On automorphism groups of Riemann double covers of Klein surfaces

E. Bujalance[*], F.J. Cirre[†] and M.D.E. Conder[‡]

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Abstract

If G is a group of automorphisms of a compact Klein surface X, then the direct product $G \times C_2$ is a group of automorphisms of the Riemann double cover X^+ of X. In this paper we analyse the relationship between G and the full groups of automorphisms $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X^+)$ of X and X^+ respectively, in the special case where the group G is uniformised by a non-Euclidean crystallographic group with quadrangular signature $(2, 2, 2, n)$. There is a difference in what happens between bordered surfaces and unbordered non-orientable surfaces, and so we consider those cases separately (including the special situation for $n = 4$ in the unbordered case).

1 Introduction

Let X be a compact Klein surface of algebraic genus $g \geq 2$, and let X^+ be its Riemann double cover. As explained in [1], this means that X^+ is a compact Riemann surface of genus g which admits an anticonformal involution $\tau : X^+ \to X^+$, such that the orbit space X^+/τ is a compact Klein surface isomorphic to X. In terms of algebraic geometry, X is a real algebraic curve and X^+ is its complexification. It is well known that the full group $\mathrm{Aut}(X)$ of automorphisms of X is isomorphic to the group of all conformal automorphisms of X^+ that commute with τ. Hence the full group $\mathrm{Aut}(X^+)$ of all automorphisms of X^+ (conformal or anticonformal) contains the direct product $\mathrm{Aut}(X) \times C_2$ where C_2 is the cyclic group generated by the anticonformal involution τ; see Proposition 2.1 below. The following question arises naturally: Is $\mathrm{Aut}(X^+)$ equal to $\mathrm{Aut}(X) \times C_2$, or does X^+ admit additional automorphisms?

This question has been considered for bordered Klein surfaces X with the largest possible number of automorphisms, namely $12(g-1)$, by May in [13], and for those with the

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second largest possible number of automorphisms, namely $8(g-1)$, by Bujalance, Costa, Gromadzki and Singerman in [5]. In both cases the authors showed that the equality $\mathrm{Aut}(X^+) = \mathrm{Aut}(X) \times C_2$ holds for almost all surfaces, with one single exception when $|\mathrm{Aut}(X)| = 12(g-1)$ and five exceptions when $|\mathrm{Aut}(X)| = 8(g-1)$.

Subsequently, Costa and Porto showed in [15] that equality also holds except in a finite number of cases when $|\mathrm{Aut}(X)| > 6(g-1)$, while if $|\mathrm{Aut}(X)| = 6(g-1)$ then there are infinitely many Klein surfaces X (with different topological types) such that $\mathrm{Aut}(X^+) \neq \mathrm{Aut}(X) \times C_2$. If $|\mathrm{Aut}(X)| > 6(g-1)$, then it follows from the Riemann-Hurwitz formula that $\mathrm{Aut}(X)$ is uniformised by a non-Euclidean crystallographic (NEC) group with quadrangular signature $(2, 2, 2, 3)$, $(2, 2, 2, 4)$ or $(2, 2, 2, 5)$. Using techniques of hyperbolic geometry, Costa and Porto investigated the case of signature $(2, 2, 2, n)$ where n is an odd prime, and showed the equality $\mathrm{Aut}(X^+) = \mathrm{Aut}(X) \times C_2$ holds with one single exception (for each odd prime n).

In this paper we consider the general case where a group G of automorphisms of X is uniformised by some NEC group with quadrangular signature $(2, 2, 2, n)$, for an arbitrary value of n greater than 2. For brevity, we will say that in this case G *acts with signature* $(2, 2, 2, n)$ on X. We will investigate the relationship between G and the full automorphism groups $\mathrm{Aut}(X)$ and $\mathrm{Aut}(X^+)$ of X and X^+, respectively. Furthermore, we will consider actions not only on bordered Klein surfaces, but also on unbordered non-orientable surfaces.

In the case of bordered surfaces, we show in Section 3 that if G acts with signature $(2, 2, 2, n)$ and $n \neq 4$ then G is the full group $\mathrm{Aut}(X)$ of automorphisms of X, and that the equality $\mathrm{Aut}(X^+) = \mathrm{Aut}(X) \times C_2$ also holds with one single exception. Accordingly, this extends the work by Costa and Porto to arbitrary values of $n \neq 4$. (When $n = 4$, the equality holds with five exceptions, as shown in [5].)

The situation is rather different for unbordered non-orientable Klein surfaces, as we show in Sections 4 and 5. In the unbordered case, G need not be the full group $\mathrm{Aut}(X)$, and we give necessary and sufficient conditions for this to happen. Even if G coincides with $\mathrm{Aut}(X)$, it can happen that $\mathrm{Aut}(X^+)$ is strictly larger than $\mathrm{Aut}(X) \times C_2$.

Here we note that groups acting on surfaces with quadrangular signature $(2, 2, 2, n)$ have occurred frequently in the literature. As partly mentioned earlier, the largest conceivable orders of automorphism groups of bordered surfaces of genus g are $12(g-1)$, $8(g-1)$ and $20(g-1)/3$, and these occur for such signatures when $n = 3$, 4 and 5, respectively.

Also group actions of type $(2, 2, 2, n)$ arise in the study of sharp upper bounds for the order of an automorphism group of a bordered Klein surface. For each $g > 1$, let $\mu(g)$ denote the maximum order of the automorphism groups of all bordered Klein surfaces of algebraic genus g. It was shown by May in [12] that in the orientable case, $\mu(g) \geq 4(g+1)$, with $\mu(g) = 4(g+1)$ for infinitely many g, and that the analogous sharp bound in the non-orientable case is $\mu(g) \geq 4g$. These theorems were refined in [4], where it was shown that a group of order $4(g+1)$ comes from an action with quadrangular signature $(2, 2, 2, g + 1)$ for all but three values of g, and that a group of order $4g$ comes from an action with quadrangular signature $(2, 2, 2, 2g)$, again for all but three values of g.

Similarly, group actions with signature $(2,2,2,n)$ were also used in [8] to obtain upper bounds for the maximum order $\nu(g)$ of the automorphism groups of unbordered non-orientable surfaces of algebraic genus $g > 1$. Specifically, it was shown by considering group actions with signature $(2,2,2,g+1)$ that if g is even then $\nu(g) \geq 4(g+1)$, and by considering group actions with signature $(2,2,2,4)$ that if g is odd then $\nu(g) \geq 8(g-1)$, and that the latter bound is sharp for infinitely many such g. (The sharpness of the former bound has been established so far for just three values of g, namely $g = 2$, 86 and 206.)

What we present in this paper can also be translated into the language of algebraic geometry. The Riemann double cover X^+ can be seen as a complex algebraic curve which admits an anti-analytic involution τ, and the group G as a group of birational transformations of X^+ that commute with τ. Our results show that in some important cases the (full) automorphism group $\mathrm{Aut}(X^+)$ is isomorphic to $G \times \langle\tau\rangle$, and that the former can be characterised by the properties of G.

2 Preliminaries

In this section we recall the main facts about NEC groups that we need, and draw some conclusions from these that will be helpful later. For a general account of NEC groups and group actions on bordered Klein surfaces, we refer the reader to [6, Sections 0.2 and 1.3].

A non-Euclidean crystallographic group (or more simply, NEC group) is a cocompact discrete subgroup of the group of orientation preserving or reversing isometries of the hyperbolic plane $\mathcal{H}$. The *signature* of an NEC group Γ, as introduced by Macbeath in [11], is a collection of symbols and non-negative integers, of the form

$$\sigma(\Gamma) \;=\; (\gamma;\, \pm;\, [m_1, \ldots, m_r];\, \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})\,. \tag{1}$$

The integers $m_1, \ldots, m_r$ are all greater than 1 and are called *proper periods*; each bracketed expression of the form $(n_{i1}, \ldots, n_{is_i})$ is a *period cycle*; and the integers n_{ij} (which are also greater than 1) are called *link periods.* An empty set of proper periods (where $r = 0$) is denoted by $[-]$, and an empty period-cycle (where $s_i = 0$) by $(-)$.

The signature of Γ gives rise to a presentation for Γ by generators and relations. For the purposes of this paper, it is enough to consider only certain kinds of signatures, but before we describe the most important of those, we need to define a few other terms.

In the action of the NEC group Γ on $\mathcal{H}$, some of the generators may preserve orientation while others reverse it. If all of them preserve orientation, then Γ is a *Fuchsian* group, and otherwise Γ is a *proper* NEC group. More generally, the orientation-preserving elements of Γ constitute what is known as the *canonical Fuchsian subgroup* Γ^+. An element of Γ lies in this subgroup if and only if it can be expressed as a word in the generators of Γ such that the total number of occurrences of orientation-reversing generators is even. Such words may be called *orientable words*, while those in which the latter number is odd are *non-orientable words.* In particular, Γ^+ is the subgroup consisting of all the elements expressible as orientable words, and has index 1 or 2 in Γ, depending on whether Γ is Fuchsian or a proper NEC group, respectively.

The two main kinds of signature we consider are *triangular* and *quadrangular* signatures. These are signatures of the form $(0; +; [-]; \{(n_1, n_2, n_3)\})$ and $(0; +; [-]; \{(n_1, n_2, n_3, n_4)\})$, which we abbreviate to simply (n_1, n_2, n_3) and (n_1, n_2, n_3, n_4) respectively.

Every NEC group with triangular signature (n_1, n_2, n_3) is generated by three elements c_0, c_1, c_2 which act as reflections and satisfy the defining relations

$$c_0{}^2 = c_1{}^2 = c_2{}^2 = (c_0c_1)^{n_1} = (c_1c_2)^{n_2} = (c_2c_0)^{n_3} = 1.$$

The abstract group with this presentation is called the *extended* (n_1, n_2, n_3) *triangle group.* The elements $x_1 = c_0c_1$, $x_2 = c_1c_2$ and $x_3 = c_2c_0$ generate the canonical Fuchsian subgroup Γ^+, and satisfy the relations $x_1x_2x_3 = x_1^{n_1} = x_2^{n_2} = x_3^{n_3} = 1$, which are defining relations for the *ordinary* (n_1, n_2, n_3) *triangle group.*

Analogously, every NEC group with quadrangular signature (n_1, n_2, n_3, n_4) is generated by four elements c_0, c_1, c_2, c_3, all of which act as reflections, subject to the defining relations

$$c_0{}^2 = c_1{}^2 = c_2{}^2 = c_3{}^2 = (c_0c_1)^{n_1} = (c_1c_2)^{n_2} = (c_2c_3)^{n_3} = (c_3c_0)^{n_4} = 1,$$

and the abstract group with this presentation is called the *extended* (n_1, n_2, n_3, n_4) *quadrangle group.* The elements $x_1 = c_0c_1$, $x_2 = c_1c_2$, $x_3 = c_2c_3$ and $x_4 = c_3c_0$ generate Γ^+, and satisfy the relations $x_1x_2x_3x_4 = x_1^{n_1} = x_2^{n_2} = x_3^{n_3} = x_4^{n_4} = 1$, which are defining relations for the *ordinary* (n_1, n_2, n_3, n_4) *quadrangle group.*

Another important type of signature we consider has the form $(0; +; [m]; \{(n)\})$. Every NEC group with this signature is generated by two elements x_1 and c_0, which preserve and reverse orientation, respectively, and satisfy the defining relations

$$x_1^m = c_0{}^2 = [c_0, x_1]^n = 1.$$

Here the elements x_1 and $x_2 = (c_0x_1c_0)^{-1}$ and $x_3 = c_0x_1c_0x_1^{-1}$ generate the canonical Fuchsian subgroup Γ^+, and satisfy the defining relations $x_1x_2x_3 = x_1^m = x_2^m = x_3^n = 1$, which imply that Γ^+ isomorphic to the ordinary (m, m, n) triangle group.

