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SCHOOL OF ENGINEERING

NUMERICAL SIMULATION OF
ARTERIAL BLOOD FLOW

A Thesis submitted as part requirement for the Degree
Master of Engineering

by

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ABSTRACT

The equations governing the flow of blood in a nonlinear viscoelastic artery are formulated and solved numerically with a finite difference technique. Flow through branching arteries and the interaction between the left ventricle and aorta are also considered.
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DEDICATION

To my wife Marnie.

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1. INTRODUCTION.

Formulation of the equations governing the motion of blood in an artery and their subsequent solution has been a subject of research since the first attempts by Euler in 1775. Until the advent of computers, effort was mainly directed towards an analytical solution and therefore a considerable simplification of the equations was required. Following the introduction of computers, numerical techniques enabled the blood flow equations to be treated with fewer simplifying assumptions and led to greater agreement between theoretical solutions and experimental measurements. The present work is a continuation of this research and in particular the application of finite difference techniques to the solution in an artery of the one dimensional nonlinear equations governing the flow of an incompressible, Newtonian fluid.

The motion of the arterial wall is particularly important and the present work includes a detailed analysis of a nonlinear viscoelastic artery. The resulting equations are coupled with those dealing with the fluid motion and the set of equations is then cast in a form suitable for solution by several finite difference methods developed for the purpose. These methods employ the Lax Wendroff approach and are all explicit techniques. Problems of stability and
accuracy and problems associated with the boundary conditions are examined numerically. The effects of changing the parameters such as those defining the arterial wall properties, are also studied numerically.

Flow through arterial branches has received little attention despite its obvious importance, so the relevant equations are formulated and solved in conjunction with the equations defining the flow in an arterial segment. Generation of an aortic pulse from the contracting left ventricle is another important application of the finite difference method and equations to describe the action of the left ventricle are developed. Finally, an approach to modelling the complete circulatory system is briefly described.

The accent in this work is on the formulation and method of solution of the relevant equations, rather than an attempt to obtain agreement between mathematical prediction and experimental measurement. No attempt is made to quantitatively compare a numerically predicted pulse with one obtained experimentally, since this would require measurements of the geometry of the artery from which the experimental measurements were obtained. Such data is not available at present and consequently accurate verification of the model is not possible. However, a
qualitative study can serve to indicate which parameters are important and when appropriate experimental measurements are performed the model can be more critically assessed.
2. MATHEMATICAL DESCRIPTION OF ARTERIAL FLOW

2.1 Properties of Blood.

The basic assumption made is that blood is approximately an incompressible, homogeneous, Newtonian fluid. In reality blood consists principally of red blood cells (erythrocytes), white cells (leukocytes) and platelets, suspended in a fluid plasma. The percentage of the total volume of blood occupied by the cells (the hematocrit) is normally from 45-50 per cent giving a relative viscosity of 3 to 4 (see Burton [1]). All calculations are made with a constant kinematic viscosity \( \nu = 0.032 \text{ cm}^2/\text{sec} \) and constant density \( \rho = 1.05 \text{ gm/cm}^3 \). The dependence of \( \nu \) on hematocrit and the extent of the non-Newtonian and nonhomogeneous nature of blood are discussed by Napolitano and Carlonagno [2]. A discussion of the effects of blood compressibility is given by Gordon and Scala [3].

2.2 Derivation of the Equation of Motion.

Considering a cyclindrical coordinate system with motion in the circumferential direction neglected, the Navier Stokes equations for an incompressible Newtonian fluid are

\[
\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + v_x \frac{\partial v_x}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} + \frac{\partial^2 v_x}{\partial x^2} \right) + F_x,
\]

and

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_x \frac{\partial v_r}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial r} = \nu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_x}{r^2} + \frac{\partial^2 v_r}{\partial x^2} \right),
\]

(2.1)

(2.2)
where \( r \) is the radial and \( x \) the axial direction and \( F_x \) includes all body forces in the axial direction.

The assumption of nonturbulent flow which has been made here is regarded as appropriate except in certain pathological conditions such as aortic insufficiency (refer to Rushmer [4]).

A conservation of mass or continuity equation is also required,

\[
\frac{\partial v_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rv_r) = 0. \tag{2.3}
\]

A parameter \( \varepsilon \) is defined by the equation \( \varepsilon = U_0/V_o \), where \( V_o \) and \( U_0 \) are typical velocities in the axial and radial directions respectively. The validity of the following derivation is dependent upon \( \varepsilon \) being small. Now following the dimensional analysis given by Barnard et al.[5], a characteristic length in the radial direction is \( R \), the artery radius, and a characteristic length in the axial direction \( \lambda \) is defined by \( \lambda = RV_o/U_0 \). Hence, designating nondimensional quantities by primes, \( r = Rr' \); \( x = \lambda x' \);
\[
v_x = V_o v_x'; \quad v_r = U_0 v_r'; \quad t = \frac{\lambda}{V_o} t'; \quad p = \rho V_o^2 p'; \quad p_x = \frac{V_o^2}{\lambda} F_x'
\]

so that (2.1), (2.2) and (2.3) can be written (dropping primes immediately for convenience)

\[
\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r} + \frac{\partial p}{\partial x} = \xi \left( \frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} + \varepsilon^2 \frac{\partial^2 v_x}{\partial x^2} \right) + F_x \tag{2.4}
\]
\[- \frac{\partial p}{\partial r} = \varepsilon^2 \left[ \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_r}{\partial r} - \xi \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \varepsilon^2 \frac{\partial^2 v_r}{\partial x^2} \right) \right] \]

\[
\frac{\partial v_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r v_r) = 0
\]

(2.5)

(2.6)

where \( \xi = \frac{\lambda v}{U_0 R^2} \).

For \( \varepsilon \ll 1 \) the terms multiplied by \( \varepsilon^2 \) are neglected. Thus (2.5) implies \( p \) is not a function of \( r \); and rearranging (2.4) and (2.6) there follows

\[
\frac{\partial}{\partial t} (r v_x) + \frac{\partial}{\partial x} (r v^2) + \frac{\partial}{\partial r} (r v_r v_x) + \frac{\partial}{\partial r} (r p) = \xi \left[ \frac{\partial}{\partial r} (r \frac{\partial v_x}{\partial r}) \right] + r F_x
\]

(2.7)

and

\[
\frac{\partial}{\partial x} (r v_x) + \frac{\partial}{\partial r} (r v_r) = 0
\]

(2.8)

Integrating (2.7) and (2.8) over \( r \) from \( r = 0 \) to \( r = R \),

\[
\frac{\partial}{\partial t} \left[ \int_0^R r v_x \, dr \right] - \left[ r v_x \right]_R \frac{\partial R}{\partial t} + \frac{\partial}{\partial x} \left[ \int_0^R r v^2 \, dr \right] - \left[ r v^2 \right]_R \frac{\partial R}{\partial x} 
\]

\[
+ \left[ r v_r v_x \right]_R + \int_0^R \frac{\partial p}{\partial x} \, dr = \xi \left[ r \frac{\partial v_x}{\partial r} \right]_R + F_x \frac{R^2}{2}
\]

(2.9)

and

\[
\frac{\partial}{\partial x} \left[ \int_0^R r v_x \, dr \right] - \left[ r v_x \right]_R \frac{\partial R}{\partial x} + \left[ r v_r \right]_R = 0
\]

(2.10)

and since the wall is a stream surface

\[ [v_r]_R = \left[ \frac{\partial r}{\partial t} \right]_R + [v_x \frac{\partial}{\partial x}]_R \]

or

\[ [r v_r v_x]_R = [r v_x]_R \frac{\partial R}{\partial t} + [r v^2]_R \frac{\partial R}{\partial x} \]
Now define a mean velocity in the axial direction by
\[ v = \frac{1}{R^2} \int_0^R 2r v_x \, dr \]
and a parameter \( \alpha \) by
\[ \alpha = \frac{1}{RV^2} \int_0^R 2r v_x^2 \, dr \, , \]
then (2.9) and (2.10) can be rewritten as
\[ \frac{3}{\partial t} \left( R^2 v \right) + \frac{3}{\partial x} \left( \alpha R^2 v^2 \right) + R^2 \frac{\partial p}{\partial x} = 2 \xi R \left[ \frac{\partial v_x}{\partial r} \right]_R + R^2 F_x \]
and
\[ \frac{3}{\partial x} \left( R^2 v \right) + 2R \frac{\partial R}{\partial t} = 0 \, . \]

or in terms of dimensional quantities and using \( S = \pi R^2 \); 
\[ \frac{\partial v}{\partial t} + \frac{v}{S} (1-\omega) \frac{\partial S}{\partial t} + \alpha v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{2v}{R} \left[ \frac{\partial v_x}{\partial r} \right]_R + F_x \] (2.11)
\[ \frac{\partial S}{\partial t} + \frac{\partial Sv}{\partial x} = 0 \, , \] (2.12)

where
\[ v = \frac{1}{R^2} \int_0^R 2r v_x \, dr \] (2.13)
and
\[ \alpha = \frac{1}{RV^2} \int_0^R 2r v_x^2 \, dr \, . \] (2.14)

The above treatment of the Navier Stokes equations eliminates the radial velocity component \( v_r \) but requires an assumption about the axial velocity profile \( v_x = v_x(r) \). Once this is specified, the parameter \( \alpha \) and the friction term \( \frac{2v}{R} \left[ \frac{\partial v_x}{\partial r} \right]_R \) can be determined.
2.3 The Velocity Profile.

For the purposes of this study it is convenient to assume a velocity profile of the form

\[ v_x = \frac{\gamma+2}{\gamma} v[1 - \left( \frac{I}{R} \right)^\gamma] , \]

where \( \gamma \) is a constant for the particular profile assumed. The constant \( \frac{\gamma+2}{\gamma} \) arises in the satisfaction of (2.13).

Equation (2.15) also satisfies the required boundary conditions \( \left[ \frac{\partial v_x}{\partial r} \right]_{r=0} = 0 \) and \( (v_x)_{r=0} = 0 \).

Substitution of (2.15) into (2.14) leads to \( \gamma = \frac{2-a}{a-1} \); and so in terms of \( a \) (2.15) becomes

\[ v_x = \frac{a}{2-a} v \left[ 1 - \left( \frac{I}{R} \right)^{a-1} \right] \]

and so the friction term becomes: \( -2\pi \frac{\alpha}{a-1} \frac{V}{\Delta} S \) for all \( a \neq 1 \).

Putting \( a = 1 \) in (2.16) results in a flat profile in which the no-slip condition is violated and all frictional effects are removed. Equation (2.11) then reduces to the standard one dimensional equation

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = F_x \]

in which the frictional effects occur only at the artery walls and so an empirical friction term such as \( -\frac{FV^2}{AR} \), as used by Streeter, Keitzer and Bohr in [6] for example, must be included in \( F_x \).
Velocity profiles in oscillatory flow are characteristically blunt and \((2.16)\) is a convenient form for representing them. An example of some experimental velocity profiles recorded at one position and at three times during one cycle are shown as circles in Figure 1 (data obtained by Ling, Atabek and Carmody [7] in the aorta of a 31 kg dog). A value of \(\alpha = 1.1\) is chosen to give a compromise fit to these profiles and the curves obtained by using this value of \(\alpha\) in \((2.16)\) and appropriately scaling the centre line velocities, are shown as continuous lines in Figure 1.

![Figure 1. Velocity profiles.](image-url)
The equations
\[ \frac{\partial V}{\partial t} + (1 - \alpha) \frac{V}{S} \frac{\partial S}{\partial t} + \alpha V \frac{\partial V}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = - \Delta p \frac{\alpha}{\alpha - 1} \frac{V}{S} \quad (2.17) \]
and
\[ \frac{\partial S}{\partial t} + \frac{\partial SV}{\partial x} = 0 \quad (2.18) \]
provide only two equations in the three unknowns and so a third equation describing the variation of pressure with cross-sectional area is required.

2.4 The Wall Equation

This is the most difficult of the three equations to formulate as the wall exhibits a nonlinear time dependent stress/strain relationship due to the combined behaviour of elastic and collagen fibres (see Burton [1]). These fibres occur in varying proportion at different locations along the artery and this changing pattern is further complicated by tethering to the surrounding tissue and longitudinal tethering at bifurcations. The presence of arterial wall muscle under vasomotor control changes both the diameter of the artery and its stiffness.

The static circumferential stress/strain behaviour of the wall can be explained in terms of the behaviour of the elastic and collagen fibres. The latter have a higher Young's modulus, but at zero stretch they are slack and the overall Young's modulus is determined by the more elastic elastin fibres. At higher transmural pressures the collagen fibres dominate and the artery becomes stiffer. This stress/strain
behaviour carries over into the distensibility curves (curves of pressure versus arterial radius measured statically) as shown in Figure 3.

The viscoelastic nature of the fibres becomes evident in the dynamic behaviour of the wall. A sudden change in the length of a strip of wall material results in an initial decrease in stress followed by a slower decay to its final value (stress relaxation). A sudden change in stress results in a gradual change in length until a constant value is reached (creep). Hysteresis is observed when a force applied to the strip of arterial wall is gradually increased and then decreased along the same loading curve. The lengths during loading and unloading do not quite correspond. Another manifestation of the wall viscosity is the frequency dependence of the Young's modulus and the frequency dependence of the phase angle between an applied oscillating force and its resulting displacement.

This nonlinear viscoelastic behaviour is incorporated in a differential equation describing the variation of transmural pressure with cross-sectional area or arterial radius. The equation could be derived by a synthetic approach using a constitutive equation for the wall material and then performing a stress analysis of the artery. However, this procedure requires many simplifying assumptions before it is feasible.
An example of the synthetic approach is the use of finite elasticity theory (see for example Green and Zerna [8]) by Mirsky [9] and Hoppmann and Lee [10]. The wall is assumed to be elastic and have constant properties over the finite length being examined and a form for the elastic potential of the wall material is proposed. Mirsky assumes the wall to behave like a Mooney material and Hoppmann and Lee use an elastic potential in the form of a Taylor series expansion of the strain invariants. Both of these assumptions imply that the wall material is homogeneous, isotropic and incompressible.

Analysing a nonlinear viscoelastic wall by a similar method is not possible at present so the best alternative is to use an empirical relationship between transmural pressure and arterial radius based directly on experimental data obtained from in vivo testing. An argument in favour of the empirical approach is that even if it was possible to analyse a viscoelastic wall with the type of synthetic approach mentioned above, it would be unnecessarily complicated as only a limited number of properties of the wall are required. For example, if the wall is considered purely elastic then knowledge of the distensibility curve at each point is sufficient to solve the flow equations. The constants involved in the constitutive equations, once calculated, would certainly define this pressure/arterial
radius relationship, but a much simpler method would be to use experimental data on the static variation of pressure with radius directly. The procedure can be further simplified by postulating a mathematical expression for the distensibility curve and including sufficient parameters in the expression to enable it to be fitted to experimental data. These parameters are similar to the constitutive constants mentioned above, but are fewer in number as they reflect only the required information (the stresses and strains in the wall of the artery are not needed in this application).

In this work we propose to use the empirical approach and an important principle adhered to is that the required experiments must be able to be performed in vivo on the same artery segment in which flow measurements are obtained. In addition to the distensibility experiment which gives all the required information on the elastic properties of the wall, the viscous behaviour of the wall can be determined by stress relaxation and creep experiments or measurements with small artificially induced sinusoidal pressure waves. The phase velocity of these signals at a range of arterial pressures provides information on both elastic and viscous wall parameters. The rate of decay of the signals also provides a check on the validity of the model. If the viscous behaviour is ignored then wave speed measurements
can replace static pressure/radius measurements in determining the elastic characteristics of the wall. This is the approach adopted by Anliker, Rockwell and Ogden [11]. The majority of wall models proposed in the literature assume the form \( p = p(R,x) \), neglecting the phasic relationship between pressure and arterial radius (see: Barnard et al. [5], Streeter et al. [6], Anliker et al. [11], Campbell and Yang [12] and Rudinger [13]). Most of the viscoelastic models proposed are in the context of electrical network models of the circulation (Westerhof and Noordergraaf [14], Beneken and DeWit [15]), or in linearized flow models (Cox [16]). Additional references can be found in the chapter on hemodynamics by Noordergraaf in "Biological Engineering" [17].

Attempts at modelling the viscoelastic nature of the arterial wall have usually assumed a Voigt or Maxwell type of model with a single time constant. However, most experimenters have found the stress relaxation behaviour of arteries to be characterized by an initial steep decrease in stress followed by a slower decay, indicating the need for more than one time constant. A survey of the various models proposed and an analysis of much of the available data on viscoelastic behaviour has been carried out by Westerhof and Noordergraaf [14] and so will not be repeated here. The conclusion these authors came to was
that a two time constant model is required to describe both stress relaxation behaviour and creep behaviour in the frequency range of interest (0 to 15 Hz). A third time constant can be included if it is desired to model the hysterisis behaviour of the wall, but as this only becomes significant at frequencies well below the fundamental heart frequency it is not necessary in a model describing viscoelastic behaviour in the frequency range of physiologic interest. Westerhof and Noordergraaf limit their analysis to the case of linear viscoelasticity. In this thesis we develop a nonlinear viscoelastic model with two time constants capable of describing the stress relaxation and creep behaviour of the artery wall in the range 0 to 15 Hz. Hysterisis behaviour is ignored for reasons given above. Additional time constants would undoubtedly be required if a wider frequency range was being modelled or if experimental data could be obtained with greater accuracy.

The simplest model incorporating two time constants and capable of describing the observed nonlinear static behaviour of the artery is shown in spring-dashpot form in Figure 2.
Figure 2. A generalized force displacement model for the arterial wall.

Displacement is expressed in terms of $e$, defined by

$$ e = \frac{R}{R_0} - 1,$$

where $R_0$ is the unstrained arterial radius ($R = R_0$ at $p = 0$). $G_3(e)$ characterizes a strain dependent elastic element; $G_1$ and $G_2$ are elastic constants and $\eta_1$ and $\eta_2$ are viscous constants. The parameters $G_1, G_2, \eta_1, \eta_2$ could also be made strain dependent but it will be shown that the model obtained by assuming they are constants is an adequate representation of available data.

The choice of a five element mechanical model follows naturally from the decision to use two time constants to model the stress relaxation and creep behaviour. The particular arrangement of the elements ensures that the model has a finite response to a constant pressure load or constant displacement applied instantaneously. By making the parallel elastic element $G_3(R)$ strain
dependent, nonlinear behaviour is incorporated.

