Correspondingly, we have chambers $c_i, d_i \in \mathrm{Res}(a)$ such that $c_i, \mathrm{proj}_b d_i$ spans $\partial A_i$; see 3.15. If two of these four chambers coincide, say $c_1 = c_2$, then $A_1$ and $A_2$ have the $x$-based Weyl chamber $\tilde{c}$ representing $c_1$ in common. Then $A_1, A_2, A'_1$ and $A'_2$ have a sub-Weyl chamber of $\tilde{c}$ in common, and therefore the two isometries $A_1 \longrightarrow A'_1$ and $A_2 \longrightarrow A'_2$ map $x$ to the same point $x'$.

If $c_1, c_2, d_1, d_2$ are pairwise different, then $x$ is also in the affine apartment determined by $c_1, d_2$ or $c_1, c_2$, because $X(a, b)$ is a product of a tree and Euclidean space. The previous argument, applied twice, shows that the various apartment isometries coincide on $x$. Thus we have a well-defined bijection which is apartment-wise an isometry. Since any two points are in some affine apartment, the map is an isometry. $\qquad\square$

**4.26** If $b'$ varies over the panels opposite $a$, we obtain a family of trees which are pairwise canonically isomorphic. This canonical isomorphism type of a tree is the *panel tree $T_a$* associated to $a$.

If $a_0 \,\rule[0.5ex]{1em}{0.4pt}\, a_1 \,\rule[0.5ex]{1em}{0.4pt}\, \cdots \,\rule[0.5ex]{1em}{0.4pt}\, a_k$ is a path in the opposition graph, then the isomorphisms $(\partial_\mathcal{A} X)(a_0, a_1) \cong (\partial_\mathcal{A} X)(a_1, a_2) \cong \cdots$ from 3.15 are accompanied by isometries $X(a_0, a_1) \cong X(a_1, a_2) \cong \cdots$. In particular, the group of projectivities $\Pi_{\partial_A X}(a)$ acts on the wall tree and on the panel tree $T_a$. If we restrict to even projectivities, the action on the Euclidean factor is trivial. If the building is thick and the type of the panel $a$ is not isolated, then the action of $\Pi^+_{\partial_A X_1}(a)$ on the ends of $T_a$ is 2-transitive by 3.14

The branch points in the panel trees correspond to the thick walls in $X$. Assuming that $\partial_\mathcal{A} X$ is irreducible, we have the following consequence of 4.21. If one panel tree $T_a$ is of type (I), then every panel tree is of type (I) and $X$ is a Euclidean cone over $|\partial_\mathcal{A} X|$ as in 4.3. If one panel tree is of type (II), then every panel tree is of type (II) and $X$ is simplicial. The remaining possibility is that every panel tree is of type (III).

**4.27 Ecological boundary isomorphisms** Let $X_1, X_2$ be irreducible Euclidean buildings. Assume that $\partial_{\mathcal{A}_1} X_1$ and $\partial_{\mathcal{A}_2} X_2$ are thick and that

$$\varphi : \partial_{\mathcal{A}_1} X_1 \longrightarrow \partial_{\mathcal{A}_2} X_2$$

is an isomorphism. Assume moreover that for every panel $a \in \partial_{\mathcal{A}_1} X_1$, there is a tree isometry

$$\varphi_a : T_a \longrightarrow T_{\varphi(a)}.$$

If for each panel $a$, the map $\partial\varphi_a : \partial T_a \longrightarrow \partial T_{\varphi(a)}$ coincides with the restriction of $\varphi$ to $\mathrm{Res}(a)$, then $\varphi$ is called *tree-preserving* or *ecological*. The following is [46, Thm. 2], see also [20, Ch. 7].

**4.28 Theorem (Tits)** *Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be Euclidean buildings. Assume that $\partial_{\mathcal{A}_1} X_1$ is thick and irreducible. If $\varphi : \partial_{\mathcal{A}_1} X_1 \xrightarrow{\cong} \partial_{\mathcal{A}_2} X_2$ is an ecological isomorphism, then $\varphi$ extends to an isomorphism $\Phi : (X_1, \mathcal{A}_1) \xrightarrow{\cong} (X_2, \mathcal{A}_2)$.* $\qquad\square$

The irreducibility is actually not important for the proof, but the result is stated in this way in [46]. For our purposes, the irreducible version suffices. We remark that the following interesting problem is open.

**4.29 Conjecture** *Let $X_1, X_2$ be Euclidean buildings of type (II) and dimension $n \geq 2$. Assume that $\partial_{cpl} X_1$ is irreducible. Then every isomorphism*

$$\partial_{cpl} X_1 \longrightarrow \partial_{cpl} X_2$$

*extends uniquely to an isometry $X_1 \longrightarrow X_2$.*

This is known to be true if $\partial_{cpl} X_1$ is Moufang [49, 27.6]. The proof is algebraic and uses the fact that a field admits at most one discrete complete valuation. The conjecture is false for Euclidean buildings of type (III), since for example $\mathbb{C}$ admits infinitely many nonisomorphic nondiscrete complete valuations, corresponding to nonisomorphic Euclidean buildings of type $\widetilde{\mathsf{A}}_m$. The boundary is always the spherical building of $\mathrm{SL}_{m+1}\mathbb{C}$.

# 5 Coarse equivalences of Euclidean buildings

In this final section we prove, among other things, Theorem II and III from the introduction. An important technical tool is the 'higher dimensional Morse Lemma' 5.9. The main step in this result, in turn, is the following topological rigidity theorem [25, 6.4.2] [28, Sec. 4].

**5.1 Theorem (Topological rigidity of affine apartments)** *Let*

$$f : X \longrightarrow Y$$

*be a homeomorphism of metrically complete Euclidean buildings. If $A \subseteq X$ is an affine apartment, then $f(A)$ is an affine apartment in the complete apartment system of $Y$.*

*Sketch of proof.* We give an outline of the sheaf-theoretic proof from [28, Sec. 6]. On any Hausdorff space $X$, there are the following two presheaves. The *local homology presheaf* assigns to every open subset $U \subseteq X$ the singular homology group $H_*(X, X - U)$. The stalk of the corresponding sheaf $\mathscr{H}_*(X)$ at a point $p \in X$ is the local homology group

$$\mathscr{H}_*(X)_p \cong H_*(X, X - \{p\}).$$

The second presheaf assigns to $U$ the poset $Cls(U) = \{A \cap U \mid A \subseteq X$ is closed $\}$ of all relatively closed subsets of $U$. Its stalk $\mathscr{Cl}_p$ consists of germs of closed sets at $p$. There is a natural transformation *supp* between these (pre)sheaves which assigns to every relative cycle $c \in H_*(X, X - U)$ its support

$$supp(c) = \{p \in U \mid c \text{ has nontrivial image in } H_*(X, X - \{p\})\}$$

[25, 6.2] [28, Sec. 4]. If $X$ is a Euclidean building, then the residue $X_p$ is by definition a sub-poset of $\mathscr{Cl}_p$. The main step in the proof of 5.1 is to show that this sub-poset can be described by means of the transformation *supp*. The reason is as follows. For any CAT(0) space $X$, the natural map

$$X - \{p\} \longrightarrow \Sigma_p X$$

which assigns to a point $q \neq p$ the direction of the geodesic segment $[q, p]$ in the (completed) *space of directions* $\Sigma_p X$ is a homotopy equivalence [28, Thm. A]. In particular, there is a natural isomorphism in homology

$$H_{*+1}(X, X - \{p\}) \xrightarrow{\cong} \tilde{H}_*(\Sigma_p X).$$

If $X$ is a Euclidean building, then the space of directions $\Sigma_p X$ is the geometric realization of the residue $X_p$. The reduced homology (of the geometric realization) of the spherical building $X_p$ is then a free $\mathbb{Z}$-module spanned by all apartments containing a given chamber, see [28, Sec. 5], the *Steinberg module* (this is essentially the contents of the Solomon-Tits Theorem). In particular, the dimension $n = \dim(X)$ can be read off from the local homology groups (alternatively, the number $n$ coincides

with the covering dimension of $X$ [28, 7.1] [30, 3.3] and hence is a topological invariant). This allows us to describe the image of the transformation

$$\mathscr{H}_*(X)_p \xrightarrow{\;supp\;} \mathscr{Cb}_{,p}.$$

If the residue $X_p$ is thick, then the simplices of $X_p$ are precisely the indecomposable elements in the distributive poset lattice generated by $supp(\mathscr{H}_*(X)_p)$, see [28, Cor. 6.6] (we call an element indecomposable if it is not a union of finitely many strictly smaller elements). This is clearly a topological invariant of the space $X$. In general, we have by 3.5 a thick reduction $X_p \cong \mathbb{S}^k * |B_0|$, with a thick spherical building $B_0$. In this case the indecomposable elements in the image of $supp$ correspond to the simplices of the thick part $B_0$, see [28, 5.6]. In particular, we can read off the number $k$ from the difference between the dimension of the simplicial complex $B_0$ and the degree $j$ for which $\tilde{H}_j(X - \{p\}) \neq 0$.