In all the above cases, we will call any set of generators for the NEC group Γ or its Fuchsian subgroup Γ^+ that satisfy the given relations a *canonical set of generators* for the corresponding group.

Next, when the NEC group Γ has signature (1), the area of a fundamental region for Γ is $2\pi\mu(\Gamma)$, where

$$\mu(\Gamma) = \alpha\gamma + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right), \tag{2}$$

with $\alpha = 2$ if the sign is $+$, and $\alpha = 1$ otherwise. The expression $\mu(\Gamma)$ is usually called the *reduced area* of Γ.

If Λ is a subgroup of finite index in Γ, then Λ is also an NEC group, and its area is given by the *Riemann-Hurwitz formula*:

$$\mu(\Lambda) = |\Gamma : \Lambda| \cdot \mu(\Gamma). \tag{3}$$

The NEC group Γ is said to be *maximal* if it is not contained in any other NEC group with finite index. The NEC group signature σ is said to be *maximal* if it has the property that for every NEC group Γ_1 containing an NEC group Γ_2 with signature σ, such that Γ_1 and Γ_2 have the same Teichmüller dimension (the dimension of their Teichmüller spaces, see [6, Section 0.3]), we have $\Gamma_1 = \Gamma_2$.

There is a curious relationship between the notions of maximal signature and maximal NEC group, as explained in [6, Theorem 5.1.2]: Given a maximal NEC group signature σ, there exists a maximal NEC group Λ with $\sigma(\Lambda) = \sigma$. But on the other hand, it can happen that some NEC group Γ whose signature is maximal is contained with finite index in another NEC group Γ'. For triangular signatures, whose Teichmüller dimension is zero, this does not happen; indeed for a given triangular signature σ, either *all* NEC groups with signature σ are maximal, or *none* of them can be.

Now let X be a compact Klein surface of topological genus γ with k boundary components. Then the *algebraic genus* g of X is defined as

$$g = \begin{cases} 2\gamma + k - 1 & \text{if } X \text{ is orientable, or} \\ \gamma + k - 1 & \text{otherwise.} \end{cases} \tag{4}$$

By the Uniformisation Theorem, if $g \geq 2$ then X is isomorphic to the orbit space $\mathcal{H}/\Delta$ for some proper NEC group Δ with signature

$$(\gamma;\ \pm;\ [-];\ \{(-), \overset{k}{\ldots}, (-)\}),$$

where the sign is '$+$' if X is orientable, and '$-$' otherwise. We will call any NEC group Δ with such a signature a *surface NEC group.* Note that the *bordered* and *unbordered* cases can be distinguished according to whether k is positive or zero.

Moreover, a finite group G acts (faithfully) as a group of automorphisms of such a surface $X = \mathcal{H}/\Delta$ if and only if there exist an NEC group Γ and an epimorphism $\theta\colon \Gamma \to G$ whose kernel is the surface NEC group Δ. In this case, we say (for short) that θ is a *smooth epimorphism*, and that *G acts on X with signature* $\sigma(\Gamma)$.

Next, let X^+ be the Riemann double cover of $X = \mathcal{H}/\Delta$, and let τ be the associated anticonformal involution. Then X^+ is isomorphic to $\mathcal{H}/\Delta^+$, where Δ^+ is the canonical Fuchsian subgroup of Δ (and this helps to explain why the superscript '$+$' can be used for both the group Δ^+ and the surface X^+). Observe that Δ^+ is a surface Fuchsian group, and is therefore torsion-free. Also it is easy to see that if G is a group of automorphisms of X, then $G \times \langle\tau\rangle \cong G \times C_2$ is a group of automorphisms of X^+. For later purposes, we prove a slight generalisation of this fact, as follows.

Proposition 2.1 *Under the above conditions, the group* $\mathrm{Aut}(X^+)$ *of all automorphisms of* X^+ *contains a subgroup* $\widetilde{G}$ *isomorphic to the direct product* $G \times C_2$*, where* G *is the subgroup of all conformal automorphisms of* X^+ *in* $\widetilde{G}$*, and the factor* C_2 *is generated by an anticonformal involution. Also* G *and* $\widetilde{G}$ *are uniformised by the same NEC group.*

Proof: Let Γ be the NEC group uniformising G, so that Δ is normal in Γ, and $\Gamma/\Delta \cong G$. Then since conjugation preserves orientability, also Δ^+ is normal in Γ. The quotient group Γ/Δ^+ contains the normal subgroups Γ^+/Δ^+ and Δ/Δ^+, the intersection of which is trivial and the product of which is the whole group Γ/Δ^+, and therefore Γ/Δ^+ is the direct product $\Gamma^+/\Delta^+ \times \Delta/\Delta^+$. The first factor Γ^+/Δ^+ represents orientation preserving elements, and by the second isomorphism theorem for groups,

$$\Gamma^+/\Delta^+ \cong \Gamma^+/(\Gamma^+ \cap \Delta) \cong \Gamma^+\Delta/\Delta = \Gamma/\Delta \cong G.$$

As Δ/Δ^+ has order 2, we find that $\widetilde{G} = \Gamma/\Delta^+$ is a group of automorphisms of $X^+ \cong \mathcal{H}/\Delta^+$ isomorphic to $G \times C_2$, and is uniformised by the same NEC group as G, namely Γ. ∎

In order to determine whether G is a proper subgroup of $\mathrm{Aut}(X)$, or $\widetilde{G} \cong G \times C_2$ is a proper subgroup of $\mathrm{Aut}(X^+)$, by Proposition 2.1 we have to determine whether or not Γ is strictly contained in another NEC group Γ' which also normalises Δ or Δ^+, respectively.

When Γ has quadrangular signature $(2, 2, 2, n)$, the possible signatures for the NEC group Γ' are given by Proposition 2.2 below. In proving it, we make use of some combinatorial group theory, and in particular, methods for finding subgroups of small finite index in finitely-presented groups, and Reidemeister-Schreier theory, which gives presentations for such subgroups and can then determine their abelianisations. Implementations of these methods are available in the MAGMA system [2], via the `LowIndexSubgroups`, `Rewrite` and `AQInvariants` commands. Before proceeding, we note that the abelianisation of the extended $(2, 2, 2, n)$ quadrangle group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if n is even, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if n is odd, while that of the ordinary $(2, 2, 2, n)$ quadrangle group is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if n is even, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if n is odd.

Proposition 2.2 *Suppose the NEC group* Γ *has quadrangular signature* $(2, 2, 2, n)$*, and is contained as a subgroup with finite index in another NEC group* Γ'*. If* $n = 4$*, then* Γ' *has triangular signature* $(2, 4, 8)$, $(2, 4, 6)$, $(2, 4, 5)$ *or* $(2, 3, 8)$*, while if* $n \neq 4$*, then* Γ' *has triangular signature* $(2, 4, 2n)$.

Proof: This is already known when $n = 3$ and $n = 4$ (see [5, Theorems 4.4 and 6.1]), and also when n is a prime greater than 3 (see [15, Lemma 1.1]). Hence we may assume that $n \geq 6$, and n is not prime. We proceed by considering the reduced areas of Γ and Γ'.

By formula (2), we know that $\mu(\Gamma) = \frac{n-2}{4n} = \frac{1}{4} - \frac{1}{2n} < \frac{1}{4}$, and then because $|\Gamma' : \Gamma| \geq 2$, it follows from the Riemann-Hurwitz formula (3) that $\mu(\Gamma') \leq \frac{\mu(\Gamma)}{2} < \frac{1}{8}$.

Now suppose Γ' has signature $(\gamma;\ \pm;\ [m_1, \ldots, m_r];\ \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\})$, so that $\mu(\Gamma') = \alpha\gamma + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right)$. In this formula, the term $\sum_{i=1}^{r}(1-\frac{1}{m_i})$ is at least $\frac{r}{2}$ (since $m_i \geq 2$ for each i). Similarly, the term $\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1-\frac{1}{n_{ij}})$ is at least $\frac{s}{4}$, where s is the total number of link periods n_{ij} in the signature of Γ'. Thus we find that

$$\alpha\gamma + k - 2 + \frac{r}{2} + \frac{s}{4} \ \leq \ \mu(\Gamma') \ < \ \frac{1}{8}\,. \tag{5}$$

Next, we note that $k \geq 1$ and $s \geq 1$, since Γ' contains Γ, which in turn contains pairs of reflections whose product has order 2. Hence the inequality (5) forces $\alpha\gamma + 1 \leq \alpha\gamma + k < \frac{1}{8} + 2 - \frac{r}{2} - \frac{1}{4} < 2$, and from this we deduce that $\gamma = 0$ and $k = 1$, and therefore $\frac{r}{2} + \frac{s}{4} < \frac{1}{8} + 2 - k \leq \frac{9}{8}$. This gives $4r + 2s < 9$, and the only solutions with $s \geq 1$ are $(r,s) = (0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(1,1)$ and $(1,2)$. From these, however, the possibilities $(0,1)$ and $(0,2)$ can be discarded, because they give $\mu(\Gamma') < 0 + 1 - 2 + 0 + \frac{1}{2}(1+1) = 0$, but $\mu(\Gamma')$ must be positive (to give genus $g \geq 2$). Also if $(r,s) = (0,4)$ or $(1,2)$, then by [6, Corollary 2.2.5] the canonical Fuchsian subgroup $(\Gamma')^+$ of Γ' has signature of the form $(0;+;[t_1,t_2,t_3,t_4];\{-\})$, and since its reduced area must satisfy $0 < \mu((\Gamma')^+) = 2\mu(\Gamma') < \frac{1}{4}$, the only possibility is $(0;+;[2,2,2,3];\{-\})$. But a group with this signature does not contain any element of order n (since $n \geq 6$), so these two cases can be eliminated as well. Thus $(r,s) = (0,3)$ or $(1,1)$.

In both of these two cases, $(\Gamma')^+$ has triangular Fuchsian signature $(0;+;[t_1,t_2,t_3];\{-\})$, where at least one period t_i is a multiple of n, and $\frac{1}{4} > \mu((\Gamma')^+) = 1 - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}$.