If $p_1$, $p_2$ and $p_3$ are the "forces" in the upper, middle and lower branches of the spring dashpot model respectively, then

$$p = p_1 + p_2 + p_3$$

$$p_1 = \frac{\partial e}{\partial t} \left( \frac{1}{G_1} \frac{\partial}{\partial t} + \frac{1}{\eta_1} \right),$$

$$p_2 = \frac{\partial e}{\partial t} \left( \frac{1}{G_2} \frac{\partial}{\partial t} + \frac{1}{\eta_2} \right),$$

and

$$p_3 = G_3(R)e.$$

hence eliminating $p_1$, $p_2$ and $p_3$,

$$\left( \frac{1}{G_1} \frac{\partial}{\partial t} + \frac{1}{\eta_1} \right) \left( \frac{1}{G_2} \frac{\partial}{\partial t} + \frac{1}{\eta_2} \right) [p - G_3(R)e] = \left( \frac{1}{G_1} + \frac{1}{G_2} \right) \frac{\partial^2 e}{\partial t^2} + \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} \right) \frac{\partial e}{\partial t}$$

and employing $\tau_1 = \eta_1/G_1$ and $\tau_2 = \eta_2/G_2$ this becomes

$$\frac{\partial^2}{\partial t^2} [p - G_3(R)e] + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \frac{\partial}{\partial t} [p - G_3(R)e] + \frac{1}{\tau_1 \tau_2} [p - G_3(R)e]$$

$$= (G_1 + G_2) \frac{\partial^2 e}{\partial t^2} + \frac{G_2}{\tau_1} + \frac{G_1}{\tau_2} \frac{\partial e}{\partial t}.$$

Now define $W = \frac{\partial}{\partial t} [p - (G_1 + G_2 + G_3(R))e]$, then the above equation can be written

$$\frac{\partial W}{\partial t} + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) W + [p - G_3(R)e]/\tau_1 \tau_2 + \frac{G_1}{\tau_1} \frac{G_2}{\tau_2} \frac{\partial e}{\partial t} = 0.$$

Substituting $e = \frac{R}{R_0} - 1$, this becomes

$$\frac{\partial W}{\partial t} + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) W + \left( \frac{G_1}{\tau_1} + \frac{G_2}{\tau_2} \right) \frac{1}{R_0} \frac{\partial R}{\partial t} = [G_3(R)(\frac{R}{R_0} - 1) - p]/\tau_1 \tau_2, \quad (2.19)$$
where
\[
W = \frac{\partial P}{\partial t} - \left[ G_1 + G_2 + G_3(R) + (R - R_0) \frac{\partial G_3(R)}{\partial R} \right] \frac{1}{R_0} \frac{\partial R}{\partial t}. \tag{2.20}
\]

Under static conditions the time derivatives reduce to zero and pressure and arterial radius are related by
\[
p = G_3(R) \left( \frac{R}{R_0} - 1 \right).
\]

This is in fact an expression of the static distensibility relationship. Denoting this "static" pressure by \( p_s \), the equation becomes
\[
p_s(R) = G_3(R) \left( \frac{R}{R_0} - 1 \right) \tag{2.21}
\]
and (2.19) and (2.20) can be written
\[
\frac{\partial W}{\partial t} + \left( \frac{1}{\tau_1^2} + \frac{1}{\tau_2^2} \right) W + \left( \tau_1^2 + \frac{1}{\tau_2^2} \right) \frac{1}{R_0} \frac{\partial R}{\partial t} = \left[ p_s(R) - p \right] / \tau_1 \tau_2, \tag{2.22}
\]
where
\[
W = \frac{\partial P}{\partial t} - \left[ \frac{G_1 + G_2}{R_0} + \frac{dp_s(R)}{dR} \right] \cdot \frac{\partial R}{\partial t}. \tag{2.23}
\]

So far no mention has been made of the form of \( G_3(R) \). Equation (2.21) defines the static elastic behaviour of the arterial wall and so \( G_3(R) \) can be chosen to match (2.21) to experimental distensibility data. As explained previously, the collagen and elastin fibres in the wall cause the distensibility curve to be concave to the pressure axis. The correct qualitative behaviour is obtained by choosing \( G_3(R) \) so that (2.21) takes the form
\[ P_s(R) = G_o \left( \frac{R}{R_o} \right)^\beta - 1 \]  

(2.24)

where \( \beta \) and \( G_o \) are constants. The constants \( \beta \) and \( G_o \) can be chosen to match this expression to experimental data. The monotonically increasing positive slope of the distensibility curve requires the parameter \( \beta \) to be greater than unity. Use of more parameters in the distensibility expression would enable more accurate modelling of experimental data but limitations to the accuracy with which data can be obtained make this pointless. An example of experimental distensibility data "fitted" by selecting suitable values of \( \beta \) and \( G_o \) in (2.24) is shown in Figure 3.

![Graph showing the pressure/arterial radius relationship](image)

**Figure 3.** The pressure/arterial radius relationship obtained statically. The experimental data (for a dog) is taken from Attinger [18] and the solid lines are obtained from equation (2.24) with \( \beta = 2 \) and \( G_o = 50 \text{ mm Hg} \) (upper curve) and \( \beta = 6 \) and \( G_o = 1.5 \text{ mm Hg} \) (lower curve).
The wall equations (2.22) and (2.23) now contain six parameters: \( \tau_1, \tau_2, \beta, G_0, G_1, \) and \( G_2 \). As described above, \( \beta \) and \( G_0 \) are calculated by matching (2.24) to experimental distensibility data and so it now remains to choose suitable experiments to evaluate the others. Two separate methods are available, the first involving stress relaxation and creep experiments and the second involving measurements with small sinusoidal pressure waves. The constants being evaluated \((\tau_1, \tau_2, G_1, G_2)\) are all concerned with the viscous properties of the wall and so if these properties are ignored then the distensibility experiments alone will provide all the required information; or alternatively these distensibility experiments can be replaced by measurements of wavespeed versus arterial pressure as mentioned previously.

Before describing the two methods, the general wavespeed will be found by examining the characteristics of the system of equations.

2.5 The System of Equations

Rewriting (2.17) and (2.18) with the arterial radius \( R \) replacing cross-sectional area \( S \) as one of the dependent variables results in

\[
\frac{\partial V}{\partial t} + 2(1-\alpha) \frac{V \partial R}{R \partial t} + \alpha V \frac{\partial V}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{2\alpha}{\alpha-1} \frac{V}{R^2}
\]  

(2.25)

and

\[
\frac{\partial R}{\partial t} + V \frac{\partial R}{\partial x} + \frac{R}{2} \frac{\partial V}{\partial x} = 0
\]  

(2.26)
The wall equations (2.22) and (2.23) are

\[
\frac{\partial W}{\partial t} + \frac{1}{R_o} \left( \frac{G_1}{\tau_1} + \frac{G_2}{\tau_2} \right) \frac{\partial R}{\partial t} = \left[ p_s(R) - p \right] / \tau_1 \tau_2 - \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) W \tag{2.27}
\]

and

\[
\frac{\partial p}{\partial t} - \left[ \frac{G_1 + G_2}{R_o} \frac{dp_s(R)}{dR} \right] \frac{\partial R}{\partial t} = W, \tag{2.28}
\]

where

\[
p_s(R) = G_0 \left[ \left( \frac{R}{R_o} \right)^\beta - 1 \right].
\]

These equations constitute a quasilinear first order system and can be written in matrix form as

\[
\begin{bmatrix}
1 & 2(1-\alpha) \frac{V}{R} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \left( \frac{G_1 + G_2}{R_o} + \frac{dp_s}{dR} \right) & 1 & 0 \\
0 & \frac{1}{R_o} \left( \frac{G_1}{\tau_1} + \frac{G_2}{\tau_2} \right) & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial R}{\partial t} \\
\frac{\partial p}{\partial t} \\
\frac{\partial W}{\partial t}
\end{bmatrix}
= \begin{bmatrix}
\frac{2\alpha \frac{V}{\alpha - 1}}{R^2} \\
0 \\
0 \\
W
\end{bmatrix}.
\]

If the characteristic directions of this system are denoted by \( \frac{dx}{dt} = \lambda \), then the characteristic equation is obtained from (Courant and Hilbert [19])

\[
\begin{vmatrix}
\alpha V - \lambda & -2(1-\alpha) \frac{V}{R} & \frac{1}{\rho} & 0 \\
R & V - \lambda & 0 & 0 \\
0 & \left( \frac{G_1 + G_2}{R_o} + \frac{dp_s}{dR} \right) & -\lambda & 0 \\
0 & -\frac{1}{R_o} \left( \frac{G_1}{\tau_1} + \frac{G_2}{\tau_2} \right) & \lambda & 0
\end{vmatrix} = 0,
\]

or \( \lambda = 0, 0, \alpha V \pm \left[ \alpha(\alpha - 1) V^2 + \frac{R}{\alpha \rho} \left( \frac{G_1 + G_2}{R_o} + \frac{dp_s}{dR} \right) \right]^{1/2} \).
\[ = 0,0,\alpha V \pm [\alpha(\alpha-1)V^2 + c^2]^{\frac{1}{2}}, \]

where

\[ c = \left[ \frac{R}{\rho} \left( \frac{G}{R_0} + \frac{dP_s}{dR} \right) \right]^{\frac{1}{2}}. \tag{2.29} \]

\( c \) is referred to as the pulse velocity or wave speed and is the velocity at which disturbances propagate when the mean velocity of the fluid is zero. For an elastic wall (2.29) reduces to

\[ c = \left( \frac{R}{\rho} \frac{dP_s}{dR} \right)^{\frac{1}{2}}. \]

2.6 Stress Relaxation and Creep Experiments.

If an artery is instantaneously distended and then held at a constant radius the subsequent pressure relaxation will be described by (2.27) and (2.28) with \( R \) constant; that is (eliminating \( W \))

\[ \frac{\partial^2 p}{\partial t^2} + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \frac{\partial p}{\partial t} + \frac{p}{\tau_1 \tau_2} = \frac{P_s}{\tau_1 \tau_2}, \]

where \( P_s \) is now a constant. This has a solution

\[ p = ae^{-t/\tau_1} + be^{-t/\tau_2} + P_s \]

and \( \tau_1 \) and \( \tau_2 \) are thus the time constants for stress relaxation.

If the arterial pressure is instantaneously raised and held constant, the transient response of the arterial radius yields another two time constants which can be
expressed in terms of $\tau_1, \tau_2, G_1$ and $G_2$. The creep response for the model could be obtained by solving (2.27) and (2.28) with constant $p$ numerically, since it is not possible analytically. To overcome this difficulty we can consider small relaxations and use the linearized versions of (2.27) and (2.28). Then substituting $R_m + \epsilon R$, $p_m + \epsilon p$ and $W_m + \epsilon W$ in place of $R$, $p$ and $W$ in (2.27) and (2.28) and expressing $p_s$ in a Taylor series expansion about $p_m$, we obtain

$$\frac{\partial W}{\partial t} + \frac{1}{R_0} \left( \frac{G_1 + G_2}{\tau_1 \tau_2} \right) \frac{\partial R}{\partial t} = \left( \frac{dp_s}{dR} \right)_m (R - p) / (\tau_1 \tau_2) - \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) W$$

and

$$\frac{\partial p}{\partial t} - \left[ \frac{G_1 + G_2}{R_0} + \left( \frac{dp_s}{dR} \right)_m \right] \frac{\partial R}{\partial t} = W.$$

These may be combined to eliminate $W$,

$$\frac{\partial^2 p}{\partial t^2} - \left[ \frac{G_1 + G_2}{R_0} + \left( \frac{dp_s}{dR} \right)_m \right] \frac{\partial^2 R}{\partial t^2} + \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) \frac{dp}{\partial t} + \left[ \frac{1}{R_0} \left( \frac{G_1 + G_2}{\tau_1 \tau_2} - \left( \frac{1}{\tau_1} \right) \left( \frac{1}{\tau_2} \right) \frac{dp_s}{dR} \right) \right] \frac{\partial R}{\partial t}$$

$$= \left[ \frac{dp_s}{dR} \right]_m (R - p) / (\tau_1 \tau_2). \tag{2.30}$$

Now for the solution to the creep experiment $p$ is a constant and the solution for $R$ has the form

$$R = ae^{-t/\xi_1} + be^{-t/\xi_2} + R_s,$$

where the two time constants in this experiment, $\xi_1$ and $\xi_2$, satisfy the relations
\[ \frac{1}{\xi_1} \cdot \frac{1}{\xi_2} = \left[ R_0 \left( \frac{dP_s}{dR} \right) \tau_1 \tau_2 / [G_1 + G_2 + R_0 \left( \frac{dP_s}{dR} \right) \tau_1] \right] \] (2.31)

and

\[ \frac{1}{\xi_1} + \frac{1}{\xi_2} = \left[ \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) R_0 \left( \frac{dP_s}{dR} \right) \tau_1 \tau_2 / [G_1 + G_2 + R_0 \left( \frac{dP_s}{dR} \right) \tau_1] \right] \] (2.32)

These expressions can be used in (2.30) to eliminate \( G_1 \) and \( G_2 \),

\[ \frac{\partial^2 P}{\partial t^2} - \xi_1 \xi_2 \left( \frac{dP_s}{dR} \right) \frac{\partial^2 R}{\partial t^2} + \left( \frac{1}{\tau_1 \tau_2} \frac{\partial P}{\partial t} - \frac{1}{\xi_1 \xi_2} \right) \frac{dP_s}{dR} \frac{\partial R}{\partial t} = \left( \frac{dP_s}{dR} m - p \right) \tau_1 \tau_2 \] (2.33)

Equation (2.33) is the linearized wall equation in terms of the stress relaxation time constants \( \tau_1 \) and \( \tau_2 \), the creep time constants \( \xi_1 \) and \( \xi_2 \) and the wall distensibility at mean pressure \( \left( \frac{dP_s}{dR} \right)_m \). By taking the transform of this equation a modulus of elasticity \( E(\omega) \), where \( p = E(\omega) \cdot R \), can be defined in the frequency domain as

\[ E(\omega) = \frac{\xi_1 \xi_2}{\tau_1 \tau_2} \left( \frac{dP_s}{dR} \right)_m \left( i \omega + \frac{1}{\xi_1} \right) \left( i \omega + \frac{1}{\xi_2} \right) \]

At zero frequency this reduces to the static modulus \( \left( \frac{dP_s}{dR} \right)_m \) and at sufficiently high frequencies the absolute value of this complex modulus reaches a constant \( \frac{\xi_1 \xi_2}{\tau_1 \tau_2} \left( \frac{dP_s}{dR} \right)_m \), in agreement with observed behaviour (see Westerhof and Noordergraaf [14]).

The following figures for the aorta are taken from Westerhof and Noordergraaf [14] who obtained the data by averaging the results of several other investigators. The
only purpose in quoting the data here is to give an idea of the magnitude of the time constants and to illustrate how they are used to calculate \( G_1 \) and \( G_2 \) once \( R_0 \left( \frac{dp_s}{dR} \right)_m \) has been evaluated from distensibility data.

\[
\begin{align*}
\tau_1 &= 0.167 \text{ secs} , \quad \xi_1 = 0.172 \text{ secs} \\
\tau_2 &= 0.022 \text{ secs} , \quad \xi_2 = 0.026 \text{ secs}
\end{align*}
\]

Now choosing \( \beta = 2 \) and \( G_0 = 50 \text{ mm Hg} \) as fairly typical of distensibility data gives \( p_s = 50 \left( \frac{R}{R_0} \right)^2 - 1 \) and hence \( R_0 \left( \frac{dp_s}{dR} \right) = 100. \frac{R}{R_0} \). Choosing \( p_m = 100 \text{ mm Hg} \), then \( R_m/R_0 = 1.73 \) and hence \( R_0 \left( \frac{dp_s}{dR} \right)_m = 173 \text{ mm Hg} \). Substituting this value into (2.31) and (2.32) along with the values of \( \tau_1, \tau_2, \xi_1 \) and \( \xi_2 \) quoted above enables \( G_1 \) and \( G_2 \) to be evaluated. The values obtained are:

\[
G_1 = 10 \text{ mm Hg} , \quad G_2 = 33 \text{ mm Hg}.
\]

These figures will later be used as typical data for solving the flow equations. No attempt will be made to compare the results obtained quantitatively with recorded pressure and flow waves as this would be meaningless unless the viscoelastic constants had been evaluated for the same artery segment used for those recordings.

2.7 Wavespeed and Attenuation Measurements.

An alternative method of evaluating \( (G_1 + G_2) \) is to measure wavespeed as a function of arterial radius. The general wavespeed at zero mean flow is given by (2.29),
that is
\[ c = \left[ \frac{R}{\varphi} \left( \frac{G_1 + G_2}{R_0} + \frac{dp_s}{dR} \right) \right]^\frac{1}{2}. \] (2.29)

Here \( \frac{dp_s}{dR} \) is known from distensibility data and so \((G_1 + G_2)\) can be chosen to match (2.29) with experimental data of \(c\) versus \(R/R_0\) (or \(c\) versus \(p_s\) since \(p_s\) is a function of \(R/R_0\) only).

A method for measuring \(c\) was pioneered by Anliker and Maxwell ([20] and [21]). The technique is to generate a small amplitude sinusoidal pressure signal at a single fixed frequency in the artery and measure its phase velocity and attenuation characteristics. The signal is of finite length, sufficiently long so that the frequency at which the wave is generated dominates the frequency spectrum associated with the finite wave, and sufficiently short so that interference from reflection effects is negligible (this question is discussed by Anliker, Histan and Ogden [22]). The assumption made is that the speed of these waves is a good approximation to their phase velocity at the given frequency. By generating the signals at a number of different mean aortic pressures a chart of \(c\) versus \(p_s\) is built up. To remove the effect of non-zero mean flow on the measured wavespeed the artery is momentarily occluded proximal to the site of measurement.
Some wavespeed measurements made by Anliker, Histand, and Ogden ([22]) are shown in Figure 4. Wavespeed has been recorded for a range of arterial pressures and a range of frequencies of the induced harmonic signal. The observation by these investigators that the wavespeed is essentially independent of the signal frequency is in agreement with the model, since the wavespeed predicted by (2.29) has no frequency dependence. Thus, although the measurements by Anliker et al. were made with signals of frequencies above 40 Hz, they can be used to evaluate \((G_1 + G_2)\) for a model designed to be valid in the range 0 to 15 Hz. The effect of non-zero mean flow is to raise the measured wavespeed, so only measurements without this added effect are shown.

Equation (2.29) can be written in terms of mean pressure \(P_s\) as

\[
c = \left[ \frac{1}{2\beta} \left( \frac{P_s}{G_o} + 1 \right) \right]^\frac{1}{\beta} + \frac{\beta}{2\beta} \left( \frac{P_s + G_o}{G_o} \right)^\frac{1}{\beta}. \tag{2.34}
\]

The curve obtained by substituting \(\beta = 2\) and \(G_o = 50\) in (2.34) and then adjusting \((G_1 + G_2)\) to give a best fit to the experimental data is shown in Figure 4. The value used is \((G_1 + G_2) = 40\) mm Hg and this is within 10\% of the value obtained from the stress relaxation and creep experiments of Westerhof and Noordergraaf (see Section 2.6) which gave \((G_1 + G_2) = 43\) mm Hg.
Figure 4. The variation of wavespeed with pressure. The experimental points are taken from Anliker et al. [22] (experiment no. 124; canine aorta; oscillator frequency: 80 Hz).

The attenuation of the wave as it propagates along the artery gives additional information on the viscous properties of the wall. Anliker et al. have measured the attenuation of signals in the frequency range 40 to 200 Hz. This is outside the frequency range of physiological interest (0 to 15 Hz) and more time constants than are available in the above model may be required to adequately cover the range 0 to 200 Hz. If the experiments were carried out at lower frequencies then they could be used to calculate the remaining parameters of the model.