It follows that the homeomorphism maps a small open ball $B_r(p) \cap A$ in the affine apartment $A \subseteq X$ to a small open set in some affine apartment in $Y$. Thus $f(A) \subseteq Y$ is a complete simply connected flat Riemannian manifold, and therefore isometric to $\mathbb{R}^n$. By 4.13, $f(A)$ is an affine apartment in the complete apartment system of $Y$. See [28, Sec. 6] for details and a more general result. If $\partial_A X$ is thick and irreducible and if $n \geq 2$, then the completeness assumption can be dropped. $\qquad\square$

We also have the following corollary of the proof.

**5.2 Corollary** *Assume that $X$ and $Y$ are metrically complete Euclidean buildings, and that $\partial_{cpl} X$ and $\partial_{cpl} Y$ are thick. If $f : X \times \mathbb{R}^{m_1} \longrightarrow Y \times \mathbb{R}^{m_2}$ is a homeomorphism, then $m_1 = m_2$.*

*Proof.* Let $p \in X$ be a point. Then the residue at $p \times v \in X \times \mathbb{R}^{m_1}$ decomposes as

$$(X \times \mathbb{R}^{m_1})_{p \times v} = |B_0| * \mathbb{S}^k,$$

where $B_0$ is a thick spherical building and $k \geq m_1$. If $p$ is thick, then $k = m_1$. The same applies to $Y$, hence $m_1 = m_2$. $\qquad\square$

Theorem 5.1 is the 'hard part' of the proof of Theorem 5.9 below. The transition from homeomorphism rigidity to coarse rigidity, on the other hand, is mainly a matter of elementary logic. See also [47] and [29].

**5.3  The language of Euclidean buildings**  Recall from 1.7 our language of metric space

$$\mathcal{L}_{\mathrm{ms}} = \{+, \cdot, \leq, 0, 1, d, X, R\}.$$

We now enlarge this language so that we can state results about Euclidean buildings. We fix a spherical Coxeter group $W = \langle I \mid (ij)^{m_{ij}} = 1 \rangle$ and add constants $i \in I$ for its generators and the $m_{i,j}$ to the language. We also add a unary predicate $W$ for the Weyl group. This allows us to include the spherical Weyl group $W$ as well as the affine Weyl group $W\mathbb{R}^n$ into our structure. Then we add function symbols for the coroots $\alpha_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, whose kernels are the reflection hyperplanes corresponding to the generators $i \in I$. This allows us to describe Weyl chambers, half spaces, walls and Weyl simplices in $\mathbb{R}^n$ in our language. The last ingredient that is missing are the coordinate charts $\varphi : \mathbb{R}^n \longrightarrow X$. The number of charts will depend very much on the Euclidean building $X$. In order to allow some flexibility, we view the charts as a family of maps

$$\{\varphi_\ell : \mathbb{R}^n \longrightarrow X\}_{\ell \in L}.$$

Thus we add a unary predicate $L$ for the indices $\ell$ that label the charts, and one single $n+1$-ary function symbol $\varphi : L \times \mathbb{R}^n \longrightarrow X$, whose first entry is the label $\ell \in L$. This is the language $\mathcal{L}_{\mathrm{eb}}$ of Euclidean buildings.

It is straightforward to see that a Euclidean building $(X, \mathcal{A})$ can be viewed as an $\mathcal{L}_{\mathrm{eb}}$-structure. The axioms (EB1)–(EB5) can be stated without difficulties in this language, and so can be most of the results about Euclidean buildings that we proved in Section 4. Below we add a few more elements to the language $\mathcal{L}_{\mathrm{eb}}$, for example the vector distance function, or a second Euclidean building $Y$ and a coarse map $f : X \longrightarrow Y$ between the two. We note also that the axioms (EB5') and (EB5$^+$) can be stated if we add one function symbol $\rho$ for the retractions,

$$\rho_\ell : X \longrightarrow \varphi_\ell(\mathbb{R}^n)$$

having as its first argument the index $\ell$ describing the target apartment $A = \varphi_\ell(\mathbb{R}^n)$. The center of the retraction is $\varphi_\ell(0)$.

**5.4 Ultraproducts of Euclidean buildings** Suppose now that $K$ is a countably infinite set and that $D$ is a nonprincipal ultrafilter on $K$. Suppose that $(X_k, \mathcal{A}_k)_{k \in K}$ is a family of Euclidean buildings, all of which are modeled on the same spherical Coxeter group $W$. We may view them as $\mathcal{L}_{\mathrm{eb}}$-structures $\mathfrak{X}_k$ and take their ultraproduct $\mathfrak{X}_D$. We note that $W_D = W$ by 1.11. By Los' Theorem 1.10, the ultraproduct satisfies the axioms (EB1)–(EB5), except that the real numbers are now replaced by the real closed field $^*\mathbb{R}$ and that the apartments are modeled on $^*\mathbb{R}^n$. The coordinate changes are now described by the nonstandard affine Weyl group $W \ltimes {}^*\mathbb{R}^n$. One can show that $\mathfrak{X}_D$ is a *generalized affine building* in the sense of Bennett [3] [20] [43]. We call $\mathfrak{X}_D$ a *nonstandard Euclidean building*, with *nonstandard charts*, *nonstandard apartments*, *nonstandard Weyl simplices*, and so on.

**5.5 Ultralimits of Euclidean buildings** Suppose that $\mathfrak{X}_D$ is an ultraproduct of of a family of Euclidean buildings $(X_k, \mathcal{A}_k)$ as in 5.3 and 5.4. Let $p \in X_D$ and let $r > 0$ be a nonstandard real. Then we have the ultralimit $C(X_D, p, r)$ of the underlying metric spaces $X_k$ as in 1.13. Recall that

$$X_D^{(p,r)} = \{x \in X_D \mid \tfrac{1}{r} d(p, x) \in {}^*\mathbb{R}_{\mathrm{fin}}\}$$

and put

$$({}^*\mathbb{R}_{\mathrm{fin}}^n)^{(0,r)} = \{v \in {}^*\mathbb{R}^n \mid \tfrac{1}{r}||v|| \in {}^*\mathbb{R}_{\mathrm{fin}}\} = \{vr \mid v \in {}^*\mathbb{R}_{\mathrm{fin}}^n\}.$$

Suppose that $\varphi_\ell$ is a nonstandard chart with $\varphi_\ell(0) \in X_D^{(p,r)}$. Then we have a commuting diagram

$$
\begin{array}{ccc}
({}^*\mathbb{R}_{\mathrm{fin}}^n)^{(0,r)} & \xrightarrow{\ \ \varphi_\ell\ \ } & X_D^{(p,r)} \\
\downarrow & & \downarrow \\
\mathbb{R}^n & \cdots\cdots\xrightarrow{\ \tilde{\varphi}_\ell\ }\cdots\cdots & C(X_D, p, r)
\end{array}
$$

where the vertical arrows identify points at infinitesimal $\frac{1}{r}d$-distance. The map $\tilde{\varphi}_\ell$ is an isometric injection, and we let $\tilde{\mathcal{A}}$ denote the set of all maps $\tilde{\varphi}_\ell$ that arise in this way. The corresponding affine apartments (the images of the $\tilde{\varphi}_\ell$) will be denoted by $\tilde{A} \subseteq C(X_D, p, r)$.

The following result is proved in [25, 5.1.1]. A more general result is proved by Schwer and Struyve in [43, 6.1]. We remark that $X_D^{(p,r)}$ is also a generalized affine building in the sense of Bennett [3].

**5.6 Theorem** *Let $(X_k, \mathcal{A}_k)_{k \in K}$ be a countably infinite family of Euclidean buildings, with a fixed spherical Coxeter group $W$. Let $C(X_D, p, r)$ and $\tilde{\mathcal{A}}$ be as in 5.5. Then the ultralimit $C(X_D, p, r)$ is a metrically complete Euclidean building, with $\tilde{\mathcal{A}}$ as its complete set of apartments.*

*Proof.* From the construction of $\tilde{\mathcal{A}}$ and Łoś' Theorem it is clear that $\tilde{\mathcal{A}}$ satisfies the axioms (EB1) and (EB2). The retractions $X_D \longrightarrow A$ onto the nonstandard affine apartments are 1-Lipschitz and descend therefore to retractions $C(X_D, p, r) \longrightarrow \tilde{A}$, hence axiom (EB5$^+$) holds. The remaining two axioms require more work.

