Abelian covers of chiral polytopes

Marston Conder and Wei-Juan Zhang

m.conder@auckland.ac.nz weijuan.zhang@auckland.ac.nz

Department of Mathematics, University of Auckland
Private Bag 92019, Auckland 1142, New Zealand

Abstract

Abstract polytopes are combinatorial structures with certain properties drawn from the study of geometric structures, like the Platonic solids, and of maps on surfaces. Of particular interest are the polytopes with maximal possible symmetry (subject to certain natural constraints). Symmetry can be measured by the effect of automorphisms on the 'flags' of the polytope, which are maximal chains of elements of increasing rank (dimension). An abstract polytope of rank n is said to be *chiral* if its automorphism group has precisely two orbits on the flags, such that two flags that differ in one element always lie in different orbits. Examples of chiral polytopes have been difficult to find and construct. In this paper, we introduce a new covering method that allows the construction of some infinite families of chiral polytopes, with each member of a family having the same rank as the original, but with the size of the members of the family growing linearly with one (or more) of the parameters making up its 'type' (Schläfli symbol). In particular, we use this method to construct several new infinite families of chiral polytopes of ranks 3, 4, 5 and 6.

Keywords: Abstract polytopes; chiral polytopes; coverings

1 Introduction

Abstract polytopes are combinatorial structures obeying certain axioms that generalise the classical properties of convex geometric polytopes. Highly symmetric examples include not only classical regular polytopes such as the Platonic solids and more exotic structures such as the 120-cell and 600-cell, but also regular maps on surfaces (such as Klein's quartic).

Roughly speaking, an abstract polytope $\mathcal{P}$ is a partially-ordered set endowed with a rank function, satisfying certain conditions that arise naturally from a geometric setting. Such objects were proposed by Grünbaum in the 1970s, and their definition (initially as 'incidence polytopes') and theory were developed by Danzer and Schulte. Every automorphism of an abstract polytope is uniquely determined by its effect on any flag, which is a maximal chain in the poset $\mathcal{P}$. The most symmetric examples are *regular*, with all

flags lying in a single orbit, and a comprehensive description of these is given in a book on the subject by McMullen and Schulte [19]. These objects are also known as 'thin residually-connected geometries with a linear diagram'.

Quite a lot is known about regular polytopes, and small examples and some infinite families are easily constructible via their automorphism groups, which are quotients of 'string' Coxeter groups (viz. Coxeter groups with a linear Coxeter-Dynkin diagram). For example, the automorphism group of a regular n-simplex is the symmetric group S_{n+1}, via its representation as a quotient of the Coxeter group $[3, 3, \ldots, 3]$ of rank n. Others are described in [19] and in other references listed there.

An interesting class of examples which are not quite regular are the *chiral* polytopes, for which the automorphism group has two orbits on flags, with any two flags that differ in just one element lying in different orbits. The study of chiral abstract polytopes was pioneered by Schulte and Weiss (see [26, 27] for example). Chiral polytopes of rank 3 are much the same as chiral maps on surfaces (see Coxeter and Moser [12]), with modest extra geometric conditions.

The first family of chiral maps was constructed by Heffter [17] in 1898; see also Doro and Wilson [15]. Contributions to the more general study of chiral polytopes were first made by Weber and Seifert [30], and also later by Coxeter [11]. After Coxeter, several families of chiral regular maps on surfaces of higher genus were found by Sherk [29], Garbe [16], and Bujalance, Conder and Costa [3]. In 2001, Conder and Dobcsanyi [7] determined all chiral regular maps on orientably surfaces of genus 7 to 15, and this list has subsequently been extended to genus 300 at [5]. Also Schulte [25] constructed three families of infinite chiral 3-polytopes in ordinary space that are geometrically chiral.

For quite some time, the only known finite examples of chiral polytopes had ranks 3 and 4 (see [13, 20, 21, 26] for example), while some infinite examples of chiral polytopes of rank 5 had been constructed by Schulte and Weiss in [28]. But then some finite examples of rank 5 were constructed about 10 years ago by Conder, Hubard and Pisanski [8]. The latter included the smallest examples in each of three classes: properly self-dual, improperly self-dual, and non-self-dual. Now quite a few such examples are known. In early 2009 Conder and Devillers devised a construction for chiral polytopes whose facets are simplices, and used this to construct examples of finite chiral polytopes of ranks 6, 7 and 8 [unpublished]. Also Breda, Jones and Schulte developed a method of 'mixing' a chiral d-polytope with a regular d-polytope to produce a larger example of a chiral polytope of the same rank d; see [2].

At about the same time, Pellicer devised a quite different method for constructing finite chiral polytopes, with given regular facets, and used this construction to prove the existence of finite chiral polytopes of every rank $d \geq 3$; see [22]. A few years later, Cunningham and Pellicer proved every finite chiral d-polytope with regular facets is itself the facet of a chiral $(d+1)$-polytope; see [14]. Then the work of Conder and Devillers was taken up by Conder, Hubard, O'Reilly Regueiro and Pellicer [9] to prove that all but finitely many alternating groups A_n and symmetric groups S_n are the automorphism group of a chiral 4-polytope of type $\{3, 3, k\}$ for some k (dependent on n). This will be extended to ranks greater than 4 by the authors of [9].

These examples are very large, however. It is still an open problem to find alternative constructions for families of chiral polytopes of relatively small order, with easily described automorphism groups. Many other questions about chiral polytopes were posed in [23]. Chiral polytopes continue to be surprisingly rare in comparison with regular polytopes, even though the latter possess a higher degree of symmetry.

In this paper, we introduce a new method for constructing chiral polytopes, as covers of a given 'base' example, with a covering group that is abelian, and sometimes cyclic. This method is similar to the 'mixing' approach of [2], in that it produces chiral polytopes of the same rank as the given one, but with different type and larger automorphism group. On the other hand, it can produce an infinite family of chiral polytopes from a given one, with the sizes of members of the family growing linearly with one (or more) of the parameters making up its 'type' (Schläfli symbol). We illustrate and apply this method in the construction of several new infinite families of chiral polytopes of ranks 3 to 6.

Before that, we give some further background on polytopes and their properties in Section 2. Then we describe our new approach in Section 3, and summarise some of the new families it produces in Section 4.

In a subsequent paper, we will take a somewhat different approach, to construct chiral polytopes as abelian covers of *regular* polytopes. In contrast to other methods, this enables the construction of chiral polytopes without needing a 'base' chiral polytope to build on.

2 Further background

Below we give some further background on abstract polytopes, especially those that are regular or chiral. Additional details may be found in [4, 8, 9, 19, 26], for example.

2.1 Abstract polytopes

An *abstract d-polytope* (or *abstract polytope of rank d*) is a partially ordered set $\mathcal{P}$, the elements and maximal totally ordered subsets of which are called *faces* and *flags* respectively, such that certain properties are satisfied, which we explain below.

First, $\mathcal{P}$ contains a minimum face F_{-1} and a maximum face F_d, and there is a rank function from $\mathcal{P}$ to the set $\{-1, 0, \ldots, d\}$ such that $\mathrm{rank}(F_{-1}) = -1$ and $\mathrm{rank}(F_d) = d$. Every flag of $\mathcal{P}$ contains precisely $d+2$ elements, including F_{-1} and F_d. The faces of rank i are called *i-faces*, the 0-faces are called *vertices*, the 1-faces are called *edges*, and the $(d-1)$-faces are called *facets* (or *co-vertices*).

If F and G are faces of ranks r and s with $F \leq G$, then we say that F and G are *incident*, and define the *section* G/F as $\{H \mid F \leq H \leq G\}$; such a section of $\mathcal{P}$ is said to have rank $s-r-1$, and may be called an $(s-r-1)$-section of $\mathcal{P}$. For any face G of $\mathcal{P}$, the section G/F_{-1} may be identified with G itself in $\mathcal{P}$; hence, for example, a facet may be viewed as a $(d-1)$-face, or as the section $\mathcal{F} = G/F_{-1}$ for some $(d-1)$-face G of $\mathcal{P}$. Similarly, if G is a j-face then the 'complementary' section F_d/G is sometimes called a *co-j-face*, or the *co-face* at G. In particular, a *co-vertex* is a vertex-figure, and a *co-edge* is a $(d-2)$-section F_d/G (where G is an edge).

One important property that has to be satisfied is the *diamond condition*, which says that whenever G/F is a 1-section, with $\operatorname{rank}(G) = \operatorname{rank}(F) + 2 = i + 2$ (say), there are precisely two intermediate faces H_1 and H_2 of rank $i+1$ with $F < H_j < G$ for $j \in \{1, 2\}$. This implies that for any flag Φ and for every $i \in \{0, \ldots, d-1\}$, there is a unique flag Φ^i that differs from Φ in precisely the i-face. We call Φ^i the *i-adjacent flag* for Φ. More generally, two flags of $\mathcal{P}$ are said to be adjacent if they differ in only one face.

The final important property is *strong connectivity*, which says that if Φ and Φ' are any two given flags of $\mathcal{P}$, then there exists a sequence $\Psi_0, \Psi_1, \ldots, \Psi_m$ of flags of $\mathcal{P}$ from $\Psi_0 = \Phi$ to $\Psi_m = \Phi'$ such that Ψ_{k-1} is adjacent to Ψ_k, and $\Phi \cap \Phi' \subseteq \Psi_k$, for $1 \leq k \leq m$.

This completes the definition of an abstract d-polytope.

2.2 Equivelar polytopes, Schläfli type, isomorphism and duality

Let $\mathcal{P}$ be any abstract regular polytope of rank $d \geq 3$, and suppose (as we will throughout this paper) that $\mathcal{P}$ is finite, in that it has only finitely many faces of each rank j.

Every 2-section G/F of $\mathcal{P}$ is isomorphic to the face lattice of a polygon, and if the number of sides of every such polygon depends only on the rank of G, and not on F or G itself, then we say that $\mathcal{P}$ is *equivelar*. When that happens, if k_i is the number of edges of every 2-section between an $(i-2)$-face and an $(i+1)$-face of $\mathcal{P}$, for $1 \leq i < d$, then the expression $\{k_1, k_2, \ldots, k_{d-1}\}$ is called the *Schläfli type* (or *Schläfli symbol*) of $\mathcal{P}$. Note that this definition carries no assumption of symmetry. Also by convention, we assume that each such polygon is non-degenerate, so has at least 3 edges, and therefore $k_i \geq 3$ for all i.