The small amplitude of the artificially induced signals enables a linear approximation to be made. Hence, substituting $V_\text{m} + \epsilon V$, $R_\text{m} + \epsilon R$, $P_\text{m} + \epsilon P$ and $W_\text{m} + \epsilon W$ in place of $V$, $R$, $p$ and $W$ in equations (2.25), (2.26), (2.27) and (2.28) and neglecting terms multiplied by powers of $\epsilon$ greater than one, yields the linearized equations (with $V_\text{m}$ set to zero):
\[
\frac{\partial V}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{2\omega}{\alpha - 1} \frac{V}{R_m^2},
\]

\[
\frac{\partial R}{\partial t} + \frac{R_m}{2} \frac{\partial V}{\partial x} = 0,
\]

\[
\frac{\partial W}{\partial t} + \frac{1}{R_o} \left( \frac{G_1 + G_2}{\tau_1 + \tau_2} \right) \frac{\partial R}{\partial t} = \left[ \frac{\partial p_s}{\partial R} \right] R(p - p) \frac{1}{\tau_1 \tau_2} - \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) W,
\]

and

\[
\frac{\partial p}{\partial t} - \left[ \frac{G_1 + G_2}{R_o} + \frac{\partial p_s}{\partial R} \right] \frac{\partial R}{\partial t} = W.
\]

The pulse velocity for this linear system has the same form as that for the nonlinear system, that is

\[
c_m = \frac{R_m}{2 \rho} \left( \frac{G_1 + G_2}{R_o} + \frac{\partial p_s}{\partial R} \right)^{1/2}.
\]

For the investigation of the propagation characteristics of harmonic waves we set

\[
\begin{bmatrix}
V \\
R \\
P \\
W
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{V} \\
\tilde{R} \\
\tilde{P} \\
\tilde{W}
\end{bmatrix} e^{i \omega t - (k + i \xi) x}
\]

Substituting these into the set of linear equations yields a set of simultaneous algebraic equations:

\[
(i \omega + \frac{2\omega}{\alpha - 1} \frac{1}{R_m^2})V - \frac{i}{\rho} (k + i \xi) P = 0,
\]

\[
-\frac{R_m}{2} i (k + i \xi) V + i \omega R = 0,
\]

\[
\left[ \frac{i \omega}{R_o} \left( \frac{G_1 + G_2}{\tau_1 + \tau_2} - \frac{\partial p_s}{\partial R} \right) \right] R + p / \tau_1 \tau_2 + (i \omega + \frac{1}{\tau_1} + \frac{1}{\tau_2}) W = 0.
\]
and
\[
-\rho c_m^2 \frac{2}{R_m} i \omega R + i \omega p - W = 0.
\]

This set of equations has a unique solution only if the determinant of the coefficient matrix is zero and this gives an expression for \((k+il)\),

\[
(k+il)^2 = [\omega^2 - i\omega \left(\frac{2\alpha}{\alpha-1} \frac{1}{R_m^2}\right) \left(\frac{\alpha}{\tau_1 \tau_2} - i\omega \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\right) / \left[\frac{c_m^2 \omega^2}{R_m^2} \frac{dR}{d\rho} \frac{\frac{1}{\tau_1} + \frac{1}{\tau_2}}{\tau_1 \tau_2}\right] + i \omega \frac{R_m}{2 \rho c^2} \left(\frac{G_1}{\tau_1} + \frac{G_2}{\tau_2}\right) - i \omega c_m^2 \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\].
\] (2.35)

The attenuation characteristics of the sinusoidal signal of frequency \(\omega\) can be defined in terms of \(k\) and \(l\). For example the attenuation of the pressure wave may be expressed in terms of the ratio of the amplitudes of two waves separated by a distance \(\Delta x\) as

\[
\frac{p(x+\Delta x)}{p(x)} = e^{-l\Delta x}.
\] (2.36)

Solving (2.35) for \(k\) and \(l\) and using measurements of the attenuation in the frequency range 0 to 15 Hz enables values to be obtained for the model parameters (in conjunction with the evaluation of \((G_1+G_2)\) from wavespeed measurements).

Fluid viscosity as well as wall viscosity has an effect on attenuation at these low frequencies and it is not possible to use the high frequency measurements of Anliker et al. by extrapolating their data to the 0 to 15 Hz range (the wavespeed measurements can be used since the
model predicts that they have no frequency dependence.
However, without attempting to evaluate any more model
parameters a qualititative check with Aniker's data can be
sought by assuming that a single dominant time constant
(\(\tau\)) adequately represents the data in a limited high
frequency range. This also implies that the order of
magnitude of \(\tau\) will be \(\frac{1}{\omega}\) where \(\omega\) is a frequency in
this range.

Considering a single time constant model then, (2.35)
becomes

\[(k+if)^2 = [\omega^2 - i\omega \left(\frac{2\alpha}{\alpha-1} \frac{1}{R_m}\right)]\left[\omega^2 - \frac{i\omega}{\tau}\right]/\left[c_m^2\omega^2 + \frac{i\omega}{\tau} \frac{R_mG}{2R_o\rho} - \frac{i\omega}{\tau} c_m^2\right],\]

where

\[c_m = \left[\frac{R_m}{2\rho} \left(\frac{G}{R_o} + \left(\frac{dp}{dR}\right)_m\right)\right]^2.\]

(2.37)

(2.38)

The attenuation of the sinusoidal wave is caused by
fluid viscosity as well as wall viscosity but the fluid
viscosity effect is small. This can be seen by substituting
\(v = 0.032 \text{ cm}^2/\text{sec}, \alpha = 1.1\) and \(R_m = 0.5\) in the first
bracket on the R.H.S of (2.37) and comparing it with the
second bracket. That is,

\[\frac{2\alpha}{\alpha-1} \frac{1}{R_m} = 2.82 \text{ sec}^{-1}\]

which is negligible compared with \(\frac{1}{\tau}\) and hence viscosity
effects can be ignored at this frequency. The signal
attenuation will also be aided by radiant energy loss but
this also is small compared with wall viscosity effects (see
Anliker, Hirstand and Ogden [22]). Hence (2.37) may be written

\[(k+i\ell)^2 = \frac{\omega^2}{c_m^2} \left(1 - \frac{i}{\omega \tau} \right) \left(1 + \frac{ia}{\omega \tau^2} \right) \left(1 + \frac{a^2}{\omega^2 \tau^2} \right)\]

Equation (2.39) may be solved for \(\ell\) and \(k\) by equating real and imaginary parts separately, so that the attenuation of a pressure wave per cm is given by (2.36) with

\[
\ell = k_1 \frac{a}{c_m} ,
\]

where

\[
k_1 = \left[ \frac{1}{k_2} \left( \frac{1}{1 + \omega^2 \tau^2} \right)^{\frac{1}{2}} \left( \frac{1}{1 + \frac{a^2}{\omega^2 \tau^2}} \right) \right]^{\frac{1}{2}}.
\]

Also from (2.38) and (2.40),

\[
a = R_o \frac{dp_s}{dR_m} / \left[ G + R_o \frac{dp_s}{dR_m} \right].
\]

Using the typical distensibility constants \(G = 2\) and \(G_o = 50\) mm Hg at \(p_s = 70\) mm Hg gives \(R_o \frac{dp_s}{dR_m} = 155\) mm Hg, and although the value of \(G\) here will certainly be different from the value of this constant calculated for the 0 to 15 Hz frequency range, 'a' will be approximately between 0.5 and 1.0. In Figure 5 \(k_1\) is plotted over a range of \(\omega \tau\) for \(a = 0.5\) and \(a = 0.75\).
If \( \omega t \) is in the range 0.5 to 1.5 then \( k_1 \) is reasonably independent of frequency and depends only upon the value of \( \alpha \).

Hence,

\[
\frac{p(x+\Delta x)}{p(x)} = e^{-k_1 \frac{\omega}{2m} \Delta x},
\]

(2.42)

where \( k_1 \) is dependent upon the mean pressure but not the frequency of the harmonic signal.

The attenuation can also be expressed as 

\[
e^{-k_2 \Delta x / \lambda},
\]

where

\[
k_2 = \lambda \ell = 2\pi \frac{c_m l}{\omega} = 2\pi k_1
\]

and hence \( k_2 \) also is reasonably independent of frequency.

This prediction of the model is in agreement with experimental evidence. Anliker, Histand and Ogden [22] found that the attenuation could be expressed as 

\[
e^{-k_\Delta x / \lambda}
\]

where \( k \) is nearly independent of frequency and has a value of 0.7 to 1.0 (based on data recorded in the thoracic aortas of eighteen
Figure 6. The attenuation of a pressure wave with the distance from its source at three different frequencies.

dogs at normal blood pressure levels). If the values $\alpha = 1$ and $a = 0.65$ (a realistic guess) are substituted in (2.40), $k_1$ is found to be 0.11 and hence $k_2 = 0.7$ and the model is in good agreement with the experimental data. Anliker et al. also found that the attenuation per cm is frequency dependent and the results of measurements on the aorta of a dog at 70 mm Hg and three different frequencies are shown in Figure 6. Also shown are three exponential decay curves obtained by substituting $k_1 = 0.11$ and $c_m = 4.3$ m/sec in (2.42) (this value of $c_m$ is that quoted by Anliker et al. for this experiment and agrees well with the theoretical value predicted by (2.34) or obtained from the curve in Figure 4 at a pressure of 70 mm Hg).
3. METHODS OF SOLUTION

3.1 Introduction.

The equations developed in the last chapter describe the pulsatile flow of blood through a nonlinear viscoelastic artery. Three related methods of solving these equations are presented in this chapter: the first two requiring considerable simplification of the equations and the third treating them in their fullest form. All three methods are new to this application and so the following brief discussion outlines some previous approaches to solving blood flow equations.

Previous analyses have usually made the assumption of an elastic wall, in which case an equation of the form

\[ R = R(p, x) \]

is employed in place of (2.27) and (2.28) and it is possible to write

\[ \frac{\partial R}{\partial t} = \left( \frac{\partial R}{\partial p} \right) \frac{\partial p}{\partial t} + \left( \frac{\partial R}{\partial x} \right) \frac{\partial x}{\partial t} \frac{\partial p}{\partial x} + \left( \frac{\partial R}{\partial x} \right) \frac{\partial x}{\partial t} \frac{\partial x}{\partial p} \cdot \]

Equation (2.25) can then be written

\[ \frac{\partial V}{\partial t} + 2(1-a) \frac{V}{R} \frac{\partial R}{\partial t} + aV \frac{\partial V}{\partial x} \frac{\partial x}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial t} = - \frac{2aV}{a-1} \frac{V}{R^2} \quad (3.1) \]

and (2.26) becomes

\[ \left( \frac{\partial R}{\partial p} \right) \frac{\partial p}{\partial t} + \left( \frac{V}{\partial p} \right) \frac{\partial V}{\partial p} + \left( \frac{\partial R}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{R}{2} \frac{\partial V}{\partial x} = 0 \quad (3.2) \]

Equations (3.1) and (3.2) are a pair of hyperbolic equations in the dependent variables \( p \) and \( V \) and may be solved
using the method of characteristics. This is the approach adopted by Anliker, Rockwell and Ogden [11] and Barnard, Hunt, Timlake and Varley [23]. Barnard et al. specify the form of $R = R(p,x)$ directly while Anliker et al. introduce experimental wavespeed data, $c(p,x)$, and so obtain a form for $R(p,x)$ by solving $\frac{R}{2} \frac{dp}{dR} = \rho c^2$. The latter authors use the simplified version of (3.1) obtained by setting $\alpha = 1$.

A further simplification is achieved by linearizing (3.1) and (3.2) and then solving them analytically with a Fourier series approach or with an analogue computer by setting up an analogy with the transmission line equations. The chapter by Noordergraaf in [17] summarizes much of this work.

Two general methods are available for solving the set of four quasilinear hyperbolic equations (2.25), (2.26), (2.27) and (2.28). One is the method of characteristics, previously used to solve the elastic wall case, and the other is a direct finite difference method. The method of characteristics has been well documented elsewhere ([5], [6] and [11]) and so will not be elaborated upon here. The disadvantage of the method is that for a nonlinear problem of this nature the characteristic curves do not fit into any regular system of mesh points so that as well as calculating the characteristic directions at each time step an interpolation procedure is required to find the values of the dependent variables at
the fixed mesh points. For this reason a finite difference method based on the Lax Wendroff approach is used.

3.2 Finite Difference Techniques.

Three explicit techniques are mentioned. The first two are the conventional Lax Wendroff techniques, one being referred to as the "one step" method and the other as the "two step" method (refer Richtmyer and Morton [24] and Ames [25]). Both methods require the equations to be written in conservative form and since this can only be done for the elastic wall case a more general technique based on a non-conservation form is used to solve the general equations. Implicit methods are not investigated since satisfactory stability can be achieved with the more easily programmed explicit methods.

A system of equations

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad (3.3) \]

where \( u = u(x,t) \) is an \( n \) component vector and \( f(u) \) is an \( n \) component nonlinear vector function of \( u \), is said to be in conservation form as this form commonly arises from the mathematical expression of some physical conservation law.

The one step Lax Wendroff method uses a truncated Taylor series expansion in time \( t \).
\[ u_{i}^{k+1} = u_{i}^{k} + \Delta t \left( \frac{\partial u}{\partial t} \right)_{i}^{k} + \frac{\Delta t^2}{2} \left( \frac{\partial^2 u}{\partial t^2} \right)_{i}^{k} \]

where \( u(x,t) \) has been replaced by \( u(i\Delta x, k\Delta t) = u_{i}^{k} \) for finite difference computations. The first time derivative is then replaced by \( \frac{\partial u}{\partial t} = -\frac{\partial f}{\partial x} \) and the second time derivative by \( \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x}(\frac{\partial f}{\partial t}) = \frac{\partial}{\partial x}(A \frac{\partial f}{\partial x}) \)

where the matrix \( A \) is the Jacobian of \( f(u) \) with respect to \( u \). That is \( A = (A_{ij}) \) where \( A_{ij} = \frac{\partial f_i}{\partial u_j} \).

Hence,

\[ u_{i}^{k+1} = u_{i}^{k} - \Delta t \left( \frac{\partial f}{\partial x} \right)_{i}^{k} + \frac{\Delta t^2}{2} \left[ \frac{\partial}{\partial x}(A \frac{\partial f}{\partial x}) \right]_{i}^{k} \]

Time derivatives have been replaced by space derivatives which are now approximated by central differences to give

\[ u_{i}^{k+1} = u_{i}^{k} - \frac{1}{2} \left( f_{i+1}^{k} - f_{i-1}^{k} \right) + \frac{\Delta t^2}{2} \left[ A_{i+\frac{1}{2}}^{k} \left( f_{i+1}^{k} - f_{i}^{k} \right) - A_{i-\frac{1}{2}}^{k} \left( f_{i}^{k} - f_{i-1}^{k} \right) \right], \]

where

\[ A_{i+\frac{1}{2}}^{k} = A \left( \frac{1}{2} u_{i+1}^{k} + \frac{1}{2} u_{i}^{k} \right) \]

The one step Lax Wendorff scheme, given by (3.4), is second order accurate in \( \Delta x \) and \( \Delta t \) (refer Ames [25]).

The two step method is simpler to implement since it does not require the evaluation of the Jacobian matrix \( A \).

The name "two step" refers to the use of an intermediate step at \((i+\frac{1}{2}, k+\frac{1}{2})\) and \((i-\frac{1}{2}, k+\frac{1}{2})\). Intermediate values are calculated at these points using the first order accurate
formula
\[ u_{i+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2}(u_{i+1}^{k} + u_{i}^{k}) - \frac{\Delta t}{\Delta x} (f_{i+\frac{1}{2}}^{k+\frac{1}{2}} - f_{i-\frac{1}{2}}^{k-\frac{1}{2}}), \]  
(3.5)

where the differential equation has been centred at \((i+\frac{1}{2}, k)\) and central space differences and forward time differences used. The final second order accurate values are obtained by central differencing both space and time derivatives around the point \((i, k+\frac{1}{2})\),

\[ u_{i}^{k+1} = u_{i}^{k} - \frac{\Delta t}{\Delta x} (f_{i+\frac{1}{2}}^{k+\frac{1}{2}} - f_{i-\frac{1}{2}}^{k-\frac{1}{2}}), \]  
(3.6)

where

\[ f_{i+\frac{1}{2}}^{k+\frac{1}{2}} = f(u_{i+\frac{1}{2}}^{k+\frac{1}{2}}). \]

Both schemes are extensively discussed by Richtmyer and Morton [24], Ames [25] and others.

The third method investigated is based on the equation

\[ \frac{\partial u}{\partial t} + A(u,x) \frac{\partial u}{\partial x} = z(u,x), \]  
(3.7)

where as before \( u = u(x,t) \) is an \( n \) component vector and now \( A(u,x) \) is an \( n \times n \) nonlinear matrix function of \( u \) and \( x \), and \( z(u,x) \) is an \( n \) component nonlinear vector function of \( u \) and \( x \). The conservation form (3.3) may always be written in the form (3.7) with \( A \) then being the Jacobian matrix already referred to, but the reverse is not always possible.
A more general equation in which the matrix $A$ and the vector $z$ may be functions of time as well as $u$ and $x$, is discussed by Gourlay and Morris [26]. The method of solution proposed by these authors is a more general form of the method to be described here, which is an extension of the two step Lax Wendroff technique.

An intermediate step is computed with first order accuracy from

$$u_{i+1} = \frac{1}{2}(u_{i+1} + u_i) - \frac{\Delta t}{2\Delta x} A_{i+\frac{1}{2}}(u_{i+1} - u_i) + \frac{\Delta t}{2} z_{i+1}, \quad (3.8)$$

where

$$A_{i+\frac{1}{2}}^k = A\left(\frac{1}{2}u_{i+1}^k + \frac{1}{2}u_i^k\right)$$

and

$$z_{i+\frac{1}{2}}^k = z\left(\frac{1}{2}u_{i+1}^k + \frac{1}{2}u_i^k\right).$$

The final second order accurate values are obtained from

$$u_i = u_i - \frac{\Delta t}{\Delta x} A_i^k (u_{i+1} - u_{i-1}) + \Delta t z_i^{k+\frac{1}{2}}, \quad (3.9)$$

where

$$A_i^{k+\frac{1}{2}} = A\left(\frac{1}{2}u_{i+1}^{k+\frac{1}{2}} + \frac{1}{2}u_i^{k+\frac{1}{2}}\right)$$

and

$$z_i^{k+\frac{1}{2}} = z\left(\frac{1}{2}u_{i+1}^{k+\frac{1}{2}} + \frac{1}{2}u_i^{k+\frac{1}{2}}\right).$$

The validity of this final step and its second order accuracy can be established by expressing all terms as Taylor series.
expansions about some arbitrary point. Taking the arbitrary point to be \((i, k + \frac{1}{2})\), (3.9) may be expanded out as

\[
\begin{align*}
&u_i^{k+\frac{1}{2}} + \frac{\Delta t}{2} \frac{\partial u}{\partial t} i^{k+\frac{1}{2}} + \frac{\Delta t}{8} \left( \frac{\partial^2 u}{\partial t^2} \right)_i^{k+\frac{1}{2}} + O(\Delta t^3) = u_i^{k+\frac{1}{2}} - \frac{\Delta t}{2} \frac{\partial u}{\partial t} i^{k+\frac{1}{2}} \\
&+ \frac{\Delta t^2}{8} \left( \frac{\partial^2 u}{\partial t^2} \right)_i^{k+\frac{1}{2}} + O(\Delta t^3) - \frac{\Delta t}{\Delta x} i^{k+\frac{1}{2}} \left[ u_i^{k+\frac{1}{2}} + \frac{\Delta x}{\Delta x} \left( \frac{\partial u}{\partial x} \right)_i^{k+\frac{1}{2}} \\
&+ \frac{\Delta x^2}{\Delta x} \left( \frac{\partial^2 u}{\partial x^2} \right)_i^{k+\frac{1}{2}} + O(\Delta x^3) \right] - u_i^{k+\frac{1}{2}} + \frac{\Delta x}{2} \left( \frac{\partial u}{\partial x} \right)_i^{k+\frac{1}{2}} - \frac{\Delta x^2}{2} \left( \frac{\partial^2 u}{\partial x^2} \right)_i^{k+\frac{1}{2}} \\
&+ O(\Delta x^3) \right] + \Delta t \epsilon_i^{k+\frac{1}{2}}. 
\end{align*}
\]

After simplifying, this becomes

\[
\frac{\partial u}{\partial t} i^{k+\frac{1}{2}} + O(\Delta t^2) + \epsilon_i^{k+\frac{1}{2}} \left[ \frac{\partial u}{\partial x} i^{k+\frac{1}{2}} + O(\Delta x^2) \right] = \epsilon_i^{k+\frac{1}{2}}
\]

and it can be seen that with the form of \(A_i^{k+\frac{1}{2}}\) and \(z_i^{k+\frac{1}{2}}\) used above the scheme retains overall second order accuracy.

The motivation for this scheme follows directly from the Lax Wendroff two step method and a scheme analogous to the one step method is also possible but is rather complicated since the usual one step method gains its simplicity from the conservation form of the equations.

