1 Introduction

Hotelling's (1929) spatial model of competition has had a large and varied influence on a number of fields. It has been applied not only in the original context of firms selecting geographic locations along "Main Street" so as to maximise their share of the market, but also to that of producers deciding on how much variety to incorporate into their products (see Chamberlin 1933). Downs (1957) adapted it with minor modifications to model an election: in particular, the ideological position-taking behaviour of political candidates in their effort to win votes.

The model in its simplest form features a number of candidates (firms) adopting positions on a one-dimensional manifold, usually taken to be the interval $[0, 1]$, along which voters' ideal positions (consumers) are distributed. Voters cast their votes for the candidate that is closest ideologically to them and candidates adopt positions so as to maximise their share of the vote (market)[1] and cannot opt out. One can imagine parties competing for votes in an election under a system of proportional representation. In the economic interpretation of this model, firms compete for market share, competition on price is excluded, and customers buy from the closest firm to minimise transportation costs.

In most of the literature, voters have only one vote, which they allocate to their favourite candidate. That is, the electoral system is taken to be plurality rule. The voters' second, third and other preferences do not come into play. Economically, this is akin to saying customers only patronise the nearest firm, with the distribution of the more distant firms being irrelevant to them.

In many situations, however, preferences other than first do indeed matter. Far from all elections are held under plurality rule: electoral systems, both in use and theoretical, are diverse. So too are the incentives that candidates are faced with under different electoral systems (Cox 1987). A thorough analysis of these incentives is important for the design of voting institutions with desirable properties. In particular, a designer should know whether candidates pursuing rational policies will cluster together, advocating identical or similar policy positions and displaying what Hotelling called "excessive sameness", or will they adopt diverse positions that appeal to different groups of voters? It is also important to know whether equilibria situations exist since their absence may lead to a permanent instability. This was the theme of Myerson's Schumpeter Lecture (1998, Berlin meetings of the European Economic Association), later published in Myerson (1999).

One very general class of electoral systems that Myerson advocates are the scoring rules. In an election held under a scoring rule, each voter submits a preference ranking of all m candidates, and then s_i points are assigned to the candidate in the ith position on the voter's ranking. The total number of points is a measure of the support for a candidate in the society. Thus, we specify a

[1] See Stigler's (1972) argumentation for this assumption and a discussion of it in Denzau et al. (1985).

scoring rule by an m-vector of real nonnegative numbers, $s = (s_1, \ldots, s_m)$, with $s_1 \geq \cdots \geq s_m \geq 0$ and $s_1 > s_m$. Well-known examples of scoring rules include plurality, Borda's rule and antiplurality, given by score vectors $s = (1, 0, \ldots, 0)$, $s = (m-1, m-2, \ldots, 0)$ and $s = (1, \ldots, 1, 0)$, respectively.

In an election—according to Stigler's thesis—the candidates would then adopt positions with the aim of maximising the total number of points received from all the electorate and hence the equilibrium strategies will depend on the particular scoring rule in use. This dependence is the focus of our investigation.

This is not the only possible interpretation of scoring rules in the context of Hotelling-Downs model; another plausible view is due to Cox (1990). For this interpretation, we normalise the score vector s so that the sum of its coordinates is 1. Then we may think that the voting rule is actually plurality and s_i is the likelihood that a voter votes for the ith nearest candidate. Indeed, ideological proximity is not the only factor at the ballot box: a single issue out of many may put a voter off a candidate who, on the whole, is of a similar ideological bent; or, it could simply be down to personal charisma, experience, prejudice or any number of other nonpolicy reasons. In the economic interpretation, it is also natural to assume that consumers buy from more distant firms with some probability—occasionally, one is simply passing through the vicinity. The probability implied by such an interpretation is of an ordinal nature—there is no dependency on absolute distance, as occurs in most of the literature on probabilistic voting (see, e.g., Coughlin 1992, Duggan 2005).