Two polytopes $\mathcal{P}$ and $\mathcal{Q}$ of the same rank d are said to be *isomorphic* (to each other) if there exists an order-preserving bijection from $\mathcal{P}$ to $\mathcal{Q}$, taking j-faces of $\mathcal{P}$ to j-faces of $\mathcal{Q}$ for $0 \leq j < d$. The automorphisms from $\mathcal{P}$ to $\mathcal{P}$ form a group denoted by $\operatorname{Aut}(\mathcal{P})$, or sometimes by $\Gamma(\mathcal{P})$. By the diamond condition and strong flag-connectivity, it is easy to see that every automorphism of $\mathcal{P}$ is uniquely determined by its effect on any given flag of $\mathcal{P}$, and it follows that the number of automorphisms of $\mathcal{P}$ is bounded above by the number of flags of $\mathcal{P}$.

Two polytopes $\mathcal{P}$ and $\mathcal{Q}$ of the same rank d are said to be *dual* (to each other) if there exists an order-*reversing* bijection from $\mathcal{P}$ to $\mathcal{Q}$, taking j-faces of $\mathcal{P}$ to $(d-1-j)$-faces of $\mathcal{Q}$ for $0 \leq j < d$. When this happens, we call $\mathcal{Q}$ the *dual* of $\mathcal{P}$ and denote it by $\mathcal{P}^*$, and vice versa, giving $(\mathcal{P}^*)^* \cong \mathcal{Q}^* \cong \mathcal{P}$. Moreover, if $\mathcal{P}$ is equivelar, with Schläfli type $\{k_1, k_2, \ldots, k_{d-2}, k_{d-1}\}$, then also $\mathcal{P}^*$ is equivelar, with Schläfli type $\{k_{d-1}, k_{d-2}, \ldots, k_2, k_1\}$. Any such order-reversing bijection from $\mathcal{P}$ to $\mathcal{Q}$ is called a *duality*, and if it has order 2 then it is called a *polarity*. The polytope $\mathcal{P}$ is *self-dual* if it is isomorphic to its dual $\mathcal{P}^*$.

2.3 Regular polytopes

A d-polytope $\mathcal{P}$ is said to be *regular* whenever $\Gamma(\mathcal{P})$ acts transitively (and therefore regularly) on the set of all flags of $\mathcal{P}$. In that case, $\Gamma(\mathcal{P})$ acts transitively on the j-faces of $\mathcal{P}$ for all j, and so $\mathcal{P}$ is equivelar.

Also when $\mathcal{P}$ is regular, its automorphism group $\Gamma(\mathcal{P})$ is generated by a canonical set

of involutions $\rho_0, \ldots, \rho_{d-1}$, where ρ_i is the unique automorphism mapping a given *base flag* Φ to its i-adjacent flag Φ^i, for $0 \leq i < d$. It is not difficult to see that these generators satisfy the relations

$$\begin{aligned} \rho_i^2 &= 1 \quad \text{for } 0 \leq i < d, & (1)\\ (\rho_{j-1}\rho_j)^{k_j} &= 1 \quad \text{for } 1 \leq j < d, & (2)\\ (\rho_i\rho_j)^2 &= 1 \quad \text{whenever } |i-j| \geq 2. & (3) \end{aligned}$$

Moreover, by the polytope axioms, they must satisfy the following *intersection condition*:

$$\langle\, \rho_i : i \in I \,\rangle \cap \langle\, \rho_j : j \in J \,\rangle = \langle\, \rho_k : k \in I \cap J \,\rangle \quad \text{for all } I, J \subseteq \{0, 1, \ldots, d-1\}. \qquad (4)$$

The relations given in (1) to (3) above are precisely the defining relations for the *Coxeter group* $[k_1, k_2, \ldots, k_{d-1}]$, and in particular, $\Gamma(\mathcal{P})$ is a smooth homomorphic image of the latter group, where 'smooth' (here) means that the orders of the generators and their pairwise products are preserved.

The *rotation group* $\Gamma^+(\mathcal{P})$ of $\mathcal{P}$ is the image of the orientation-preserving subgroup of the Coxeter group, or in other words, the subgroup of $\Gamma(\mathcal{P})$ consisting of words of even length in the generators $\rho_0, \rho_1, \ldots, \rho_{d-1}$. In particular, $\Gamma^+(\mathcal{P})$ is generated by the *abstract rotations* $\sigma_j = \rho_{j-1}\rho_j$ for $1 \leq j < d$, and has index at most 2 in $\Gamma(\mathcal{P})$. We say that $\mathcal{P}$ is *directly regular* (or sometimes *orientably-regular*) when this index is 2.

Also the stabiliser in $\Gamma(\mathcal{P})$ of the i-face of the base flag Φ is the subgroup generated by $\{\rho_0, \rho_1, \ldots, \rho_{d-1}\} \setminus \{\rho_i\}$, for $0 \leq i < d$. In particular, the stabiliser of the vertex (0-face) of Φ is $\langle\, \rho_1, \rho_2, \ldots, \rho_{d-1} \,\rangle$, while the stabiliser of the facet of Φ is $\langle\, \rho_0, \rho_1, \ldots, \rho_{d-2} \,\rangle$.

2.4 Chiral polytopes

A d-polytope $\mathcal{P}$ is said to be *chiral* if its automorphism group $\Gamma(\mathcal{P})$ has two orbits on flags, with every two adjacent flags lying in different orbits. In this case, for a given base flag Φ, and for $1 \leq j < d$, the polytope $\mathcal{P}$ admits an automorphism σ_j that takes Φ to the flag $(\Phi^j)^{j-1}$ which differs from Φ in precisely its $(j-1)$- and j-faces. This automorphism σ_j is the analogue of the abstract rotation $\rho_{j-1}\rho_j$ in the regular case, for each j, and in particular, it follows that $\mathcal{P}$ is equivelar. Moreover, $\mathcal{P}$ has maximum possible 'rotational' symmetry (because it admits the analogues of all the abstract rotations), but on the other hand, it admits none of the 'reflections' ρ_i.

Every non-degenerate *orientably-regular map* on a surface can be regarded as an abstract 3-polytope, and each one is either regular or chiral depending on whether or not it admits reflections. In fact 3 is the smallest rank of a chiral polytope, because every abstract 2-polytope is combinatorially isomorphic to a regular convex polygon with at least 3 sides (by our non-degeneracy assumption), and hence is regular. The facets and vertex-figures of a chiral d-polytope $\mathcal{P}$ may be regular or chiral, but the $(d-2)$-faces and the co-edges are always regular, by a nice argument given in [26, Proposition 9].

If $\mathcal{P}$ has Schläfli type $\{k_1, k_2, \ldots, k_{d-1}\}$, then its automorphism group is a smooth quotient of the orientation-preserving subgroup of the Coxeter group $[k_1, \ldots, k_{d-1}]$. Indeed the elements $\sigma_1, \sigma_2, \ldots, \sigma_{d-1}$ satisfy the relations

$$\sigma_j^{k_j} = 1 \quad \text{for } 1 \le j < d, \tag{5}$$

$$(\sigma_i\sigma_{i+1}\dots\sigma_j)^2 = 1 \quad \text{for } 1 \le i < j < d. \tag{6}$$

It also follows that σ_i commutes with σ_j whenever $j - i > 2$, since if $w = \sigma_{i+1}\sigma_{i+2}\dots\sigma_{j-1}$ then each of w, $\sigma_i w$, $w\sigma_j$ and $\sigma_i w\sigma_j$ is an involution, and therefore $1 = (\sigma_i w\sigma_j)^2 = \sigma_i w\sigma_j\sigma_i w\sigma_j = \sigma_i\sigma_j^{-1}w^{-1}w^{-1}\sigma_i^{-1}\sigma_j = \sigma_i\sigma_j^{-1}w^{-2}\sigma_i^{-1}\sigma_j = \sigma_i\sigma_j^{-1}\sigma_i^{-1}\sigma_j$.

The stabiliser in $\Gamma(\mathcal{P})$ of the i-face of the base flag Φ is the subgroup generated by

$$\begin{cases} \sigma_2, \sigma_3, \dots, \sigma_{d-1} & \text{when } i = 0 \\ \sigma_1, \sigma_2, \dots, \sigma_{i-1}, \sigma_i\sigma_{i+1}, \sigma_{i+2}, \dots, \sigma_{d-1} & \text{when } 1 \le i \le d-2 \\ \sigma_1, \sigma_2, \dots, \sigma_{d-2} & \text{when } i = d-1. \end{cases}$$

Moreover, these generators σ_j must satisfy the following chiral form of the intersection condition, which is provable using [26, Proposition 7] and [26, Lemma 10]:

$$\langle\, \sigma_1, \sigma_2, ..., \sigma_i \,\rangle \cap \langle \sigma_j, \sigma_{j+1}, ..., \sigma_k \rangle = \langle \sigma_j, ..., \sigma_i \rangle \quad \text{for } \ 1 \le i < k \ \text{ and } \ 2 \le j \le k < d. \tag{7}$$

Here we note that chiral polytopes occur in pairs (or *enantiomorphic forms*), such that each member of the pair is the 'mirror image' of the other. If one of them, say $\mathcal{P}$, has Schläfli type $\{k_1, k_2, \dots, k_{d-1}\}$, and ψ is the corresponding epimorphism to $\Gamma(\mathcal{P})$ from the orientation-preserving subgroup of the Coxeter group $[k_1, \dots, k_{d-1}]$, then the kernel K of ψ is not normal in the full Coxeter group, but is conjugated by any orientation-reversing element (in the full Coxeter group) to another subgroup K^c which is the kernel of the epimorphism ψ^c corresponding to the mirror image $\mathcal{P}^c$ of $\mathcal{P}$. In fact the automorphism groups of $\mathcal{P}$ and $\mathcal{P}^c$ are the same, but have different canonical generating sets: a base flag of $\mathcal{P}^c$ can be chosen such that $\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\sigma_1^{-1}, \sigma_3, \sigma_4, \dots, \sigma_{d-2}, \sigma_{d-1}$ are the canonical generators for $\Gamma(\mathcal{P}^c)$. Note that these would be the conjugates of $\sigma_1, \sigma_2, \dots, \sigma_{d-1}$ by the reflection ρ_0 if $\mathcal{P}$ were regular.