Now let $d = |(\Gamma')^+ : \Gamma^+|$, and note that Γ^+ has Fuchsian signature $(0;+;[2,2,2,n];\{-\})$, so that $\mu(\Gamma^+) = \frac{1}{2} - \frac{1}{n}$, and therefore $\frac{1}{2} - \frac{1}{n} = \mu(\Gamma^+) = d\mu((\Gamma')^+) = d(1 - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_3}) < \frac{d}{4}$. We may also suppose that $t_1 \leq t_2 \leq t_3$, and then a straightforward arithmetical exercise gives only the possibilities below:

- (a) $[t_1,t_2,t_3] = [2,3,3n]$, with $d = 3$ for arbitrary non-prime $n \geq 6$;
- (b) $[t_1,t_2,t_3] = [2,4,2n]$, with $d = 2$ for arbitrary non-prime $n \geq 6$;
- (c) $[t_1,t_2,t_3] = [2,3,8]$, with $d = 9$ for $n = 8$;
- (d) $[t_1,t_2,t_3] = [2,3,9]$, with $d = 7$ for $n = 9$;
- (e) $[t_1,t_2,t_3] = [2,3,10]$, with $d = 6$ for $n = 10$;
- (f) $[t_1,t_2,t_3] = [2,3,12]$, with $d = 4$ for $n = 6$;
- (g) $[t_1,t_2,t_3] = [2,3,12]$, with $d = 5$ for $n = 12$;
- (h) $[t_1,t_2,t_3] = [2,3,18]$, with $d = 4$ for $n = 18$;
- (i) $[t_1,t_2,t_3] = [2,4,6]$, with $d = 4$ for $n = 6$;
- (j) $[t_1,t_2,t_3] = [2,4,8]$, with $d = 3$ for $n = 8$;
- (k) $[t_1,t_2,t_3] = [2,5,10]$, with $d = 2$ for $n = 10$;
- (l) $[t_1,t_2,t_3] = [2,6,6]$, with $d = 2$ for $n = 6$;
- (m) $[t_1,t_2,t_3] = [3,3,6]$, with $d = 2$ for $n = 6$.

All but the second of these cases can be eliminated with the help of some group theory.

First, we show that $(r,s) \neq (1,1)$. For suppose the contrary. Then Γ' has signature of the form $(0;+;[m];\{(n)\})$, which forces the group $(\Gamma')^+$ to have Fuchsian triangular signature $(0;+;[m,m,n];\{-\})$. The only such possibilities in the above list are $[2,6,6]$ and $[3,3,6]$, with $d=2$ for $n=6$. In the former case, $(\Gamma')^+$ is isomorphic to the ordinary $(6,6,2)$ triangle group. This group has three subgroups of index 2, but their abelianisations are $\mathbb{Z}_3\oplus\mathbb{Z}_6$, $\mathbb{Z}_3\oplus\mathbb{Z}_6$ and $\mathbb{Z}_6$, and so $(\Gamma')^+$ contains no subgroup of index 2 isomorphic to Γ^+ (whose abelianisation is $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$). And in the latter case, Γ^+ is isomorphic to the ordinary $(3,3,6)$ triangle group, which has no subgroup of index 2 at all.

Thus $(r,s)=(0,3)$, and so Γ' has triangular signature.

Next, in some cases the group $(\Gamma')^+$ has no subgroup of index d, or the group Γ' itself has no subgroup of index $|\Gamma':\Gamma|=|(\Gamma')^+:\Gamma^+|=d$. This happens in cases (c), (d), (g), (j) and (m), which can all therefore be ruled out. On the other hand, in cases (e), (f), (h), (i), (k) and (l), both Γ' and $(\Gamma')^+$ have one or more subgroups of the required index d, but none of those subgroups has the same abelianisation as the extended or ordinary $(2,2,2,n)$ quadrangle group, respectively.

In case (a), where $[t_1,t_2,t_3]=[2,3,3n]$, we know that Γ' is isomorphic to the extended $(2,3,3n)$ triangle group. When n is odd, this group has just one conjugacy class of subgroups of index $d=3$, all of which are generated by three involutions, and have abelianisation $\mathbb{Z}_2\oplus\mathbb{Z}_2$, which means they cannot be isomorphic to the extended $(2,2,2,n)$ quadrangle group. On the other hand, when n is even, the extended $(2,3,3n)$ triangle group has two conjugacy classes of such subgroups, all generated by three involutions, and these all have abelianisation $\mathbb{Z}_2\oplus\mathbb{Z}_2$ or $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$, and so once again they cannot be isomorphic to the extended $(2,2,2,n)$ quadrangle group.

Finally we are left with case (b), where $[t_1,t_2,t_3]=[2,4,2n]$. In this case, we find that for each n there is just one subgroup of index $d=2$ in the extended $(2,4,2n)$ triangle NEC group Γ' isomorphic to the extended $(2,2,2,n)$ quadrangle group, namely the subgroup generated by c_0', c_1', $c_2'c_1'c_2'$ and $c_2'c_0'c_2'$, where $\{c_0',c_1',c_2'\}$ is a canonical set of generators for Γ'. (When n is odd, there is one other subgroup of index 2 with the abelianisation $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ required (for odd n), namely the one generated by c_2', $c_1'c_2'c_1'$ and c_0', but this is isomorphic to the extended $(2,2n,2n)$ triangle group.) This completes the proof. ∎

Remark 2.3 The above proof shows that if Γ' is any NEC group with signature $(2,4,2n)$ for $n\geq 2$, then Γ' contains a unique subgroup Γ with signature $(2,2,2,n)$. Here we note that this is the unique subgroup of index 2 containing the first two but not the third of the three canonical generating reflections for Γ'.

Remark 2.4 If G acts on $X=\mathcal{H}/\Delta$ with signature $(2,2,2,n)$ where $n\neq 4$, then the NEC group Γ realising such an action (that is, with $\sigma(\Gamma)=(2,2,2,n)$ and $G\cong\Gamma/\Delta$) is unique. Indeed if the normaliser Γ' of Δ in $\mathrm{PGL}(2,\mathbb{R})$ is not Γ, then Γ' has signature $(2,4,2n)$ by Proposition 2.2, but then Γ' contains a unique NEC group with signature $(2,2,2,n)$, by Remark 2.3.

3 Bordered Klein surfaces with a group acting with signature $(2, 2, 2, n)$

The maximum number of boundary components of a compact Klein surface of algebraic genus $g \geq 2$ is $g + 1$. This is known as Harnack's bound, and surfaces attaining it are called *M-curves.* The Riemann double covers of M-curves were studied by Natanzon in [14]. If X is any M-curve, then by (4) its topological genus satisfies $\alpha\gamma = g - k + 1 = 0$, and so topologically, X is a sphere with $g + 1$ holes. We will say that X is a *regular sphere with $g + 1$ holes* if it admits a group of automorphisms isomorphic to the direct product $D_{g+1} \times C_2$. Except for a small number of values of g, this is the largest group of automorphisms that a sphere with $g + 1$ holes may admit; see [3]. Also a regular sphere with holes is unique within its algebraic genus, and so it makes sense to speak about *the* regular sphere with $g + 1$ holes. We can now prove the following.

Theorem 3.1 *Let X be a compact bordered Klein surface of algebraic genus $g \geq 2$ that admits an automorphism group G acting with signature $(2, 2, 2, n)$, where $n \geq 3$. Also let X^+ be the Riemann double cover of X, and let $\mathrm{Aut}(X^+)$ be the full group of conformal and anticonformal automorphisms of X^+. If $n \neq 4$, then $\mathrm{Aut}(X^+) \cong G \times C_2$ except when X is the regular sphere with $g + 1$ holes, in which case X^+ is the Accola-Maclachlan surface of genus g. On the other hand, if $n = 4$ then also $\mathrm{Aut}(X^+) \cong G \times C_2$, but now there are five exceptions: the regular sphere with holes, and four more surfaces, which topologically are a projective plane with two holes, and a torus with one, two or four holes.*

Proof: This is already known when $n = 3$ and $n = 4$ (see [5, Theorems A and B]), and also when n is a prime greater than 3 (see [15, Theorem 2.1]). Hence we may assume that $n \geq 6$, and n is not prime.

Now let us write $X = \mathcal{H}/\Delta$, where Δ is a surface NEC group. Then $X^+ = \mathcal{H}/\Delta^+$ and $G \cong \Gamma/\Delta$, where Γ is an NEC group with quadrangular signature $(2, 2, 2, n)$, and Δ^+ is the canonical Fuchsian subgroup of Δ. By Proposition 2.1, the full group $\mathrm{Aut}(X^+)$ contains a copy of $\Gamma/\Delta^+ \cong G \times C_2$. We have to show that if $\mathrm{Aut}(X^+)$ strictly contains $G \times C_2$, then X is the regular sphere with $g + 1$ holes.

So suppose that $\mathrm{Aut}(X^+)$ strictly contains $G \times C_2$. Then $\mathrm{Aut}(X^+) \cong \Gamma'/\Delta^+$ where Γ' is an NEC group containing Γ and normalising Δ^+. By Proposition 2.2, we find that Γ' has triangular signature $(2, 4, 2n)$, and in particular, $\mu(\Gamma') = \frac{1}{8} - \frac{1}{4n}$.

Next, let $\{c'_0, c'_1, c'_2\}$ be a canonical set of generating reflections of Γ', so that $(c'_0c'_1)^2 = (c'_1c'_2)^4 = (c'_2c'_0)^{2n} = 1$. Then by our proof of Proposition 2.2, taking

$$c_0 = c'_0, \quad c_1 = c'_1, \quad c_2 = c'_2c'_1c'_2 \quad \text{and} \quad c_3 = c'_2c'_0c'_2 \tag{6}$$

gives a canonical set of generating reflections of an NEC group with quadrangular signature $(2, 2, 2, n)$, satisfying $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^n = 1$. Moreover, by Remark 2.3, this NEC group has to be Γ, since Γ' contains only one subgroup with signature $(2, 2, 2, n)$.

Next, let $\theta : \Gamma \to \Gamma/\Delta$ and $\theta' : \Gamma' \to \Gamma'/\Delta^+$ be the canonical smooth epimorphisms. Since $\Delta = \ker\theta$ is a bordered surface NEC group, it must contain at least one of the generating reflections c_i of Γ. If $c_0 \in \Delta$ then also $(c_0c_3)^2 = c_0\, c_0^{c_3} \in \Delta$, which is impossible because Δ has no non-trivial orientation preserving elements of finite order. Thus $c_0 \notin \Delta$, and similarly $c_3 \notin \Delta$. Hence Δ contains either c_1 or c_2, but not both (since otherwise Δ contains c_1c_2, of order 2). Without loss of generality we may assume that $c_1 \in \Delta$, since the other possibility $c_2 \in \Delta$ can be achieved by conjugating by c_2' (and then gives an isomorphic surface). Then since $c_1 \in \Delta$ we have $(c_1c_3)^2 = c_1\, c_1^{c_3} \in \Delta$, indeed $(c_1c_3)^2 \in \Delta^+$, and so by (6) we have

$$(c_1'c_2'c_0'c_2')^2 \in \Delta^+. \tag{7}$$

Now let a, b and c be the images in Γ'/Δ^+ of c_0', c_1' and c_2' respectively. Since Δ^+ is torsion-free, these elements are involutions, and ab, bc and ac have orders 2, 4 and $2n$ respectively. Also $[b, cac] = (bcac)^2 = 1$ by (7), but $bcac \neq 1$ since otherwise $ab = (ac)^2$, which is impossible because ac has order $2n > 4$. Thus $bcac$ is an involution in Γ'/Δ^+.