For the frictionless equations when \(\alpha = 1\) and the wall is elastic the Lax Wendroff one and two step methods can be used. Then (2.25) and (2.26) become (using cross-section area \(S = \pi R^2\) as a dependent variable rather than arterial radius \(R\))

\[
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left( \frac{v^2}{2} + E \right) = 0
\]
and
\[
\frac{\partial S}{\partial t} + \frac{\partial SV}{\partial x} = 0 ,
\]
or
\[
\frac{\partial}{\partial t} \left[ V \right] + \frac{\partial}{\partial x} \left[ \frac{V^2}{2} + \frac{P}{\rho} \right] = 0 ,
\]
(3.10)

the required conservation form. A relation \( p = p(S, x) \) is also needed. The form introduced in section 2.4 is
\[
p = G_0 \left( \frac{S}{S_0} \right)^2 - 1 .
\]
(3.11)

Equation (3.11) is used to evaluate \( p \) at each time step after \( V \) and \( S \) have been obtained by solving (3.10) with (3.4) (one step) or (3.5) and (3.6) (two step).

The third technique mentioned can be used to solve (2.25), (2.26), (2.27) and (2.28) without simplification and since it appeared to have no disadvantages relative to the other two schemes, after some preliminary comparative calculations it was used exclusively. Substitution of (2.26) into (2.25), (2.27) and (2.28) yields the set
\[
\frac{\partial V}{\partial t} + (2\alpha-1)V \frac{\partial V}{\partial x} + 2(\alpha-1) \frac{V^2}{R} \frac{\partial R}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = - \frac{2\alpha}{\alpha-1} \frac{V}{R^2} ,
\]
\[
\frac{\partial R}{\partial t} + R \frac{\partial V}{\partial x} + V \frac{\partial R}{\partial x} = 0 ,
\]
\[
\frac{\partial P}{\partial t} + \left( \frac{G_1 + G_2}{R_0} + \frac{dpS}{dR} \right) V \frac{\partial R}{\partial x} + \left( \frac{G_1 + G_2}{R_0} + \frac{dpS}{dR} \right) R \frac{\partial V}{\partial x} = W
\]
and
\[
\begin{align*}
\frac{\partial W}{\partial t} - \frac{1}{R_{\infty}} \left( \frac{G_1 + G_2}{R_{\infty}} \right) V \frac{\partial R}{\partial x} - \frac{1}{2} \left( \frac{G_1 + G_2}{R_{\infty}} \right) \frac{R}{R_{\infty}} \frac{\partial V}{\partial x} = \left( p_s - p \right) / \tau_1 \tau_2 - \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} \right) W,
\end{align*}
\]

where

\[
p_s = G_0 \left( \frac{R}{R_{\infty}} \right)^\beta - 1,
\]

which can be written in matrix form as

\[
\begin{bmatrix}
V \\
R \\
P \\
W
\end{bmatrix}
= \begin{bmatrix}
(2\alpha-1)V & 2(\alpha-1)\frac{V^2}{R} & \frac{1}{\rho} & 0 \\
\frac{R}{2} & V & 0 & 0 \\
\frac{G_1 + G_2}{R_{\infty}} \frac{dP_s}{dR} & \left( \frac{G_1 + G_2}{R_{\infty}} \frac{dP_s}{dR} \right) V & 0 & 0 \\
-\frac{1}{2} \left( \frac{G_1 + G_2}{R_{\infty}} \right) \frac{R}{\tau_1 \tau_2} - \left( \frac{G_1 + G_2}{R_{\infty}} \right) \frac{V}{\tau_2} & \frac{G_1 + G_2}{R_{\infty}} \frac{V}{\tau_1} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
V \\
R \\
P \\
W
\end{bmatrix}
\]

\[
\frac{\partial W}{\partial t} = \begin{bmatrix}
-\frac{2\alpha}{\alpha-1} \frac{V^2}{R} & 0 \\
0 & 0
\end{bmatrix}
\]

This has the required form, \( \frac{\partial u}{\partial t} + A(u, x), \frac{\partial u}{\partial x} = 2(u, x) \) and may be solved using (3.5) and (3.6).

3.3 Stability Considerations.

The question of stability and errors is most easily visualized in terms of the characteristics which define the paths of propagation of disturbances in the \((x, t)\) plane.

The characteristic directions for the system of equations were found in Section 2.5. Two characteristics are vertical \( \left( \frac{dx}{dt} = 0 \right) \) and the other two can be drawn from points \((i-1, k)\) and \((i+1, k)\) (see Figure 7) to define a domain of dependence for the point of their intersection. The sloping characteristics \(XZ\) and \(YZ\) are given by

\[
\frac{dx}{dt} = \alpha V \pm \left[ \alpha(\alpha-1)V^2 + \frac{R}{2} \left( \frac{G_1 + G_2}{R_{\infty}} + \frac{dP_s}{dR} \right) \frac{1}{2} \right]
\]
and only approximate to straight lines for small $\Delta t$ and $\Delta x$.

![Figure 7](image)

All the information needed to compute the vector $u$ at $Z$ is contained on the line $XY$ (see Ames [35]). When the point $(i,k+1)$ is exactly at $Z$ then the computation is stable and has minimal errors. If an attempt is made to compute $u(i,k+1)$ at $Z'$, ahead of $Z$, with information given on $XY$ then the computation will be unstable since not enough information is used (extending characteristics back from $Z'$ will define a domain of dependence wider than $XY$). On the other hand, if $u(i,k+1)$ is computed at $Z''$ then too much information is used and an error is introduced. The argument presented here is only approximate since the characteristics are curves rather than straight lines and vary considerably from node to node. The only method of finding accurate stability limits and error magnitudes is to perform numerical calculations with a decreasing time step (this is done in the next chapter). Reference to a more rigorous stability and error analysis will be found in
3.4 Boundary Conditions.

Using any of the above finite difference schemes one may compute the values of \( u_1^{k+1} \) for \( i = 2, N-1 \), where \( N \) is the total number of spatial nodes defining the artery segment, from values of \( u_1^k \) prescribed on \( i = 1, N \). The question then arises as to how the end point values \( u_1^{k+1} \) and \( u_N^{k+1} \) are obtained.

Although (3.12) is a set of four first order equations, it was shown in Section 2.5 that only two of the roots of the characteristic equation are non-zero and hence the system requires the specification of only two boundary conditions. One boundary condition is required at each end of the artery since each boundary has one characteristic sloping away from it (refer Courant and Hilbert [19]). The initial conditions require specification of all four dependent variables since four characteristics leave the initial row of mesh points. A factor to be taken into account when deciding a suitable method of computing the non-prescribed end point components of \( u \), is the region of dependence of \( u_1^{k+1} \) and \( u_N^{k+1} \). The following argument applies to \( u_1^{k+1} \) but a completely analogous argument holds for \( u_N^{k+1} \). The domain of dependence is shown shaded in figure 8 (in the diagram it is assumed that the numerical velocity \( \Delta x/\Delta t \) is exactly equal
to the physical velocity so that the diagonal characteristic passes through the mesh points, which in practice it will not quite do).

Ideally, information lying outside the shaded region should not enter into the computation of \( u_1^{k+1} \) (the prescribed component is at \((1, k+1)\) and so lies within this region). In the method of characteristics, relationships between the variables are defined along the characteristics so that there are three equations in four unknowns and specification of one of these variables enables the others to be found. However, the characteristic sloping away from the mesh point \((1, k+1)\) does not pass exactly through the mesh point \((2, k)\) and an interpolation procedure is required. The method proposed here uses \( u_1^k \), \( u_2^k \) and \( u_2^{k+1} \), this last vector having been previously found by the general scheme. Since the use of \( u_2^{k+1} \) generates a small error and since only three equations are needed to solve for the three unknown components of \( u_1^{k+1} \), then one component of \( u_2^{k+1} \) can be eliminated from the four equations available and so reduce the error.
The scheme is obtained by centring the equation

\[ \frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = z \] at the point \( \left( \frac{3}{2}, k + \frac{1}{2} \right) \), hence

\[ u_{k+1}^{\frac{1}{2}} - u_{k}^{\frac{1}{2}} \frac{k}{\Delta t} + A_{k+1}^{\frac{1}{2}} \frac{k+1}{\Delta x} - u_{k}^{\frac{1}{2}} = z_{\frac{3}{2}}^{\frac{1}{2}} \]

and then using \( u_{k}^{\frac{1}{2}} = \frac{1}{2} u_{k} + \frac{1}{2} u_{k+1} \) and \( u_{k+1}^{\frac{1}{2}} = \frac{1}{2} u_{k+1} + \frac{1}{2} u_{k+1} \)

and referring to the point \( \left( \frac{3}{2}, k + \frac{1}{2} \right) \) with the superscript "*," \( u_{1}^{k+1} \) is obtained from

\[ u_{1}^{k+1} + u_{2}^{k+1} - u_{1} - u_{2} \frac{\Delta t}{\Delta x} A^{*}(u_{2}^{k+1} - u_{1} - u_{2} - u_{1} - u_{1}) = z_{2}^{*} \]

where \( A^{*} = A(u_{2}^{k+1}) \) and \( z^{*} = z(u_{2}^{k+1}) \).

\( u_{2}^{k+1} \) is computed to first order accuracy with the general scheme (3.8) and \( u_{2}^{k+1} \) is computed to second order accuracy with the general scheme (3.9), both vectors being at interior points, all of which are calculated before the end points.

The second order accuracy of the scheme can be illustrated by expressing each term as a Taylor series expansion about the point \( \left( \frac{3}{2}, k + \frac{1}{2} \right) \). Thus,

\[ u_{1}^{k+1} = u^{*} \frac{\Delta t}{2} \left( \frac{\partial u^{*}}{\partial t} \right) + \frac{\Delta x^{*}}{8} \left( \frac{\partial^{2} u^{*}}{\partial x^{2}} \right) - \frac{\Delta x^{*} \Delta t}{4} \left( \frac{\partial^{2} u^{*}}{\partial t \partial x} \right) \]

\[ u_{2}^{k+1} = u^{*} \frac{\Delta t}{2} \left( \frac{\partial u^{*}}{\partial t} \right) + \frac{\Delta x^{*}}{8} \left( \frac{\partial^{2} u^{*}}{\partial x^{2}} \right) - \frac{\Delta x^{*} \Delta t}{4} \left( \frac{\partial^{2} u^{*}}{\partial t \partial x} \right) \]

\[ u_{1}^{k} = u^{*} \frac{\Delta t}{2} \left( \frac{\partial u^{*}}{\partial t} \right) + \frac{\Delta x^{*}}{8} \left( \frac{\partial^{2} u^{*}}{\partial x^{2}} \right) - \frac{\Delta x^{*} \Delta t}{4} \left( \frac{\partial^{2} u^{*}}{\partial t \partial x} \right) \]

\[ u_{2}^{k} = u^{*} \frac{\Delta t}{2} \left( \frac{\partial u^{*}}{\partial t} \right) + \frac{\Delta x^{*}}{8} \left( \frac{\partial^{2} u^{*}}{\partial x^{2}} \right) - \frac{\Delta x^{*} \Delta t}{4} \left( \frac{\partial^{2} u^{*}}{\partial t \partial x} \right) \]
so that
\[ u_{1}^{k+1} + u_{2}^{k+1} - u_{1}^{k} - u_{2}^{k} = 2\Delta t \frac{\partial u_{1}^{k}}{\partial t} \]
and
\[ u_{2}^{k+1} + u_{2}^{k} - u_{1}^{k+1} - u_{1}^{k} = 2\Delta x \frac{\partial u_{1}^{k}}{\partial x} \]

Now (3.13) becomes
\[ (\frac{\partial u}{\partial t})^{*} + O(\Delta t^{2}) + A^{*} [(\frac{\partial u}{\partial x})^{*} + O(\Delta x^{2})] = z^{*} \]
and \( A^{*} \) and \( z^{*} \) need only be calculated with first order accuracy to retain the overall second order accuracy of the scheme.

For the general set of equations, \( u = (V,R,p,W)^{T} \) and (3.13) represents four equations. One of the components of \( u \) is prescribed at the boundary, so there are only three unknowns and the four equations can be reduced to three by eliminating one component of \( u_{2}^{k+1} \). It is not obvious which component should be eliminated but the easiest one to eliminate is \( V_{2}^{k+1} \). An alternative to eliminating one of the four components of \( u_{2}^{k+1} \) is to ignore one equation completely and hence solve for the three unknown components of \( u_{1}^{k+1} \) with the other three equations. The effect these different boundary procedures have on the solution of \( V_{1}^{k+1} \) from a prescribed pressure \( p_{1}^{k+1} \) will be examined numerically in Section 4.5.
4. FINITE DIFFERENCE COMPUTATIONS

4.1 Introduction.

Three finite difference techniques have been proposed for the solution of the blood flow equations developed in Chapter 2. Of these, two require the equations to be limited to the elastic wall case and so will not be developed any further. The third technique can deal with the full nonlinear viscoelastic wall and since it can be used for the elastic wall case with little alteration, the method is used for these calculations as well.

It was shown in Section 3.3 that the choice of the ratio $\Delta x/\Delta t$ affects both the stability of the numerical scheme and the introduction of errors. Computations involving this choice are illustrated in Section 4.3. The choice of $\Delta x$ and $\Delta t$ for a given ratio $\Delta x/\Delta t$ is determined by physical considerations (for example how rapidly the rheological properties of the wall change down the artery), by the accuracy required and by available computer time. The length of a heart beat is approximately one second and the rapid changes in pressure and flow occurring in that period necessitate a large number of time steps to accurately define the pulse. The reduction in error achieved by halving the distance and time steps (retaining the same ratio $\Delta x/\Delta t$) is an indication of the time step needed for a desired
accuracy. Computations along these lines are presented in Section 4.4 and the problem of errors introduced at a boundary is discussed in the following section. The effects of parameter changes on flow, velocity and pressure pulses are illustrated in Section 4.6.

In all calculations the boundary condition at the proximal end of the artery consists of a prescribed pressure pulse. This is used in preference to a prescribed velocity or flow pulse because it has been found by others (for example Anliker, Rockwell and Ogden [11]) that the velocity or flow calculated at the proximal end of the artery is a more sensitive test of the model. The pressure pulse is less responsive to changes in model parameters and is therefore a more logical choice for a boundary condition. The distal end of the artery is sufficiently far away to ensure that there are no reflection effects. All computations are begun with the artery at a pressure of 80 mm Hg and zero flow everywhere (diastolic conditions).

Before any of these computations can proceed, however, the numerical scheme given by (3.8) and (3.9) must be applied to the set of equations (3.12) and this is the subject of the next section.

4.2 Finite Difference Equations.

Applying (3.8) and (3.9) to (3.12), the following equations are obtained for the calculation of \( u_{i}^{k+1} \) for
i = 2, N-1 given \( u_{i}^{k} \), i = 1; N. Intermediate values are computed with

\[
k_{i+\frac{1}{2}}^{+} = \frac{1}{2}(V_{i+1}^{1} + V_{i}^{1}) - \Delta t \left[ \frac{2\alpha - 1}{2} \left( V_{i+1}^{1} - V_{i-1}^{1} \right) + (\alpha - 1) \frac{(V_{i+1}^{1} + V_{i-1}^{1})^2}{(R_{i+1}^{1} + R_{i-1}^{1})} (R_{i+1}^{1} - R_{i-1}^{1}) \right] + \frac{1}{\rho} (p_{i+1}^{1} - p_{i}^{1}) \right)^k - 2 \Delta t \frac{\alpha}{\alpha - 1} \left[ \frac{(V_{i+1}^{1} + V_{i}^{1})}{(R_{i+1}^{1} + R_{i}^{1})^2} \right]^k + (4.1)
\]

\[
k_{i+\frac{1}{2}}^{-} = \frac{1}{2}(R_{i+1}^{1} + R_{i}^{1}) - \Delta t \left[ \frac{1}{2} (R_{i+1}^{1} + R_{i}^{1}) (V_{i+1}^{1} - V_{i}^{1})^2 + \frac{1}{2} (V_{i+1}^{1} + V_{i}^{1}) (R_{i+1}^{1} - R_{i}^{1}) \right]^k , \quad (4.2)
\]

\[
p_{i+\frac{1}{2}}^{1} = \frac{1}{2}(p_{i+1}^{1} + p_{i}^{1}) - \Delta t \left[ \frac{1}{2} (p_{i+1}^{1} + p_{i}^{1}) - \beta \left( V_{i+1}^{1} - V_{i}^{1} \right) + \beta \left( V_{i+1}^{1} + V_{i}^{1} \right) (R_{i+1}^{1} - R_{i}^{1}) \right]^k \times 2 \left( \frac{G_{0}^{1} + G_{0}^{2}}{(G_{0}^{1} + G_{0}^{2})^2} \left( \frac{R_{i+1}^{1} + R_{i}^{1}}{R_{i+1}^{1} + R_{i}^{1}} \right)^{\beta - 1} \right) \left( R_{i+1}^{0} + R_{i}^{0} \right)^{\beta - 1} + \frac{\alpha t}{4} (w_{i+1}^{1} + w_{i}^{1})^k , \quad (4.3)
\]

and

\[
w_{i+\frac{1}{2}}^{1} = \frac{1}{2}(w_{i+1}^{1} + w_{i}^{1}) + \Delta t \left[ \frac{1}{2} (w_{i+1}^{1} + w_{i}^{1}) - \beta \left( V_{i+1}^{1} - V_{i}^{1} \right) + \beta \left( V_{i+1}^{1} + V_{i}^{1} \right) (R_{i+1}^{1} - R_{i}^{1}) \right]^k \times 2 \left( \frac{G_{0}^{1} + G_{0}^{2}}{(G_{0}^{1} + G_{0}^{2})^2} \left( \frac{R_{i+1}^{1} + R_{i}^{1}}{R_{i+1}^{1} + R_{i}^{1}} \right)^{\beta - 1} \right) \left( R_{i+1}^{0} + R_{i}^{0} \right)^{\beta - 1} - \Delta t \left( 1 + \frac{1}{\alpha} \right) \left( \frac{1}{R_{i+1}^{1}} + \frac{1}{R_{i}^{1}} \right) (w_{i+1}^{1} + w_{i}^{1})^k . \quad (4.4)
\]

Final values are computed with

\[
v_{i}^{k+1} = v_{i}^{1} - \Delta t \left[ \frac{2\alpha - 1}{2} \left( V_{i+\frac{1}{2}}^{1} - V_{i-\frac{1}{2}}^{1} \right) + (\alpha - 1) \frac{(V_{i+\frac{1}{2}}^{1} + V_{i-\frac{1}{2}}^{1})^2}{(R_{i+\frac{1}{2}}^{1} + R_{i-\frac{1}{2}}^{1})} (R_{i+\frac{1}{2}}^{1} - R_{i-\frac{1}{2}}^{1}) \right] + \frac{1}{\rho} (p_{i+\frac{1}{2}}^{1} - p_{i-\frac{1}{2}}^{1}) \right)^{k+\frac{1}{2}} - 4 \Delta t \frac{\alpha}{\alpha - 1} \left[ \frac{(V_{i+\frac{1}{2}}^{1} + V_{i-\frac{1}{2}}^{1})}{(R_{i+\frac{1}{2}}^{1} + R_{i-\frac{1}{2}}^{1})^2} \right]^{k+\frac{1}{2}} , \quad (4.5)
\]

\[
r_{i}^{k+1} = r_{i}^{1} - \Delta t \left[ \frac{1}{2} (R_{i+\frac{1}{2}}^{1} + R_{i-\frac{1}{2}}^{1}) (V_{i+\frac{1}{2}}^{1} - V_{i-\frac{1}{2}}^{1}) + \frac{1}{2} (V_{i+\frac{1}{2}}^{1} + V_{i-\frac{1}{2}}^{1}) (R_{i+\frac{1}{2}}^{1} - R_{i-\frac{1}{2}}^{1}) \right]^{k+\frac{1}{2}} \right) , \quad (4.6)
\]
\[ p_{i+\frac{1}{2}}^{k+1} = p_{i+\frac{1}{2}}^k - \frac{\Delta t}{\Delta x} \left[ \frac{1}{4} (R_{i+\frac{1}{2}} + R_{i-\frac{1}{2}}) (V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}) + \frac{1}{4} (V_{i+\frac{1}{2}} + V_{i-\frac{1}{2}}) (R_{i+\frac{1}{2}} - R_{i-\frac{1}{2}}) \right]^{k+\frac{3}{2}} \]

\[ \times \left[ G_1 + G_2 + \beta G_{O1} \left( \frac{R_{i+\frac{1}{2}} + R_{i-\frac{1}{2}}}{2R_{O1}} \right)^{\beta - 1} \right] / R_{O1} + \frac{\Delta t}{2} \left( \frac{1}{w_{i+\frac{1}{2}} + w_{i-\frac{1}{2}}} \right)^{k+\frac{1}{2}} \]

(4.7)

and

\[ w_{i+\frac{1}{2}}^{k+1} = w_{i+\frac{1}{2}}^k + \frac{\Delta t}{\Delta x} \left[ \frac{1}{4} (R_{i+\frac{1}{2}} + R_{i-\frac{1}{2}}) (V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}) + \frac{1}{4} (V_{i+\frac{1}{2}} + V_{i-\frac{1}{2}}) (R_{i+\frac{1}{2}} - R_{i-\frac{1}{2}}) \right]^{k+\frac{3}{2}} \]

\[ \times \left( \frac{G_1 + G_2}{\tau_1 + \tau_2} \right) / R_{O1} + \frac{\Delta t}{2} \left( \frac{R_{i+\frac{1}{2}} + R_{i-\frac{1}{2}}}{2R_{O1}} \right)^{\beta - 1} \] - \frac{\Delta t}{2} \left( \frac{1}{\tau_1 + \tau_2} \right) \left( w_{i+\frac{1}{2}} + w_{i-\frac{1}{2}} \right)^{k+\frac{1}{2}} \]

(4.8)

In these equations, \( G_1, G_2, \tau_1, \tau_2 \) and \( \beta \) are assumed to be independent of \( x \). The term \( G_{O1} \) is prescribed as a function of \( x \) so that the effect of increasing the wall stiffness along the artery can be examined. The unstressed arterial radius \( R_{O1} \) is given \( x \) dependence to allow for a tapering artery.