*Axiom (EB3) holds.* We note that $C(X_D, p, r)$ is CAT(0) by 1.15. Since the affine apartments are convex, their intersections are also convex. Now we add the vector distance $\Theta$ to the structure. It is clear that this gives us a nonstandard vector distance on $X_D^{(p,r)} \subseteq X_D$. Suppose that $x, x', y \in X_D^{(p,r)}$ and that $x$ and $x'$ have infinitesimal $\frac{1}{r}d$-distance. Then $\Theta(y, x)$ and $\Theta(y, x')$ also have infinitesimal $\frac{1}{r}d$-distance. From the symmetry of the vector distance 4.6, we conclude that $\Theta(x, y)$ and $\Theta(x', y)$ have infinitesimal distance. This implies that the nonstandard vector distance descends to an ordinary vector distance $\tilde{\Theta} : C(X_D, p, r) \times C(X_D, p, r) \longrightarrow a_0$. It now follows from 4.7 that coordinate changes between charts in $C(X_D, p, r)$ are given by elements of $W\mathbb{R}^n$, hence (EB3) holds.

*Axiom (EB4) holds.* We first show two auxiliary results.

*Claim 1. Let $c, c' \subseteq X_D$ be nonstandard Weyl chambers with $\partial c = \partial c'$. If the tips $z$ and $z'$ of $c$ and $c'$ are in $X_D^{(p,r)}$, then there exists a nonstandard Weyl chamber $a \subseteq c \cap c'$ whose tip $w$ is also in $X_D^{(p,r)}$.*

Let $[z, \zeta_c)$ be the nonstandard central ray of $c$ (we may add these rays to our structure and language). This ray intersects the Weyl chamber $c'$ and there is a unique point[2] $w$ with $[z, \zeta_c) \cap c' = [w, \zeta_c)$. Let

$$s = \tfrac{1}{r}d(w, z) \in {}^*\mathbb{R}.$$

We claim that $s$ is a finite nonstandard real. Suppose to the contrary that $s$ is infinite. Let $[x, \zeta_c) \subseteq c'$ be the maximal ray in $c'$ extending $[w, \zeta_c)$. Then the three points $z, z', x$ have mutually infinitesimal $\frac{1}{rs}d$-distance, while $w$ and $z$ have $\frac{1}{rs}d$-distance 1 (mod ${}^*\mathbb{R}_{\inf}$). It follows that the three nonstandard Alexandrov angles at $w$ between the segments $[w, z], [w, z'], [w, x]$ are infinitesimally small (we may also add the Alexandrov angles to our structure and language, so they are defined in $X_D$).
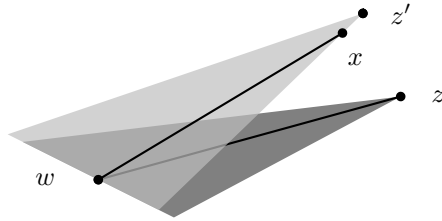


This contradicts 4.15. Thus $s$ is finite and hence $w \in X_D^{(p,r)}$. Now let $a$ be the unique nonstandard $w$-based Weyl chamber with $\partial a = \partial c$. Thus Claim 1 is proved.

We note that every Weyl chamber $\tilde{c} \subseteq C(X_D, p, r)$ arises from some nonstandard Weyl chamber $c$ whose tip is in $X_D^{(p,r)}$. Suppose that $\tilde{c}, \tilde{c}' \subseteq C(X_D, p, r)$ are Weyl chambers. We choose corresponding nonstandard Weyl chambers $c, c' \subseteq X_D$ in such a way that $\partial c, \partial c'$ have minimal gallery distance $m \in \mathbb{N}$ in the spherical building at infinity of $X_D$.

---

[2]The point exists by Łoś' Theorem because it is *definable* in the structure, hence it does not matter that the ordered field ${}^*\mathbb{R}$ is non-archimedean.

*Claim 2. In this situation, there is an nonstandard affine apartment $A$ containing a point $w \in X_D^{(p,r)}$ and a gallery of $w$-based nonstandard Weyl chambers $c_0, \ldots, c_m$, with $\partial c = \partial c_0$ and $\partial c' = \partial c_m$.*

We proceed by induction on $m$, the case $m = 0$ being trivial. Assume now that $c_0, \ldots, c_{m-1}$ are $w$-based Weyl chambers contained in a nonstandard affine apartment $A$, with $w \in X_D^{(p,r)}$. Let $a$ be the unique $w$-based Weyl chamber with $\partial a = \partial c'$. If $a$ is contained in $A$ we are done, with $c_m = a$. If $a$ is not contained in $A$, then we let $M \subseteq A$ denote the nonstandard wall that separates $a$ and $c_{m-1}$. Let $H \subseteq A$ denote the closed half space bounded by $M$ whose boundary at infinity contains $\partial c_{m-1}$. If $H$ and $X_D^{(p,r)}$ have a point $w'$ in common, then we replace $c_0, \ldots, c_{m-1}$ by their $w'$-based translates in $A$, and we let $c_m$ denote the unique $w'$-based Weyl chamber with $\partial c_m = \partial c'$.
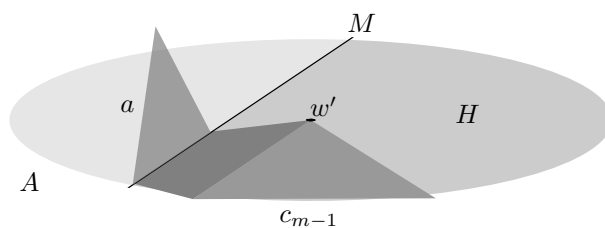


Let $\bar{a}$ be the unique $w'$-based Weyl chamber in $H$ whose germ near $w'$ is opposite to the germ of $a$. Then $a, \bar{a}$ are contained in a unique nonstandard affine apartment $A'$ containing $c_0 \cup \cdots \cup c_m$, hence we are done.
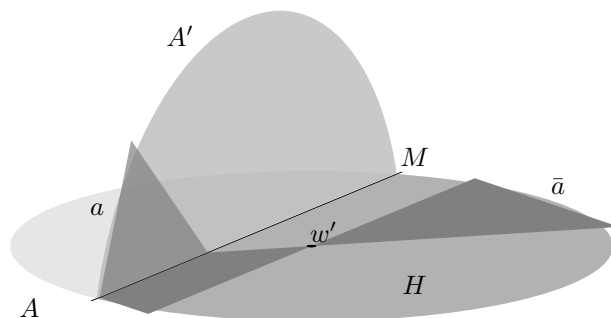


Suppose that $H \cap X_D^{(p,r)} = \varnothing$. Then $M$ separates $w$ from $\partial c_{m-1}$, hence $a$ has the same germ near $w$ as $c_{m-1}$.
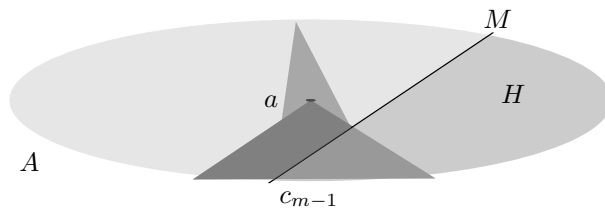


This situation is excluded by our assumptions. Thus we have proved Claim 2.

In the setting of Claim 2, it follows from Claim 1 that $\tilde{c} \cap \tilde{c}_0$ contains a Weyl chamber, and that $\tilde{c}' \cap \tilde{c}_m$ contains a Weyl chamber. Therefore (EB4) holds.