Here we note that b commutes with cac and hence also with $a(cac) = (ac)^2$, and it follows that the cyclic group N of order n generated by $(ac)^2$ is normal in Γ'/Δ^+ (with each of a and c conjugating $(ac)^2$ to its inverse). The factor group $(\Gamma'/\Delta^+)/N$ is generated by three involutions, such that one (namely Na) commutes with each of the other two, and the product of those two (Nb and Nc) has order 4. Thus $(\Gamma'/\Delta^+)/N$ is isomorphic to $D_4 \times C_2$, of order 16, and it follows that $\mathrm{Aut}(X^+) \cong \Gamma'/\Delta^+$ has order $16n$, and presentation

$$\Gamma'/\Delta^+ = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^4 = (ac)^{2n} = (bcac)^2 = 1 \,\rangle. \tag{8}$$

Since Δ^+ is a Fuchsian surface group, we have $\mu(\Delta^+) = 2g-2$, where g is the algebraic genus of X, and then by the Riemann-Hurwitz formula (3) we find that

$$2g-2 = 16n \cdot \mu(\Gamma') = 16n \cdot (\frac{1}{8} - \frac{1}{4n}) = 2(n-2),$$

and therefore $g = n-1$. Also by [6, Theorem 2.3.3] and since $acbc$ is conjugate to $bcac$ and therefore has order 2, the number of boundary components of X is

$$\frac{1}{2} \cdot \frac{|\Gamma/\Delta|}{\text{order of } \theta(c_0c_2)} = \frac{1}{2} \cdot \frac{4(g+1)}{\text{order of } (acbc)} = g+1.$$

Thus X is a sphere with holes, as required. To determine the full group $\mathrm{Aut}(X)$, we observe that

$$X \cong \mathcal{H}/\Delta \cong (\mathcal{H}/\Delta^+)/(\Delta/\Delta^+) \cong X^+/\langle c_1\Delta^+\rangle \cong X^+/\langle c_1'\Delta^+\rangle \cong X^+/\langle b\rangle.$$

In particular, it follows that $\mathrm{Aut}(X)$ consists of the conformal automorphisms in $\mathrm{Aut}(X^+)$ that commute with b. It is easy to see that $(ac)^2$, ab and $(bc)^2$ commute with b. Also since $ab(ac)^2 = abacac = bcac$ which has order 2, the first two of these generate a dihedral subgroup of order $2n = 2(g+1)$, while the third centralises $(ac)^2$ (by observations made above about the normal subgroup N), and the third also centralises ab (by an easy exercise).

Hence $(ac)^2$, ab and $(bc)^2$ generate a subgroup of order $4(g+1)$ isomorphic to $D_{g+1} \times C_2$, and of index 4 in $\langle a, b, c\rangle = \Gamma'/\Delta^+$. On the other hand, ac does not commute with b, since otherwise $1 = [b, ca] = bcabac = bcbc = (bc)^2$, contradicting the fact that bc has order 4.

We conclude that $\mathrm{Aut}(X) \cong \langle (ac)^2, ab, (bc)^2\rangle \cong D_{g+1} \times C_2$, and hence that X is the regular sphere with $g+1$ holes.

Finally, the double cover X^+ of X is a compact Riemann surface of genus g with $\mathrm{Aut}(X^+)$ having order $16(g+1)$ and presentation (8) with $n = g+1$. Any such surface is isomorphic to the Accola-Maclachlan surface of genus g, given by the algebraic equation $w^2 = z^{2g+2} - 1$; see [10]. This completes the proof. ∎

Another consequence of part of the above proof is the following.

Corollary 3.2 *Let X be a compact bordered Klein surface of algebraic genus $g \geq 2$ which admits an automorphism group G acting with signature $(2, 2, 2, n)$ where $n \geq 3$. Then G is the full automorphism group of X.*

Proof: With the same notation as in the proof of Theorem 3.1, suppose that $G = \Gamma/\Delta$ is not (isomorphic to) the full group $\mathrm{Aut}(X)$ of $X = \mathcal{H}/\Delta$. Then $\mathrm{Aut}(X) \cong \Gamma'/\Delta$, where Γ' is an NEC group containing Γ with finite index, and normalising Δ. By Proposition 2.2, the group Γ' would have triangular signature $(2, 4, 2n)$, $(2, 4, 6)$, $(2, 4, 5)$ or $(2, 3, 8)$. Repeating the arguments in the beginning of the proof of Theorem 3.1, however, we see that for each of these signatures, none of the generating reflections c'_i of Γ' can lie in the surface NEC group Δ. This contradicts the fact that X is a bordered surface. ∎

In contrast, we will see in Example 4.4 below that for unbordered non-orientable Klein surfaces, the full group $\mathrm{Aut}(X)$ can sometimes be strictly larger than G.

4 Unbordered Klein surfaces with a group acting with signature $(2, 2, 2, n)$, where $n \neq 4$

We now turn to the case of *unbordered* non-orientable Klein surfaces. Let $X = \mathcal{H}/\Delta$ be such a surface, where Δ is an unbordered non-orientable surface NEC group, and let $G = \Gamma/\Delta$ be a group of automorphisms of X acting with signature $(2, 2, 2, n)$. Then G admits a (partial) presentation of the form

$$G = \langle\, a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = (ad)^n = w = \ldots = 1 \,\rangle \qquad (9)$$

where $w = w(a, b, c, d)$ is a 'non-orientable' word, with odd length greater than 1, in the generators a, b, c, d. (Note that if c_0, c_1, c_2 and c_3 constitute a canonical set of generating reflections for Γ, then their images a, b, c and d under the smooth epimorphism $\theta : \Gamma \to G$ must satisfy the same relations as the reflections c_i, because Δ is torsion free. Also since

Δ is non-orientable, it must contain a non-orientable word of Γ, and so such a word has odd length greater than 1 in the generators c_0, c_1, c_2, c_3, and then its image under θ is a word w of odd length greater than 1 in a, b, c, d.)

A presentation of G of the form (9) induced by a smooth epimorphism $\theta : \Gamma \to G$ will be called a (partial) *monodromy presentation.* For any such presentation, we will consider the effect of interchanging the generators a and b with the generators d and c respectively, which we will refer to simply as the assignment

$$a \leftrightarrow d, \qquad b \leftrightarrow c. \tag{10}$$

Theorem 4.1 *Let X be a compact unbordered non-orientable Klein surface of algebraic genus $g \geq 2$, which admits an automorphism group G acting with signature $(2, 2, 2, n)$, where $n \neq 4$. Let X^+ be the Riemann double cover of X. Then the condition*

(i) $\mathrm{Aut}(X)$ *strictly contains* G

is equivalent to the combination of the two conditions

(ii) $\mathrm{Aut}(X^+)$ *strictly contains* $G \times C_2$,

(iii) *every monodromy presentation* (9) *for G is preserved by the assignment* (10).

Proof: First, we have $X \cong \mathcal{H}/\Delta$ and $G \cong \Gamma/\Delta$ with $\sigma(\Gamma) = (2, 2, 2, n)$, and also we know that Γ is unique, by Remark 2.4.

Now suppose that $\mathrm{Aut}(X)$ strictly contains G. Then $\mathrm{Aut}(X^+)$ strictly contains $G \times C_2$, by Proposition 2.1. Hence (i) implies (ii).

Moreover, $\mathrm{Aut}(X) \cong \Gamma'/\Delta$ where Γ' is an NEC group containing Γ and normalising Δ, and then Γ' also normalises Δ^+, since conjugation preserves orientability. By Proposition 2.2, we find that that Γ' has triangular signature $(2, 4, 2n)$. Also if $\{c'_0, c'_1, c'_2\}$ is a canonical set of generating reflections of Γ' satisfying $(c'_0c'_1)^2 = (c'_1c'_2)^4 = (c'_2c'_0)^{2n} = 1$, then

$$c_0 = c'_0, \quad c_1 = c'_1, \quad c_2 = c'_2c'_1c'_2 \quad \text{and} \quad c_3 = c'_2c'_0c'_2 \tag{11}$$

make up a canonical set of generating reflections for an NEC group with quadrangular signature $(2, 2, 2, n)$, and this NEC group has to be Γ, by Remark 2.3. The respective images a, b, c and d of c_0, c_1, c_2 and c_3 under the smooth epimorphism $\theta : \Gamma \to \Gamma/\Delta = G$ satisfy the relations in (9), and so give a monodromy presentation of G. Also both Γ and Δ are normal subgroups in Γ', and therefore conjugation by any element of Γ' induces an automorphism of $G = \Gamma/\Delta$. In particular, conjugation by c'_2 induces an automorphism of G that takes a, b, c and d respectively to $\theta(c'_2c_0c'_2) = \theta(c_3) = d$, $\theta(c'_2c_1c'_2) = \theta(c_2) = c$, $\theta(c'_2c_2c'_2) = \theta(c_1) = b$ and $\theta(c'_2c_3c'_2) = \theta(c_0) = a$. Hence this monodromy presentation is preserved by the assignment (10).

Next, consider an arbitrary monodromy presentation of G. For this, there exists a canonical set of generating reflections $\hat{c}_0$, $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ of Γ, such that their images in G satisfy the same form of relations as in the partial presentation (9). But there exists a unique embedding of Γ in Γ', up to conjugacy in Γ', and so there exists some element $\gamma' \in \Gamma'$

such that $\hat{c}_i = \gamma'^{-1} c_i \gamma'$ for $0 \leq i \leq 3$, where the reflections c_i are given by (11). It then follows that conjugation by $\gamma'^{-1} c_2' \gamma'$ induces an automorphism of G which interchanges the images of $\hat{c}_0$ and $\hat{c}_1$ with the images of $\hat{c}_3$ and $\hat{c}_2$, and so the given monodromy presentation is preserved by the assignment (10) as well.

(In fact, since conjugation by γ' takes the generators c_i for Γ to the generators $\hat{c}_i$ for Γ, the given monodromy presentation for G satisfied by the images of the $\hat{c}_i$ is equivalent to the one satisfied by a, b, c and d.)

Conversely, suppose that (ii) and (iii) both hold. We have to prove that $\mathrm{Aut}(X)$ strictly contains $G = \Gamma/\Delta$. By (ii), we know that $\mathrm{Aut}(X^+)$ strictly contains $G \times C_2 \cong \Gamma/\Delta^+$, and so there exists an NEC group Γ' strictly containing Γ and normalising Δ^+. It now suffices to show that Γ' also normalises Δ, because in that case, Γ'/Δ will be a group of automorphisms of X strictly larger than Γ/Δ. Here we note that $\Gamma' = \langle \Gamma, c_2' \rangle$, and that Γ normalises Δ; hence it suffices to show that c_2' normalises Δ. So with the same notation as used above, let $\delta = w(c_0, c_1, c_2, c_3) \in \Delta$ be a non-orientable word (of odd length greater than 1) in c_0, c_1, c_2, c_3, such that its image $\theta(\delta) = w = w(a, b, c, d)$ is trivial in G. Then $\Delta = \langle \Delta^+, \delta \rangle$, and since $c_2' \in \Gamma'$ normalises Δ^+, all we have to do is show that $\delta^{c_2'} \in \Delta$, or equivalently, that $\theta(\delta^{c_2'}) = 1$. To do this, observe that

$$\delta^{c_2'} = w(c_0, c_1, c_2, c_3)^{c_2'} = w(c_0^{c_2'}, c_1^{c_2'}, c_2^{c_2'}, c_3^{c_2'}) = w(c_3, c_2, c_1, c_0),$$

and hence that

$$\theta(\delta^{c_2'}) = \theta(w(c_3, c_2, c_1, c_0)) = w(\theta(c_3), \theta(c_2), \theta(c_1), \theta(c_0)) = w(d, c, b, a).$$

But the latter word is the image of $w(a, b, c, d)$ under the assignment (10), and hence must be trivial. Thus $\theta(\delta^{c_2'}) = 1$, so $\delta^{c_2'} \in \Delta$, and therefore Γ' normalises Δ, as required. ∎

Remark 4.2 Condition (iii) in Theorem 4.1 does not imply condition (i), or condition (ii). Indeed in *most* cases, G is the full group $\mathrm{Aut}(X)$, and $G \times C_2$ is the full group $\mathrm{Aut}(X^+)$, regardless of any automorphisms that G might have, because in most cases an NEC group with signature $(2, 2, 2, n)$ is maximal.