If a purely elastic wall is being modelled then (4.3) and (4.4) are replaced by

\[ p_{i+\frac{1}{2}}^{k+\frac{3}{2}} = \frac{1}{2} (G_{O1} + G_{O1}) \left( \frac{R_{i+\frac{1}{2}} + R_{i-\frac{1}{2}}}{R_{O1}} \right)^{\beta} \]

and (4.7) and (4.8) are replaced by

\[ p_{i+\frac{1}{2}}^{k+\frac{3}{2}} = G_{O1} \left( \frac{R_{i+\frac{1}{2}} + R_{i-\frac{1}{2}}}{2R_{O1}} \right)^{\beta} \]

Applying the boundary scheme (3.13) to the system of equations (3.12) gives the following set of equations (as
before the superscript "*" refers to the point \((\frac{3}{2}, k+\frac{1}{2})\):

\[
V_{k}^{k+1} + V_{2}^{k+1} - V_{1}^{k} - V_{2}^{k} + \frac{\Delta t}{\Delta x} [(2\epsilon - 1) V^{*} (V_{2}^{k+1} + V_{2}^{k-1} - V_{1}^{k})
+ 2(a-1) \left( \frac{V^{*}}{R^{*}} \right)^{2} (R_{2}^{k+1} - R_{2}^{k-1} - R_{1}^{k}) + \frac{1}{\rho} (p_{2}^{k+1} - p_{2}^{k-1} - p_{1}^{k})]
= - 4\Delta t \frac{a}{a-1} \frac{V^{*}}{(R^{*})^{2}},
\]

\[
\left. \begin{array}{c}
R_{1}^{k+1} + R_{2}^{k+1} - R_{1}^{k} - R_{2}^{k} + \frac{\Delta t}{\Delta x} \left[ \frac{1}{2} R^{*} (V_{2}^{k+1} + V_{2}^{k-1} - V_{1}^{k}) + V^{*} (R_{2}^{k+1} - R_{2}^{k-1} - R_{1}^{k}) \right]
\end{array} \right\} = 0,
\]

\[
\left. \begin{array}{c}
p_{1}^{k+1} + p_{2}^{k+1} - p_{1}^{k} - p_{2}^{k} + \frac{\Delta t}{\Delta x} \left[ \frac{1}{2} R^{*} (V_{2}^{k+1} + V_{2}^{k-1} - V_{1}^{k}) + V^{*} (R_{2}^{k+1} - R_{2}^{k-1} - R_{1}^{k}) \right]
\end{array} \right\} = 2 \Delta t W^{*},
\]

and

\[
W_{1}^{k+1} + W_{2}^{k+1} - W_{1}^{k} - W_{2}^{k} + \frac{\Delta t}{\Delta x} \left[ \frac{1}{2} R^{*} (V_{2}^{k+1} + V_{2}^{k-1} - V_{1}^{k}) + V^{*} (R_{2}^{k+1} - R_{2}^{k-1} - R_{1}^{k}) \right]
\]

\[
\times 2 (G_{1} + G_{2}) / (R_{O1} + R_{O2}) = \frac{\Delta t}{\tau_{1} \tau_{2}} (G_{O1} + G_{O2}) \left[ \left( \frac{2 R^{*}}{R_{O1} + R_{O2}} \right)^{\beta-1} - 1 \right]
- 2 \frac{\Delta t}{\tau_{1} \tau_{2}} p^{*} - 2 \Delta t \left( \frac{1}{\tau_{1}} + \frac{1}{\tau_{2}} \right) W^{*}.
\]

Using (4.10) in (4.9), (4.11) and (4.12) to eliminate \(V_{2}^{k+1}\) results in a set of three equations in three unknowns. If \(p_{1}^{k+1}\) is the prescribed variable then \(R_{1}^{k+1}\) is obtained directly in terms of \(p_{1}^{k+1}\) from

\[
R_{1}^{k+1} = R_{1}^{k} + R_{2}^{k+1} - R_{2}^{k} + (p_{1}^{k+1} - p_{1}^{k} - p_{2}^{k} - 2 \Delta t W^{*})
\]

\[
\times \left[ \left( \frac{2 R^{*}}{R_{O1} + R_{O2}} \right)^{\beta-1} \right] / (G_{1} + G_{2} + \frac{p}{2} (G_{O1} + G_{O2}) (\frac{2 R^{*}}{R_{O1} + R_{O2}})^{\beta-1})
\]

(4.13)
and \( W_{1}^{k+1} \) is then expressed in terms of \( R_{1}^{k+1} \) as

\[
W_{1}^{k+1} = W_{1}^{k} + \frac{k}{k+1}W_{2}^{k+1} - 2\left(\frac{G_{1} + G_{2}}{R_{1}^{k} + R_{2}^{k+1}} - \frac{k}{k+1} \frac{G_{1} + G_{2}}{R_{1}^{k} - R_{2}^{k}}\right) / \left(\frac{R_{1}^{k+1}}{R_{1}^{k}} + \frac{R_{2}^{k+1}}{R_{2}^{k}}\right) \]

\[
+ \frac{\Delta t}{\tau_{1} \tau_{2}} \left(\frac{2 R_{1}^{*}}{R_{O_{1}} + R_{O_{2}}} - 1\right) - \frac{2 \Delta t}{\tau_{1} \tau_{2}} \frac{\Delta V^{*}}{V^{*}} - 2 \Delta t \left(\frac{1}{\tau_{1}} + \frac{1}{\tau_{2}}\right) W^{*}. \tag{4.14}
\]

Finally \( V^{k+1} \) is calculated with

\[
V_{1}^{k+1} = V_{2}^{k} + \frac{\Delta t}{\tau_{X}} \left(\frac{(2 \alpha - 1) \frac{v^{*} R_{1}^{k+1} - R_{2}^{k+1} - R_{1}^{k+1} + R_{2}^{k}}{R_{2}^{k+1} + R_{2}^{k}}}{R_{2}^{k+1} + R_{2}^{k}}\right) - \frac{\alpha}{\rho} \left[ \frac{v^{*} R_{1}^{k+1} - R_{2}^{k+1} + R_{2}^{k}}{R_{2}^{k+1} + R_{2}^{k}} \right] \]

\[
- 2 \alpha \Delta V^{*} \left(\frac{\Delta V^{*}}{V^{*}} + 2 \frac{\Delta V^{*}}{V^{*}}\right) \left(\frac{R_{1}^{k+1} + R_{2}^{k+1} - R_{1}^{k} - R_{2}^{k}}{R_{2}^{k+1} + R_{2}^{k}}\right) - \frac{2 V^{*}}{R^{*}} \left(\frac{R_{1}^{k+1} - R_{2}^{k+1}}{R_{2}^{k+1} + R_{2}^{k}}\right). \tag{4.15}
\]

For the case of the purely elastic wall, (4.13) and (4.14) are replaced by

\[
R_{1}^{k+1} = R_{O_{1}} \left(\frac{P_{1}^{k+1}}{G_{O_{1}} + 1}\right)^{\frac{1}{2}}.
\]

Equations (4.1) to (4.8) and the boundary equations (4.13), (4.14), and (4.15) are translated into FORTRAN IV in Appendix A. The boundary equations appear in the subroutine entitled "BDRY" and the interior equations in "LAX".

### 4.3 Stability and Errors for a Given Time Step

The discussion in section 3.3 demonstrated that the numerical scheme is stable for \( \frac{\Delta x}{\Delta t} > \frac{dx}{dt} \), that is

\[
\frac{\Delta x}{\Delta t} > |a V^{*} [\alpha (\alpha - 1) V^{2} + \frac{R}{2 \rho} (G_{1} + G_{2} \frac{d P_{S}}{d R})^{1/2} + c^{2}]^{1/2} |,
\]

or

\[
\frac{\Delta x}{\Delta t} > |a V^{*} [\alpha (\alpha - 1) V^{2} + c^{2}]^{1/2} | \tag{4.16}
\]
where \( c \) is the wavespeed (at zero mean flow).

The velocity of blood is seldom greater than 1 m/sec while \( c \) is approximately 5 m/sec, so that (with \( \alpha \) approximately 1.1) the stability criterion is approximately

\[
\frac{\Delta x}{\Delta t} > \alpha |V| + c
\]

Increasing \( \Delta x/\Delta t \) away from the critical value \((\alpha |V| + c)\) results in a loss of accuracy. In the computations illustrated in Figure 9, use is made of the elastic wall equation with \( R_0 = 1.5 \text{ cm} \), \( \beta = 1 \) and \( G_0 = 160 \text{ mm Hg} \), and the frictionless flow equation with \( v = 0 \) and \( \alpha = 1 \). The time step is fixed at \( \Delta t = 0.1 \text{ secs} \) and the proximal boundary condition is a pressure pulse defined over sixty four time steps, beginning at a diastolic pressure of 80 mm Hg, rising to a systolic peak of 120 mm Hg and falling off to 80 mm Hg again. The initial conditions consist of the artery being at 80 mm Hg with zero flow everywhere. The initial arterial radius is calculated for a pressure of 80 mm Hg. These conditions are very close to those that exist at the completion of one cycle, so that subsequent cycles are the same as the first.

The graphs show the pressure and velocity pulses as they pass a point 90 cm from the proximal end of the artery for three values of \( \Delta x : \Delta x = 10 \text{ cm}, 7.5 \text{ cm} \) and \( 6.92 \text{ cm} \). The values are chosen so that an integral value results when
Figure 9. The velocity and pressure pulses after travelling 90 cm, for three values of Δx (elastic wall model).
they are divided into 90 and hence data is always computed at the 90 cm mark. These values of $\Delta x$ correspond to $\Delta x/\Delta t = 10 \text{ m/sec}$, $7.5 \text{ m/sec}$ and $6.92 \text{ m/sec}$; smaller values of $\Delta x$ result in the computations becoming unstable. The predominant error produced by $\Delta x/\Delta t$ being greater than the critical value is a small decrease in wavespeed shown by the wave appearing at the 90 cm point later in time. This is particularly apparent at the higher pressure portion of the pulse and the region immediately behind the peak shows the largest discrepancy.

The choice of $\Delta x/\Delta t$ for a given $\Delta t$ was made in subsequent calculations by finding the $\Delta x$ at which the peak of the pressure pulse advances approximately one distance step ($\Delta x$) for every time step. The inaccuracy resulting from having too large a value of $\Delta x/\Delta t$ is exaggerated in Figure 9 because the pulse is shown after it has travelled 90 cm, a distance two or three times the length of the human or dog aorta.

![Figure 10. The velocity pulse after travelling 90 cm, for three values of $\Delta x$ (viscoelastic wall model).](image-url)
A similar check with the viscoelastic model is shown in Figure 10. The parameters: \( \alpha = 1.1, \beta = 2, \)
\( G_0 = 50 \text{ mm Hg}, \ G_1 = 10 \text{ mm Hg}, \ G_2 = 33 \text{ mm Hg}, \) are used with
\( \Delta t = .01 \) and three values of \( \Delta x: \Delta x = 8.18 \text{ cm}, 6.43 \text{ cm} \) and
\( 6.0 \text{ cm}. \) At \( \Delta x = 6.0 \text{ cm} \) the computations become unstable,
giving rise to the small oscillations seen at systolic pressure (the instability will always first occur at the
high pressure portion of the pulse since this portion moves with the highest wavespeed). The velocity pulse is shown
as it passes the 90 cm mark and so the errors incurred by \( \Delta x/\Delta t \) being greater than the physical wavespeed are
exaggerated. However, an appreciation of these errors is essential if the scheme is to give accurate results.

4.4. An Accuracy Check for a Given Ratio of \( \Delta x \) to \( \Delta t \)

Figure 11 shows the effect on the velocity pulse of using small time and distance steps but retaining the same ratio \( \Delta x/\Delta t. \) One computation is made with \( \Delta t = .01 \text{ secs} \)
and \( \Delta x = 8.18 \text{ cm} \) and the other is made with one third of these values. The elastic wall equation with \( R_0 = 1.5 \text{ cm}, \)
\( \beta = 3 \) and \( G_0 = 23 \text{ mm Hg}, \) and the frictionless flow
equation with \( v = 0 \) and \( \alpha = 1 \) are employed for both computations. Boundary data is obtained from the same
continuous curve with the second case requiring three times as many time steps to cover the same time interval as the
first case. The pulses are recorded at the 90 cm mark to emphasise the difference between the two. The pressure pulses show a similar comparison to the velocity pulses and are therefore not shown.

The increased accuracy obtained by computing at the smaller time and distance steps is similar to that obtained by computing with a value of $\Delta x/\Delta t$ closer to the physical wave speed. Thus with very small time and distance steps the choice of $\Delta x/\Delta t$ has less effect on the accuracy, provided it is greater than the critical value.

1. $\Delta t = 0.01$ secs; $\Delta x = 8.18$ cm
2. $\Delta t = 0.01/3$ secs; $\Delta x = 8.18/3$ cm

Figure 11. The velocity pulse after travelling 90 cm with $\Delta x/\Delta t = 8.18$ m/sec, for two mesh sizes.
4.5 Errors Introduced at the Boundary.

It was pointed out in Section 3.4 that there is one more equation than there are unknowns at a boundary. It was also mentioned that use of the vector $u_{k+1}$ in the computation of $u_{k+1}$ leads to errors and that the magnitude of these errors can be reduced by eliminating one component of $u_{k+1}$ and hence leaving three equations in the three unknowns. Figure 12 shows the effect of three different schemes on the velocity pulse computed at the boundary. One uses the method of eliminating the component $v_{k+1}$ from the boundary equations as mentioned above (graph no 1 in Figure 12); another scheme leaves the equation of motion out of the boundary computations altogether and uses the continuity equation alone (graph no 2), while the third scheme uses the equation of motion and ignores the continuity equation (graph no 3).

![Figure 12. Three boundary schemes.](image)
The third scheme shows a physically unrealistic kink and so it may be inferred that the continuity equation cannot be left out of the boundary calculations. The second scheme is more reasonable and the only reason to suppose that the first scheme is the most accurate of the three is that it yields values which lie between the values predicted by the second and third schemes. Computations with branching arteries mentioned in Section 5.3 provide a better illustration of the errors incurred by ignoring the equation of motion.

If for any reason it is decided to use only one boundary equation then the calculations here indicate that the continuity equation will provide the best results.

4.6 Parameter Changes.

In this section several of the parameters are varied to find the effect they have on the solution of the equations. In all cases a pressure pulse is prescribed at the proximal end of the artery and the shape of the resulting pressure and velocity or flow pulses are observed as they propagate past a fixed point unless otherwise indicated, 90 cm from the source. The pressure pulse at the first mesh point will of course be unaffected by parameter changes, but the velocity or flow pulse computed at the same point shows up these changes very well.
In the computations illustrated in Figures 13 and 14 the elastic wall equation has been used with the frictionless flow equations \((v = 0, \alpha = 1\) and \(R_0 = 1.5\) cm). Figure 13 shows the effect of three values of the parameter \(G_o\) \((G_o = 11.5, 23\) and \(46\) mm Hg) and Figure 14 shows the result obtained when \(\beta\) and \(G_o\) are varied such that the distensibility curves pass through a common point \((p_s = 160\) mm Hg at \(R/R_0 = 2\)). The interesting result is that doubling the stiffness parameter \(G_o\) has little effect (Figure 13) while increasing the parameter \(\beta\) (appropriately modifying \(G_o\)) and thereby altering the degree of nonlinearity in the distensibility curve has a pronounced effect (Figure 14).

Figure 15 is an illustration of the effect the viscosity of the fluid has on the pressure and velocity pulses. Computations were made with the elastic wall equation with \(\beta = 3\), \(G_o = 23\) mm Hg, \(R_0 = 1.5\) cm and \(v = 0.032\) cm²/sec. These curves can be compared with the centre curves in the previous two graphs which have \(\beta = 3\), \(G_o = 23\) mm Hg and no friction. The difference is very slight and arises from the different shape of the velocity profile rather than the frictional effect at the wall (a computation with a flat profile but with frictional effects at the wall included, showed very little deviation from the no-friction case). The friction term becomes more important in small diameter arteries.
Figure 13. Changes in the stiffness parameter
Figure 14. Changes in the nonlinearity of the distensibility curve.
Figure 15. Changes in the velocity profile.
Steepening of the wavefront as the pulse propagates along the artery is evident in Figures 13, 14 and 15. This is a result of the wavespeed being an increasing function of pressure, namely

\[ c = \frac{\beta}{2\rho} (p + G_o). \]

(This relationship is obtained by inserting (2.24) in (2.29) for the elastic walled case). Thus the higher pressure portions of the pulse propagate with a greater velocity than the lower pressure portions and the wavefront steepens. Alternatively, the high pressure portion moves more rapidly because it encounters a more rigid tube. If the pulse travels far enough a shock wave develops and in the pathological condition known as aortic insufficiency, the elevated pressure enables a shock to develop within the length of the body and is heard in a stethoscope placed over the femoral artery as a "pistol shot" (for more details refer to Anliker et al. [11]).