*The atlas $\tilde{\mathcal{A}}$ is maximal.* We have to show that every geodesic line and ray is contained in some affine apartment [35, 2.18]. Let $\gamma : \mathbb{R} \longrightarrow C(X_D, p, r)$ be a geodesic line. We choose points $x_k \in X_D^{(p,r)}$ corresponding to $\gamma(k) \in C(X_D, p, r)$, for $k = 0, \pm 1, \pm 2, \ldots$. For $k \geq 0$, let $A_k$ be a nonstandard affine

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apartment containing $\{x_{-k}, x_k\}$. Then $x_\ell$ is $\frac{1}{r}d$-infinitesimally close to $A_k$ for all $\ell = -k, \ldots, k$. Hence we have the following property of the set $T = \{x_k \mid k \in \mathbb{Z}\}$. For every finite subset $S \subseteq T$ there exists a nonstandard affine apartment $A_S$ such that all elements of $S$ are $\frac{1}{r}d$-infinitesimally close to $A_S$. By 1.12 there exists a nonstandard affine apartment $A_T$ such that all members of $T$ are $\frac{1}{r}d$-infinitesimally close to $A_T$. Thus $\gamma(k) \in \tilde{A}_T$ holds for all $k \in \mathbb{Z}$. Since $\tilde{A}_T$ is convex, we have $\gamma(\mathbb{R}) \subseteq \tilde{A}_T$. The reasoning for rays is completely analogous. $\quad\square$

The following consequence is often useful. Note that the metric completion of a non-complete Euclidean building need not be a Euclidean building [28, 6.9], see also [32].

**5.7 Corollary** *Let* $(X, \mathcal{A})$ *be a Euclidean building. Then there exists a complete Euclidean building* $(\bar{X}, \bar{\mathcal{A}})$ *of the same type, with* $X \subseteq \bar{X}$ *and* $\mathcal{A} \subseteq \bar{\mathcal{A}}$. *The action of the automorphism group of* $(X, \mathcal{A})$ *extends to an action on* $(\bar{X}, \bar{\mathcal{A}})$.

*Proof.* Let $X_D$ be the ultrapower of $X$ and put $\bar{X} = C(X_D, p, r)$. The diagonal embedding $X \longrightarrow X_D$ extends to an embedding $X \longrightarrow \bar{X}$. The automorphism group of $(X, \mathcal{A})$ acts diagonally on $X_D$, on $X_D^{(p,r)}$, and hence on $\bar{X}$ in a natural way. $\quad\square$

**5.8 The structure $\mathfrak{C}$** Suppose that $f : X \longrightarrow Y$ is a coarse equivalence of Euclidean buildings. We consider the structure $\mathfrak{C}$ consisting of the two Euclidean buildings and the map $f$, and we take the ultrapower $\mathfrak{C}_D$ of this structure. If $p \in X_D$ is any basepoint and if $r \in {}^*\mathbb{R}$ is infinitely large, then $f$ induces a bi-Lipschitz homeomorphisms

$$C(f) : C(X_D, p, r) \longrightarrow C(Y_D, f(p), r)$$

by 1.16. By 5.6, the asymptotic cones $C(X_D, p, r)$ and $C(Y_D, f(p), r)$ are metrically complete Euclidean buildings with complete apartment systems, and $C(f)$ maps affine apartments to affine apartments by 5.1. We also note the following. The ordered field ${}^*\mathbb{R}$ is non-archimedean, so a bounded set of nonstandard reals will in general not have a supremum. Nevertheless, the quantity

$$Hd_{B_r(y)}(f(A), A') = Hd(f(A) \cap B_r(y), A' \cap B_r(y))$$

is defined for all nonstandard affine apartments $A \subseteq X_D$, $A' \subseteq Y_D$, points $y \in Y_D$ and positive nonstandard reals $r > 0$. The reason for this is that we can add $Hd_{B_r(y)}(f(A), A')$ as an $\mathbb{R} \cup \{\infty\}$-valued function (depending on arguments $\ell, \ell', y, r$, with $A = \varphi_\ell(\mathbb{R}^n)$, $A' = \varphi'_{\ell'}(\mathbb{R}^n)$) to the structure $\mathfrak{C}$ and then take its ultrapower. Łos' Theorem 1.10 guarantees that this function has exactly the intended meaning in $\mathfrak{C}_D$, namely that of an ${}^*\mathbb{R}$-valued Hausdorff distance[3]. These observations are the main steps in our proof of the following theorem [25, 7.1.5].

**5.9 Theorem (Higher dimensional Morse Lemma)** *Let*

$$f : X \longrightarrow Y$$

*be a coarse equivalence between Euclidean buildings. There exists a real constant* $r_f > 0$ *such that the following holds. For every affine apartment* $A \subseteq X$ *there is a unique affine apartment* $A'$ *in the complete apartment system of* $Y$ *with*

$$Hd(f(A), A') \leq r_f.$$

---

[3]The point is that we measure only the Hausdorff distance between so-called *internal* or *definable* sets.

We first prove a weaker statement in the ultraproduct.

**5.10 Lemma** *Let $\mathfrak{C}$ be as in 5.8. Let $A \subseteq X_D$ be a nonstandard affine apartment and let $y \in f(A)$. Let $r \geq 1$ be a nonstandard real. There exist a nonstandard affine apartment $A' \subseteq Y_D$ and a finite nonstandard real number $s \geq 0$ such that*

$$Hd_{B_{\varepsilon r}(y)}(f_D(A), A') \leq s$$

*holds for all nonstandard $\varepsilon$ with $\frac{1}{2} \leq \varepsilon \leq 1$.*

*Proof.* This is clear if $r$ is finite: we may choose any nonstandard affine apartment $A'$ containing $y$ and choose any real number $s \geq 2r$. So suppose that $r$ is infinite. We choose a preimage $x \in A$ of $y$. Since $r$ is infinite, we have by Lemma 1.16 a bi-Lipschitz map

$$C(f) : C(X_D, x, r) \longrightarrow C(Y_D, y, r).$$

Let $\tilde{A} \subseteq C(X_D, x, r)$ denote the affine corresponding to $A$. By Theorem 5.1 $C(f)$ maps $\tilde{A}$ onto an affine apartment $\tilde{A}'$ in the complete apartment system of the Euclidean building $C(Y_D, y, r)$. By 5.6 there is a nonstandard affine apartment $A' \subseteq Y_D$ corresponding to $\tilde{A}'$. Suppose now that $\varepsilon$ is a nonstandard real with $\frac{1}{2} \leq \varepsilon \leq 1$. Let

$$s_\varepsilon = Hd_{B_{\varepsilon r}(y)}(f(A), A').$$

Since we have in $C(Y_D, y, r)$ that

$$B_\varepsilon(\tilde{y}) \cap C(f)(\tilde{A}) = B_\varepsilon(\tilde{y}) \cap \tilde{A}',$$

the quotient $s_\varepsilon/r$ is infinitesimally small. We claim that $s_\varepsilon$ is finite. Suppose to the contrary that $s_\varepsilon > 0$ is infinite. There is either a point $z \in A' \cap B_{\varepsilon r}(y)$ such that $B_{s_\varepsilon/2}(z) \cap f(A) = \varnothing$ or a point $z \in f(A) \cap B_{\varepsilon r}(y)$ such that $B_{s_\varepsilon/2}(z) \cap A' = \varnothing$. Because $s_\varepsilon$ is infinite, we have a bi-Lipschitz map

$$C(f) : C(X_D, w, s_\varepsilon) \longrightarrow C(Y_D, z, s_\varepsilon)$$

(for a suitably chosen point $w \in X_D$). In $C(Y_D, z, s_\varepsilon)$, the sets $C(f)(\tilde{A})$ and $\tilde{A}'$ are affine apartments, and $C(f)(\tilde{A}) \neq \tilde{A}'$ by the choice of $z$. It is also clear from the construction that $C(f)(\tilde{A})$ and $\tilde{A}'$ have Hausdorff distance at most 1 in $C(Y_D, z, s_\varepsilon)$. This is impossible by [25, 4.6.4] [35, p. 10], or by 4.16. Thus $s_\varepsilon$ has to be finite.

Now we claim that the set $\{s_\varepsilon \mid \frac{1}{2} \leq \varepsilon \leq 1\} \subseteq {}^*\mathbb{R}$ has a finite upper bound. Suppose that this is false. Then we find for every $k \in \mathbb{N}$ an $\varepsilon_k$ with $\frac{1}{2} \leq \varepsilon_k \leq 1$ such that $s_{\varepsilon_k} \geq k$. By 1.12 we find an $\varepsilon$ with $\frac{1}{2} \leq \varepsilon \leq 1$ such that $s_\varepsilon$ is infinite, a contradiction. Hence $\{s_\varepsilon \mid \frac{1}{2} \leq \varepsilon \leq 1\} \subseteq {}^*\mathbb{R}$ has a finite upper bound $s$. $\qquad\square$

The next lemma says that the numbers $s$ occurring in the previous lemma can be bounded uniformly from above. This is another application of 1.12.