Remark 4.3 For $n \neq 4$, the triangular NEC signature $(2, 4, 2n)$ is maximal, since the same is true of its canonical Fuchsian signature $(0; +; [2, 4, 2n]; \{-\})$, as shown in [16] and [6, Remark 5.1.1]. Accordingly, any NEC group Γ' with signature $(2, 4, 2n)$ where $n \neq 4$ is not contained with finite index in any other NEC group. Hence if Γ is as above, with $|\Gamma' : \Gamma| = 2$, we find that if $\mathrm{Aut}(X)$ strictly contains G then $\mathrm{Aut}(X)$ is a C_2-extension of G, and in fact, $\mathrm{Aut}(X) \cong G \rtimes_\phi C_2$ where ϕ is the automorphism of G given by (10). Analogously, if $\mathrm{Aut}(X^+)$ strictly contains $G \times C_2$, then $\mathrm{Aut}(X^+) \cong (G \times C_2) \rtimes_\phi C_2$ where ϕ acts trivially on the C_2-factor of $G \times C_2$.

In Examples 4.4 and 4.5 below, we see that there exist cases where a group G acts with signature $(2, 2, 2, n)$ on an unbordered non-orientable surface, for which the assignment (10) gives an automorphism of G, and other cases for which it does not.

Example 4.4 This example comes from a family of groups acting on non-orientable surfaces given in [7, Example 3.1] (and used again in [8, §3.3.3]).

For each positive integer m, there exists an extension G_m of a cyclic group of order m by the symmetric group S_4, with presentation

$$G_m = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^{3m} = (bc)^4 = (ac)^2 = (ab)^{-1}(cb)^2abc = 1\,\rangle,$$

such that the cyclic normal subgroup of order m is generated by $(ab)^3$. Now in this group, define $\hat{a} = b$, $\hat{b} = (bc)^2$, $\hat{c} = c$ and $\hat{d} = a$. Then G_m has the alternative presentation

$$\langle\, \hat{a}, \hat{b}, \hat{c}, \hat{d} \mid \hat{a}^2 = \hat{b}^2 = \hat{c}^2 = \hat{d}^2 = (\hat{a}\hat{b})^2 = (\hat{b}\hat{c})^2 = (\hat{c}\hat{d})^2 = (\hat{a}\hat{d})^{3m} = \hat{b}(\hat{a}\hat{c})^2 = \hat{c}(\hat{b}\hat{d})^2 = 1\,\rangle.$$

The relator $\hat{b}(\hat{a}\hat{c})^2$ has odd length in the generators $\hat{a}, \hat{b}, \hat{c}, \hat{d}$, and so the group G_m acts with signature $(2, 2, 2, 3m)$ on an unbordered non-orientable Klein surface X. This surface has algebraic genus $g = 6m - 3$, by the Riemann-Hurwitz formula (3).

It is easy to see from the second presentation for G_m that the assignment $\hat{a} \leftrightarrow \hat{d}$, $\hat{b} \leftrightarrow \hat{c}$ gives an automorphism ϕ of G_m. Hence by Theorem 4.1, the action of G_m on X extends to the action of a larger group if and only if the same happens for the action of $G_m \times C_2$ on the Riemann double cover X^+ of X. In turn, this happens if and only if the NEC group Γ with signature $(2, 2, 2, 3m)$ uniformising the action of G_m on X is contained in an NEC group Γ' with signature $(2, 4, 6m)$. If this is the case, then the full group $\mathrm{Aut}(X)$ will be the semidirect product $G' = G \rtimes \langle\phi\rangle$, and $\mathrm{Aut}(X^+)$ will be $G' \times C_2$.

As noted above, this contrasts with the case of bordered surfaces, where the action of the given group G cannot extend (by Corollary 3.2).

Example 4.5 For each integer $n > 1$, let H_n be the group with presentation

$$H_n = \langle\, a, c, d \mid a^2 = c^2 = d^2 = (ac)^2 = (cd)^2 = (ad)^n = 1\,\rangle.$$

Then H_n is isomorphic to the direct product $D_n \times C_2$, of order $4n$, where the D_n factor is generated by a and d, and the C_2 factor is generated by c. Next, let $b = ac$, which is an involution that commutes with a and c. Then H_n has alternative presentation

$$\langle\, a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = (ad)^n = abc = 1\,\rangle,$$

and since the final relator abc has odd length, it follows that H_n acts as a group of automorphisms with signature $(2, 2, 2, n)$ on an unbordered non-orientable Klein surface X. This surface has algebraic genus $g = n - 1$, by the Riemann-Hurwitz formula. Here the assignment $a \leftrightarrow d$, $b \leftrightarrow c$ does not give an automorphism of H_n, since otherwise the relation $b = ac$ would imply that $c = db = dac$, and therefore $ad = 1$, which is impossible Hence H_n is the full group $\mathrm{Aut}(X)$, by Theorem 4.1.

The next example is a family for which $\mathrm{Aut}(X^+)$ strictly contains $G \times C_2$, but $\mathrm{Aut}(X)$ coincides with G. In particular, this shows that in Theorem 4.1, condition (ii) does not imply condition (i).

Example 4.6 For each integer $m \neq 2$, let J_m be the group with presentation

$$J_m = \langle\, a, b, z \mid a^2 = b^2 = z^2 = (ab)^2 = (bz)^4 = (az)^{4m} = (bzaz)^2 = 1\,\rangle.$$

In this group, take $c = zbz$ and $d = zaz$. Then a, b, c and d generate a subgroup K_m of index 2 (containing a and b but not z), with presentation

$$K_m = \langle\, a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (ad)^{2m} = (bd)^2 = (cd)^2 = (ac)^2 = 1\,\rangle.$$

This group is isomorphic to the direct product $D_{2m} \times C_2 \times C_2$, with a and d generating a dihedral subgroup of order $4m$, and b and c being central involutions. In particular, K_m has order $16m$, and so J_m has order $32m$.

Now let Γ' be an NEC group with triangular signature $(2, 4, 4m)$, with canonical generating reflections c_0', c_1' and c_2'. Then the assignment

$$c_0' \mapsto a, \qquad c_1' \mapsto b, \qquad c_2' \mapsto z$$

induces an epimorphism $\theta : \Gamma' \to J_m$ whose kernel Λ is a surface Fuchsian group, and the orbit space $Y = \mathcal{H}/\Lambda$ is a compact Riemann surface, admitting J_m as a group of automorphisms. In fact J_m is the full group $\mathrm{Aut}(Y)$ of all conformal and anticonformal automorphisms of Y, since any NEC group with triangular signature $(2, 4, 4m)$ is maximal. The subgroup of Γ' generated by the reflections

$$c_0 = c_0', \quad c_1 = c_1', \quad c_2 = c_2'c_1'c_2' \quad \text{and} \quad c_3 = c_2'c_0'c_2'$$

is an NEC group Γ with quadrangular signature $(2, 2, 2, 2m)$, whose image under θ is precisely the subgroup $D_{2m} \times C_2 \times C_2$ generated by $a = \theta(c_0)$, $b = \theta(c_1)$, $c = \theta(c_2)$ and $d = \theta(c_3)$. Also the element $t = c(ad)^m$ is an anticonformal involution in $\mathrm{Aut}(Y)$, and so the orbit space $X = Y/\langle t\rangle$ is a compact Klein surface.

Next, let $X = \mathcal{H}/\Delta$ where $\Delta = \langle \Lambda, c_2(c_0c_3)^m\rangle$, which is the pre-image of $\langle t\rangle$ under θ. Then Δ is a surface NEC group, and we claim that X is unbordered and non-orientable. To see this, we note that $t = c(ad)^m$ is not conjugate in J_m to a, b, c or d, and therefore $\delta = c_2(c_0c_3)^m$ is not conjugate to a reflection of Γ. Hence δ is a glide reflection in Γ, and it follows that $\Delta = \langle \Lambda, \delta\rangle$ is an unbordered non-orientable surface NEC group.

It also follows that the compact Riemann surface Y is the double cover X^+ of X, as Λ is the canonical Fuchsian subgroup Δ^+ of Δ. Moreover, $\Delta = \langle \Delta^+, \delta\rangle$ is normal in Γ, since Δ^+ itself is normal in Γ, and the fact that $\theta(\delta) = t = c(ad)^m$ is central in $K_m = \theta(\Gamma)$ implies that $\theta(\delta^\gamma) = t^{\theta(\gamma)} = t$ and therefore $\delta^\gamma \in \Delta$, for each $\gamma \in \Gamma$.

The quotient group $G = \Gamma/\Delta$ can be obtained by adding the relation $t = 1$ to the above presentation for K_m. This forces the image $\bar{c}$ of c to be equal to the image of $(ad)^m$ and hence redundant, giving

$$\Gamma/\Delta \cong \langle\, \bar{a}, \bar{b}, \bar{d} \mid \bar{a}^2 = \bar{b}^2 = \bar{d}^2 = (\bar{a}\bar{b})^2 = (\bar{a}\bar{d})^{2m} = (\bar{b}\bar{d})^2 = 1\,\rangle \cong \langle \bar{a}, \bar{d}\rangle \times \langle \bar{b}\rangle \cong D_{2m} \times C_2.$$

Thus X is a compact unbordered non-orientable Klein surface which admits $G = \Gamma/\Delta \cong D_{2m} \times C_2$ as a group of automorphisms, acting with signature $(2,2,2,2m)$, such that its double cover $X^+ = Y$ admits $G \times C_2 \cong D_{2m} \times C_2 \times C_2$.

This example was constructed so that $G \times C_2$ is strictly contained in the full group $\mathrm{Aut}(X^+) \cong J_m$, but G is not strictly contained in $\mathrm{Aut}(X)$. In fact $G = \mathrm{Aut}(X)$ by Theorem 4.1, since there is clearly no automorphism of $G = \Gamma/\Delta$ interchanging $\bar{a}$ with $\bar{d}$ and $\bar{b}$ with $\bar{c} = (\bar{a}\bar{d})^m$.

5 Unbordered Klein surfaces with a group acting with signature $(2,2,2,4)$

In this final section we consider the exceptional case of quadrangular signature $(2,2,2,n)$ with $n = 4$. By Proposition 2.2, if an NEC group Γ with quadrangular signature $(2,2,2,4)$ is contained in another NEC group Γ', then $\sigma(\Gamma') = (2,4,8)$, $(2,4,6)$, $(2,4,5)$ or $(2,3,8)$.

For these four triangular signatures, the index $|\Gamma' : \Gamma|$ is 2, 3, 5 and 6 respectively. When $\sigma(\Gamma') = (2,4,8)$, the subgroup Γ is normal in Γ', but in the other cases it is not. (In fact the index in Γ' of the core of Γ is 6, 10 and 24 respectively.) This makes the study of the $(2,2,2,4)$ case more tricky, but still possible. We will maintain previous notation.

A group G acting with signature $(2,2,2,4)$ on an unbordered non-orientable Klein surface X admits the following partial monodromy presentation:

$$G = \langle\, a,b,c,d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = (ad)^4 = w = \ldots = 1\,\rangle \qquad (12)$$

where $w = w(a,b,c,d)$ is a non-orientable word — that is, a word of odd length greater than 1 in the generators a,b,c,d. Also G has order $8(g-1)$, where g is the algebraic genus of X, by the Riemann-Hurwitz formula (3).