A chiral polytope $\mathcal{P}$ can sometimes be self-dual, but there are two kinds of self-duality. If $\delta : \mathcal{P} \to \mathcal{P}$ is a duality, and Φ is a base flag for $\mathcal{P}$, then we say that $\mathcal{P}$ is *properly self-dual* if δ takes Φ to a flag in the same orbit as Φ under the automorphism group $\Gamma(\mathcal{P})$, or *improperly self-dual* if Φ^δ lies in the other orbit of $\Gamma(\mathcal{P})$.

2.5 Construction of regular and chiral polytopes from groups

Some of the properties of the automorphism group of a regular or chiral polytope described above can be turned around to give constructions for regular and chiral polytopes from particular kinds of generating sets for groups.

If Γ is any finite group generated by d involutions $\rho_0, \rho_1, \dots, \rho_{d-1}$ that satisfy the relations (1) to (3) above, as well as the intersection condition (4), then we may construct a regular d-polytope $\mathcal{P}$ with automorphism group $\Gamma(\mathcal{P})$ isomorphic to Γ, by taking as its i-faces the (right) cosets of the subgroup generated by $\{\rho_0, \rho_1, \dots, \rho_{d-1}\} \setminus \{\rho_i\}$, for $0 \le i < d$, and defining incidence by non-empty intersection; see [19, Theorem 2E11].

Similarly, if Γ is any finite group generated by $d-1$ elements $\sigma_1, \sigma_2, \ldots, \sigma_{d-1}$ that satisfy the relations (5) and (6) and the intersection condition (7), then we may construct a directly regular or chiral d-polytope $\mathcal{P}$ with rotation group $\Gamma^+(\mathcal{P})$ isomorphic to Γ, by taking as its j-faces the (right) cosets of the subgroup generated by

$$\begin{cases} \sigma_2, \sigma_3, \ldots, \sigma_{d-1} & \text{when } j = 0 \\ \sigma_1, \sigma_2, \ldots, \sigma_{j-1}, \sigma_j\sigma_{j+1}, \sigma_{j+2}, \ldots, \sigma_{d-1} & \text{when } 1 \leq j \leq d-2 \\ \sigma_1, \sigma_2, \ldots, \sigma_{d-2} & \text{when } j = d-1, \end{cases}$$

and defining incidence by non-empty intersection.

We will denote this polytope by $\mathcal{P}(\sigma_1, \sigma_2, \ldots, \sigma_{d-1})$. If F and G are incident faces of $\mathcal{P}$ of ranks $i-2$ and $j+1$ with $i \leq j$, then the section G/F is isomorphic to the $(j-i+2)$-polytope $\mathcal{P}(\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$. Also we observe that $\mathcal{P}(\sigma_1, \sigma_2, \ldots, \sigma_{d-1})$ is regular if and only if the group Γ has an automorphism ρ that takes $(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots, \sigma_{d-2}, \sigma_{d-1})$ to $(\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\sigma_1^{-1}, \sigma_3, \sigma_4, \ldots, \sigma_{d-2}, \sigma_{d-1})$, and in that case, the polytope is directly regular.

Similarly, if $\mathcal{P}(\sigma_1, \sigma_2, \ldots, \sigma_{d-1})$ is chiral, then it is properly self-dual if Γ has an automorphism δ that takes $(\sigma_1, \sigma_2, \ldots, \sigma_{d-2}, \sigma_{d-1})$ to $(\sigma_{d-1}^{-1}, \sigma_{d-2}^{-1}, \ldots, \sigma_2^{-1}, \sigma_1^{-1})$, and improperly self-dual if Γ has an automorphism δ that takes $(\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\sigma_1^{-1}, \sigma_3, \sigma_4, \ldots, \sigma_{d-2}, \sigma_{d-1})$ to $(\sigma_{d-1}^{-1}, \sigma_{d-2}^{-1}, \ldots, \sigma_2^{-1}, \sigma_1^{-1})$. See [18] for further details.

2.6 Flatness and tightness

An abstract polytope is said to be *flat* if each of its facets contains every vertex. An easy example is the hemicube. A regular polytope $\mathcal{P}$ is flat if and only if its automorphism group $\Gamma(\mathcal{P})$ is the set-theoretic product of the stabilisers of a vertex and an incident facet (see [19, Proposition 4E4]), and the same holds also for a chiral polytope $\mathcal{P}$ (see the remarks following Lemma 1.2 in [24]).

If $\mathcal{P}$ is a regular polytope of type $\{k_1, k_2, \ldots, k_{d-1}\}$, then by multiple applications of the intersection condition it is easy to prove that $|\Gamma(\mathcal{P})| \geq 2k_1k_2\ldots k_{d-1}$, and then $\mathcal{P}$ is called *tight* if this lower bound on $|\Gamma(\mathcal{P})|$ is attained; see [4]. Similarly, if $\mathcal{P}$ is a chiral polytope of type $\{k_1, k_2, \ldots, k_{d-1}\}$, then $\mathcal{P}$ is called tight if the corresponding lower bound $|\Gamma(\mathcal{P})| \geq k_1k_2\ldots k_{d-1}$ is attained. In both cases, the order of $\Gamma(\mathcal{P})$ is equal to k_1 times the order of the stabiliser of a vertex, and also to k_{d-1} times the order of the stabiliser of a facet, and it follows that every tight regular or chiral polytope is flat.

3 Coverings

Let $\mathcal{P}$ and $\mathcal{Q}$ be any two polytopes of the same rank. Then a function $\gamma : \mathcal{Q} \mapsto \mathcal{P}$ is called a *covering* if it preserves incidence, rank and adjacency of flags. By flag-connectivity, we note that any such γ is surjective. Also we say that $\mathcal{Q}$ *covers* $\mathcal{P}$ if there exists such a covering $\gamma : \mathcal{Q} \mapsto \mathcal{P}$. This terminology is adopted from the theory of maps and surfaces.

Next, we note that if $\mathcal{P}$ and $\mathcal{Q}$ are polytopes of the same rank d that are either chiral or directly regular, then their rotation groups are both quotients of the orientation-preserving subgroup W^+ of the rank $d-1$ Coxeter group $[\infty, \cdots, \infty]$. This group W^+ is generated

by $d-1$ elements $\sigma_1, \sigma_2, \ldots, \sigma_{d-1}$, subject to the defining relations $(\sigma_i\sigma_{i+1}\ldots\sigma_j)^2 = 1$ for $1 \le i < j \le d-1$. Also if J and K are the corresponding kernels, with $\Gamma^+(\mathcal{P}) \cong W^+/J$ and $\Gamma^+(\mathcal{Q}) \cong W^+/K$, then it is easy to see that $\mathcal{Q}$ covers $\mathcal{P}$ if and only if $K \le J$. Indeed in that case, we have $\Gamma^+(\mathcal{Q})/(J/K) \cong (W^+/K)/(J/K) \cong W^+/J \cong \Gamma^+(\mathcal{P})$, and then we may call the quotient J/K the *covering group*, or the group of covering transformations.

We now introduce an approach for constructing covers of chiral polytopes with cyclic covering group. The main idea is to take a chiral d-polytope $\mathcal{P}$ of type $\{k_1, k_2, \ldots, k_{d-1}\}$ and construct an infinite family $\{\mathcal{Q}^{(n)} : n = 1, 2, 3, \ldots\}$ of chiral polytopes of the same rank d, such that each member $\mathcal{Q}^{(n)}$ of this family is a cover of $\mathcal{P}$ having almost the same type as $\mathcal{P}$, with just one of the k_i replaced by nk_i. In particular, $\mathcal{Q}^{(1)} = \mathcal{P}$. For example, below we will exhibit such a family of chiral 4-polytopes having types $\{3n, 6, 9\}$ for every positive integer n, and covering a particular chiral 4-polytope $\mathcal{P}$ of type $\{3, 6, 9\}$.

Our approach is based on the following key theorem:

Theorem 3.1 *Let U be a group generated by $d-1$ elements $x_1, x_2, \ldots, x_{d-1}$, with the property that for some $\ell \in \{1, 2, \ldots, d-1\}$, the following hold:*

(a) $(x_i x_{i+1} \ldots x_j)^2 = 1$ *for* $1 \le i < j < d$,

(b) x_i *has finite order* $k_i \ge 3$ *for all* $i \ne \ell$, *while* x_ℓ *has infinite order,*

(c) $x_\ell^{k_\ell}$ *generates a cyclic normal subgroup* N *of* U, *for some integer* $k_\ell \ge 3$,

(d) *the intersection of* N *with the subgroup generated by all the* x_i *other than* x_ℓ *is trivial,*

(e) *the images of the generators* $x_1, x_2, \ldots, x_{d-1}$ *in the factor group* $U/N = U/\langle x_\ell^{k_\ell}\rangle$ *satisfy the intersection condition* (7), *and make* U/N *the automorphism group of a chiral* d*-polytope* $\mathcal{P}$ *of type* $\{k_1, k_2, \ldots, k_{d-1}\}$.

Then for every positive integer n, *the factor group* $U^{(n)} = U/\langle x_\ell^{nk_\ell}\rangle$ *is the automorphism group of a chiral* d*-polytope* $\mathcal{Q}^{(n)}$ *of type* $\{k_1, \ldots, k_{\ell-1}, nk_\ell, k_{\ell+1}, \ldots, k_{d-1}\}$, *covering the chiral polytope* $\mathcal{Q}^{(1)} \cong \mathcal{P}$. *Moreover, if* $\mathcal{P}$ *is flat, then so is* $\mathcal{Q}^{(n)}$ *for all* n, *and if* $\mathcal{P}$ *is tight, then so is* $\mathcal{Q}^{(n)}$ *for all* n.