**5.11 Lemma** *Let $\mathfrak{C}$ be as in 5.8. There exists a finite real constant $s \geq 0$ such that the following holds. For every $r \geq 1$, every nonstandard affine apartment $A \subseteq X_D$ and every $x \in A$ there exists a nonstandard affine apartment $A' \subseteq Y_D$ such that $Hd_{B_{\varepsilon r}(f(x))}(f(A), A') \leq s$ holds for all $\frac{1}{2} \leq \varepsilon \leq 1$.*

*Proof.* Assume that this is false. Then we can find for every $k \in \mathbb{N}$ a positive nonstandard real $r_k \geq 1$, an $\varepsilon_k$ with $\frac{1}{2} \leq \varepsilon_k \leq 1$, a nonstandard affine apartment $A_k \subseteq X_D$, a point $x_k \in A_k$ such that

for every nonstandard affine apartment $A' \subseteq Y_D$ we have $Hd_{B_{\varepsilon_k r_k}(f(x_k))}(f(A_k), A') \geq k$. By 1.12 there exist a nonstandard affine apartment $A \subseteq X_D$, a point $x \in A$, a nonstandard real $r \geq 1$ and an $\varepsilon$ with $\frac{1}{2} \leq \varepsilon \leq 1$ such that $Hd_{B_{\varepsilon r}(f(x))}(f(A), A')$ is infinite for all nonstandard affine apartments $A' \subseteq Y_D$. This contradicts Lemma 5.10. $\qquad \square$

By Łos' Theorem 1.10 we have the following immediate consequence for the coarse equivalence that we started with.

**5.12 Corollary** *Let*

$$f : X \longrightarrow Y$$

*be a a coarse equivalence between Euclidean buildings. There exists a real constant $r_f \geq 0$ such that the following holds. For every $r \geq 1$, every affine apartment $A \subseteq X$, every $x \in A$ there exists an affine apartment $A' \subseteq Y$ such that*

$$Hd_{B_{\varepsilon r}(f(x))}(f(A), A') \leq r_f$$

*holds for all $\frac{1}{2} \leq \varepsilon \leq 1$.* $\qquad \square$

*Proof of the higher dimensional Morse Lemma 5.9.* Let $A \subseteq X$ be an affine apartment, let $x \in A$ and put $Z = f(A)$ and $z = f(x)$. Let $r_f$ be as in 5.12. For every $s \geq 1$ we may by 5.12 choose an affine apartment $A_s \subseteq Y$ such that

$$Hd_{B_{\varepsilon s}(z)}(Z, A_s) \leq r_f$$

holds for all $s \in [\frac{1}{2}, 1]$. Let $c = c_{2r_f}$ be the constant from 4.16, corresponding to Hausdorff distance $2r_f$. For $s \geq 1$ we have $Hd_{B_{s+c}(z)}(Z, A_{s+c}) \leq r_f$ and $Hd_{B_{s+c}(z)}(Z, A_{2(s+c)}) \leq r_f$, whence

$$Hd_{B_{s+c}(z)}(A_{s+c}, A_{2(s+c)}) \leq 2r_f$$

and, by 4.16,

$$B_s(z) \cap A_{s+c} = B_s(z) \cap A_{2(s+c)}.$$

Let $w = \pi_{A_{s+c}}(z) = \pi_{A_{2(s+c)}}(z)$. We have then, for $s > r_f$,

$$B_{s-r_f}(w) \cap A_{s+c} = B_{s-r_f}(w) \cap A_{2(s+c)}.$$

Then $A_{r_f+c}, A_{2(r_f+c)}, A_{4(r_f+c)}, \ldots, A_{2^k(r_f+c)}, \ldots$ gives us a nested sequence of metric open $n$-balls (where $n = dim(X)$) with centers $w$ and of radii $2^k(r_f + c) - c$. Their union

$$A' = \bigcup_{k=0}^{\infty} \left( B_{2^k(r_f+c)-c}(w) \cap A_{2^k(r_f+c)} \right)$$

is isometric to $\mathbb{R}^n$ and hence by 4.13 an affine apartment in the complete apartment system of $Y$. From the construction of $A'$ it is clear that

$$Hd(Z, A') \leq r_f.$$

$\qquad \square$

We also have the following.

**5.13 Proposition** *Let $X, Y$ be Euclidean buildings whose spherical buildings at infinity are thick. Let $f : X \times \mathbb{R}^{m_1} \longrightarrow Y \times \mathbb{R}^{m_2}$ be a coarse equivalence. Then $m_1 = m_2$.*

*Proof.* We choose an infinite positive nonstandard real $r$ and a thick point $p \in X$, see 4.20. We then consider the asymptotic cone

$$C((X \times \mathbb{R}^{m_1})_D, r, p \times 0) \cong C(X_D, r, p) \times \mathbb{R}^{m_1}$$

with respect to the constant families $X_k = X$ and $p_k = p$. It is clear that the constant sequence $(p_k)_{k \in K}$ represents a thick point in the ultraproduct $X_D$, and hence also in the asymptotic cone $C(X_D, r, p)$. In particular, the spherical building at infinity of the Euclidean building $C(X_D, r, p)$ is thick. Now we may apply Corollary 5.2 to the continuous map $C(f)$. $\square$

Now we prove that a coarse equivalence of Euclidean buildings induces an isomorphism between the spherical buildings at infinity. In order to show this, we have to describe the spherical building at infinity by coarse data.

**5.14 Lemma** *Let $A_1, \ldots, A_k$ be affine apartments in a Euclidean building $X$. The following are equivalent.*
   *(i) $\partial A_1 \cap \cdots \cap \partial A_k \neq \varnothing$.*
   *(ii) There is an unbounded set which is dominated by each of the affine apartments $A_1, \ldots, A_k$.*

*Proof.* To see that (i) implies (ii), let $a \subseteq X$ be a Weyl simplex representing an element $\partial a \in \partial A_1 \cap \cdots \cap \partial A_k$. For each $j$ there is a Weyl simplex $a_j \subseteq A_j$ representing $\partial a$. Since $a$ and $a_j$ have finite Hausdorff distance, each $A_j$ dominates $a$.

Before we prove the converse implication, we note the following. If an affine apartment $A$ dominates a set $Z \subseteq X$, then $\pi_A Z$ has finite Hausdorff distance from $Z$. If $Z$ is unbounded, then there exists a Weyl simplex $a \subseteq A$ that contains an unbounded subset of $\pi_A Z$. In particular, there exists then a Weyl simplex $a \subseteq A$ that dominates an unbounded subset of $Z$ (or $\pi_A Z$).

Now we assume that the unbounded set $Z \subseteq X$ is dominated by the affine apartments $A_1, \ldots, A_k$. Consider the unbounded set $Y = \pi_{A_1} Z$. There exists a Weyl simplex $a \subseteq A_1$ of minimal dimension which dominates an unbounded subset $Y_1 \subseteq Y$. We claim that

$$\partial a \in \partial A_1 \cap \cdots \cap \partial A_k.$$

Let $j > 1$. In $A_j$ we find a Weyl chamber $c_j$ which dominates an unbounded subset $Y_j$ of $\pi_{A_j}(Y_1)$. Let $A'_j$ be an affine apartment containing representatives $a'$ and $c'_j$ of $\partial a$ and $\partial c_j$. Then $Y'_j = \pi_{A'_j}(Y_j)$ has finite Hausdorff distance from $Y_j$. Since both $a'$ and $c'_j$ dominate the unbounded set $Y'_j$, and since $a'$ is also a Weyl simplex of minimal dimension which dominates an unbounded subset of $Y$, we have $\partial a \subseteq \partial c_j \in \partial A_j$. $\square$

Suppose that $X$ and $Y$ are Euclidean buildings and that $f : X \times \mathbb{R}^{m_1} \longrightarrow Y \times \mathbb{R}^{m_2}$ is a coarse equivalence. We may view $\mathbb{R}^{m_1}$ as a Euclidean building. Then every affine apartment in $X \times \mathbb{R}^{m_1}$ is of the form $A \times \mathbb{R}^{m_1}$, where $A \subseteq X$ is an affine apartment. By 5.9 there is a map $f_*$ from the set of all affine apartments of $X$ to the set of all affine apartments of $Y$, such that $f(A \times \mathbb{R}^{m_1})$ has finite Hausdorff distance from $f_* A \times \mathbb{R}^{m_2}$. If $g$ is a coarse inverse of $f$, then $g_*$ is an inverse of $f_*$.