If the action of G extends to an action of a larger group with signature $(2,4,8)$, then that could extend further to the action of an even larger group with signature $(2,3,8)$, since every NEC group with triangular signature $(2,4,8)$ is contained as a subgroup of another NEC group with signature $(2,3,8)$, by [9, Table 4] or an easy extension of [16, Thm 2].

On the other hand, if the action of G extends to an action of a larger group with signature $(2,4,6)$, $(2,4,5)$ or $(2,3,8)$, then the latter group has order $24(g-1)$, $40(g-1)$ or $48(g-1)$ respectively, and moreover, must be the full group $\mathrm{Aut}(X)$, because these three signatures are maximal. Analogously, if the action of $G \times C_2$ on the Riemann double cover X^+ extends to an action with signature $(2,4,6)$, $(2,4,5)$ or $(2,3,8)$, then the full group $\mathrm{Aut}(X^+)$ has order $48(g-1)$, $80(g-1)$ or $96(g-1)$ respectively.

We consider each of these four possible extensions in turn. In the case of the three non-normal extensions, we have to look deeper into the NEC group Γ and to consider its core Λ in Γ' (that is, the largest subgroup of Γ which is normal in Γ', or equivalently, the intersection of all conjugates of Γ by elements of Γ'). The case of triangular signature $(2,4,6)$ will be described in detail, and the other three possibilities explained only briefly.

5.1 $\sigma(\Gamma') = (2, 4, 8)$

In this case we can repeat the arguments from the previous Section 4 to obtain the following analogue of Theorem 4.1:

Theorem 5.1 *With G and X and X^+ as above, the condition*

(i) *the action of G on X extends to an action with signature* $(2, 4, 8)$

is equivalent to the combination of the two conditions

(ii) *the action of $G \times C_2$ on X^+ extends to an action with signature* $(2, 4, 8)$,

(iii) *every monodromy presentation* (12) *for G is preserved by the assignment* (10).

5.2 $\sigma(\Gamma') = (2, 4, 6)$

This is the case we describe in detail, in the following variant of Theorems 4.1 and 5.1:

Theorem 5.2 *With G and X and X^+ as above, the condition*

(i) $|\mathrm{Aut}(X)| = 24(g-1)$

is equivalent to the combination of the two conditions

(ii) $|\mathrm{Aut}(X^+)| = 48(g-1)$,

(iii) *for every monodromy presentation* (12) *for G, there exists a non-orientable word in G expressible as a word w_0 in the four elements a, b, dad and cd such that $w_0(a, b, dad, cd) = w_0(a, dad, b, cd) = 1$ in G.*

Proof: First we have $X \cong \mathcal{H}/\Delta$ where Δ is the kernel of a smooth epimorphism $\theta\colon \Gamma \to G$.

Now suppose that (i) holds, so that $\mathrm{Aut}(X)$ has order $24(g-1)$. Then there exists an NEC group Γ' containing Γ with index 3, such that Δ is normal in Γ' and $\mathrm{Aut}(X) \cong \Gamma'/\Delta$, and by Proposition 2.2, the group Γ' has triangular signature $(2, 4, 6)$. Also Δ^+ is normal in Γ' (since conjugation preserves orientability), and therefore Γ'/Δ^+ extends the action of $\Gamma/\Delta^+ \cong G \times C_2$ on $X^+ \cong \mathcal{H}/\Delta^+$. Moreover, since $(2, 4, 6)$ is a maximal signature, Γ'/Δ^+ is in fact the full group $\mathrm{Aut}(X^+)$, which therefore has order $48(g-1)$. Thus (ii) holds.

Next, if $\{c'_0, c'_1, c'_2\}$ is a canonical set of generating reflections for Γ', which satisfy $(c'_0c'_1)^2 = (c'_1c'_2)^4 = (c'_0c'_2)^6 = 1$, then an embedding of Γ in Γ' is given by

$$c_0 = c'_1, \quad c_1 = c'_0c'_2c'_1c'_2c'_0, \quad c_2 = c'_0c'_2c'_0c'_2c'_0, \quad c_3 = c'_2. \tag{13}$$

This embedding is unique up to conjugation, since the extended $(2, 4, 6)$ triangle group has just one conjugacy class of subgroups of index 3. Verifying this is an easy exercise — for example using MAGMA [2] — and a little further computation shows that the core Λ of Γ in Γ' has index 6, and is generated by $x = c'_1 = c_0$, $y = c'_2c'_1c'_2 = c_3c_0c_3$, $z = c'_0c'_2c'_1c'_2c'_0 = c_1$ and $u = (c'_0c'_2)^3 = c_2c_3$, which satisfy the defining relations

$$x^2 = y^2 = z^2 = u^2 = (xy)^2 = (xz)^2 = (yuzu)^2 = 1.$$

The signature of Λ is $(0;+;[2];\{(2,2,2)\})$, and the quotient Γ'/Λ is isomorphic to D_3, of order 6. Also Λ contains Δ, since Λ is the core of Γ in Γ', and $\Delta = \ker\theta$ is contained in Γ and is normal in Γ', and so Λ contains a non-orientable word $\delta = w(c_0, c_1, c_2, c_3)$ of Δ. The respective images a, b, c and d of c_0, c_1, c_2 and c_3 under the epimorphism θ satisfy the relations in (12), and so give a monodromy presentation of G.

Moreover, $K = \Lambda/\Delta$ is the core of $\Gamma/\Delta \cong G$ in $\Gamma'/\Delta = \mathrm{Aut}(X)$, and K is generated by a, dad, b and cd (the images under θ of $x = c_0$, $y = c_3c_0c_3$, $z = c_1$ and $u = c_2c_3$), and with $\mathrm{Aut}(X)/K \cong (\Gamma'/\Delta)/(\Lambda/\Delta) \cong \Gamma'/\Lambda \cong D_3$. The image $\theta(\delta) = w(a,b,c,d)$ in G of the non-orientable word $\delta = w(c_0, c_1, c_2, c_3)$ is trivial, so lies in $K = \Lambda/\Delta$, and so δ can be expressed as a word $w_0(c_0, c_3c_0c_3, c_1, c_2c_3)$ in the generators of Λ. In particular, $w_0(c_0, c_3c_0c_3, c_1, c_2c_3)$ lies in Δ, with trivial image $w_0(a, dad, b, cd)$ in G. Also it is easy to see from the relations (13) that conjugation by the reflection $c_0' \in \Gamma' \setminus \Gamma$ fixes $c_0 = c_1'$ and $c_2c_3 = (c_0'c_2')^3$ and interchanges $c_3c_0c_3 = c_2'c_1'c_2'$ with $c_1 = c_0'c_2'c_1'c_2'c_0'$. Thus $w_0(c_0, c_1, c_3c_0c_3, c_2c_3) = w_0(c_0, c_3c_0c_3, c_1, c_2c_3)^{c_0'}$ lies in Δ, and it follows that $1 = \theta(w_0(c_0, c_1, c_3c_0c_3, c_2c_3)) = w_0(a, b, dad, cd)$, so this monodromy presentation satisfies (iii).

Next, consider an arbitrary monodromy presentation of G. For this, there exists a canonical set of generating reflections $\hat{c}_0$, $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ of Γ, whose images in G satisfy the same form of relations as in the partial presentation (12). But we know there exists a unique embedding of Γ in Γ', up to conjugacy in Γ', and so there exists some element $\gamma' \in \Gamma'$ such that $\hat{c}_i = \gamma'^{-1}c_i\gamma'$ for $0 \le i \le 3$, where the reflections c_i are given by (13). It follows that conjugation by $\gamma'^{-1}c_0'\gamma'$ induces an automorphism of the normal subgroup K of $\Gamma'/\Delta = \mathrm{Aut}(X)$, fixing the images of $\hat{c}_0$ and $\hat{c}_2\hat{c}_3$, and interchanging the image of $\hat{c}_3\hat{c}_0\hat{c}_3$ with the image of $\hat{c}_1$. Hence the given monodromy presentation satisfies (iii) as well.

Conversely, suppose that (ii) and (iii) both hold. Then by (ii), we know there exists an NEC group Γ' with triangular signature $(2,4,6)$ such that Γ' contains Γ, and Δ^+ is normal in Γ', and we need to show that Δ is also normal in Γ' (so that the group Γ'/Δ extends the action of $G = \Gamma/\Delta$ on X). Since $\Gamma' = \langle \Gamma, c_0' \rangle$, and Γ normalises Δ, we need only show that c_0' normalises Δ. By (iii), there exists a non-orientable word in Γ expressible as a word w_0 in the four elements c_0, c_1, $c_3c_0c_3$ and c_2c_3 such that both $w_0(c_0, c_1, c_3c_0c_3, c_2c_3)$ and $w_0(c_0, c_3c_0c_3, c_1, c_2c_3)$ lie in $\ker\theta = \Delta$. In particular, if $\delta = w_0(c_0, c_1, c_3c_0c_3, c_2c_3)$ then $\Delta = \langle \Delta^+, \delta \rangle$, and then since Δ^+ is normal in Γ', all we have to do is show that $\delta^{c_0'} \in \Delta$. But this is easy: again since conjugation by c_0' fixes c_0 and c_2c_3 and interchanges $c_3c_0c_3$ with c_1, we find that

$$\delta^{c_0'} = w_0(c_0, c_1, c_3c_0c_3, c_2c_3)^{c_0'} = w_0(c_0^{c_0'}, c_1^{c_0'}, (c_3c_0c_3)^{c_0'}, (c_2c_3)^{c_0'}) = w_0(c_0, c_3c_0c_3, c_1, c_2c_3),$$

and we know that $w_0(c_0, c_3c_0c_3, c_1, c_2c_3)$ lies in Δ, by (iii). Thus (i) holds, as required. ∎

Remark 5.3 Suppose the smooth epimorphism $\theta : \Gamma \to G$ above can be extended to $\theta' : \Gamma' \to \mathrm{Aut}(X)$, with $|\mathrm{Aut}(X)| = 24(g-1)$. Then with the embedding of Γ in Γ' given in (13), we can take $a = \theta(c_0) = \theta'(c_1')$ and $d = \theta(c_3) = \theta'(c_2')$, and then if we take $\alpha = \theta'(c_0')$, we see that the group $\mathrm{Aut}(X)$ is generated by a, d and α, which satisfy the relations $\alpha^2 = a^2 = d^2 = (\alpha a)^2 = (ad)^4 = (\alpha d)^6 = 1$. Note that α does not normalise G, but it normalises the core K of G in $\mathrm{Aut}(X)$, which has index 6 in $\mathrm{Aut}(X)$, and is generated by a, dad, b and cd, where $b = \theta(c_1) = \alpha dad\alpha$ and $c = \theta(c_2) = \alpha d\alpha d\alpha$. Also $\mathrm{Aut}(X^+) \cong \mathrm{Aut}(X) \times C_2$, by Proposition 2.1 and the maximality of the signature $(2, 4, 6)$.