Proof. First, note that $N^{(n)} = \langle x_\ell^{nk_\ell}\rangle$ is the only subgroup of index n in $\langle x_\ell^{k_\ell}\rangle = N$, and so is characteristic in N and hence normal in U. Also $N^{(n)}$ intersects trivially the subgroup generated by $\{x_i \mid i \ne \ell\}$, by (b), and therefore the images of $x_1, \ldots, x_{\ell-1}, x_\ell, x_{\ell+1}, \ldots, x_{d-1}$ in the factor group $U/N^{(n)} = U/\langle x_\ell^{nk_\ell}\rangle = U^{(n)}$ have orders $k_1, \ldots, k_{\ell-1}$, nk_ℓ, $k_{\ell+1}, \ldots, k_{d-1}$ respectively. Furthermore, $N^{(1)} = \langle x_\ell^{k_\ell}\rangle = N$, so $U^{(1)} = U/N \cong \Gamma^+(\mathcal{P})$.

Next, we show that the images $\bar{x}_1, \ldots, \bar{x}_{d-1}$ in $U^{(n)}$ of the generators of U satisfy the intersection condition (7). To do this, let $I = \{1, 2, \ldots i\}$ and $J = \{j, j+1, \ldots, k\}$ where $1 \le i < k$ and $2 \le j \le k < d$, and then define $\overline{A} = \langle\, \bar{x}_r : r \in I \,\rangle$ and $\overline{B} = \langle\, \bar{x}_s : s \in J \,\rangle$ and $\overline{C} = \langle\, \bar{x}_t : t \in I \cap J \,\rangle$, and also $\overline{U} = U^{(n)} = U/N^{(n)}$ and $\overline{N} = N/N^{(n)}$. The intersection condition requires $\overline{A} \cap \overline{B} = \overline{C}$, but as usual, it is easy to see that $\overline{A} \cap \overline{B}$ contains $\overline{C}$, and hence all we need do is prove the reverse inclusion. Next, if we let $\overline{N} = N/N^{(n)}$, then by (e) we know that the images of $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{d-1}$ in the quotient $U^{(1)} = U/N \cong \overline{U}/\overline{N}$ satisfy the intersection condition, and therefore $\overline{A} \cap \overline{B} \subseteq \overline{C}\,\overline{N}$. Now if $\ell \in I \cap J$, then $\overline{N} = \langle \bar{x}_\ell^{k_\ell}\rangle \subseteq \langle \bar{x}_\ell\rangle \subseteq \langle\, \bar{x}_t : t \in I \cap J \,\rangle = \overline{C}$, so $\overline{A} \cap \overline{B} \subseteq \overline{C}\,\overline{N} = \overline{C}$; while on the other

hand, if $\ell \notin I \cap J$, then by (d) either $\langle\, x_r : r \in I \,\rangle$ or $\langle\, x_s : s \in J \,\rangle$ intersects N trivially, so $\overline{A} \cap \overline{N} = \emptyset$ or $\overline{B} \cap \overline{N} = \emptyset$, and therefore $(\overline{A} \cap \overline{B}) \cap \overline{N} = \emptyset$, so $\overline{A} \cap \overline{B} \subseteq \overline{C}$.

Hence the intersection condition is satisfied, making $U^{(n)}$ the orientation-preserving subgroup of the automorphism group of a chiral or directly regular d-polytope $\mathcal{Q}^{(n)}$ of type $\{k_1, \ldots, k_{\ell-1}, nk_\ell, k_{\ell+1}, \ldots, k_{d-1}\}$.

In fact, $\mathcal{Q}^{(n)}$ is chiral. For suppose the contrary, namely that $\mathcal{Q}^{(n)}$ is directly regular. Then there exists an automorphism θ of the group $\overline{U} = U^{(n)}$ that takes $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{d-1})$ to $(\bar{x}_1^{-1}, \bar{x}_1\bar{x}_2^{-1}\bar{x}_1^{-1}, \bar{x}_3, \ldots, \bar{x}_{d-1})$. Now if $\ell \neq 2$, then θ takes $\bar{x}_\ell$ to $\bar{x}_\ell^{\pm 1}$ and so preserves $\langle \bar{x}_\ell \rangle = \overline{N}$, while on the other hand if $\ell = 2$, then θ takes $\bar{x}_\ell$ to $\bar{x}_1\bar{x}_2^{-1}\bar{x}_1^{-1} = \bar{x}_1\bar{x}_\ell^{-1}\bar{x}_1^{-1}$, which generates $\overline{N}$ (because $\overline{N}$ is cyclic and normal in $\overline{U}$), and so again θ preserves $\overline{N}$. But then θ induces an analogous automorphism of $\overline{U}/\overline{N} = U/N = U^{(1)}$, making the polytope $\mathcal{Q}^{(1)} = \mathcal{P}$ reflexible, a contradiction.

Finally, we consider flatness and tightness. If $\mathcal{P}$ is flat, then $U^{(1)} = U/N$ is expressible as the product of the images of $\langle x_1, x_2, \ldots, x_{d-2} \rangle$ and $\langle x_2, x_3, \ldots, x_{d-1} \rangle$. When we move from $\mathcal{P}$ to its cover $\mathcal{Q}^{(n)}$, the analogues of these two subgroups are $\langle \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{d-2} \rangle$ and $\langle \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{d-1} \rangle$. At least one of these contains $\bar{x}_\ell$ and hence contains $\overline{N}$, so their product must be $U^{(n)}$. Thus $\mathcal{Q}^{(n)}$ is also flat. Also if $\mathcal{P}$ is tight, then $|U^{(1)}| = |\Gamma(\mathcal{P})| = k_1 k_2 \ldots k_{d-1}$, and so $|\Gamma(\mathcal{Q}^{(n)})| = |U^{(n)}| = |U/N^{(n)}| = |U/N^{(1)}||N^{(1)}/N^{(n)}| = |U^{(1)}|n = nk_1k_2 \ldots k_{d-1}$, which is the product of the entries of the Schläfli type $\{k_1, \ldots, k_{\ell-1}, nk_\ell, k_{\ell+1}, \ldots, k_{d-1}\}$ of $\mathcal{Q}^{(n)}$, and so $\mathcal{Q}^{(n)}$ is tight as well. $\square$

As our first application of this theorem, we have the following:

Example 3.2 *An infinite family of chiral* 4*-polytopes of type* $\{3n, 6, 9\}$

To construct this family, take U as the group with presentation

$$\langle\, u, v, w \mid (uv)^2 = (vw)^2 = (uvw)^2 = v^6 = w^9 = (v^{-1}u^2)^2 = [w, u^3] = vw^2v^{-3}w^{-1} = 1 \,\rangle.$$

Note that $(v^{-1}u^2)^2 = 1$ can be rewritten as $1 = v^{-1}u^3u^{-1}v^{-1}u^2 = v^{-1}u^3vuu^2 = v^{-1}u^3vu^3$, which implies that $v^{-1}u^3v = u^{-3}$, and it follows that the cyclic subgroup N generated by u^3 is centralised by u and w and normalised by v. Adding the relation $u^3 = 1$ gives the quotient U/N, which by a relatively easy calculation in Magma [1] is a group of order 486, and is the automorphism group of the mirror image $\mathcal{P}$ of the chiral 4-polytope of type $[3, 6, 9]$ with 486 automorphisms listed at [6]. Moreover, the Reidemeister-Schreier process, implemented via the `Rewrite` command in Magma, shows that the subgroup N is infinite cyclic. Hence the hypotheses (a), (b), (c) and (e) in the above theorem are satisfied, for $(x_1, x_2, x_3) = (u, v, w)$.

But also v and w satisfy the relations $(vw)^2 = v^6 = w^9 = vw^2v^{-3}w^{-1} = 1$, which by another Magma computation define a group of order 54. Moreover, in the factor group U/N of order 486, the image of the subgroup generated by v and w has order 54, and has trivial intersection with the image of the cyclic subgroup generated by u. (Indeed U/N is the complementary product of the images of $\langle u \rangle$ and $\langle v, w, u^{-1}wu \rangle$, which have orders 3 and 162 respectively.) Then since the order of the subgroup generated by v and w in U cannot be greater than 54, it follows that the intersection $\langle u \rangle \cap \langle v, w \rangle$ is trivial in U, and therefore $\langle x_1^3 \rangle \cap \langle x_2, x_3 \rangle = \langle u^3 \rangle \cap \langle v, w \rangle = N \cap \langle v, w \rangle$ is trivial as well, so (d) holds too.

Hence by Theorem 3.1 we obtain an infinite family $\{\mathcal{Q}^{(n)} : n = 1, 2, 3, \dots\}$ of chiral 4-polytopes of type $\{3n, 6, 9\}$, as indicated earlier.

The 'base' polytope $\mathcal{P} = \mathcal{Q}^{(1)}$ has 9 vertices, 27 edges, 81 2-faces and 9 facets (and 972 flags), with each facet $\mathcal{F}$ being a directly regular 3-polytope of type $\{3, 6\}$ having full automorphism group of order 108, and each vertex-figure $\mathcal{V}$ being a chiral 3-polytope of type $\{6, 9\}$ having automorphism group of order 54 (isomorphic to $\langle v, w\rangle$). It follows that $\mathcal{Q}^{(n)}$ has $9n$ vertices, $27n$ edges, 81 2-faces, 9 facets and $972n$ flags. Moreover, each facet $\mathcal{F}^{(n)}$ of $\mathcal{Q}^{(n)}$ is a directly regular 3-polytope of type $\{3n, 6\}$, with $9n$ vertices, $27n$ edges and 18 faces, and full automorphism group of order $108n$, while each vertex-figure $\mathcal{V}^{(n)}$ is isomorphic to the same tight chiral 3-polytope of type $\{6, 9\}$ as for $\mathcal{P}$, with 6 vertices, 27 edges and 9 faces, and automorphism group of order 54.