**5.15 Lemma** *Suppose that $X$ and $Y$ are Euclidean buildings. Let*

$$f : X \times \mathbb{R}^m \longrightarrow Y \times \mathbb{R}^m$$

be a coarse equivalence. Then $f_*$ induces an isomorphism between the apartment complexes $AC(\partial_{cpl}X)$ and $AC(\partial_{cpl}Y)$.

Proof. Let $A_1, \ldots, A_k \subseteq X$ be affine apartments with $\partial A_1 \cap \cdots \cap \partial A_k \neq \varnothing$. Let $a$ be a 1-dimensional Weyl simplex representing a vertex $\partial a \in \partial A_1 \cap \cdots \cap \partial A_k$. Choose $s > 0$ such that $a \subseteq B_s(A_1) \cap \cdots \cap B_s(A_k)$. Let $r_f > 0$ be as in 5.9 and let

$$Z = B_{r_f + \rho(s)}(f_*A_1) \cap \cdots \cap B_{r_f + \rho(s)}(f_*A_k) \subseteq Y,$$

where $\rho$ is the control function for $f$. Then $f$ restricts to a quasi-isometric embedding of $a \times \mathbb{R}^m$ into $Z \times \mathbb{R}^m$. If $Z$ is bounded, we get a quasi-isometric embedding of $a \times \mathbb{R}^m \cong [0, \infty) \times \mathbb{R}^m$ into $\mathbb{R}^m$, which is impossible (by topological dimension invariance, applied to the asymptotic cones, see 1.17 and the proof of 2.16). Hence $Z$ is unbounded. Therefore $\{f_*A_1, \ldots, f_*A_k\}$ is a simplex in $AC(\partial_{cpl}X_2)$ by 5.14. It follows that $f_*$ is a simplicial map $f_* : AC(\partial_{cpl}X_1) \longrightarrow AC(\partial_{cpl}X_2)$. If $g$ is a coarse inverse of $f$, then $g_*$ is a simplicial inverse of $f_*$. □

Combining these results, we have the following first main result about coarse equivalences between Euclidean buildings. This is Theorem II in the introduction.

**5.16 Theorem** Let $X_1$ and $X_2$ be Euclidean buildings whose spherical buildings at infinity $\partial_{cpl}X_1$ and $\partial_{cpl}X_2$ are thick. Let

$$f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2}$$

be a coarse equivalence. Then $m_1 = m_2$ and the map $f_*$ on the affine apartments extends uniquely to a simplicial isomorphism $f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_2$.

Proof. By 5.13 we have $m_1 = m_2$. By 5.15 the map $f_*$ induces a simplicial isomorphism between $AC(\partial_{cpl}X_1)$ and $AC(\partial_{cpl}X_2)$. By 3.10 $f_*$ induces a simplicial isomorphism $f_* : \partial_{cpl}X_1 \xrightarrow{\cong} \partial_{cpl}X_2$. □

The thickness of the spherical buildings is essential for the argument. However, the Euclidean factors which are allowed in the theorem lead also to a result for the case that the spherical buildings at infinity are weak buildings.

**5.17 Corollary** Let $f : X_1 \longrightarrow X_2$ be a coarse equivalence between Euclidean buildings. Then the induced map on the affine apartments induces an isomorphism between the thick building factors in the reductions of $\partial_{cpl}X_1$ and $\partial_{cpl}X_2$.

Proof. This follows from 4.12, and 5.16. □

Note that we do not claim that $f$ induces directly a map $\partial f : \partial X_1 \longrightarrow \partial X_2$ between the Tits boundaries in the sense of CAT(0) geometry. This will in general not be the case; for example, $f$ could be a bi-Lipschitz homeomorphism of the Euclidean cone $EB(B)$ over a thick spherical building $B$. Such a self homeomorphism can be rather wild at infinity. Our combinatorial construction of $f_*$ applies nevertheless.

**5.18** It remains to prove Theorem III. We fix some notation. We assume that $X_1$ and $X_2$ are metrically complete Euclidean buildings whose spherical buildings at infinity $\partial_{cpl}X_1$ and $\partial_{cpl}X_2$ are thick. Furthermore, we assume that

$$f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2}$$

is a coarse equivalence. By 5.16, $f$ induces an isomorphism

$$f_* : \partial_{cpl}X_1 \xrightarrow[\cong]{} \partial_{cpl}X_2$$

which is characterized by the fact that for every affine apartment $A \subseteq X_1$, the image $f(A \times \mathbb{R}^{m_1})$ has finite Hausdorff distance from $f_*A \times \mathbb{R}^{m_2}$. We put

$$n = \dim(X_1) = \dim(X_2).$$

The Euclidean factors have by 5.13 the same dimension, which we denote by

$$m = m_1 = m_2.$$

Finally, we put
$$f(x \times v) = f_1(x \times v) \times f_2(x \times v).$$

Let $a, b \in \partial_{cpl}X_1$ be opposite panels. As in 4.24, we denote by $X_1(a, b)$ the union of all affine apartments in $X_1$ whose boundary contains $a$ and $b$. Consider the closed convex subsets

$$Y_1 = X_1(a, b) \times \mathbb{R}^m \subseteq X_1 \times \mathbb{R}^m \quad \text{and} \quad Y_2 = X_2(f_*a, f_*b) \times \mathbb{R}^m \subseteq X_2 \times \mathbb{R}^m.$$

The composite $\pi_{Y_2} \circ f : Y_1 \longrightarrow Y_2$ is a controlled map. If $A \subseteq X_1(a, b)$ is an affine apartment, then $f_*A$ is an affine apartment in $X_2(f_*a, f_*b)$; see 5.15. By 5.9, there is a uniform constant $r_f > 0$ such $d(f(y), \pi_{Y_2}(f(y))) \le r_f$ for all $y \in Y_1$, so $f|Y_1$ and $\pi_{Y_2} \circ f|Y_1$ have finite distance. If $g$ is a coarse inverse of $f$, then $\pi_{Y_1} \circ g|_{Y_1}$ is therefore a coarse inverse of $\pi_{Y_2} \circ f|_{Y_2}$, and we obtain a coarse equivalence

$$\pi_{Y_2} \circ f : Y_1 \longrightarrow Y_2.$$

By 4.24, $Y_i$ factors for $i = 1, 2$ as a metric product $T_i \times \mathbb{R}^{n-1+m}$ of a metrically complete leafless tree $T_i$, the wall tree, and a Euclidean space.

The ends of the wall tree $T_1$ correspond bijectively to the chambers of $\partial_{cpl}X_1$ containing the panel $a$. If the type of the panel $a$ is not isolated in the Coxeter diagram of $\partial_{cpl}X_1$, then $\Pi^+_{\partial_{cpl}X_1}(a)$ acts 2-transitively on the ends of $T_1$. The boundary map $f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_2$ is clearly equivariant with respect to the isomorphism $\Pi^+_{\partial_{cpl}X_1}(a) \longrightarrow \Pi^+_{\partial_{cpl}X_2}(f_*a)$. In this situation we may apply 2.18 and conclude that $f_*$ extends to an isometry from $T_1$ to $T_2$, possibly after rescaling.

**5.19 Lemma** *Assume that*
$$f : X_1 \times \mathbb{R}^m \longrightarrow X_2 \times \mathbb{R}^m$$

*is as in 5.18. If $\partial_{cpl}X_1$ is irreducible and $\dim(X_1) = n \ge 2$ and if some wall tree of $X_1$ is of type (I), then there is an isometry $\bar{f} : X_1 \longrightarrow X_2$ with $(\bar{f} \times \mathrm{id}_{\mathbb{R}^m})_* = f_*$.*

*Proof.* If the wall tree $T_1$ is of type (I), then $X_1 = \mathrm{EB}(\partial_{cpl}X_1)$ by 4.26. Since $\partial_{cpl}X_1$ is thick and irreducible, $\Pi^+_{\partial_{cpl}X_1}(a)$ acts 2-transitively on the ends of $T_1$ by 3.14. By 2.18, the wall trees of $X_2$ are also of type (I). Therefore $X_2 = \mathrm{EB}(\partial_{cpl}X_2)$. The isomorphism $f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_1$ extends to an isometry $\bar{f} : X_1 \longrightarrow X_2$ of the respective Euclidean cones. $\square$

**5.20 Lemma** *Assume that*
$$f : X_1 \times \mathbb{R}^m \longrightarrow X_2 \times \mathbb{R}^m$$

is as in 5.18. If $\partial_{cpl}X_1$ is irreducible and $\dim(X_1) = n \geq 2$ and if no wall tree of $X_1$ is of type (I), then the metric on $X_2$ can be rescaled so that there is an isometry

$$\bar{f} : X_1 \longrightarrow X_2$$

with $(\bar{f} \times \mathrm{id}_{\mathbb{R}^m})_* = f_*$. This isometry $\bar{f}$ is unique and there is a constant $r > 0$ such that

$$d(f_1(x \times v), \bar{f}(x)) \leq r \quad \text{for all} \quad x \times v \in X_1 \times \mathbb{R}^m.$$

*Proof.* Let $(a, b)$ be a pair of opposite panels in $\partial_{cpl}X_1$. Then $(a, b)$ determines a *wall* (a sphere of dimension $n - 2$) in the spherical building $\partial_{cpl}X_1$. $\partial_{cpl}X_1$ is irreducible, there are at most 2 types of walls in $\partial_{cpl}X_1$.