Remark 5.4 It follows from the proof of Proposition 2.1 that the surface kernel epimorphism $\theta^+ : \Gamma \to \Gamma/\Delta^+ \cong G \times C_2$ is given by $\theta^+(c_0) = at$, $\theta^+(c_1) = bt$, $\theta^+(c_2) = ct$ and $\theta^+(c_3) = dt$, where a, b, c and d represent orientation-preserving automorphisms that generate G, and t is an orientation-reversing involution that generates the factor C_2. Moreover, if θ^+ can be extended to a surface kernel epimorphism $\Gamma' \to \mathrm{Aut}(X^+)$, with $|\mathrm{Aut}(X^+)| = 48(g-1)$, then $\mathrm{Aut}(X^+)$ is generated by at, dt and an element β such that $\beta^2 = (at)^2 = (dt)^2 = (\beta at)^2 = (ab)^4 = (\beta dt)^6 = 1$, with $bt = \beta dadt\beta$ and $ct = \beta dt\beta dt\beta$.

5.3 $\sigma(\Gamma') = (2, 4, 5)$

Suppose that the NEC group Γ with signature $(2, 2, 2, 4)$ is contained as a non-normal subgroup of index 5 in some NEC group Γ' with signature $(2, 4, 5)$, and that $\{c_0, c_1, c_2, c_3\}$ and $\{c_0', c_1', c_2'\}$ are canonical sets of generating reflections for Γ and Γ' respectively, with $(c_0'c_1')^2 = (c_1'c_2')^4 = (c_0'c_2')^5 = 1$. Then an embedding of Γ in Γ' is given by

$$c_0 = c_2', \quad c_1 = c_0'c_2'c_0'c_2'c_1'c_2'c_0'c_2'c_0', \quad c_2 = c_0'c_2'c_1'c_2'c_0', \quad c_3 = c_1',$$

and this is unique up to conjugation because Γ' has just one conjugacy class of subgroups of index 5 isomorphic to Γ. (There is one other class of subgroups of index 5, but those are isomorphic to NEC groups with signature $(0; +; [2]; \{(2, 4)\})$.)

The core Λ of Γ in Γ' has index 10 in Γ', with quotient Γ'/Λ isomorphic to D_5, and is generated by the five elements $x = c_1' = c_3$, $y = c_2'c_1'c_2' = c_0c_3c_0$, $z = c_0'c_2'c_1'c_2'c_0' = c_2$, $u = c_2'c_0'c_2'c_1'c_2'c_0'c_2' = c_0c_2c_0$ and $v = (c_0'c_2')^2c_1'(c_2'c_0')^2 = c_1$, which satisfy defining relations

$$x^2 = y^2 = z^2 = u^2 = v^2 = (xy)^2 = (xz)^2 = (yu)^2 = (zv)^2 = (uv)^2 = 1.$$

In particular, Λ has signature $(0; +; [-]; \{(2, 2, 2, 2, 2)\})$. Also conjugation by the reflection $c_0' \in \Gamma' \setminus \Gamma$ fixes c_3 and interchanges c_1 and c_2 with $c_0c_2c_0$ and $c_0c_3c_0$ respectively. All of this gives the following analogue of Theorem 5.2, the proof of which is omitted.

Theorem 5.5 *With G and X and X^+ as above, $|\mathrm{Aut}(X)| = 40(g-1)$ if and only if $|\mathrm{Aut}(X^+)| = 80(g-1)$ and for every monodromy presentation* (12) *for G, there exists a non-orientable word in G expressible as a word w_0 in the five elements b, c, d, aca and ada such that $w_0(b, c, d, aca, ada) = w_0(aca, ada, d, b, c) = 1$ in G.*

Remark 5.6 If the epimorphism $\theta : \Gamma \to G$ can be extended to $\theta' : \Gamma' \to \mathrm{Aut}(X)$, with $|\mathrm{Aut}(X)| = 40(g-1)$, then $\mathrm{Aut}(X)$ is generated by $a = \theta(c_0) = \theta'(c_2')$, $d = \theta(c_3) = \theta'(c_1')$ and $\alpha = \theta'(c_0')$, which satisfy the relations $\alpha^2 = d^2 = a^2 = (\alpha d)^2 = (da)^4 = (\alpha a)^5 = 1$.

Again α does not normalise G, but normalises the core K of G in $\mathrm{Aut}(X)$, which has index 10 in $\mathrm{Aut}(X)$ and is generated by b, c, d, aca and ada, where $b = \theta(c_1) = (\alpha a)^2 d(a\alpha)^2$ and $c = \theta(c_2) = \alpha ada\alpha$. Also $\mathrm{Aut}(X^+) \cong \mathrm{Aut}(X) \times C_2$, by Proposition 2.1 and maximality of the signature $(2, 4, 5)$.

Remark 5.7 If $|\mathrm{Aut}(X^+)| = 80(g-1)$, and t is an orientation-reversing involution that generates the factor C_2 in the image of the epimorphism $\theta^+ : \Gamma \to \Gamma/\Delta^+ \cong G \times C_2$, then $\mathrm{Aut}(X^+)$ is generated by at, dt and an element β such that $\beta^2 = (dt)^2 = (at)^2 = (\beta dt)^2 = (da)^4 = (\beta at)^5 = 1$, with $bt = (\beta at)^2 dt(at\beta)^2$ and $ct = \beta adat\beta$.

5.4 $\sigma(\Gamma') = (2, 3, 8)$

In this last case, we suppose that the NEC group Γ with signature $(2, 2, 2, 4)$ is contained as a non-normal subgroup of index 6 in an NEC group Γ' with signature $(2, 3, 8)$, and that $\{c_0, c_1, c_2, c_3\}$ and $\{c'_0, c'_1, c'_2\}$ are canonical sets of generating reflections for Γ and Γ' respectively, with $(c'_0c'_1)^2 = (c'_1c'_2)^3 = (c'_0c'_2)^8 = 1$. An embedding of Γ in Γ' is given by

$$c_0 = c'_1, \quad c_1 = c'_0, \quad c_2 = c'_2c'_0c'_2c'_0c'_2c'_0c'_2, \quad c_3 = c'_2c'_0c'_2c'_1c'_2c'_0c'_2,$$

and this is unique up to conjugation because Γ' has just one conjugacy class of subgroups of index 6 isomorphic to Γ. (There are six other classes of subgroups of index 6, but those are isomorphic to NEC groups with signatures $(0; +; [2, 4, 8]; \{-\})$, $(0; +; [4]; \{(4)\})$, $(0; +; [8]; \{(2)\})$, $(0; +; [2]; \{(2, 4)\})$, $(0; +; [-]; \{(2, 8, 8)\})$ and $(0; +; [-]; \{(4, 4, 4)\})$.)

The core Λ of Γ in Γ' has index 24 in Γ', with quotient $\Gamma'/\Lambda \cong S_4$, and is generated by $x = (c'_0c'_2)^4 = c_1c_2$, $y = c'_0c'_2c'_0c'_1c'_2c'_1c'_0c'_2c'_1 = c_1c_3c_0$ and $z = c'_0c'_1c'_2c'_0c'_1c'_2c'_1c'_0c'_2 = c_0c_1c_3$, which satisfy the defining relations $x^2 = (yz^{-1})^2 = (xyz)^2 = 1$. These can be rewritten as

$$d^2x_1x_2x_3 = x_1^2 = x_2^2 = x_3^2 = 1,$$

where $d = y^{-1}$, $x_1 = x$, $x_2 = xyz$ and $x_3 = z^{-1}y$, so Λ has signature $(1; -; [2, 2, 2]; \{-\})$. Conjugation by the reflection $c'_2 \in \Gamma' \setminus \Gamma$ fixes x and interchanges y and z with xy and z^{-1} respectively, and using this, we obtain the following analogue of Theorem 5.2:

Theorem 5.8 *With G and X and X^+ as above, $|\mathrm{Aut}(X)| = 48(g-1)$ if and only if $|\mathrm{Aut}(X^+)| = 96(g-1)$ and for every monodromy presentation (12) for G, there exists a non-orientable word in G expressible as a word w_0 in the three elements bc, bda and abd such that $w_0(bc, bda, abd) = w_0(bc, cda, dba) = 1$ in G.*

Remark 5.9 If the epimorphism $\theta : \Gamma \to G$ can be extended to $\theta' : \Gamma' \to \mathrm{Aut}(X)$, with $|\mathrm{Aut}(X)| = 48(g-1)$, then $\mathrm{Aut}(X)$ is generated by $a = \theta(c_0) = \theta'(c'_1)$, $b = \theta(c_1) = \theta'(c'_0)$ and $\alpha = \theta'(c'_2)$, which satisfy the relations $b^2 = a^2 = \alpha^2 = (ba)^2 = (a\alpha)^3 = (\alpha b)^8 = 1$. Once again α does not normalise G, but normalises the core K of G in $\mathrm{Aut}(X)$, which has index 24 in $\mathrm{Aut}(X)$ and is generated by bc, bda and abd, where $c = \theta(c_2) = \alpha b\alpha b\alpha b\alpha$ and $d = \theta(c_3) = \alpha b\alpha a\alpha b\alpha$. Also $\mathrm{Aut}(X^+) \cong \mathrm{Aut}(X) \times C_2$, by Proposition 2.1 and maximality of the signature $(2, 3, 8)$.

Remark 5.10 If $|\mathrm{Aut}(X^+)| = 96(g-1)$, and t is an orientation-reversing involution that generates the factor C_2 in the image of the epimorphism $\theta^+ : \Gamma \to \Gamma/\Delta^+ \cong G \times C_2$, then $\mathrm{Aut}(X^+)$ is generated by at, bt and an element β such that $(bt)^2 = (at)^2 = \beta^2 = (ba)^2 = (at\beta)^3 = (\beta bt)^8 = 1$, with $ct = \beta bt\beta bt\beta bt\beta$ and $dt = \beta bt\beta at\beta bt\beta$.

5.5 Examples with exceptional signature (2,2,2,4)

To complete our study of unbordered Klein surfaces admitting the action of a group G with exceptional signature $(2,2,2,4)$, we show in Examples 5.11 to 5.15 below that there are cases where the action can extend, and others where it cannot be extended.

Example 5.11 Let G be the symmetric group $G = S_8$, and take $a = (1,2)(3,4)(5,6)(7,8)$, $b = (5,6)$, $c = (4,7)$ and $d = (1,3)(2,5)(4,7)(6,8)$. These elements satisfy the relations for the extended $(2,2,2,4)$ quadrangle group, and it is easy to show that they generate S_8, for example using transitivity and the fact that b is a single 2-cycle. Also the subgroup generated by $ab = (1,2)(3,4)(7,8)$, $bc = (5,6)(4,7)$ and $cd = (1,3)(2,5)(6,8)$ is S_8, and hence there exists a non-orientable word on the generators a, b, c and d. It follows that the corresponding action of G (with signature $(2,2,2,4)$) is on a non-orientable surface X, which by the Riemann-Hurwitz formula has algebraic genus $g = |G|/8 + 1 = 5041$.

Next, conjugation by the involutory permutation $(2,3)(4,5)(6,7)$ is an automorphism of G which interchanges a with d, and b with c. Hence by Theorem 5.1, the action of S_8 on X extends to the action with signature $(2,4,8)$ of a larger group G' if and only if the same happens for the action of $S_8 \times C_2$ on its Riemann double cover X^+ of X. In turn, this happens if and only if the NEC group Γ with signature $(2,2,2,4)$ uniformising the action of G on X is contained in an NEC group Γ' with signature $(2,4,8)$. If that happens, then since the automorphism is inner, the group G' will be $S_8 \times C_2$.