Next, the above approach can be extended to produce families of covers for which the covering group is abelian but not necessarily cyclic:

Theorem 3.3 *Let U be a group generated by $d-1$ elements $x_1, x_2, \dots, x_{d-1}$, with the property that for some subset L of $\{1, 2, \dots, d-1\}$, the following hold:*

(a) $(x_i x_{i+1} \dots x_j)^2 = 1$ *for* $1 \le i < j < d$,

(b) x_i *has finite order* $k_i \ge 3$ *for all* $i \notin L$, *while* x_ℓ *has infinite order for all* $\ell \in L$,

(c) *for all* $\ell \in L$, *there exists an integer* $k_\ell \ge 3$ *such that* $x_\ell^{k_\ell}$ *generates a cyclic normal subgroup* N_ℓ *of* U *that intersects* $\langle\, x_i : i \ne \ell \,\rangle$ *trivially,*

(d) *the normal subgroup* $N = \langle\, x_\ell^{k_\ell} : \ell \in L \,\rangle = \prod_{\ell \in L} N_\ell$ *intersects* $\langle\, x_i : i \notin L \,\rangle$ *trivially,*

(e) *the images of the generators* $x_1, x_2, \dots, x_{d-1}$ *in* U/N *satisfy the intersection condition* (7), *and make* U/N *the automorphism group of a chiral* d*-polytope* $\mathcal{P}$ *of type* $\{k_1, k_2, \dots, k_{d-1}\}$.

Then for every indexed sequence $S_L = (n_\ell)_{\ell \in L}$ *of positive integers, the factor group* $U^{(S_L)} = U/\langle\, x_\ell^{n_\ell k_\ell} : \ell \in L \,\rangle$ *is the automorphism group of a chiral* d*-polytope* $\mathcal{Q}^{(S_L)}$ *that covers* $\mathcal{P}$ *and has type* $\{s_1, s_2, \dots, s_{d-1}\}$, *where* $s_i = k_i$ *for all* $i \notin L$ *and* $s_\ell = n_\ell k_\ell$ *for all* $\ell \in L$, *and the covering group for* $\mathcal{Q}^{(S_L)}$ *over* $\mathcal{P}$ *is isomorphic to the abelian group* $\prod_{\ell \in L} C_{n_\ell}$. *Moreover, if* $\mathcal{P}$ *is flat, then so is* $\mathcal{Q}^{(S_L)}$, *and if* $\mathcal{P}$ *is tight, then so is* $\mathcal{Q}^{(S_L)}$. *Also if* $\mathcal{P}$ *is properly* (*resp. improperly*) *self-dual, then* $\mathcal{Q}^{(S_L)}$ *is properly* (*resp. improperly*) *self-dual if and only if* $d - \ell \in L$ *and* $n_{d-\ell} = n_\ell$ *whenever* $\ell \in L$.

Proof. Most of this follows easily from Theorem 3.1, by induction on $|L|$. Note that $N = \langle\, x_\ell^{k_\ell} : \ell \in L \,\rangle$ is the product of the normal subgroups $N_\ell = \langle x_\ell^{k_\ell} \rangle$ for $\ell \in L$, which are cyclic and have trivial pairwise intersections, and hence N is abelian. In turn, this implies that the covering group is the direct product of the quotients $N_\ell / \langle x_\ell^{n_\ell k_\ell} \rangle \cong C_{n_\ell}$ for $\ell \in L$. For the final claims about duality, necessity follows from the fact that the type of the dual of an equivelar polytope is the reverse of the given type, while sufficiency can be proved by showing that when $d - \ell \in L$ and $n_{d-\ell} = n_\ell$ whenever $\ell \in L$, any duality of $\mathcal{P}$ can be extended to a duality of $\mathcal{Q}^{(S_L)}$. $\square$

As an application of this more general theorem, we have the following:

Example 3.4 *An infinite family of chiral* 3*-polytopes of type* $\{4m, 4n\}$

To construct this family, take U as the group with presentation

$$\langle\, u, v \mid (uv)^2 = (v^{-1}u^3)^2 = (u^{-1}v^3)^2 = uv^{-1}uv^{-1}u^2v^2u^{-2}v^{-2} = 1\,\rangle.$$

Note that $(v^{-1}u^3)^2 = 1$ can be rewritten as $1 = v^{-1}u^4v^{-1}v^{-1}u^3 = v^{-1}u^4vuu^3 = v^{-1}u^4vu^4$, which implies that $v^{-1}u^4v = u^{-4}$, and hence that the cyclic subgroup generated by u^4 is centralised by u and normalised by v. Similarly, the relation $(u^{-1}v^3)^2 = 1$ implies that $u^{-1}v^4u = v^{-4}$, and hence that the cyclic subgroup generated by v^4 is normalised by u and centralised by v. Thus $N = \langle u^4, v^4\rangle$ is normal in U. Adding the relations $u^4 = v^4 = 1$ gives the quotient U/N, which by a calculation in Magma is a group of order 80, and is the automorphism group of a chiral 3-polytope of type $\{4, 4\}$ listed at [6]. In particular, the intersection of the images of $\langle u\rangle$ and $\langle v\rangle$ in U/N is trivial. Moreover, the Reidemeister-Schreier process shows that the subgroup N is free abelian of rank 2 (with just a single defining relation $[u^4, v^4] = 1$), and it follows that the cyclic subgroups generated by u and v have trivial intersection in U. Hence the hypotheses (a), (b), (c) and (e) in the above theorem are satisfied, for $(x_1, x_2) = (u, v)$. The hypothesis (d) is vacuous.

By Theorem 3.3, we obtain for every ordered pair of positive integers m and n a chiral 4-polytope $\mathcal{Q}^{(m,n)}$ of type $\{4m, 4n\}$, with automorphism group of order $80mn$. The 'base' polytope $\mathcal{P} = \mathcal{Q}^{(1,1)}$ is an improperly self-dual chiral polytope of type $\{4, 4\}$ mentioned above, with 20 vertices, 40 edges, 20 2-faces and automorphism group of order 80, and it follows that the covering polytope $\mathcal{Q}^{(m,n)}$ is also improperly self-dual, with $20m$ vertices, $40mn$ edges, $20n$ 2-faces, and $160mn$ flags.

Before giving more examples, we note that analogues of Theorems 3.1 and 3.3 can be proved also in finite cases — where one or more of the selected generators x_ℓ has finite order, and then the subgroup $\langle x_\ell^{k_\ell}\rangle$ has order s_ℓ (divisible by k_ℓ), and the integer n or n_ℓ is restricted to divisors of s_ℓ/k_ℓ. The proofs are essentially the same, and applications appear in §§ 4.2, 4.14, 4.19, 4.22, 4.23, 4.26, 4.27 and 4.29–4.31 below.

4 Infinite and finite families

This section exhibits further applications of Theorems 3.1 and 3.3, to the construction of infinite families of chiral polytopes of ranks 3 to 6, and some additional finite families in the rank 6 case.

We use much the same notation as above, but for simplicity we write group presentations in the form $\langle X \mid \mathcal{R}\rangle$ where X is the generating set and $\mathcal{R}$ is the set of defining relators (with a relation of the form $w = z$ written as the relator wz^{-1}. Also we use $\mathcal{R}(d)$ as an abbreviation for the set of relators $(x_ix_{i+1}\ldots x_j)^2$ for $1 \le i < j < d$ and the implied relators $[x_i, x_j]$ when $j - i > 2$, and we use the symbols u, v, w, x and y in place of x_1, x_2, x_3, x_4 and x_5.

In each case we give the finitely-presented group U and indicate the relevant normal subgroup N, and then summarise particular properties of the polytopes in the resulting family, but without the kind of detail given in Examples 3.4 and 3.2.

4.1 Chiral 3-polytopes of type $\{4m, 4n\}$

- $U = \langle\, u, v \mid \mathcal{R}(3), (v^{-1}u^3)^2, (u^{-1}v^3)^2, uv^{-1}uv^{-1}u^2v^2u^{-2}v^{-2} \,\rangle$
- $N = \langle\, u^4, v^4 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ (free abelian of rank 2), with quotient U/N of order 80
- $\mathcal{Q}^{(m,n)}$ has $20m$ vertices, $40mn$ edges, $20n$ 2-faces, and $160mn$ flags, for all $m, n \geq 1$
- $\mathcal{Q}^{(n,n)}$ is improperly self-dual, for all $n \geq 1$.

4.2 Chiral 4-polytopes of type $\{3,4m, 4n\}$ for $m = 1$, 2, 3 or 6

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^{24}, u^3, [u, v^4], (w^{-1}v^3)^2, [u, w^4], (v^{-1}w^3)^2, uv^{-1}uw^2v^{-1}wuw \,\rangle$
- $N = \langle\, v^4, w^4 \,\rangle \cong \mathbb{Z}_6 \oplus \mathbb{Z}$, with quotient U/N of order 480
- $\mathcal{Q}^{(m,n)}$ has 6 vertices, $60m$ edges, $80mn$ 2-faces, $20n$ facets, and $960mn$ flags, for all $n \geq 1$, whenever $m \in \{1, 2, 3, 6\}$
- Each facet is a directly regular 3-polytope of type $\{3, 4m\}$ with $48m$ automorphisms
- Each vertex-figure is a chiral 3-polytope of type $\{4m, 4n\}$ with $80mn$ automorphisms, isomorphic to the one of type $\{4m, 4n\}$ in §4.1 above when $m = 1$ or 3.

4.3 Chiral 4-polytopes of type $\{3n, 6, 9\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^6, w^9, (v^{-1}u^2)^2, [w, u^3], vw^2v^{-3}w^{-1} \,\rangle$
- $N = \langle\, u^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 486
- $\mathcal{Q}^{(n)}$ has $9n$ vertices, $27n$ edges, 81 2-faces, 9 facets, and $972n$ flags, for all $n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3n, 6\}$ with $108n$ automorphisms
- Each vertex-figure is a tight chiral 3-polytope of type $\{6, 9\}$ with 54 automorphisms.