The group $\Pi^+_{\partial_{cpl}X_1}(a)$ acts 2-transitively on the ends of the wall tree $T_1$ by 3.14. We may apply 2.18 to the coarse equivalence $T_1 \times \mathbb{R}^{n-1+m} \longrightarrow T_2 \times \mathbb{R}^{n-1+m}$, since the equivariance condition is satisfied by our previous discussion. Once and for all, we rescale the metric on $X_2$ in such a way that $f_*$ extends to an equivariant isometry $\tau : T_1 \longrightarrow T_2$.

If $(a', b')$ is any other pair of panels in $\partial_{cpl}X_1$ of the same type as $(a, b)$, with $X_1(a', b') = T'_1 \times \mathbb{R}^{n-1}$, then there is some projectivity which induces an isometry $\varphi_1 : T_1 \longrightarrow T'_1$. Pushing this projectivity forward via $f_*$, we obtain an isometry $\varphi_2 : T_2 \longrightarrow T'_2$, where $X_2(f_*a', f_*b') = T'_2 \times \mathbb{R}^{n-1}$. By construction, the maps $\varphi_2 \circ \tau \circ \varphi_1^{-1}$ and $f_*$ induce the same map $\partial T'_1 \longrightarrow \partial T'_2$. By 2.18, the map $\varphi_2 \circ \tau \circ \varphi_1^{-1}$ is the unique equivariant tree isometry which accompanies the coarse equivalence $X_1(a', b') \longrightarrow X_2(f_*a', f_*b')$. This shows that with respect to our metrics on $X_1$ and $X_2$, the map $f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_2$ is ecological for all wall trees whose wall in $\partial_{cpl}X_1$ is of the same type as the wall determined by $(a, b)$.

Suppose that $z \in X_1$ is a thick point in an affine apartment $A \subseteq X_1$. Since $X_1$ is irreducible, $z$ is the intersection of $n$ walls $M_1, \ldots, M_n \subseteq A$ which are of the same type as the wall determined by $(a, b)$. To see this, we note that the $W$-orbit of any nonzero vector spans the ambient Euclidean space. Each of these walls determines a branch point in a panel tree. Let $M_{1,*}, \ldots, M_{n,*}$ be the corresponding walls in $f_*A$, defined by the isometries between the corresponding wall trees. The intersection $M_{1,*} \cap \cdots \cap M_{n,*}$ is a point $z_* \in f_*A$. By 2.18 there is a uniform constant $s > 0$ such that $d(f_1(z \times v), z_*) \leq s$. If $z' \in A$ is another thick point and if $z'_*$ is constructed in the same way, then the $n$ tree isometries yield $d(z, z') = d(z_*, z'_*)$, whence $d(f_1(z \times v), f_1(z' \times v)) \leq d(z, z') + 2s$. The thick points are cobounded in $X_1$, so the map $x \longmapsto f_1(x \times v)$ is a rough isometry $X_1 \longrightarrow X_2$. This implies by 2.18 that for no wall tree of $X_1$, the accompanying isometry requires any rescaling of $X_2$. The accompanying tree isometries therefore fit together with $f_*$ to an ecological building isomorphism. By Tits' result 4.28, $f$ is accompanied by an isometry $\bar{f} : X_1 \longrightarrow X_2$. This isometry maps the thick point $z$ precisely to the point $z_*$ described above, $z_* = \bar{f}(z)$. It follows that there is a constant $r > 0$ such that $d(f_1(x \times v), \bar{f}(x)) \leq r$ holds for all $x \times v \in X_1 \times \mathbb{R}^m$. $\qquad\square$

We now decompose the Euclidean building $X_1$ into a product $X_1^I \times X_1^{II} \times X_1^{III}$ of Euclidean buildings of types (I), (II) and (III), respectively. Similarly, we decompose $X_2$. The next result implies Theorem III.

**5.21 Theorem** *Let*

$$f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2}$$

*be a coarse equivalence of Euclidean buildings. Assume that $\partial_{cpl}X_1$ and $\partial_{cpl}X_2$ are thick and that $X_1$ splits off no tree factors. Then the following hold.*

*(o) There are numbers $m, n$ with $m_1 = m_2 = m$ and $\dim(X_1) = \dim(X_2) = n$.*

*(i) The irreducible factors of $X_2$ can be rescaled in such a way that there is an isometry*

$$\bar{f} : X_1 \longrightarrow X_2 \quad \text{with} \quad f_* = (\bar{f} \times \mathrm{id}_{\mathbb{R}^m})_*.$$

*(ii) If*

$$X_1 = X_1^I \times X_1^{II} \times X_1^{III} \text{ and } X_2 = X_2^I \times X_2^{II} \times X_2^{III}$$

*are decomposed as above, then $\bar{f}$ factors as a product,*

$$\bar{f}(x_I \times x_{II} \times x_{III}) = \bar{f}_I(x_I) \times \bar{f}_{II}(x_{II}) \times \bar{f}_{III}(x_{III}).$$

*There is a constant $r > 0$ such that*

$$d(\bar{f}_N(x_N), \pi_{X_N} f(x_I \times x_{II} \times x_{III} \times p)) \leq r,$$

*for $N \in \{II, III\}$ and for all*

$$x_I \times x_{II} \times x_{III} \times v \in X_1^I \times X_1^{II} \times X_1^{III} \times \mathbb{R}^m.$$

*(iii) The maps $f_{II}$ and $f_{III}$ are unique.*

*Proof.* We proceed by induction on the number irreducible factors of $X_1$. The case of one irreducible factor is covered by 5.19 and 5.20. In general, we decompose $X_1 = Y_1 \times Z_1$, with $Z_1$ irreducible. As $f_*$ is an isomorphism, we have a corresponding decomposition $X_2 = Y_2 \times Z_2$. Fix opposite chambers $a, b$ in $\partial_{cpl} Y_1$, and let $A \subseteq Y_1$ be the corresponding affine apartment. Then $A \times Z_1$ is the union of all affine apartments in $Y_1 \times Z_1$ which contain $a, b$ at infinity. This relation is preserved by $f_*$. So if $y \times z \times v \in A \times Z_1 \times \mathbb{R}^m$, then $f(y \times z \times v)$ has uniform distance from $f_* A \times Z_1 \times \mathbb{R}^m$. It follows that $\pi_{f_* A \times Z_2 \times \mathbb{R}^m} \circ f$ is a coarse equivalence between $A \times Z_1 \times \mathbb{R}^m$ and $f_* A \times Z_2 \times \mathbb{R}^m$. By the induction hypothesis we can rescale the irreducible factors of $Y_2$ in such a way that there is an isometry between $Y_1$ and $Y_2$, and 5.19 and 5.20 give us isometries between $Z_1$ and $Z_2$, possibly after rescaling $Z_2$. These isometries fit together to an isometry $\bar{f} : X_1 \longrightarrow X_2$, with $f_* = \bar{f}_*$. This gives (i). If $Z_1$ is of type (II) or (III), then the claims (ii) and (iii) follow, by applying 5.20 to $A \times Z_1 \times \mathbb{R}^m$ and $f_* A \times Z_2 \times \mathbb{R}^m$. $\qquad\square$

# Appendix to: 'Coarse equivalences of Euclidean buildings'

by Jeroen Schillewaert[*] and Koen Struyve[†]

The purpose of this appendix is to provide a generalization of the main results of the preceding paper 'Coarse equivalences of Euclidean buildings' by Linus Kramer and Richard Weiss. We will refer to this paper as [KW]. The paper in question proves rigidity results for coarse equivalences of metrically complete Euclidean buildings. We will show that the hypothesis of completeness can be omitted.