---

**Diagram:**

- **Elastic wall**
- **Viscoelastic wall**

**Graph:**

- Y-axis: Velocity (cm/sec)
- X-axis: Time (secs)

**Legend:**

- 1 Elastic wall
- 2 Viscoelastic wall

**Note:** The effects of arterial wall viscosity.
Viscoelasticity of the arterial wall causes energy dissipation and hence damping of the pulse, but this only becomes significant when the pressure and arterial radius are changing rapidly. High frequency components of the pulse are therefore the most affected. The computations shown in Figure 16 are an illustration of the effect of the arterial wall viscoelasticity on a relatively smooth pulse, but nevertheless the slight damping caused by the viscoelastic wall can be seen. Both computations were made with the viscoelastic wall equation, one with \( \tau_1 = .157 \) secs, \( \tau_2 = .022 \) secs, \( G_1 = 10 \) mm Hg and \( G_2 = 33 \) mm Hg (viscoelastic wall) and the other with \( \frac{1}{\tau_1} = \frac{1}{\tau_2} = G_1 = G_2 = 0 \) (elastic wall). In both cases the parameters \( G_0 = 23 \) mm Hg, \( \beta = 3, \ R_0 = 1.5 \) cm, \( \alpha = 1 \) and \( \nu = 0 \) were employed. A comparison between Graph 1 in Figure 16 and the curve obtained with these parameters but with the elastic wall equation rather than the viscoelastic equation with \( \frac{1}{\tau_1} = \frac{1}{\tau_2} = G_1 = G_2 = 0 \), shows agreement within 1%. 

So far the changes have been confined to altering the parameters of the artery as a whole. The effect of tapering parameters is important since the distal end of the aorta is two or three times as rigid and considerably smaller in diameter than the proximal end. These two effects are examined separately in Figures 17 and 18. The effects of tapering the other parameters are not
nearly as marked and so are not illustrated here.

Figure 17 compares the pressure and flow from a parallel-walled artery of radius 1.0 cm (Graphs labelled "1") with those from an artery which narrows to 0.5 cm radius over a distance of 35 cm (Graphs "2") and from an artery which expands to 1.5 cm radius over the same distance (Graphs "3"). The taper begins 63 cm from the source of prescribed pressure and consequently the reflected wave resulting from the change of cross-section is also reflected from the proximal end of the artery. These reflected components (see Graphs "2" and "3") are responsible for the effects superimposed on the basic wave shape illustrated by Graphs "1". Diagrams labelled "A" show the pressure and flow recorded at the proximal end of the artery and Diagrams "B", "C" and "D" refer to points spaced at 42 cm intervals further on. Thus "B" is on a straight portion of artery proximal to the change of cross-section while "C" is at the midpoint and "D" distal to the change of cross-section.

The effect of a narrowing artery is to cause a pressure rise and a flow decrease while an expanding artery has the opposite effect. Measurements of the human or canine aortic pulse, providing no pathological condition is present, invariably show a pressure increase and flow decrease as the wave propagates down the aorta and it is reasonable to suppose that this is largely a result of the tapering of the
aorta away from the heart.

An added effect is provided by the distally decreasing distensibility of the aorta as seen in Figure 18. This too has the effect of raising the pressure and decreasing the flow. Figure 18 shows pressure and flow recorded at intervals of 42 cm, with Diagrams "A" and "B" referring to points proximal to the section of increasing stiffness, Diagram "C" lying within this region and Diagram "D" distal to it. The source of pressure is sufficiently removed to avoid the secondary reflections that appear in Figure 17.
Figure 17. The effect of a tapering artery.
Figure 18: The effect of a stiffening artery.
5. BRANCHING ARTERIES

5.1 Introduction.

Previous chapters have dealt with the development and testing of a mathematical model of blood flow in a single segment of artery. The modelling of flow in a network of arteries, or the study of flow down the aorta with its many large branches, requires further equations describing the flow through a junction of two or more arteries. Whereas previously a pressure pulse was prescribed at the end of the artery segment and the three boundary equations (for the viscoelastic artery) were then solved for the remaining three variables, now there are three artery segments issuing from one junction and an additional three branch equations are needed to supplement the three lots of three boundary equations. There are therefore twelve equations to be solved in the twelve unknowns (p, V, R and W for each segment).

Section 5.2 deals with the description of the branch equations and Section 5.3 examines the solution of these in conjunction with the other boundary equations. Finally the reflection and transmission of a wave entering a bifurcation is studied numerically in Section 5.4.

5.2 The Branch Equations.

Two general approaches to the formulation of the branch equations are possible. One is to write equations conserving
energy, mass and momentum across the junction and the other
uses the integrated form of these equations. The energy
equation is a statement that the rate of change of kinetic
energy due to both time variation and convection, is
caused by the rate at which work is done on the element
under consideration by body forces and by the pressure and
viscous stresses acting on it. When integrated over the
volume between two cross-sections a specified distance apart,
the Bernoulli equation results and includes terms such as
the work done by the viscous stresses on the fluid and wall
and the energy transmitted to the wall by the fluid. This
latter approach is described by Brown in Biomedical
Engineering [27], but as with the first approach there is
no way of evaluating many of the coefficients that arise in
the equations. Experimental measurements of flow through
artery junctions indicate a very complex situation in which
the velocity profiles are certainly not symmetrical and
turbulence often occurs during systole (Ling, Atabek and
Carmody [7]; Zeller Talukder and Lorenz [28]). The amount
of energy lost in secondary flow effects and the energy
absorbed by the surrounding tissue in the expansion of the
joint cannot be determined from any tractable theoretical
analysis so that we are compelled to take the following
more empirical approach.

The junction is approximated by three short elastic
tubes meeting at a common point. If the tubes are
sufficiently short then all the fluid in a tube can be regarded as moving at the same velocity and the tube as remaining parallel sided. Thus, \( V_a \) is the fluid velocity and \( S_a \) the cross-sectional area of tube "a". At one end tube "a" joins artery segment "a", at which point the pressure is \( p_a \), and at the other end it meets tubes "b" and "c" where the pressure is \( p_c \). The conservation of axial momentum for tube "a" (of length \( l_a \)), is therefore given by

\[
S_a p_a - S_a p_o = k_a \frac{\partial}{\partial t} (\rho l_a S_a V_a),
\]

where the resultant force has been equated to the rate of change of momentum of the fluid within the tube. The dimensionless factor \( k_a \) is included to account for the difference between the idealized situation and the real situation. This factor (and the similar ones for the other two tubes) cannot be calculated from first principles and is therefore an empirical factor to be evaluated by experimental measurement. The momentum flux entering the tube is equal to that leaving so a momentum flux term does not appear in (5.1). Fluid viscosity effects are regarded as being negligibly small.

Two similar equations can be written for the other tubes:

\[
S_b p_o - S_b p_b = k_b \frac{\partial}{\partial t} (\rho l_b S_b V_b)
\]

\[
S_c p_o - S_c p_c = k_c \frac{\partial}{\partial t} (\rho l_c S_c V_c).
\]
The sign convention used here is consistent with positive flow entering tube "a" and leaving via tubes "b" and "c". Finally, a conservation of mass equation is written for the junction as a whole with "Q" representing the fluid stored in the expansion of the junction:

\[
\frac{\partial Q}{\partial t} = S_v a - S_v b - S_v c,
\] (5.4)

where

\[
Q = c p_0
\] (5.5)

for an elastic walled junction with elastic constant \( c \).

So far we have described the equations conserving mass and momentum and by combining them an equation of energy conservation can be obtained. Hence, multiplying (5.1) by \( v_a \), (5.2) by \( v_b \) and (5.3) by \( v_c \) and adding them together yields the following equation:

\[
S_v a p_a - S_v b b_b - S_v c c_c - p_0 (S_v a - S_v b - S_v c) = k v_a \frac{\partial}{\partial t} (p_a S_v a) + k v_b \frac{\partial}{\partial t} (p_b S_v b) + k v_c \frac{\partial}{\partial t} (p_c S_v c).
\] (5.6)

Now each term on the R.H.S of this equation can be written as follows:

\[
k v_a \frac{\partial}{\partial t} (p_a S_v a) = k_a \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho a S_v a v_a^2 \right) + \frac{1}{2} \rho a v_a^2 \frac{\partial S_a}{\partial t} \right].
\]

Using the continuity equation valid for each artery segment (for example: \( \frac{\partial S_a}{\partial t} = - \frac{\partial S_a v_a}{\partial x} \)), (5.6) becomes
Making use of the divergence theorem and simplifying the L.H.S with (5.4) and (5.5) this becomes

\[ S_{V{p}_a a} a - S_{V{p}_b b} b - S_{V{p}_c c} c = \frac{1}{2} c \frac{\partial}{\partial t} \frac{\partial Q^2}{\partial x} + k_a \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \frac{S V^2}{a a} \right) - \rho \frac{S V^3}{a a} \]

\[ + k_b \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \frac{S V^2}{b b} \right) - \rho \frac{S V^3}{b b} \]

\[ + k_c \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \frac{S V^2}{c c} \right) - \rho \frac{S V^3}{c c} \]

(5.7)

This is a conservation of energy equation in which the rate at which work is done by the pressure forces (terms on the L.H.S. of (5.7)) is equated to the rate of change of kinetic energy of the fluid within the junction and the resultant energy flux of fluid entering the junction. That is,

kinetic energy = \( k_a \frac{1}{2} \rho \frac{S V^2}{a a} \) + \( k_b \frac{1}{2} \rho \frac{S V^2}{b b} \) + \( k_c \frac{1}{2} \rho \frac{S V^2}{c c} \) + \( \frac{1}{2} \frac{\partial}{\partial t} \) \( \frac{\partial Q^2}{\partial x} \)

resultant energy flux = \(- k_a \rho \frac{S V^3}{a a} - k_b \rho \frac{S V^3}{b b} - k_c \rho \frac{S V^3}{c c} \).

The factors \( k_a \), \( k_b \) and \( k_c \) can be adjusted to allow for the losses not accounted for and to correct for the oversimplification made by assuming all the fluid in tube "a" to have velocity \( V_a \) etc.

The idealized junction may be conveniently compared with the electrical network shown in Figure 19.
Pressure in the fluid mechanics system is analogous to voltage in the electrical system. Similarly, fluid flow becomes current and fluid volume becomes charge. The inductances offer an impedance to changing current and are analogous to the inertia of the fluid while the capacitance stores change in a manner analogous to the storage of fluid by an expanding junction. Also shown in Figure 19 are the resistances $R_a$, $R_b$ and $R_c$ which represent fluid viscosity and the resistance $R_o$ representing wall viscosity. Using the terminology of Figure 19 where current or fluid flow is denoted by $F$ ($=SV$), voltage or pressure by $p$ and fluid volume or charge by $Q$, the following equations apply to the network:

$$P_a - P_o = L_a \frac{\partial F_a}{\partial t} + R_a F_a,$$  \hspace{1cm} (5.8)
\[ P_0 - P_b = L_b \frac{\partial F_c}{\partial t} + R_b F_b \]  
\[ (5.9) \]
\[ P_0 - P_c = L_c \frac{\partial F_c}{\partial t} + R_c F_c \]  
\[ (5.10) \]
\[ P_0 = Q_c + R_c \frac{\partial Q}{\partial t} \]  
\[ (5.11) \]
\[ \frac{\partial Q}{\partial t} = F_a - F_b - F_c \]  
\[ (5.12) \]

Equation (5.12) is the same as (5.4) and the other four equations are equivalent to (5.1), (5.2), (5.3) and (5.5) with viscosity terms added. Hence, comparing (5.1) and (5.5),

\[ L_a = k_a \frac{\rho \Delta x}{S_a} \]

with similar expressions for \( L_b \) and \( L_c \). The capacitance \( C \) in (5.11) is equivalent to the elastic constant \( C \) in (5.5).

### 5.3 Finite Difference Equations

Ignoring the effects of fluid and wall viscosity, equations (5.8) to (5.10) can be expanded with a central difference about the \((k+\frac{1}{2})^{th}\) time step:

\[ p_{a_1}^{k+1} + p_{a_1} - p_0^{k+1} - p_0^k = \frac{2L_a}{\Delta t} (F_{a_1}^{k+1} - F_{a_1}^k), \]  
\[ (5.13) \]

\[ p_0^{k+1} + p_0 - p_{b_1}^{k+1} - p_{b_1}^k = \frac{2L_b}{\Delta t} (F_{b_1}^{k+1} - F_{b_1}^k), \]  
\[ (5.14) \]

and

\[ p_0^{k+1} + p_0 - p_{c_1}^{k+1} - p_{c_1}^k = \frac{2L_c}{\Delta t} (F_{c_1}^{k+1} - F_{c_1}^k). \]  
\[ (5.15) \]
The subscripts "a₁", "b₁" and "c₁" refer to the boundary mesh points in artery segments "a", "b" and "c" respectively and in subsequent equations the subscripts "a₂", "b₂" and "c₂" will refer to the adjacent mesh points in the respective artery segments.

The amount of fluid stored in the expanding junction is small and as a first approximation these terms will be ignored. This decision is made primarily through lack of adequate experimental data to evaluate "C". Hence the above three equations are supplemented with,

\[ F_{a₁} = F_{b₁} + F_{c₁} \quad (5.16) \]

and there are now four equations across the junction rather than three as mentioned in Section 5.1, because of the additional variable \( p_o \).

Adding (5.13), (5.14) and (5.15) and applying (5.16) at time steps \( k \) and \( k+1 \) yields an expression for \( p_o^{k+1} \) in terms of \( P_{a₁}^{k+1}, P_{b₁}^{k+1} \) and \( P_{c₁}^{k+1} \):

\[ p_o^{k+1} = -p_o^k + [(p_{a₁}^{k+1} + p_{a₁}^k)/L_a + (p_{b₁}^{k+1} + p_{b₁}^k)/L_b + (p_{c₁}^{k+1} + p_{c₁}^k)/L_c]/(1/L_a + 1/L_b + 1/L_c) \quad (5.17) \]

Now combining the boundary equations (2.26) and (2.28) and rewriting with \( F = \pi R^2 V \), yields
\[
\frac{\partial p}{\partial t} + \frac{1}{2\pi R} \left[ \frac{G_1 + G_2}{R_0} + \frac{dP_s}{dR} \right] \frac{\partial F}{\partial x} = W,
\]

which can be written as three equations, one for each artery segment, by expanding about the point \( \left( \frac{3}{2}, k+\frac{1}{2} \right) \) for segments "b" and "c" and about the point \( (N-\frac{1}{2}, k+\frac{1}{2}) \) for segment "a" \( (N \text{ is the end mesh point for segment "a"}) \), that is,

\[
k^{+1}\left[ \frac{k+1}{2} - p_{a_1} - p_{a_2} - A^*_a \frac{\Delta t}{\Delta x} (F^{k+1}_{a_2} - F^{k-1}_{a_1}) \right] = 2\Delta t W^*_a, \tag{5.18}
\]

\[
k^{+1}\left[ \frac{k+1}{2} - p_{b_1} - p_{b_2} + A^*_b \frac{\Delta t}{\Delta x} (F^{k+1}_{b_2} - F^{k-1}_{b_1}) \right] = 2\Delta t W^*_b, \tag{5.19}
\]

and

\[
k^{+1}\left[ \frac{k+1}{2} - p_{c_1} - p_{c_2} + A^*_c \frac{\Delta t}{\Delta x} (F^{k+1}_{c_2} - F^{k-1}_{c_1}) \right] = 2\Delta t W^*_c, \tag{5.20}
\]

where

\[
A^*_a = \left[ G_1 + G_2 + \frac{\beta}{2} \left( c_{a_1} + c_{a_2} \right) \left( \frac{2R_{N-\frac{1}{2}}}{R_{N-\frac{1}{2}}} \right) \right]^{-1} \left[ \frac{k+\frac{1}{2}}{2} \right] \left( R_{a_1} + R_{a_2} \right)
\]

and similarly \( A^*_b \) and \( A^*_c \) refer to the term,

\[
\left( \frac{G_1 + G_2}{R_0} + \frac{dP_s}{dR} \right) / 2\pi R
\]

evaluated at the point \( \left( \frac{3}{2}, k+\frac{1}{2} \right) \) in artery segments "b" and "c" respectively \( (W^* \text{ is also evaluated at these points}). The interior equations are always solved before the boundary equations and branch equations so that all variables are known at the \( (k+1) \text{th} \) time step and all variables except those on the boundary are known at the \( (k+\frac{1}{2}) \text{th} \) time step before the boundary and branch equations are solved.
Now eliminating \( p_{a_1}^{k+1} \) from (5.13) and (5.18) yields

\[
p_{a_1}^k + p_{a_2}^k - p_{a_2}^{k+1} + A_a^* \Delta t \left[ F_{a_2}^{k+1} + F_{a_2}^k - 2F_{a_1}^k \frac{\Delta t}{2L_a} (p_{a_1}^k - p_{a_0}^k - p_o^k) \right] \\
+ 2\Delta t w^x(1 + A_a^* \frac{\Delta t^2}{2\Delta x L_a})
\]

Similar expressions can be obtained for \( p_{b_1}^{k+1} \) and \( p_{c_1}^{k+1} \) and if these pressure terms are substituted in (5.17), an expression for \( p_{o}^{k+1} \) is obtained in terms of information known at the \( k^{th} \) and \( (k+\frac{1}{2})^{th} \) time steps as well as the variables \( F_{a_2}^{k+1}, F_{b_2}^{k+1} \) and \( F_{c_2}^{k+1} \) which are known from the previous solution of the interior equations.

Once \( p_{o}^{k+1} \) has been evaluated then \( p_{a_1}^{k+1} \) is obtained from (5.21) and \( p_{b_1}^{k+1} \) and \( p_{c_1}^{k+1} \) from similar expressions. With these boundary pressures evaluated, the procedure to obtain the other variables is identical to the boundary scheme mentioned in Section 4.2 (instead of prescribing the boundary pressures, these have now been calculated). However, use of the variables \( F_{a_2}^{k+1}, F_{b_2}^{k+1} \) and \( F_{c_2}^{k+1} \) gives rise to an error as mentioned previously in the discussion of boundary schemes. The error can best be exhibited by computing flow through a junction with the above explicit scheme and evaluating \( (F_{a_1} - F_{b_1} - F_{c_1}) \) at each time step. This expression should be zero (equation (5.16)) but is only used implicitly in the above scheme and in fact grows to a value exceeding the flow itself.
The problem can be overcome by using an iterative procedure in which the above explicit scheme provides the first estimate of \( p_{a1}^{k+1}, p_{b1}^{k+1} \) and \( p_{c1}^{k+1} \). The first estimates of \( R^{k+1}, V^{k+1} \) and \( W^{k+1} \) are then obtained with the boundary equations (4.13), (4.14) and (4.15) (contained in the subroutine "BDRY" in Appendix A) and with these values, equations (5.13), (5.14), (5.15) and (5.17) yield new estimates for \( p_{a1}^{k+1}, p_{b1}^{k+1} \) and \( p_{c1}^{k+1} \). Second estimates of \( R^{k+1}, V^{k+1} \) and \( W^{k+1} \) at "a1", "b1" and "c1" are again obtained with the boundary equations and then equations (5.13), (5.14), (5.15) and (5.17) yield third estimates of \( p_{a1}^{k+1}, p_{b1}^{k+1} \) and \( p_{c1}^{k+1} \). The iterations can be continued until \( (F_{a1}^{k+1} - F_{b1}^{k+1} - F_{c1}^{k+1}) \) is sufficiently close to zero. Thus the explicit scheme requiring use of \( F_{a2}^{k+1}, F_{b2}^{k+1} \) and \( F_{c2}^{k+1} \) has been replaced with an iterative scheme not involving these variables. Convergence to an error of less than 1\% in equation (5.16) (that is, \( (F_{a1}^{k+1} - F_{b1}^{k+1} - F_{c1}^{k+1}) \) is less than 1\% of \( F_{a1} \)) is obtained after a few iterations.

5.4 Computations at a Bifurcation.

The iterative scheme developed in Section 5.3 is presented in FORTRAN IV in Appendix A in the subroutine entitled "BIFUR". In the computations in this section the following parameters are used:

\( \alpha = 1.1 \); \( \beta = 2 \); \( G_0 = 50 \text{ mm Hg} \); \( \tau_1 = .167 \text{ secs} \); \( \tau_2 = .022 \text{ secs} \);

\( G_1 = 10 \text{ mm Hg} \); \( G_2 = 33 \text{ mm Hg} \)
A pressure pulse is prescribed at the beginning of artery segment "a" (the same pulse used for computation in Chapter 4). The segment extends for 210 cm with an unstressed radius of 1 cm to a bifurcation into equal-sized arteries (segments "b" and "c") sufficiently long so that reflection of the pulse from their distal ends can be ignored. Computations are begun with the whole artery at 80 mm Hg and zero flow. Mesh lengths are $\Delta t = .01$ secs and $\Delta x = 7.0$ cm.