4.4 Chiral 4-polytopes of type $\{4, 3n, 6\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), u^4, w^6, (u^{-1}v^2)^2, (w^{-1}v^2)^2, (uwv^{-1}w)^2 \,\rangle$
- $N = \langle\, v^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 576
- $\mathcal{Q}^{(n)}$ has 8 vertices, $48n$ edges, $72n$ 2-faces, 24 facets, and $1152n$ flags, for all $n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{4, 3n\}$ with $48n$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{3n, 6\}$ with $144n$ automorphisms.

4.5 Chiral 4-polytopes of type $\{3m, 4, 6n\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^4, (v^{-1}u^2)^2, (v^{-1}w)^4, u^{-1}wvw^{-1}vuw^{-2} \,\rangle$
- $N = \langle\, u^3, w^6 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 576
- $\mathcal{Q}^{(m,n)}$ has $6m$ vertices, $48m$ edges, $96n$ 2-faces, $24n$ facets, and $1152mn$ flags, for all $m, n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3m, 4\}$ with $48m$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{4, 6n\}$ with $192n$ automorphisms.

4.6 Chiral 4-polytopes of type $\{3m, 8, 3n\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^8, (v^{-1}u^2)^2, (v^{-1}w^2)^2, (w^{-1}v^3)^2, uwvuv^{-1}uwv^{-1}wuwv \,\rangle$
- $N = \langle\, u^3, w^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 576
- $\mathcal{Q}^{(m,n)}$ has $12m$ vertices, $96m$ edges, $96n$ 2-faces, $12n$ facets, and $1152mn$ flags, for all $m, n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3m, 8\}$ with $96m$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{8, 3n\}$ with $96n$ automorphisms
- $\mathcal{Q}^{(n,n)}$ is properly self-dual, for all $n \geq 1$.

4.7 Chiral 4-polytopes of type $\{4, 4, 3n\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), u^4, v^4, (v^{-1}w^2)^2, u^{-1}vu^{-1}vu^2v^2 \,\rangle$
- $N = \langle\, w^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 720
- $\mathcal{Q}^{(n)}$ has 30 vertices, 120 edges, $90n$ 2-faces, $18n$ facets, and $1440n$ flags, for all $n \geq 1$
- Each facet is a chiral 3-polytope of type $\{4, 4\}$ with 40 automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{4, 3n\}$ with $48n$ automorphisms.

4.8 Chiral 4-polytopes of type $\{3n, 8, 8\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^8, w^8, (v^{-1}u^2)^2, (v^2w^{-2})^2, uwv^{-1}uw^{-2}vw^{-1} \,\rangle$
- $N = \langle\, u^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 768
- $\mathcal{Q}^{(n)}$ has $6n$ vertices, $48n$ edges, 128 2-faces, 16 facets, and $1536n$ flags, for all $n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3n, 8\}$ with $96n$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{8, 8\}$ with 256 automorphisms.

4.9 Chiral 4-polytopes of type $\{4, 4, 4n\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), u^4, v^4, uv^{-1}uv^{-1}u^{-2}v^2, uv^{-1}u^{-2}w^{-1}u^{-1}w^2v^{-1}w \,\rangle$
- $N = \langle\, w^4 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 800
- $\mathcal{Q}^{(n)}$ has 10 vertices, 100 edges, $100n$ 2-faces, $20n$ facets, and $1600n$ flags, for all $n \geq 1$
- Each facet is a chiral 3-polytope of type $\{4, 4\}$ with 40 automorphisms
- Each vertex-figure is a chiral 3-polytope of type $\{4, 4n\}$ with $80n$ automorphisms, isomorphic to the mirror image of the one of type $\{4, 4n\}$ in §4.1 above.

4.10 Chiral 4-polytopes of type $\{3n, 6, 18\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^6, (v^{-1}u^2)^2, v^{-1}wv^3w^{-5}, uv^{-1}w^{-1}v^2uwv^{-3} \,\rangle$
- $N = \langle\, u^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 972
- $\mathcal{Q}^{(n)}$ has $9n$ vertices, $27n$ edges, 162 2-faces, 18 facets, and $1944n$ flags, for all $n \geq 1$

- Each facet is a directly regular 3-polytope of type $\{3n, 6\}$ with $108n$ automorphisms
- Each vertex-figure is a chiral 3-polytope of type $\{6, 18\}$ with 108 automorphisms.

4.11 Chiral 4-polytopes of type $\{3m, 18, 3n\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), (v^{-1}u^2)^2, (v^{-1}w^2)^2, [u, v^6], [w, v^6], u^{-1}v^2u^{-1}v^2u^2v^2,$
$(uv^{-1}wuw)^2, v^2w^{-1}v^3wv^3w^2 \,\rangle$
- $N = \langle\, u^3, w^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 1458
- $\mathcal{Q}^{(m,n)}$ has $9m$ vertices, $243m$ edges, $243n$ 2-faces, $9n$ facets, and $2916mn$ flags, for all $m, n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3m, 8\}$ with $324m$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{8, 3n\}$ with $324n$ automorphisms
- $\mathcal{Q}^{(n,n)}$ is properly self-dual, for all $n \geq 1$.

4.12 Chiral 4-polytopes of type $\{3n, 6, 6\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^6, w^6, [u, v^2], (v^{-1}w)^4, u^{-1}vw^3u^{-1}w^2vw^{-1} \,\rangle$
- $N = \langle\, u^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 1728
- $\mathcal{Q}^{(n)}$ has $3n$ vertices, $144n$ edges, 288 2-faces, 96 facets, and $3456n$ flags, for all $n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3n, 6\}$ with $36n$ automorphisms
- Each vertex-figure is a chiral 3-polytope of type $\{6, 6\}$ with 576 automorphisms.

4.13 Chiral 4-polytopes of type $\{3m, 24, 3n\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), (v^{-1}u^2)^2, (v^{-1}w^2)^2, [u, v^4], [w, v^4], uwvuwuwvuwv^{-2} \,\rangle$
- $N = \langle\, u^3, w^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 1728
- $\mathcal{Q}^{(m,n)}$ has $12m$ vertices, $288m$ edges, $288n$ 2-faces, $12n$ facets, and $3456mn$ flags, for all $m, n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3m, 24\}$ with $288m$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{24, 3n\}$ with $288n$ automorphisms
- $\mathcal{Q}^{(n,n)}$ is properly self-dual, for all $n \geq 1$.

4.14 Chiral 4-polytopes of type $\{4n, 4s, 4t\}$ for $s, t = 1$ or 2

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^8, w^8, (v^{-1}u^3)^2, [w, u^4], (u^{-1}v^3)^2, (w^{-1}v^3)^2, [u, w^4], (v^{-1}w^3)^2,$
$uv^{-1}uv^{-1}u^2v^2u^{-2}v^{-2}, u^{-1}w^{-2}v^{-2}u^2w^{-1}u^{-1}w^{-1} \,\rangle$
- $N = \langle\, u^4, v^4, w^4 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with quotient U/N of order 3200
- $\mathcal{Q}^{(n,s,t)}$ has $20n$ vertices, $400ns$ edges, $400st$ 2-faces, $40t$ facets, and $6400nst$ flags, for all $n \geq 1$, whenever $s, t \in \{1, 2\}$
- Each facet is a chiral 3-polytope of type $\{4n, 4s\}$ with $80ns$ automorphisms, as in §4.1 above (with (n, s) in place of (m, n)).

- Each vertex-figure is a chiral 3-polytope of type $\{4s, 4t\}$ with $160st$ automorphisms
- $\mathcal{Q}^{(n)}$ is never self-dual.

4.15 Chiral 4-polytopes of type $\{3n, 6, 12\}$

- $U = \langle\, u, v, w \mid \mathcal{R}(4), v^6, w^{12}, [u, v^2], (v^{-1}w)^4, u^{-1}vw^3u^{-1}w^2vw^{-1} \,\rangle$
- $N = \langle\, u^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 6912
- $\mathcal{Q}^{(n)}$ has $3n$ vertices, $288n$ edges, 1152 2-faces, 384 facets, and $13824n$ flags, for all $n \geq 1$
- Each facet is a directly regular 3-polytope of type $\{3n, 6\}$ with $36n$ automorphisms
- Each vertex-figure is a chiral 3-polytope of type $\{6, 12\}$ with 2304 automorphisms.

4.16 Chiral 5-polytopes of type $\{3n, 4, 6, 3\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), v^4, w^6, x^3, (v^{-1}u^2)^2, (v^{-1}w)^4, (vu)^{-1}w^2vuw^2, vx^{-1}v^2wx^{-1}v^{-1}xw \,\rangle$
- $N = \langle\, u^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 2304
- $\mathcal{Q}^{(n)}$ has $6n$ vertices, $48n$ edges, 128 2-faces, 48 3-faces, 4 facets, and $4608n$ flags, for all $n \geq 1$
- Each facet is a chiral 3-polytope of type $\{3n, 4, 6\}$ with $576n$ automorphisms
- Each vertex-figure is a directly regular 3-polytope of type $\{4, 6, 3\}$ with 768 automorphisms.

4.17 Chiral 5-polytopes of type $\{3, 8, 8, 4n\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), u^3, v^8, w^8, (u^{-1}v^3)^2, (v^2w^{-2})^2, uwv^{-1}uw^{-2}vw^{-1},$
  $v^{-1}x^{-1}vxw^2, (vxw^{-1})^2 \,\rangle$
- $N = \langle\, x^4 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 3072
- $\mathcal{Q}^{(n)}$ has 6 vertices, 48 edges, 128 2-faces, $32n$ 3-faces, $4n$ facets, and $6144n$ flags, for all $n \geq 1$
- Each facet is a chiral 4-polytope of type $\{3, 8, 8\}$ with 768 automorphisms, isomorphic to the one of type $\{3, 8, 8\}$ in §4.8 above
- Each vertex-figure is a directly regular 4-polytope of type $\{8, 8, 4n\}$ with $1024n$ automorphisms.