The proof for this generalization is certainly not independent of the original proof. In fact, we only discuss where and how the proof of Kramer and Weiss should be altered to allow for non-complete Euclidean buildings. As our proof is an extension of the Kramer and Weiss argument, we refer and rely on it for a detailed introduction, details and definitions.

Here are the theorems that we obtain.

**Theorem A.I** *Let $G$ be a group acting isometrically on two leafless trees $T_1, T_2$. Assume that there is a coarse equivalence $f : T_1 \longrightarrow T_2$, that $T_1$ has at least $3$ ends and that the induced map $\partial f : \partial T_1 \longrightarrow \partial T_2$ between the ends of the trees is $G$-equivariant. If the $G$-action on $\partial T_1$ is $2$-transitive, then (after rescaling the metric on $T_2$) there is a $G$-equivariant isometry $\bar{f} : T_1 \longrightarrow T_2$ with $\partial f = \partial \bar{f}$. If $T_1$ has at least two branch points, then $\bar{f}$ is unique and has finite distance from $f$.*

**Theorem A.III** *Let $f : X_1 \times \mathbb{R}^m \longrightarrow X_2 \times \mathbb{R}^m$ be as in Theorem II in [KW] and assume in addition that $X_1$ has no tree factors. Then there is (possibly after rescaling the metrics on the de Rham factors of $X_2$) an isometry $\bar{f} : X_1 \longrightarrow X_2$ with $(\bar{f} \times \mathrm{id}_{\mathbb{R}^m})_* = f_*$. Put $f(x \times v) = f_1(x \times v) \times f_2(x \times v)$. If none of the de Rham factors of $X_1$ is a Euclidean cone over its boundary, then $\bar{f}$ is unique and $d(f_1(x \times y), \bar{f}(x))$ is bounded as a function of $x \in X_1$.*

We now discuss the modifications one has to make in order to avoid the assumption of metrical completeness in the results on coarse rigidity of Linus Kramer and Richard Weiss in [KW]. Completeness is used at two places in their proof, which we will discuss separately. All references are to [KW] unless mentioned otherwise.

## Recovering the tree from the $G$-action

The first problem that occurs is that the Bruhat-Tits Fixed Point Theorem requires completeness, so a bounded isometry group acting on a metrically non-complete $\mathbb{R}$-tree $T$ does not necessarily have a fixed point (although it does in the metric completion $\overline{T}$ of $T$). Consequently also Proposition 2.8 no longer holds. The next lemma and propositions offer a way to deal with this observation. For a geodesic segment $[x, y]$ in a tree we put $(x, y) = [x, y] - \{x, y\}$.

**A.1 Lemma** *Let $T$ be an $\mathbb{R}$-tree and $x, y \in \overline{T}$. Then the open segment $(x, y)$ lies in $T$.*

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*Proof.* First of all note that the metric completion of an $\mathbb{R}$-tree is again an $\mathbb{R}$-tree by [23] (see also [33, Cor. II.1.10]), hence it makes sense to speak about the open segment $(x, y)$. (Note however that the metric completion of a leafless metrically non-complete $\mathbb{R}$-tree is never leafless.)

Given $\varepsilon > 0$, we choose points $x', y' \in T$ with $d(x, x'), d(y, y') < \varepsilon$. The geodesic segment $[x', y']$ is contained in $T$. We put $x'' = \pi_{[x', y']}(x)$ and $y'' = \pi_{[x', y']}(y)$. The unique geodesic from $x$ to $x'$ then passes through $x''$. In particular, $d(x, x'') < \varepsilon$, and similarly $d(y, y'') < \varepsilon$. By (T2), we have that $[x, y] = [x, x''] \cup [x'', y''] \cup [y'', y]$. The claim follows now, because $[x'', y''] \subseteq T$ and because $\varepsilon$ was arbitrarily small. $\qquad\square$

For the purpose of this appendix, we restate condition $(2\text{-}\partial)$ without the completeness assumption:

**$(2\text{-}\partial)$** The group $G$ acts isometrically on the leafless tree $T$, and this action is 2-transitive on $\partial T$.

The next two propositions serve as replacements of Propositions 2.7 and 2.8 of [KW].

**A.2 Proposition** *Assume that the tree $T$ satisfies $(2\text{-}\partial)$ and that the set of branch points is dense. Then every point $x \in T$ is $G$-isolated in $\overline{T}$.*

*Proof.* Let $x \in T$. Suppose that $G_x$ fixes another point $y \neq x$ in $\overline{T}$. Let $y' \in (x, y)$. Then $y'$ lies in $T$ by Lemma A.1 and $G_x$ fixes the geodesic segment $[x, y']$. There is a branch point $z$ between $x$ and $y'$, so Proposition 2.3 implies that $y'$ is not a fixed point of $G_{x,z}$. $\qquad\square$

This modification together with the Bruhat-Tits Fixed Point Theorem allows us, as in [KW], to conclude that the stabilizers of $G$-isolated points in $T$ are maximal bounded subgroups.

**A.3 Proposition** *Assume $(2\text{-}\partial)$ and that $P \subseteq G$ is a maximal bounded subgroup. Then $P$ is the stabilizer of a $G$-isolated point, or it only fixes exactly one point in the completion $\overline{T}$ and none in $T$.*

*Proof.* Note that we may assume that the tree is metrically non-complete, and hence that the tree $T$ is of type (III) with a dense set of branch points (see Corollary 2.6).

Assume that what we want to prove is false. As in Proposition 2.8, this implies that $\overline{T}^P$ contains a geodesic segment $[x, y] \subseteq \overline{T}$ with $x \neq y$. Applying Lemma A.1 one has that $(x, y) \subset T$, so there exists a subsegment $[x', y'] \subset (x, y) \subset T^P$ with $x' \neq y'$. This situation was proved to be impossible in Proposition 2.8. $\qquad\square$

We are still left with the possibility that there are more maximal bounded subgroups than only those corresponding to $G$-isolated points of $T$. However those corresponding to $G$-isolated points can be recognized as follows:

**A.4 Lemma** *A maximal bounded subgroup $H$ is the stabilizer of a $G$-isolated point of $T$ if and only if either $T$ is not of type (III), or if $T$ is of type (III) and for each two ends $u, v \in \partial T$, $H$ is the only maximal bounded subgroup of $G$ containing $\langle (G_u \cup G_v) \cap H \rangle$.*

*Proof.* Let $H$ be a maximal bounded subgroup and let $x$ be the point stabilized by $H$. If the tree is not of type (III), then the tree is metrically complete and no extra maximal bounded subgroups appear. If the tree is of type (III) then Lemma 2.11 states that for each two ends $u, v \in \partial T$, the group $H$ is the only maximal bounded subgroup of $G$ containing $\langle (G_u \cup G_v) \cap H \rangle$ if and only if $x$ lies in some apartment. This holds if and only if $x$ is $G$-isolated. $\qquad\square$

Note that one can still distinguish between types (0)-(III) using Lemma A.1 and 2.12.

Now we can modify the last part of Section 2 (starting directly after Proposition 2.8), and define $i_G(T)$ to correspond with the set of maximal bounded subgroups of $G$ satisfying the conditions of the above lemma. We conclude that Proposition 2.13 still holds in the metrically non-complete case.

Similarly one has to alter Proposition 2.17 and Theorem 2.18 restricting to the maximal bounded subgroups with the above property. Observe that this property is preserved by the $G$-equivariance.

### The use of retraction maps $\pi$

In 1.1 one refers to [7, II.2.4] for a 1-Lipschitz retraction map $\pi_K : X \longrightarrow K$ when $K$ is a metrically complete convex subset in a CAT(0)-space $X$. This kind of retraction maps is used in various places of [KW]. In most of these places, $K$ is a closed convex subset of an apartment and there is no problem. The only place where this is not the case is at the beginning of 5.18.

For each point $x$ in the completion of a set $K$ and for each $\varepsilon > 0$, one can find a point $x' \in K$ such that $d(x, x') < \varepsilon$. Combining this with the above mentioned result for complete convex subsets of CAT(0)-spaces one obtains the following lemma:

**A.5 Lemma** *Given a closed convex subset $K$ of a CAT(0)-space $X$ and an $\varepsilon > 0$, there exists a controlled map $\pi'_K : X \longrightarrow K$ with control function $\rho(t) = t + \varepsilon$.*

These maps $\pi'_K$ are sufficient for the purposes of 5.18. Indeed, one only uses 1-Lipschitz to obtain that certain compositions of maps are controlled. This conclusion remains valid with the weaker control function of $\pi'_K$.

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