Initial computations were performed with artery segment "a" having an unstressed radius of 1 cm and the two branching arteries with a combined cross-sectional area equal to that of the parent artery. The inductances $L_a$, $L_b$ and $L_c$ were taken to be equal and were varied over a wide range to find the effect on the solution.

For an inductance value of $2.5 \times 10^{-4}$ mm Hg/sec$^2$/cm$^3$ and above, the iterative branch scheme becomes unstable and for values below $10^{-4}$ mm Hg/sec$^2$/cm$^3$ the process takes a long time to converge. Between these extremes the branch scheme converges in a few iterations. An approximate explanation of this phenomena is as follows: The inductance of section "a" of the junction can be expressed as

$$L_a = k_a \frac{\rho A_a}{S_a},$$

where $l_a$ and $S_a$ are the length and cross-sectional area
of that section and $k_a$ is the empirical correction factor, representing the difference between an idealized junction and a real junction. The fluid density $\rho$ has a value of $7.9 \times 10^{-4}$ mm Hg. sec$^2$/cm$^3$ and at a pressure of 80 mm Hg, $S_a = 5.5$ cm$^2$ so that a value of $L_a = 1.5 \times 10^{-4}$ mm Hg. sec$^2$/cm$^3$ (the approximate optimal value for convergence) results in $k_a L_a$ being approximately 1 cm. This is one-seventh the spatial mesh length used for the calculations. If $k_a L_a$ becomes too large (for example when it is comparable to the spatial mesh length) then the idealized form of junction we have assumed is no longer valid and the approximation breaks down. This is reflected in the non-convergence of the iteration scheme. Slow convergence at small values of $k_a L_a$ is a result of the small correction made to the previous estimates of pressure when equations (5.13), (5.14), (5.15) and (5.17) are implemented.

The effect on the transmission and reflection of an incident pulse wave at inductance values below the critical value for convergence is slight and so in the following computations the values of $L_a L_b$ and $L_c$ are kept equal and varied until optimum convergence is found. This lack of any significant effect on the flow, by varying the inductance values, is to be expected physically since it merely corresponds to altering the length of the outlets from the junction and this should have little effect.
Figure 20 shows the effect of a bifurcation into arteries of combined cross-sectional area equal to, smaller than and greater than the parent artery (Graphs "1", "2" and "3"). Diagrams "A" and "B" are recorded 70 cm and 35 cm proximal to the bifurcation respectively and Diagrams "C" and "D" immediately before and after the junction. The source of pressure is sufficiently far removed to avoid secondary reflections from the boundary. The effect of the bifurcation is very similar to that obtained when there is a cross-sectional area increase or decrease in a single length of artery. A bifurcation into a smaller total area results in an upstream pressure increase and flow decrease consistent with a compression wave travelling upstream. The reverse occurs for an area increase and a rarefraction wave propagates upstream. A more interesting observation is that for a bifurcation into arteries of total cross-section equal to that of the parent artery, there is very little pressure drop across the junction and very little reflection of the incident pulse.

Most arterial branching is observed to result in a slight area increase and hence the results here predict a slight upstream pressure decrease and flow increase for the average physiological branch.
Figure 20. The effect of a bifurcating artery.
6. HEART-AORTA MODEL

6.1 Introduction.

To examine the formation of the aortic pulse we require a set of equations describing the action of the left ventricle and the flow through the aortic valve into the ascending aorta. The nature of the pulse will also be determined by the downstream conditions and ideally the flow equations should be solved for the entire aorta and its branches. However, without including all the relevant factors, we intend to examine the importance of a few factors influencing the formation of the pulse.

The aorta is assumed to have no leakage from branching arteries and with the exception of the radius, all other physical properties are assumed to be constant along its length. The parametric studies in Section 4.6 indicated the very large upstream effects that result from a tapering artery and since the aorta is observed to have considerable taper, particularly in the region of the aortic arch, the heart-aorta model presented in this chapter includes a tapering aorta. Actual values for the radii at different points in the ascending aorta and aortic arch are taken from Wright [29]. Any comparison between computed and measured flow or pressure pulses would require this information to be obtained in the same aorta from which the flow
and pressure measurements had been made. The purpose of
the present work is not to make any such detailed comparisons,
but rather to provide the means for doing so when the
experimental data becomes available.

With the method developed for computing flow in
arterial segments and through junctions of arteries, a
considerably more complex and thorough analysis of the
generation of the aortic pulse than we are considering here
could be performed. The decreased distensibility of the
more distal aorta is easily incorporated by entering different
values for the parameter $G_0$ at each mesh point of the
modelled aorta. However, this and other parameters appear
to affect the pulse considerably less than a tapering cross-
section, so the present analysis concentrates on this alone.
The artery is still considered to be viscoelastic and the
same viscoelastic parameters that were used in Chapter 4
will be employed here, namely: $\tau_1 = .167$ secs;
$\tau_2 = .022$ secs; $G_1 = 10$ mm Hg; $G_2 = 33$ mm Hg. Other
parameters are $\alpha = 1.1$, $\beta = 2$ and $G_0 = 50$ mm Hg. The
aorta begins with an unstressed radius of $R_0 = 1.25$ cm,
tapering to 1.2 cm, 1.15 cm and 1.0 cm at 3 cm intervals
and subsequently tapering over a 45 cm length to 0.5 cm.

The aortic valve consists of three symmetrical cusps
attached to a fibrous ring at the base of the aorta. When
the contracting left ventricle raises the left ventricular
pressure to a value exceeding that in the aortic root, the valves open and blood is ejected into the aorta. As the muscles of the ventricles relax the ventricular pressure drops and a reverse pressure gradient decelerates and finally reverses the flow, at which point the valve closes. It is suggested in many physiology texts that the valve cusps respond directly to a reverse pressure gradient and close immediately the ventricular pressure falls below the aortic pressure. This description neglects the momentum of the ejecting blood and it seems more probable that a delay occurs while the blood flow decelerates to zero and that the valve does not close until a reverse flow forces it to.

6.2 A Model of the Left Ventricle.

The starting point for a model of the heart is usually a description of the mechanical model used to represent the heart muscle. Prescribing the length-force behaviour of the various elements of this mechanical model is then a convenient way of introducing experimental measurements. The model will usually contain a contractile element and the relationship between the force provided by this element and the resulting velocity of contraction or "shortening velocity", is also empirical.
An approach such as this is adopted by Beneken and DeWit in [15]. The description of their mechanical model provides eight simultaneous equations which are solved to give a relationship between muscle length and the force developed by a unit cross-section of muscle. Two more equations, one relating muscle length to ventricular volume and the other relating muscle force to ventricular pressure supplement the above equations and when solved with them result in a series of isometric pressure-volume curves. These yield the volume of the ventricle at a given pressure and at a given "activation factor" (the state of contraction of the heart muscle).

By prescribing the level of activation throughout the heart cycle and using the pressure-volume curves derived from the more fundamental analysis, it is possible to describe the behaviour of the ventricle by employing a relationship of the form

\[ P_v = P_v(Q_v, \delta), \]  

(6.1)

where \( P_v \) and \( Q_v \) are ventricle pressure and volume and \( \delta \) is the activation factor.

In general, this relationship may be obtained either from a series of direct measurements of pressure versus volume at various activation levels, or indirectly from measurements of muscle behaviour. Either way, it is
assumed that the properties of the left ventricle are defined by this relationship.

Additional equations are now required to relate the ventricular pressure to aortic pressure and to relate the change in volume to flow out of the ventricle. The first equation could either be obtained from a momentum balance across the outflow tract or an energy balance equation. The problem with using an equation relating the total momentum influx into a control volume with the forces on that volume is that the area of the control volume acted upon by ventricular pressure forces is indeterminate. A similar argument holds for an energy equation dealing with gross quantities. On the other hand, an energy equation satisfying averaged quantities along a streamline is more applicable. This takes the form of an unsteady Bernoulli equation:

\[ \frac{P_v}{\rho} = \frac{P_a}{\rho} + \frac{1}{2} V_a^2 + \int_{V}^{A} \frac{\partial V(s,t)}{\partial t} \, ds, \quad (6.2) \]

where \( V_a \) is the mean velocity at the aortic end of the outflow tract and \( P_a \) the mean pressure. The streamline is assumed to pass from a region of the ventricle where the velocity is zero and pressure represented by \( P_v \).

Integration along the streamline results in the integral term in (6.2), where \( V(s,t) \) is the mean velocity of flow at a point \( s \) on the streamline. By assuming this term can be represented by \( k_s \frac{\partial V_a}{\partial t} \), then (6.2) becomes
\[ p_v = p_a + \frac{1}{2} \rho v_a^2 + \rho k_a \frac{\partial v_a}{\partial t}, \quad (6.3) \]

where \( k_a \) has dimensions of length and is proportional to the length of the streamline. \( k_a \) is considered here as an empirical factor and in addition to performing the function just mentioned, it incorporates the energy losses not included in (6.2). These losses to viscous effects in the fluid and in the surrounding tissue cannot be determined from a tractable theoretical analysis, so that any equation describing an energy balance across the outflow tract must include empirical terms.

The equations described here deal only with the ventricle and not the atrium. It is assumed the mitral valve remains closed and the filling of the ventricle is not considered. This is satisfactory for an analysis of the ejection characteristics of the ventricle where the volume of blood existing in the ventricle prior to contraction is a prescribed boundary condition.

Finally, if \( Q_v \) is the volume of the left ventricle then a conservation of mass equation takes the form

\[ \frac{\partial Q_v}{\partial t} = -\pi R_a^2 v_a \quad (6.4) \]

for the period during which the aortic valve is open \( (R_a \) is the radius of the aorta at the end of the outflow tract).
Equations (6.1), (6.2) and (6.4) supplement the three boundary equations for the aorta given by (4.13), (4.14) and (4.15). Thus there are six simultaneous equations in the variables $p_v, q_v, p_a, r_a, v_a$ and $w_a$ to be solved at every step of the scheme solving the flow equations in the aorta.

When the aortic valve is closed equations (6.3) and (6.4) are no longer applicable and another boundary condition for the aorta must be considered. An obvious choice is to specify zero flow at the valve and so, $v_a = 0$ replaces (6.3) and (6.4) when the valve is closed. Equation (6.1) will continue to yield the ventricular pressure with the added condition that $q_v$ is a constant.

Beneken and DeWit [15] found from their solution of the equations describing muscle behaviour that (6.1) could be approximated by a linear relationship of the form

$$p_v = c_v(t)(q_v - q_{uv}),$$

where $q_{uv}$ is the unstressed ventricular volume and $c_v(t)$ is a factor dependent only on the level of activation of the muscles. $c_v(t)$ can be described as the time dependent elastance of the ventricle and is prescribed throughout the heart cycle. The description of $c_v$ as a function of time is a boundary condition on the solution of all the equations and replaces the boundary condition used in
previous chapters of this thesis, namely the prescription of a proximal pressure pulse.

6.3 Finite Difference Equations

The approach followed here is similar to that used to solve the branch equations in Section 5.3. Once a value has been obtained for the pressure $p_a$, then the boundary equations given by (4.13), (4.14) and (4.15) (subroutine "BDRY" in Appendix A) yield the other boundary variables $R_a$, $V_a$ and $W_a$.

Equations (6.3), (6.4) and (6.5) in finite difference notation become

$$
\frac{1}{2}(p_v^{k+1} + p_v^k) = \frac{1}{2}(p_a^{k+1} + p_a^k) + \frac{1}{2}(\frac{v^{k+1} + v^k}{a^2}) + \rho \frac{k}{a} \frac{v^{k+1} - v^k}{\Delta t},
$$

(6.6)

$$
\frac{Q_v^{k+1} - Q_v^k}{\Delta t} = -\pi (\frac{a}{2} + \frac{R_a}{2}) (\frac{a}{2} + \frac{V_a}{2}),
$$

(6.7)

and

$$
p_v^{k+1} = \frac{Q_v^{k+1} - Q_{uv}}{C_v^k}.
$$

(6.8)

An initial prediction for $p_a^{k+1}$ is obtained by applying forward differences in both $x$ and $t$ to the boundary equation,

$$
\frac{\partial p_a}{\partial t} + \frac{1}{2R_a^2} \left[ G_{1} + G_{o} + \frac{dp_s}{dr} \right] \frac{\partial}{\partial x} \left( \frac{R_a^2 \phi}{a} \right) = 0
$$

(6.9)
obtained by combining equations (2.26) and (2.28), so that

\[ p_{a}^{k+1} = p_{a}^{k} - \Delta t \frac{R_{o}^{k} G_{a} - G_{a}^{2} + \beta G_{o}^{2} \left( \frac{R_{a}}{R_{o}} \right)}{2 R_{a}^{k} R_{o}^{k}} \left[ \frac{R_{a}^{k} R_{o}^{k}}{R_{a}^{k} + R_{o}^{k}} \left( v_{a}^{k} - v_{a}^{k} \right) + \Delta t w_{a}^{k} \right], \]

where the subscript "as" refers to the mesh point adjacent to the boundary mesh point "a". The boundary equations (4.13) to (4.15) then yield first estimates for \( R_{a}^{k+1}, v_{a}^{k+1}, \) and \( w_{a}^{k+1} \). Equation (6.7) is rearranged to obtain \( Q_{v}^{k+1} \) from these values; that is

\[ Q_{v}^{k+1} = Q_{v}^{k} - \frac{\Delta t}{2} (R_{a}^{k+1} + R_{a}^{k}) (v_{a}^{k+1} - v_{a}^{k}). \]

Substituting \( Q_{v}^{k+1} \) in (6.8) yields \( p_{v}^{k+1} \) and finally a new value is obtained for \( p_{a}^{k+1} \) from (6.6),

\[ p_{a}^{k+1} = p_{a}^{k+1} + k_{a}^{k} - k_{a}^{k} - \frac{\Delta t}{2} (v_{a}^{k+1} + v_{a}^{k}) + \frac{2 \rho k_{a}}{\rho} (v_{a}^{k+1} - v_{a}^{k}). \]

Second estimates for \( R_{a}^{k+1}, v_{a}^{k+1}, \) and \( w_{a}^{k+1} \) follow from the boundary equations and the iterate procedure continues until the values have converged.

When the aortic valve is closed the boundary condition becomes \( v_{a}^{k} = 0 \) and \( p_{a}^{k}, R_{a}^{k} \) and \( W_{a}^{k} \) are obtained by rearranging equations (4.13), (4.14) and (4.15) (the equations appear in subroutine "BDRY2" in Appendix A).

6.4 Computation with the Model

The time dependent elastance, \( C_{v}(t) \) determines the performance of the ventricular muscles and is therefore an
important factor in the formation of the pulse. Since we do not have accurate data on this parameter, the analysis is again qualitative rather than quantitative. Using the work of Beneken and DeWit [15] as a guide, we prescribed a cubic variation of the elastance beginning with 1.0 mm Hg/cm³ at diastole, rising to 4.7 mm Hg/cm³ in .09 secs and falling to 1.0 mm Hg/cm³ again in another .18 secs, with an unstressed ventricular volume of 80 cm³. Computations were begun with the aorta at 80 mm Hg, the aortic valve closed and the left ventricle containing 150 cm³ of blood at a pressure of 70 mm Hg.

Initial computations established the effect of the parameter $k_a$. Too large a value led to the nonconvergence of the iterative scheme mentioned in Section 6.3. Too small a value had the effect of producing physically unrealistic fluctuations in the computed variables. Smaller time steps reduced this problem and smoothly varying results were obtained with a time step of $\Delta t = .001$ secs at a value of $k_a = .1$ cm. This required a spatial mesh length of .7 cm. Thus $k_a$ was found to be one-seventh the spatial mesh length for stable computations, the same ratio that was found necessary for the similar factor included in the branch calculations in Chapter 5.

Using these mesh lengths and value of $k_a$, the effect of a tapering aorta was then examined. In all computations, with or without a tapering aorta, the aortic pressure was found to have a very large influence on the ventricular pressure. However, only with the taper could we obtain a reasonable pulse
and smooth operation of the aortic valve. The taper produces an upstream pressure rise and flow decrease in agreement with the computations in Section 4.6. This maintains the systolic pressure for a longer period than would otherwise occur and produces the flow reversal necessary to close the valve. Figure 21 illustrates the ventricular pressure during systole that occurs with a tapering aorta (Graph "1") and with a non-tapering aorta (Graph "2"). Without the taper, the ventricular pressure falls off very rapidly, producing a very thin pulse and no flow reversal.

The aortic pressure follows slightly below the ventricular pressure until the valve closes, when the ventricular pressure continues to drop while the aortic pressure rises to a small secondary peak. This secondary peak which
is experimentally observed in pressure recordings made
in the aorta, was also found to occur only when the aorta
was tapered.

All the above computations were made with all
parameters other than the unstressed arterial radius, having
a constant value, but the experimentally observed increase
of wall stiffness down the aorta will also aid the upstream
pressure rise mentioned above.
7. ASPECTS OF THE CIRCULATORY SYSTEM.

7.1 Introduction.

A method of solving the blood flow equations in arterial segments and through junctions of these segments has been developed in Chapters 2 to 5. Chapter 6 considered the left ventricle and its relation to the aorta and illustrated a technique which can be readily applied to a more general model of the arterial side of the circulatory system. This consists of solving the equations through each of the larger arteries and their junctions and then whenever the configuration of arteries becomes too complex the artery segments join onto "black boxes" or lumped parameter systems which represent groups of arteries, arterioles and capillaries. The equations describing the action of a lumped parameter system form the boundary condition for the adjacent arterial segment.

A further extension to include the venous side of the circulatory system can be made if it is assumed that blood flow in veins can be described with the same flow equations, but with a different wall equation. At zero transmural pressure while an artery remains approximately cylindrical, a vein collapses and so when the pressure is raised slightly above zero the cross-sectional area of the vein increases by a much greater amount than that of the artery. As the transmural pressure continues to rise the venous cross-section becomes an
ellipse and finally a circle, at which stage the wall
distensibility is very low. The equation of motion for
arterial flow was derived in Chapter 2 with the assumption
that the artery was cylindrical and to model any other
shape requires a two or three dimensional approach. However,
if flow through an elliptically shaped vein can be approxi-
mated by flow through a cylindrical vein of the same
cross-sectional area, then the equations of Chapter 2 are
applicable. The statically determined pressure/radius
relationship (2.21) needs to be prescribed in a form
considerably different from (2.24) and the velocity profile
given by (2.16) also needs some modification. Before being
accepted as valid, however, the approximation requires
experimental testing with cylindrical and elliptical
synthetic tubing.

An approach to modelling the complete circulation
system, employing the assumption mentioned above, is briefly
described in Section 7.2. As a first step towards the
modelling of control systems involved in the circulatory
system, the temperature of blood and the concentrations of
oxygen and carbon dioxide may be calculated by solving the
one-dimensional heat equation and appropriate diffusion
equations. The solution of these equations in conjunction
with the solution of the blood flow equations is described
in Section 7.3. Finally, the general circulation model
mentioned in Section 7.2 provides a framework for modelling
a number of the control systems operating on the individual
organs of the body and a brief discussion of this method
appears in Section 7.4.

7.2 A General Model of the Circulatory System.

The following discussion is not intended to be a
description of a working model since, although such a model
has been developed and run on the computer, we consider
the core size (16K) too small to enable enough blood
vessels to be treated individually and the descriptions of
the organs not sufficiently accurate to give results worth
reproducing here. Rather it is intended to illustrate an
approach which, if employed with enough accuracy and with
detailed models of the various organs, will provide a
comprehensive model of the complete circulation.