On the other hand, the possible extensions described in Subsections 5.2, 5.3 and 5.4 to actions with signatures $(2,4,6)$, $(2,4,5)$ and $(2,3,8)$ respectively cannot occur. In the first case, the subgroup generated by a, dad, b and cd has index 1 in S_8, rather than index 2 (see Remark 5.3); in the second case, the subgroup generated by d, ada, c, aca and b has index 1 in S_8, rather than index 2 (see Remark 5.6); and in the third case, the subgroup generated by bc, bda and abd has index 1 in S_8, rather than index 4 (see Remark 5.9). Hence if the initial extension occurs, then no further extension is possible, and $G' = S_8 \times C_2$ is the full group $\mathrm{Aut}(X)$, and $G' \times C_2$ is the full group $\mathrm{Aut}(X^+)$.

Example 5.12 In the direct product $S_4 \times C_2$, let z be the generator of the central C_2 factor, and then define $a = (1,2)(3,4)z$, $b = (1,3)(2,4)z$, $c = (2,4)z$ and $d = (2,4)$. These elements satisfy the relations for the extended $(2,2,2,4)$ quadrangle group, and it is easy to prove that they generate a group G isomorphic to $D_4 \times C_2$. Also conjugation by a takes $bd = (1,3)z$ to c, so $abdac$ is trivial. This gives a non-orientable word, and it follows that the corresponding action of $D_4 \times C_2$ with signature $(2,2,2,4)$ is on a non-orientable surface X, with algebraic genus $g = 3$. Moreover, b commutes with $(1,4)(2,3)z = dad$, and so the non-orientable word $w_0(a, b, dad, cd) = abdac = ab(dad)cd$ can be expressed equally as $w_0(a, dad, b, cd) = a(dad)bcd$ in G. (This also follows from conjugation by $(1,2)$ in $S_4 \times C_2$, which fixes a and cd $(= z)$ and interchanges b with dad.)

Hence, by Theorem 5.2, if the NEC group Γ with signature $(2,2,2,4)$ uniformising the action of G on X is contained in some NEC group Γ' with signature $(2,4,6)$, then the actions of G and $G \times C_2$ on X and X^+ respectively will both extend to larger actions with signature $(2,4,6)$. Indeed, if this happens then the action of G can be extended to an action of the group $G' = S_4 \times C_2$ with signature $(2,4,6)$ via the generators $A = (3,4)z$, $B = (1,2)(3,4)z$ and $C = (2,4)$, which satisfy the relations for the extended $(2,4,6)$ triangle group. In that case, an embedding of G in G' is given by $a = B$, $b = ACBCA$, $c = ACACA$ and $d = C$; also no further extension is possible since the triangular NEC signature $(2,4,6)$ is maximal, and therefore $\mathrm{Aut}(X) = G'$ and $\mathrm{Aut}(X^+) = G' \times C_2$.

Example 5.13 Let G be the subgroup of S_{16} generated by the permutations

$$\begin{aligned} a &= (2,3)(4,7)(6,8)(10,13)(11,14)(12,15), \\ b &= (1,9)(2,12)(3,15)(4,11)(5,16)(6,8)(7,14)(10,13), \\ c &= (1,4)(2,6)(3,13)(5,14)(7,16)(8,12)(9,11)(10,15), \\ d &= (1,2)(3,5)(4,6)(7,10)(8,11)(9,12)(13,14)(15,16). \end{aligned}$$

These four elements satisfy the relations for the extended $(2,2,2,4)$ quadrangle group. Also letting $e = [a,c] = (ac)^2 = (1,16)(2,15)(3,12)(4,7)(5,9)(6,10)(8,13)(11,14)$, we see that b, c, d and e are four commuting involutions, which generate an elementary abelian 2-group of order 16, and furthermore, that conjugation by the involution a takes (b,c,d,e) to (b, ce, bde, e). Thus G is isomorphic to a semi-direct product of $(C_2)^4$ by C_2, of order 32. Also $b = (acd)^2$, hence $b(acd)^2$ is a non-orientable word, and it follows that the corresponding action of G with signature $(2,2,2,4)$ is on a non-orientable surface X, with algebraic genus $g = 5$. Moreover, the non-orientable word $b(acd)^2$ is expressible as $b \cdot aca \cdot ada \cdot c \cdot d = w_0(b, c, d, aca, ada)$, and the relations $b = (acd)^2$ and $cdc = d$ give also

$$w_0(aca, ada, d, b, c) = aca \cdot b \cdot c \cdot ada \cdot d = aca(acd)^2 cadad = (ad)^4 = 1.$$

Hence, by Theorem 5.5, if the NEC group Γ with signature $(2,2,2,4)$ uniformising the action of G on X is contained in an NEC group Γ' with signature $(2,4,5)$, then the actions of G and $G \times C_2$ on X and X^+ respectively will both extend to larger actions with signature $(2,4,5)$. Indeed, if this happens then the action of G can be extended to an action of the subgroup G' of S_{16} of order 160 generated by

$$\begin{aligned} A &= (3,4)(5,6)(7,9)(8,11)(10,12)(15,16), \\ B &= (1,2)(3,5)(4,6)(7,10)(8,11)(9,12)(13,14)(15,16), \\ C &= (2,3)(4,7)(6,8)(10,13)(11,14)(12,15), \end{aligned}$$

which satisfy the relations for the extended $(2,4,5)$ triangle group. In that case, an embedding of G in G' is given by $a = C$, $b = ACACBCACA$, $c = ACBCA$ and $d = B$. The group G' itself is a semi-direct product of the abelian group of order 16 generated by b, c, d and e, by the dihedral subgroup of order 10 generated by A and C.

Example 5.14 Let G be the direct product $\mathrm{PGL}(2,7) \times C_2$, of order 672, and in this group, let u be the central involution, and define elements a, b, c and d in G such that

au is the transformation $z \mapsto z/(5z-1)$ in $\mathrm{PGL}(2,7)$,
bu is the transformation $z \mapsto (4z+4)/(z-4)$ in $\mathrm{PSL}(2,7)$,
cu is the transformation $z \mapsto 3/z$ in $\mathrm{PSL}(2,7)$,
d is the transformation $z \mapsto (z-5)/(4z-1)$ in $\mathrm{PGL}(2,7)$.

Then a, b, c and d are involutions that generate G, and satisfy the relations for the extended $(2,2,2,4)$ quadrangle group. Also it is easy to see that the elements $bda = bu \cdot d \cdot au$ and $cda = cu \cdot d \cdot au$ both have order 7. (In fact, these are the transformations $z \mapsto (6z+3)/(z-5)$ and $z \mapsto (4z-1)/(z-1)$ in $\mathrm{PSL}(2,7)$.) Therefore both $(bda)^7$ and $(cda)^7$ are non-orientable words in G, and so the corresponding group action with signature $(2,2,2,4)$ is on a non-orientable surface X, with algebraic genus $g = 85$. Furthermore, since $bcb = c$ we can write $(cda)^7$ as $(bc \cdot bda)^7 = w_0(bc, bda, abd)$, and the corresponding word $w_0(bc, cda, dba)$ is $(bc \cdot cda)^7 = (bda)^7$, which is trivial as well.

Hence, by Theorem 5.8, if the NEC group Γ with signature $(2,2,2,4)$ uniformising the action of G on X is contained in an NEC group Γ' with signature $(2,3,8)$, then the actions of G and $G \times C_2$ on X and X^+ respectively will both extend to larger actions with signature $(2,4,5)$. Indeed, if this happens then the action of G can be extended to an action of the group G' defined as the semi-direct product of $\mathrm{PSL}(2,7)$ by S_4, where every even permutation in S_4 centralises $\mathrm{PSL}(2,7)$, and every odd permutation acts in the same way as conjugation in $\mathrm{PGL}(2,7)$ by the linear fractional transformation $v : z \mapsto 1/z$ (which lies outside $\mathrm{PSL}(2,7)$). The group G can be embedded into G', via a homomorphism that takes u and v to the permutations $(1,2)(3,4)$ and $(1,2)$, respectively. (This is possible because $G = \mathrm{PGL}(2,7) \times C_2$ is isomorphic to a semi-direct product of $\mathrm{PSL}(2,7)$ by $\langle u, v\rangle \cong C_2 \times C_2$.) Moreover, the action of G extends to one of G' with signature $(2,3,8)$, via the generators

$$A = b = (bu)(1,2)(3,4), \qquad B = a = (auv)(3,4), \qquad C = (dv)^{-1}(1,4),$$

with $a = B$, $b = A$, $c = CACACAC$ and $d = CACBCAC$, and for this action, $(ABC)^{21}$ is a non-orientable word. Note also that the given action of G with signature $(2,2,2,4)$ extends partially to one of a subgroup of order $2|G| = 1344$ and index 3 in G', isomorphic to the obvious semi-direct product of $\mathrm{PSL}(2,7)$ by D_4, and acting with signature $(2,4,8)$.

Our final example is actually a whole family of examples, in which the given group action with signature $(2,2,2,4)$ does not extend in any of the ways we have considered.

Example 5.15 For each positive integer n we consider the group G_n with presentation

$$G_n = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = (ad)^2c = (ac)^2 = (bd)^{2n} = 1\rangle.$$

This group is an extension of its cyclic normal subgroup of order n generated by $(bd)^2$ by the group $D_4 \times C_2$ of order 16, and as such, G_n has order $16n$. It acts with signature $(2,2,2,4)$ on an unbordered surface X_n, which is non-orientable (since $(ad)^2c$ is a non-orientable word) and has algebraic genus $g = 2n+1$. We claim that the action of G_n on X_n does not extend to any larger action with triangular signature $(2,4,8)$, $(2,4,6)$, $(2,4,5)$ or $(2,3,8)$, and so in fact G_n is the full automorphism group $\mathrm{Aut}(X_n)$ of X_n, for all n.

To begin with, there is no extension to an action with signature $(2,4,8)$ as described in Subsection 5.1, because any automorphism of G_n that interchanges a with d would have to fix the involution $c = (ad)^2$, and so cannot interchange c with b. Next, the possible extensions described in Subsections 5.2 and 5.4 to actions with signatures $(2,4,6)$ and $(2,3,8)$ respectively cannot occur, because in the first case the subgroup generated by a, dad, b and cd has index 1 in G_n, rather than index 2 (see Remark 5.3), and in the second case the subgroup generated by bc, bda and abd has index 2 in G_n, rather than index 4 (see Remark 5.9). Finally, suppose there is an extension to an action with signature $(2,4,5)$, as described in Subsection 5.3. Then the index 2 subgroup of G_n generated by d, ada, c, aca and b is preserved by some automorphism φ of $\mathrm{Aut}(X)$ outside G_n that fixes d and interchanges b and c with aca and ada respectively. But the defining relations for G_n give $aca = c$ and $ada = cd$, so it follows that φ interchanges b with c, and hence that $b = ada$. Thus G_n is generated by the involutions a and d, so G_n is dihedral, and hence cannot have $D_4 \times C_2$ as a quotient, contradiction. This proves our claim, for all n.

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Emilio Bujalance: Departamento de Matemáticas Fundamentales,
Facultad de Ciencias, UNED, c/ Senda del Rey s/n, 28040 Madrid, Spain
Email: eb@mat.uned.es

Francisco-Javier Cirre: Departamento de Matemáticas Fundamentales,
Facultad de Ciencias, UNED, c/ Senda del Rey s/n, 28040 Madrid, Spain
Email: jcirre@mat.uned.es

Marston Conder: Department of Mathematics, University of Auckland,
Private Bag 92019, Auckland 1142, New Zealand
Email: m.conder@auckland.ac.nz