4.18 Chiral 5-polytopes of type $\{3, 4, 4, 3n\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), u^3, v^4, w^4, (w^{-1}x^2)^2, v^{-1}wv^{-1}wv^2w^2, uwuwu^{-1}v^{-2}w^{-2} \,\rangle$
- $N = \langle\, x^3 \,\rangle \cong \mathbb{Z}$, with quotient U/N of order 4320
- $\mathcal{Q}^{(n)}$ has 6 vertices, 90 edges, 240 2-faces, $90n$ 3-faces, $18n$ facets, and $8640n$ flags, for all $n \geq 1$
- Each facet is a chiral 4-polytope of type $\{3, 4, 4\}$ with 240 automorphisms
- Each vertex-figure is a chiral 4-polytope of type $\{4, 4, 3n\}$ with $720n$ automorphisms, as in §4.7 above.

4.19 Chiral 5-polytopes of type $\{3k, 3m, 4, 6n\}$ for $k, m = 1, 2, 4$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), u^{12}, v^{12}, w^4, [v, u^3], [u, v^3], (w^{-1}x)^4, (w^{-1}v^2)^2, vx^2wvx^2w \,\rangle$
- $N = \langle\, u^3, v^3, x^6 \,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$, with quotient U/N of order 4608
- $\mathcal{Q}^{(k,m,n)}$ has $8k$ vertices, $24km$ edges, $128m$ 2-faces, $192n$ 3-faces, $24n$ facets, and $9216kmn$ flags, for all $n \geq 1$ whenever $k, m \in \{1, 2, 4\}$
- Each facet is a directly regular 4-polytope of type $\{3k, 3m, 4\}$ with $384km$ automorphisms
- Each vertex-figure is a chiral 4-polytope of type $\{3m, 4, 6n\}$ with $576mn$ automorphisms, as in §4.5 above (but with m restricted to $\{1, 2, 4\}$).

4.20 Chiral 5-polytopes of type $\{3m, 6, 6, 3n\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), v^6, w^6, [u, v^2], [x, w^2], (v^{-1}w)^4, u^{-1}vw^3u^{-1}w^2vw^{-1} \,\rangle$
- $N = \langle\, u^3, x^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 5184
- $\mathcal{Q}^{(m,n)}$ has $3m$ vertices, $144m$ edges, 288 2-faces, $144n$ 3-faces, $3n$ facets, and $10368mn$ flags, for all $m, n \geq 1$,
- Each facet is a chiral 4-polytope of type $\{3m, 6, 6\}$ with $1728m$ automorphisms, as in §4.12 above (with m in place of n)
- Each vertex-figure is a chiral 4-polytope of type $\{6, 6, 3n\}$ with $1728n$ automorphisms, dual to the mirror image of the one of type $\{3n, 6, 6\}$ in §4.12 above
- $\mathcal{Q}^{(n,n)}$ is improperly self-dual, for all $n \geq 1$.

4.21 Chiral 5-polytopes of type $\{3m, 4, 6, 3n\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), v^4, w^6, (v^{-1}u^2)^2, (w^{-1}x^2)^2, (v^{-1}w)^4,$
  $u^{-1}v^{-1}w^2vuw^2, wx^{-1}w^{-2}xw^3xw^{-2}x^{-1} \,\rangle$
- $N = \langle\, u^3, x^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 6912
- $\mathcal{Q}^{(m,n)}$ has $6m$ vertices, $48m$ edges, 384 2-faces, $144n$ 3-faces, $12n$ facets, and $13824mn$ flags, for all $m, n \geq 1$,
- Each facet is a chiral 4-polytope of type $\{3m, 4, 6\}$ with $576m$ automorphisms, isomorphic to the one of type $\{3m, 4, 6\}$ in §4.5 above
- Each vertex-figure is a directly regular 4-polytope of type $\{4, 6, 3n\}$ with $2304n$ automorphisms.

4.22 Chiral 5-polytopes of type $\{3, 3m, 8, 3n\}$ for $m = 1, 2, 4$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), u^3, v^{12}, w^8, (w^{-1}v^2)^2, (w^{-1}x^2)^2, [u, v^3], (v^{-1}w^3)^2, (x^{-1}w^3)^2,$
  $vxwvxwvw^{-1}vxw^{-1}x \,\rangle$
- $N = \langle\, v^3, x^3 \,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}$, with quotient U/N of order 9216
- $\mathcal{Q}^{(m,n)}$ has 16 vertices, $96m$ edges, $512m$ 2-faces, $384n$ 3-faces, $24n$ facets, and $18432mn$ flags, for all $n \geq 1$ when $m = 1, 2$ or 4
- Each facet is a directly regular 4-polytope of type $\{3, 3m, 8\}$ with $768m$ automorphisms

- Each vertex-figure is a chiral 4-polytope of type $\{3m, 8, 3n\}$ with $576mn$ automorphisms, as in §4.6 above (but with m restricted to $\{1, 2, 4\}$).

4.23 Chiral 5-polytopes of type $\{3m, 3, 8, 3n\}$ for $m = 1, 2, 4$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), v^3, w^8, [v, u^3], (w^{-1}x^2)^2, (x^{-1}w^3)^2, vxwvxwvw^{-1}vxw^{-1}x \,\rangle$
- $N = \langle\, u^3, x^3 \,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}$, with quotient U/N of order 9216
- $\mathcal{Q}^{(m,n)}$ has $16m$ vertices, $96m$ edges, 512 2-faces, $384n$ 3-faces, $24n$ facets, and $18432mn$ flags, for all $n \geq 1$ when $m = 1, 2$ or 4
- Each facet is a directly regular 4-polytope of type $\{3m, 3, 8\}$ with $768m$ automorphisms
- Each vertex-figure is a chiral 4-polytope of type $\{3, 8, 3n\}$ with $576n$ automorphisms, isomorphic to the one of type $\{3, 8, 3n\}$ in §4.6 above.

4.24 Chiral 5-polytopes of type $\{3m, 4, 4, 3n\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), v^4, w^4, (v^{-1}u^2)^2, (w^{-1}x^2)^2, vw^{-1}vw^2v^2w^{-1}, uwvxw^{-1}vu^{-1}vxw^2v \,\rangle$
- $N = \langle\, u^3, x^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 12960
- $\mathcal{Q}^{(m,n)}$ has $18m$ vertices, $270m$ edges, 720 2-faces, $270n$ 3-faces, $18n$ facets, and $25920mn$ flags, for all $m, n \geq 1$
- Each facet is a chiral 4-polytope of type $\{3m, 4, 4\}$ with $720m$ automorphisms, dual to the mirror image of the one of type $\{4, 4, 3m\}$ in §4.7 above
- Each vertex-figure is a chiral 4-polytope of type $\{4, 4, 3n\}$ with $720n$ automorphisms, as in §4.7 above
- $\mathcal{Q}^{(n,n)}$ is improperly self-dual for all $n \geq 1$.

4.25 Chiral 5-polytopes of type $\{3m, 4, 12, 3n\}$

- $U = \langle\, u, v, w, x \mid \mathcal{R}(5), v^4, w^{12}, (v^{-1}u^2)^2, (w^{-1}x^2)^2, (v^{-1}w)^4, [x, w^6], v^{-1}w^{-1}vxwv^{-2}x,$
  $u^{-1}wvw^{-1}vuw^{-2}, wx^{-1}w^2x^{-1}w^3x^{-1}w^2x^{-1} \,\rangle$
- $N = \langle\, u^3, x^3 \,\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, with quotient U/N of order 13824
- $\mathcal{Q}^{(m,n)}$ has $6m$ vertices, $48m$ edges, 768 2-faces, $288n$ 3-faces, $12n$ facets, and $27648mn$ flags, for all $m, n \geq 1$
- Each facet is a chiral 4-polytope of type $\{3m, 4, 12\}$ with $1152m$ automorphisms, isomorphic to the one of type $\{3m, 4, 12\}$ in §4.5 above
- Each vertex-figure is a directly regular 4-polytope of type $\{4, 12, 3n\}$ with $4608n$ automorphisms.

4.26 Chiral 6-polytopes of type $\{4m, 3, 8, 8, 4n\}$ for $m = 1$ or 2

- $U = \langle\, u, v, w, x, y \mid \mathcal{R}(6), v^3, w^8, x^8, (v^{-1}u^3)^2, u^2w^2u^{-2}w^2, vxw^{-1}vx^{-2}wx^{-1}, (w^2x^{-2})^2,$
  $(wyx^{-1})^2, w^{-1}y^{-1}wyx^2 \,\rangle$
- $N = \langle\, u^4, y^4 \,\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}$, with quotient U/N of order 24576
- $\mathcal{Q}^{(m,n)}$ has $8m$ vertices, $24m$ edges, 96 2-faces, 128 3-faces, $32n$ 4-faces, $4n$ facets, and $49152mn$ flags, for all $n \geq 1$, when $m = 1$ or 2

- Each facet is a chiral 5-polytope of type $\{4m, 3, 8, 8\}$ with $6144m$ automorphisms
- Each vertex-figure is a chiral 5-polytope of type $\{3, 8, 8, 4n\}$ with $3072n$ automorphisms, as in §4.17 above.

4.27 Chiral 6-polytopes of type $\{3k, 3m, 4, 6, 3n\}$ for $k, m = 1, 2, 4$

- $U = \langle\, u, v, w, x, y \mid \mathcal{R}(6), u^{12}, v^{12}, w^4, x^6, (w^{-1}v^2)^2, (x^{-1}y^2)^2, [v, u^3], [u, v^3], (w^{-1}x)^4, v^{-1}w^{-1}x^2wvx^2, xy^{-1}x^{-2}yx^3yx^{-2}y^{-1}\,\rangle$
- $N = \langle\, u^3, v^3, y^3\,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$, with quotient U/N of order 55296
- $\mathcal{Q}^{(k,m,n)}$ has $8k$ vertices, $24km$ edges, $128m$ 2-faces, 768 3-faces, $144n$ 4-faces, $12n$ facets, and $110592kmn$ flags, for all $n \geq 1$ when $k = 1, 2$ or 4 and $m = 1, 2$ or 4
- Each facet is a chiral 5-polytope of type $\{3k, 3m, 4, 6\}$ with $4608km$ automorphisms
- Each vertex-figure is a chiral 5-polytope of type $\{3m, 4, 6, 3n\}$ with $6912mn$ automorphisms, as in §4.21 above (but with m restricted to $\{1, 2, 4\}$).