Cox (1987) provides valuable results on the existence of convergent Nash equilibria (CNE), that is, equilibria in which all candidates adopt the same policy position. He identifies three classes to which a scoring rule may belong: "best-rewarding" rules which never possess CNE; "worst-punishing" rules, to the contrary, do allow CNE in which all candidates adopt a common position located within a certain interval; and, an intermediate case between the two, for which a unique CNE exists at the median voter's ideal position.

As one infers from the nomenclature, under each of these three classes of scoring rules candidates are faced with different incentives. Myerson (1999, p. 677)—who introduced the terminology[2]—explains the competing incentives as follows: under a best-rewarding rule, "the candidate gains more from moving up in the preferences of voters who currently rank her near the top"; on the contrary, under a worst-punishing rule "the candidate gains more from moving up in the preferences of voters who currently rank her near the bottom."

Cox stopped short of investigating the existence of nonconvergent Nash equilibria (NCNE) for general scoring rules, i.e., those equilibria in which the positions adopted by the candidates are not all the same. For plurality, however, the situation is well-studied and a full characterisation of NCNE was given by Eaton and Lipsey (1975) and Denzau et al. (1985). The nonexistence of NCNE for antiplurality was also known, as Cox (1987, p. 93) pointed out. For general scoring rules the picture was unclear, and little work has previously been done in this direction. Cox conjectured (1987, p. 93) that "nonconvergent equilibria

[2] Cox (1990) refers to "first-place rewarding", "intermediate" and "last-place punishing" rules.

were at best rare for [worst-punishing] scoring functions, but fairly common for [best-rewarding] scoring functions." We look to fill this gap in the literature.

Why should we care about nonconvergent Nash equilibria? First, a degree of differentiation in policy platforms (firm locations) is socially desirable as it provides a better representation of voters by candidates (transportation costs, respectively). On the other hand, one may want to avoid a rule which leads to excessively extreme positions. A rule for which equilibria do not exist is also undesirable insofar as it entails inherent strategic instability and a lack of predictability.

Second, nonconvergent equilibria are often observed. For example, fast food restaurants of various chains are usually located in clusters at various points along a main street. This effect was called the 'Principle of Local Clustering' by Eaton and Lipsey (1975): "When a new firm enters a market, or when an existing firm relocates, there is a strong tendency for that firm to locate as close as possible to another firm. This behaviour tends to create local clusters of firms in many equilibrium and disequilibrium situations." The standard Hotelling-Downs model cannot explain this phenomenon since nonconvergent Nash equilibria in this model cannot have more than two firms in the same location (and there are often more in the case of fast food restaurants). One of the aims of this paper is to generalise the Hotelling-Downs model to explain this principle.

Since a general characterisation of NCNE appears to be intractable, our approach is to consider the problem in large classes of rules satisfying various additional conditions. For two broad classes of scoring rules, we find that NCNE do not exist at all or seldom exist. For rules having scores that are "convex", NCNE are impossible except for some derivatives of Borda rule (Theorem 6.3). For rules with "symmetric" scores we find that NCNE do not exist either.

On the other hand, we identify two broad classes of rules that do allow NCNE. The rules of the first class allow NCNE with multiple candidates clustered more or less evenly along the issue space (Theorem 7.1). The rules of the second class allow bipositional NCNE in which all candidates are split into two symmetrically located clusters with the same number of candidates (Theorem 7.6).

In the special cases of four- and five-candidate elections, matters are simpler and we provide a complete characterisation NCNE (Theorems 5.1 and 5.3). We also examine six-candidate elections (Theorem 5.4).

The paper is organised as follows. Firstly, we briefly review the literature in Section 2. In Section 3 we introduce our model and we derive some preliminary results in Section 4. In Section 5, we examine the special cases of four-, five- and six-candidate elections, and then we investigate rules that do not allow NCNE in Section 6. In Section 7, we look at rules that do admit NCNE including those that explain the Principle of Local Clustering. We conclude in Section 8. For better readability, several proofs are relegated to the appendix.