The 16K core on the IBM 1130 computer we were using
limited individual modelling of arteries and veins to the
major vessels supplying each organ. The "organs" are
considered to be the following: The left heart (blood
entering via a single pulmonary vein and leaving via the
aorta); the right heart (entry via inferior and superior
vena cavae, exit via pulmonary artery); the liver (entry
via the hepatic artery and portal vein, exit via the hepatic
vein); the kidney (entry via the renal artery, exit via
the renal vein); the spleen (entry via the splenic artery, exit via the splenic vein); the ascending colon (entry via the superior mesenteric artery, exit via superior mesenteric vein); the descending colon (entry via inferior mesenteric artery, exit via the inferior mesenteric vein); the rectum (entry via the interior iliac artery, exit via the interior iliac vein); the leg (entry via the external iliac artery, exit via the external iliac vein); the lung (entry via the pulmonary artery, exit via a single pulmonary vein); the arm (entry via the subclavian artery, exit via the subclavian vein); the brain (entry via the internal carotid artery, exit via the internal jugular vein); the face (entry via the external carotid artery, exit via the facial vein).

Several of the "organs" represent an area of the body with its blood predominantly supplied by a single artery and drainage predominantly from a single vein. For example, the "ascending colon" represents that portion of the body supplied by the superior mesenteric artery and since this overlaps with the area supplied by the inferior mesenteric, the division is an oversimplification. Wherever possible symmetry is used to reduce the amount of computation. Thus the right lung and left lung are considered to have a similar blood supply and only one is specifically modelled (this is not strictly accurate as one pulmonary artery is shorter than the other). A diagram
of the arteries, veins and organs mentioned appears in Jacob and Francone [30]. For a more detailed study, see Michels [31].

In our simplified model of the human circulatory system the equations describing the behaviour of the organs other than the heart are equivalent to the equations obtained from the electrical network shown in Figure 22. The network is shown with two arteries and two veins, but for organs such as the spleen with only one artery and one vein one of the resistances at either end is equated to zero.

![Figure 22](image)

The equations are as follows:

$$p_a - p_o = R_a a + L_a \frac{\partial F_a}{\partial t},$$

(7.1)
\[ p_b - p_o = R_b F_b + L_b \frac{\partial F_b}{\partial t}, \]  
\[ p_o - p_c = R_c F_c + L_c \frac{\partial F_c}{\partial t}, \]  
\[ p_o - p_d = R_d F_d + L_d \frac{\partial F_d}{\partial t}, \]  
\[ \frac{\partial Q}{\partial t} = p_a + F_b - F_c - F_d, \]  
and
\[ p_o = \frac{Q}{C} + R_o \frac{\partial Q}{\partial t}, \]

where \( Q \) is the blood volume stored in the organ, \( C \) is a measure of the organ's capability of accepting an increase in stored blood and \( R_o \) represents the energy loss associated with this increase. The resistances \( R_a, R_b, R_c \) and \( R_d \) represent the energy loss from frictional effects in the arterioles, capillaries and venules. The inductances \( L_a, L_b, L_c \) and \( L_d \) represent the inertia of the blood.

The solution of these equations in conjunction with the boundary equations from each of the arterial and venous segments is an iterative procedure similar to that mentioned in the context of arterial junctions and the heart-aorta model. Apart from being longer the method involves no new techniques and so will not be elaborated upon again.

An electrical analogy is not possible for the heart equations but these are a simple extension of the equations derived in Chapter 6 for the heart-aorta model. The equations for the left atrium are solved in conjunction with
those of the left ventricle. They take the same form as the ventricular equations with an added equation relating ventricular pressure ($P_V$) to atrial pressure ($P_A$) across an open mitral valve, of the form

$$P_A - P_V = R_{AV} F_{AV},$$

where $F_{AV}$ is the flow through the valve and $R_{AV}$ a proportionality constant representing the resistance to flow. A similar set of equations applies to the right heart. The prescription of the time dependent elastance (see Chapter 6) is different for each chamber.

The set of equations (7.1) to (7.6) is too simple a representation of the effect of an organ on the blood flow through it to be of much value. However, employing a more detailed set of equations does not alter the general approach described here and we hope that future work will improve the equations.

7.3 Diffusion Equations.

The general one dimensional heat equation, assuming that there is no loss to the surroundings, can be expressed as

$$\frac{dT}{dt} = \kappa \frac{\partial^2 T}{\partial x^2} \quad (7.7)$$

where $T(x,t)$ is the temperature and $\kappa$ the thermal diffusivity, or
\[ \kappa = \frac{k}{\rho c_p} \],

where \( k \) is the thermal conductivity, \( \rho \) the density and \( c_p \) the specific heat of blood. The total derivative in (7.7) may be written as time variation and convection terms to give

\[ \frac{\partial T}{\partial t} + V \frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial x^2} \].

If the equation is expressed in finite difference form by expanding about the point \((i,k)\) with central differences in the space derivatives and a forward difference in the time derivative, then

\[ \frac{1}{\Delta t} (T_{i+1}^{k+1} - T_i^k) + \frac{V_i^k}{2\Delta x} (T_{i+1}^k - T_{i-1}^k) = \frac{\kappa}{\Delta x^2} (T_{i+1}^k - 2T_i^k + T_{i-1}^k) \].

Hence, \( T_i^{k+1} \) is obtained with first order accuracy from

\[ T_i^{k+1} = T_i^k + \left[ \frac{\kappa \Delta t}{2\Delta x^2} \right] (T_{i+1}^k - 2T_i^k + T_{i-1}^k) - \left[ \frac{\Delta v_i^k}{2\Delta x} \right] (T_{i+1}^k - T_{i-1}^k) \]. (7.8)

Now for stability this equation requires each of the terms in square brackets to be less than .5 (refer Richtmyer and Morton [24]). The thermal diffusivity for blood at normal body temperature is \( .002 \text{ cm}^2/\text{sec} \) (refer Bergel et al. [32]) so that with a temp step of \( \Delta t = .004 \text{ secs} \) and distance step of \( \Delta x = 3 \text{ cm} \), the first term is approximately \( 10^{-6} \) and the stability criterion is obviously satisfied. The second term is dependent on the blood velocity \( V_i^k \) and
for stability, $v_i^k$ is required to be less than 750 cm/sec. This is well within the probable velocities attained anywhere in the circulation system so that the stability criterion is again satisfied.

The extremely small magnitude of the first term in square brackets in (7.8) indicates that advection is considerably more important in the transfer of heat along the artery than diffusion and the diffusion term may in fact be ignored for a one dimensional analysis. The central difference applied to the term $\frac{dT}{dx}$ can sometimes lead to premature instability and use of the one sided difference $(T_i^k - T_{i-1}^k)$ when $v_i^k$ is positive and $(T_{i+1}^k - T_i^k)$ when $v_i^k$ is negative gives improved stability to the solution of equation (7.8).

Equation (7.8) is solved at each time step along with the flow equations so that use can be made of $v_i^k$ computed by these equations. The boundary conditions necessary for the solution of (7.8) in a blood vessel segment are the initial conditions at all points in the segment and the specification of the temperature at either end for all time. If a more general model of the circulatory system is being considered such as that described in Section 7.2, then sets of equations describing the production and absorption of heat in each organ may be defined and solved in conjunction with the solution of (7.8) in each segment. Heat losses through the walls of the blood vessels necessitate an additional
for stability, $V_1^k$ is required to be less than 750 cm/sec. This is well within the probable velocities attained anywhere in the circulation system so that the stability criterion is again satisfied.

The extremely small magnitude of the first term in square brackets in (7.8) indicates that advection is considerably more important in the transfer of heat along the artery than diffusion and the diffusion term may in fact be ignored for a one-dimensional analysis. The central difference applied to the term $\frac{\partial T}{\partial x}$ can sometimes lead to premature instability and use of the one sided difference $(T_i^k - T_{i-1}^k)$ when $V_1^k$ is positive and $(T_{i+1}^k - T_i^k)$ when $V_1^k$ is negative gives improved stability to the solution of equation (7.8).

Equation (7.8) is solved at each time step along with the flow equations so that use can be made of $V_1^k$ computed by these equations. The boundary conditions necessary for the solution of (7.8) in a blood vessel segment are the initial conditions at all points in the segment and the specification of the temperature at either end for all time. If a more general model of the circulatory system is being considered such as that described in Section 7.2, then sets of equations describing the production and absorption of heat in each organ may be defined and solved in conjunction with the solution of (7.8) in each segment. Heat losses through the walls of the blood vessels necessitate an additional
term in (7.7) and hence in (7.8).

The transfer of oxygen or carbon dioxide through a blood vessel is described by a diffusion equation similar to (7.7) that is,

\[ \frac{dC}{dt} = D \nabla^2 C, \]

where \( C \) is the concentration of the oxygen or carbon dioxide and \( D \) is the diffusivity. As with the heat equation, advection is the predominant transfer mechanism along the artery and the diffusion term may be ignored. The solution of this equation follows the procedure described for the heat equation above.

7.4 Control Mechanisms.

The enormous number of factors controlling various parameters of the circulation system and often operating simultaneously, make the task of modelling even some of the control systems seem hopelessly complex. However, one important mechanism which can be modelled with reasonable results is the carotid sinus, aortic arch reflex (refer Burton [1]). This is a means by which the blood supply to the brain is kept approximately constant under fairly severe fluctuations in the supply to the rest of the body. Stretch receptors situated in the region of the bifurcation
into the internal and external carotid arteries and in
the region of the aortic arch, respond to a change of mean
pressure by effecting an alteration of both the rate and
strength of contraction of the left ventricle. In addition
they affect the central blood pressure by producing a
reduction in diameter of the peripheral blood vessels,
thereby altering their resistance to flow.

All of these responses can be modelled by changing
various parameters of the general circulation model
described in Section 7.2. Thus the rate and strength of
ventricular contraction is controlled by the time varying
elastance parameter $C_V$ (see Chapter 6). Similarly the
resistances included in the various organ models can be
continually modified by instructions from the stretch
receptors. The receptors may be modelled to respond to
mean pressure and rate of change of mean pressure or any
other pressure change felt to be important. A transfer
relationship between these variables and the heart rate
and strength of contraction is described by Beneken and
DeWit [15].
APPENDIX A. COMPUTER PROGRAMS

The following programs are written in FORTRAN IV and were run on an IBM 1130 computer with 16 k storage. The only local feature employed is the use of "x" to indicate a following statement.

The subroutines appear in the order they are referred to in the text. Subroutine "LAX" contains the equations for computing the flow in the interior of an arterial segment with "IA" and "IB" in the parameter list referring to the end mesh points of the segment and "x" being the spatial mesh length. Subroutine "BDRY" computes the variables R, W and V at a boundary where the pressure p is prescribed. The boundary mesh point is denoted by "IA", the adjacent mesh point by "IAA" and the adjacent half-point by "IH". "XA" is the spatial mesh length of the segment whose boundary variables are being calculated. Subroutine "BIFUR" computes the boundary variables at a junction of three arteries, with "RLA", "RLB" and "RLC" being the three inductance parameters. Subroutine "BDRY2" computes the variables R, p and W at a boundary where the velocity V is prescribed and Subroutine "HEART" contains the equations relating the left ventricle to the aorta.

The notation used in the subroutines directly follows that given in the text.
**SUBROUTINE LAX**

CCMKCN PI,T,RHU,TAU1,TAL2,G1,G2,ALFA,BETA,GC(20C),RC(20C),V(20C),
R(20C),P(20C),RCLC(20C),WHC(20C),WHH(20C)

IAA=IA+1

IF=IF+1

DC 1 I=1A,1BB

Z=(R(I)+R(I-1))/2.+4.+(V(I)-V(I-1))/4.+*(V(I)+V(I-1))/4.*((R(I)+R(I-1))/2.)

V(I)=V(I-1)*X**(ALFA-5)*V(I-1)/V(I-1)+V(I-1)/ALFA-1.)*V(I-1) +*(PH(I-1)+PH(I-1))

V(I)/RC(I)-4.*V(I)+V(I-1)/RC(I-1)+V(I-1)/RC(I-1)+R(I+1)/RC(I-1)+4.*R(I-1)/RC(I-1)

R(I)=R(I)+2.*X*Z

P(I)=P(I)+2.*X*Z*(G1+G2+BETA+GC(I))*((R(I)+R(I-1))/2.+RC(I-1))

RETURN

END

**SUBROUTINE BDRY**

CCMKCN PI,T,RHU,TAU1,TAL2,G1,G2,ALFA,BETA,GC(20C),RC(20C),V(20C),
R(20C),P(20C),RCLC(20C),WHH(20C),WHH(20C)

RC(I)=-RCLC(I)+RCLC(I-1)+2.(G1+G2+BETA+GC(I))

RETURN

END
SUBROUTINE BIFUR

SUBROUTINE EIFUR(IA, IE, IC, XA, XC, RC, RLA, RLE, RLC)
CMMCN PI, T, RHL, TAU, TAU2, GI, G2, ALFA, ETA, GC(2CC), RC(2CC), V(2CC),
$R(2CC), P(2CC), (2CC), VHLC(2CC), HOLE(2CC), PHGCL(2CC), VHLC(2CC), AH(2CC),
$PC, VIC(2CC), RH(2CC), PH(2CC), AH(2CC), IA = IA-1
1 IE = IE+1
1 IC = IC+1
A = (GI + G2 + BETA/2) * (GC(IA) + Go(IAA)) * (2 * R(IAB) / (RC(IA) + RC(IAA))) **
$(BET A-1.)) * (P1 * R(IAB) / (RC (IA) + Rc (IAB)))
B = (GI + G2 + BETA/2) * (GC(IB) + Go(IEB)) * (2 * R(IB) / (RC(IB) + RC(IEB))) **
$(BET A-1.)) * (P1 * R(IB) / (RC(IB) + RC(IEB)))
C = (GI + G2 + BETA/2) * (GC(IC) + Go(ICC)) * (2 * R(IC) / (RC(IC) + RC(ICC))) **
$(BET A-1.)) * (P1 * R(IC) / (RC(IC) + RC(ICC)))
$(I AA) + RHL(IAA)**2) - 2. + VHLC(IA) + RHL(IAA)**2) - T/2. + RLA*(PHLC(IA)
$-PQ) + 2. * T*H(IA) / (RLA + A*T**2 / 2. / XA)
ZB = (PHLC(IE) + PHLC(IEB) - P(IE) + E**2/2) * (PI*(V(IEB) + R(IEB)**2 + VHLC
$(IEB) + RHL(IEB)**2) - 2. + VHLC(IEB) + RHL(IEB)**2) - T/2. + RLE*(PHLC(IE)
$-PQ) + 2. * T*H(E) / (RLE + B*T**2 / 2. / XE)
ZC = (PHLC(IC) + PHLC(ICC) - P(ICC) + C**2/2) * (PI*(V(ICC) + R(ICC)**2 + VHLC
$(ICC) + RHL(ICC)**2) - 2. + VHLC(ICC) + RHL(ICC)**2) - T/2. + RLC*(PHLC(IC)
$-PQ) + 2. * T*H(C) / (RLC + C*T**2 / 2. / XC)
PNENK = (ZK + ZC + PHLC(IA) / RLA + PHLC(IE) / RLE + PHLC(ICC) / RLC)*PC*
$+(1. / RLA + 1. / RLE + 1. / RLC)) / (1. / RLA + 1. / RLE + 1. / RLC - 1. / (2.*X/C/(RLC/T)**2 + RLC))
R(IA) = ZK + RLA + PNENK / (1. + 2.*X*C + RLA/T)
R(IE) = ZK + RLE + PNENK / (1. + 2.*X*C + RLE/T)
R(IC) = ZK + RLC + PNENK / (1. + 2.*X*C + RLC/T)
KCNT = 1
1 CONTINUE
KCN = KCNT + 1
CALL ECY(IA, IAA, IA, -X)
CALL ECY(EE, IE, IE, X)
CALL ECY(IC, IC, IC, X)
FCDIFF = PI*(R(IA)**2 + V(IA) - R(IB)**2) * V(IE)**2 + R(IE)**2 * V(IE)**2)
IF(AES(FCDIFF) - 2.) 2, *
IF(KCUNT = 2) *, *, 2
PNENK = -PC + (PI + PCLC(IA)) / RLA + (PI + PCLC(IE)) / RLE + (PI + PCLC(ICC)) / RLC
$+PCLC(IA) / RLC)
R(IA) = PC*PI + PCLC(IA) / RLA + R(IA) + R(IA) + 2*V(IA) - RHLCL(IA)**2
$+VHLC(IA)
R(IE) = PC*PI + PCLC(IE) / RLE + R(IE) + R(IE) + 2*V(IE) - RHLCL(IE)**2
$+VHLC(IE)
R(IC) = PC*PI + PCLC(ICC) / RLC + R(IC) + R(IC) + 2*V(IC) - RHLCL(ICC)**2
$+VHLC(ICC)
GC 10
2 CONTINUE
PC = PNENK
RETURN
END
SUBROUTINE ECRY2

SUBROUTINE ECRY2(IA, XA)
CCMN, P1, T, RH0, TAU1, TAU2, G1, G2, ALFA, ETA, CC(20C), RC(20C), V(20C),
$R(20C), P(20C), H(20C), VHCLC(20C), RHCLL(20C), PHCLC(20C), KFCLC(20C),
$PG, VISC, VH(20C), RH(20C), PH(20C), W(20C)
IAA=IA+1
GG=(G1+G2+BETA2/2.*(CC(IAA)+CC(IA))+2.*RH(IA)/(RC(IA)+RC(IAA)))*
$(ETA-1)/((2.*RC(IAA)+RC(IAA))
R(IAA)=VHCLC(IAA)-V(IA)+VH(IA)/RH(IA)*(T/XA*(ALFA-5.)*VH(IA)+
$R(IAA)+*RHCLC(IAA)-2.*T*VISC*ALFA/(ALFA-1.)*RH(IA)
1+2.*ALFA*(R(IAA)-RHCLC(IAA)))*X/A/RH(IA)+2.*RHCLC(IAA))/(T/XA*(ALFA-5.)*VH(IA)+
$R(IAA)-R*RHCLC(IAA)))+T/XA/RH(IA)-P(IAA)*T*KH(IA)+
$ALFA-1.)*VH(IA)-X/A/RH(IA)*CG)/(T/XA*(ALFA-5.)*VH(IA)+
RETURN
END

SUBROUTINE HEART

SUBROUTINE HEART(IA, XA, K, CV, RKA)
CCMN, P1, T, RH0, TAU1, TAU2, G1, G2, ALFA, ETA, CC(20C), RC(20C), V(20C),
$R(20C), P(20C), H(20C), VHCLC(20C), RHCLL(20C), PHCLC(20C), KFCLC(20C),
$PG, VISC, VH(20C), RH(20C), PH(20C), W(20C)
$CV=CV, CV, PV, NV, LV
KCNT=1
IAA=IA+1
WRITE(5,1) CV
FCMAT(1X, , , , 3)
IF(PV-P(IAA)) *, 3
IF(V(IAA)-.01) *, 3
NV=I5
V(IAA)=0.
CALL ECRY2(IA, XA)
CVNEW=CV
PVNEW=CV*(CVNEW-CV)
GC TC 2
3 CONTINUE
NV=I5
$(ETA-1.)/2.*(RC(IAA)+RHCLC(IAA))-(RHCLC(IAA)+2.*VHCLC(IAA)-
1 CONTINUE
KCNT=KCNT+1
PIA=P(IAA)
CALL ECRY2(IA, IAA, IA, XA)
$V/H(IAA)-VHCLC(IAA)
1IF(KCNT=20) *, 2
IF(AES(P(IAA)-PIA)-1.) *, 1
2 CONTINUE
CV=CVNEW
PV=PVNEW
RETURN
REFERENCES.