4.28 Chiral 6-polytopes of type $\{3, 3, 4, 12, 3n\}$

- $U = \langle\, u, v, w, x, y \mid \mathcal{R}(6), u^3, v^3, w^4, x^{12}, [v, u^3], (w^{-1}x)^4, (x^{-1}y^2)^2, [y, x^6], v^{-1}w^{-1}x^2wvx^2, xy^{-1}x^2y^{-1}x^3y^{-1}x^2y^{-1}\,\rangle$
- $N = \langle\, y^3\,\rangle \cong \mathbb{Z}$, with quotient U/N of order 110592
- $\mathcal{Q}^{(n)}$ has 8 vertices, 24 edges, 128 2-faces, 1536 3-faces, $288n$ 4-faces, $12n$ facets, and $221184n$ flags, for all $n \geq 1$
- Each facet is a chiral 5-polytope of type $\{3, 3, 4, 12\}$ with 9216 automorphisms
- Each vertex-figure is a chiral 5-polytope of type $\{3, 4, 12, 3n\}$ with $13824n$ automorphisms, isomorphic to the one of type $\{3, 4, 12, 3n\}$ in §4.25 above.

4.29 Chiral 6-polytopes of type $\{3m, 3, 8, 3, 3n\}$ for $m, n = 1, 2, 4$

- $U = \langle\, u, v, w, x, y \mid \mathcal{R}(6), u^{12}, v^3, w^8, x^3, y^{12}, [v, u^3], (x^{-1}w^3)^2, [x, y^3], vxwvxwvw^{-1}vxw^{-1}x\,\rangle$
- $N = \langle\, u^3, y^3\,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$, with quotient U/N of order 294912
- $\mathcal{Q}^{(m,n)}$ has $32m$ vertices, $384m$ edges, 4096 2-faces, 4096 3-faces, $384n$ 4-faces, $32n$ facets, and $589824mn$ flags, for all $m, n \in \{1, 2, 4\}$
- Each facet is a chiral 5-polytope of type $\{3m, 3, 8, 3\}$ with $9216m$ automorphisms, as in §4.23 above (with $n = 1$)
- Each vertex-figure is a chiral 5-polytope of type $\{3, 8, 3, 3n\}$ with $9216n$ automorphisms, dual to the polytope of type $\{3n, 3, 8, 3\}$ from §4.23 above
- $\mathcal{Q}^{(n,n)}$ is properly self-dual for all $n \in \{1, 2, 4\}$
- $\mathcal{Q}^{(1,1)}$ is currently the smallest known self-dual chiral polytope of rank 6; see [10].

4.30 Chiral 6-polytopes of type $\{3, 3m, 8, 3n, 3\}$ for $m, n = 1, 2, 4$

- $U = \langle\, u, v, w, x, y \mid \mathcal{R}(6), u^3, v^{12}, w^8, x^{12}, y^3, [u, v^3], (w^{-1}v^2)^2, [v, w^4], [x, w^4], (w^{-1}x^2)^2, [y, x^3], vxwvxwvw^{-1}vxw^{-1}x\,\rangle$

- $N = \langle\, v^3, x^3 \,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$, with quotient U/N of order 294912
- $\mathcal{Q}^{(m,n)}$ has 32 vertices, $384m$ edges, $4096m$ 2-faces, $4096n$ 3-faces, $384n$ 4-faces, 32 facets, and $589824mn$ flags, for all $m, n \in \{1, 2, 4\}$
- Each facet is a chiral 5-polytope of type $\{3, 3m, 8, 3n\}$ with $9216mn$ automorphisms, as in §4.22 above
- Each vertex-figure is a chiral 5-polytope of type $\{3m, 8, 3n, 3\}$ with $9216mn$ automorphisms, dual to the polytope of type $\{3, 3n, 8, 3m\}$ from §4.22 above
- $\mathcal{Q}^{(n,n)}$ is properly self-dual for all $n \in \{1, 2, 4\}$.

4.31 Chiral 6-polytopes of type $\{3m, 3, 8, 3n, 3\}$ for $m, n = 1, 2, 4$

- $U = \langle\, u, v, w, x, y \mid \mathcal{R}(6), u^{12}, v^3, w^8, x^{12}, y^3, [v, u^3], (w^{-1}v^2)^2, [x, w^4], (w^{-1}x^2)^2, [y, x^3], vxwvxwvw^{-1}vxw^{-1}x \,\rangle$
- $N = \langle\, u^3, x^3 \,\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$, with quotient U/N of order 294912
- $\mathcal{Q}^{(m,n)}$ has $32m$ vertices, $384m$ edges, 4096 2-faces, $4096n$ 3-faces, $384n$ 4-faces, 32 facets, and $589824mn$ flags, for all $m, n \in \{1, 2, 4\}$
- Each facet is a chiral 5-polytope of type $\{3m, 3, 8, 3n\}$ with $9216mn$ automorphisms, as in §4.23 above
- Each vertex-figure is a chiral 5-polytope of type $\{3, 8, 3n, 3\}$ with $9216n$ automorphisms, dual to the polytope of type $\{3, 3n, 8, 3\}$ from §4.22 above.

Acknowledgements

We are grateful to Dimitri Leemans and Daniel Pellicer for helpful discussions, and to the N.Z. Marsden Fund for financial support (via grants UOA1218 and UOA1323). We also acknowledge the extensive use of the Magma system [1] in helping conduct experiments, find examples and test conjectures in much of the work that produced the outcomes described in this paper.

References

[1] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: the user language, *J. Symbolic Comput.* 24 (1997), 235–265.

[2] A. Breda, G. Jones and E. Schulte, Constructions of chiral polytopes of small rank, *Canad. J. Math.* 63 (2011), 1254–1283.

[3] E. Bujalance, M. Conder and A. Costa, Psendo-real Riemann surfaces and chiral regular maps, *Trans. Amer. Math. Soc.* 7 (2010) 3365–3376.

[4] M. D. E. Conder, The smallest regular polytopes of given rank, *Adv. Math.* 236 (2013), 92–110.

[5] M. D. E. Conder, 'Chiral rotary maps of genus 2 to 301', available at `www.math.auckland.ac.nz/~conder/ChiralMaps301.txt`.

[6] M. D. E. Conder, 'Chiral polytopes with up to 4000 flags', available at `www.math.auckland.ac.nz/~conder/ChiralPolytopesWithUpTo4000Flags-ByOrder.txt`.

[7] M. D. E. Conder and P. Dobcsányi, Determination of all regular maps of small genus, *J. Combin. Theory Ser. B* 81 (2001), 224–242.

[8] M. D. E. Conder, I. Hubard and T. Pisanski, Construction for chiral polytopes, *J. London Math. Soc.* 77 (2008), 115–129.

[9] M. D. E. Conder, I. Hubard, E. O'Reilly Regueiro and D. Pellicer, Construction of chiral 4-polytopes with an alternating or symmetric group as automorphism group, *J. Algebraic Combinatorics* 42 (2015), 225–244.

[10] M. D. E. Conder and W.-J. Zhang, The smallest chiral 6-polytopes, *preprint*.

[11] H. S. M. Coxeter, Configurations and maps, *Rep. Math. Colloquium* 2 (1948), 18–38.

[12] H. S. M. Coxeter and W.O.J. Moser, *Generators and Relations for Discrete Groups*, 4th edition, Springer, Berlin, 1980.

[13] H. S. M. Coxeter and A.I. Weiss, Twisted honeycombs $\{3,5,3\}_t$ and their groups, *Geom. Dedicata* 17 (1984), 169–179.

[14] G. Cunningham and D. Pellicer, Chiral extensions of chiral polytopes, *Discrete Math.* 330 (2014), 51–60.

[15] S. Doro and S. Wilson, Rotary maps of type $\{6,6\}_4$, *Quarterly J. Math.* 2 (180), 403–414.

[16] D. Garbe, Über die Regularen Zerlegungen geschlossener orientierbarer Flächen, *J. Reine. Angw. Math.* 237 (1969), 39–55.

[17] L. Heffter, Über Metacyklische Gruppen und Nachbarconfigurationen, *Math. Ann.* 50 (1898), 261–268.

[18] I. Hubard and A. I. Weiss, Self-duality of chiral polytopes, *J. Combin. Theory Ser. A* 111 (2005), 128–136.

[19] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, Encyclopedia Math. Appl., vol. 92, Cambridge University Press, 2002.

[20] B. R. Monson and A.I. Weiss, Regular 4-polytopes related to general orthogonal groups, *Mathematika* 37 (1990), 106–118.

[21] B. Nostrand and E. Schulte, Chiral polytopes from hyperbolic honeycombs, *Comput. Geom.* 13 (1995), 17–39.

[22] D. Pellicer, A construction of higher rank chiral polytopes. *Discrete Math.* 310 (2010), 1222–1237.

[23] D. Pellicer, Developments and open problems on chiral polytopes, *Ars Math. Contemp.* 5 (2012), 333–354.

[24] D. Pellicer and A. I. Weiss, Generalized CPR-graphs and applications, *Contrib. Discrete Math.* 5 (2010), 76–105.

[25] E. Schulte, Chiral polyhedra in ordinary space I, *Discrete and Computational Geometry* 32 (2004), 55–99.

[26] E. Schulte and A. I. Weiss, Chiral polytopes, in: *Applied Geometry and Discrete Mathematics*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., Vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 493–516.

[27] E. Schulte and A. I. Weiss, Chirality and projective linear groups, *Discrete Math.* 131 (1994), 221–261.

[28] E. Schulte and A. I. Weiss, Free extensions of chiral polytopes, *Canad. J. Math.* 47 (1995), 641–654.

[29] F. A. Sherk, A family of regular maps of type $\{6,6\}$, *Canad. Math. Bull.* 5 (1962), 13–20.

[30] C. Weber and H. Seifert, Die beiden Dodekaederräume, *Math. Z.* 37(1933), 237–253.