2 Related literature

There are several closely connected but distinct variants of the Hotelling-Downs model in the literature. This model determines the general framework for competition but does not model candidates themselves. In particular, according to Cox (1987), the candidates can be viewed as share maximisers, plurality maximisers or complete plurality maximisers. Share maximisers aim to maximise their support in the society as calculated by the voting rule. Share maximization is a direct reformulation of Hotelling's (1929) assumption that firms are competing for customers, i.e., for market share. The original Downs (1957) paper talks about competition of parties for voters, meaning election of a parliament under some form of proportional representation. Stigler (1972, p. 98) strongly argues for the "share maximisation" paradigm against the "winning the election" paradigm.

A plurality maximiser seeks to maximise (minimise) the margin by which he or she wins (loses) the election. Complete plurality maximisers do the same, except they also care about their position relative to more than just the first- or second-placed candidate. Osborne (1993) brought a new dimension into this type of model by allowing agents to stay out of the competition if they have no chances of winning. This changes the whole game and the equilibria. This line of research was further pursued in Osborne (1993, 1995), Sengupta and Sengupta (2008), Brusco et al. (2012), and others.

Assuming that candidates are share maximisers, this paper is most closely related to Eaton and Lipsey (1975) and Denzau et al. (1985). The former consider plurality rule, and the latter extend these results to "generalised rank functions". That is, the candidates' objectives depend to some degree both on vote share and rank (say, the number of candidates with a larger share). For a review emphasising multicandidate competition, see Shepsle (2001).

Another important assumption we adopt is the sincerity of the voters, which results in the only strategic agents in our model being the candidates. Due to the Gibbard-Satterthwaite theorem (Gibbard 1973, Satterthwaite 1975), stating that in the case of more than two candidates all non-dictatorial single-winner elections are manipulable by a single voter, this assumption is quite restrictive in the "winning the election" paradigm. However it is benign in the "share maximisation" paradigm. Indeed, as Slinko and White (2010) show, such a voting system becomes manipulable only if a threshold for entry into parliament is introduced.

The last important assumption to be mentioned is the uniform distribution of voters along the ideological spectrum. This is, of course, a simplifying assumption. To its defense, however, as has been noted in other papers such as Aragonès and Xefteris (2012), the distribution need not literally be uniform. Indeed, it is enough that the agents believe the distribution to be uniform or are taking this as a simplifying assumption in their calculations. Also, generalizing previous results to arbitrary scoring rules, it is natural to keep the uniform distribution as the first step.

Scoring rules and similar voting systems have appeared in spatial models before. However, apart from Cox (1987), all the work has been done in somewhat different contexts. Myerson (1999) looks at the incentives inherent in different scoring rules and the political implications for such matters as corruption, barriers to entry and strategic voting. His model consists of a simpler issue space, with candidates deciding between two policy positions—"yes" or "no". In Myerson (1993), he compares various scoring rules with respect to the campaign promises they encourage and, in Myerson (2002), he investigates scoring rules from the voter's perspective in Poisson voting games. Laslier and Maniquet (2010) look at multicandidate elections under approval voting when the voters are strategic. Myerson and Weber (1993) introduce the concept of a "voting equilibrium", where voters take into account not only their personal preferences but also whether contenders are serious, and compared plurality and approval voting in a three-candidate positioning game similar to ours. We focus only on candidate strategies.

Many other more realistic refinements of the Hotelling-Downs model have been constructed, incorporating uncertainty, incomplete information, incumbency and underdog effects, and so on. However, the price of added realism is that the models becomes more complicated, and considering more than two candidates is often intractable. For a survey focusing on variations of the two candidate case, see Duggan (2005) or Osborne (1995).

In light of the probabilistic interpretation of the scoring rule mentioned in the introduction, we feel the need to emphasise the differences between our approach and that taken in the probabilistic voting literature. For an introduction to the field, see Coughlin (1992) or, for a survey, Duggan (2005). These works usually involve some distance dependent function to give the probabilities as, for example, in multinomial logit choice models. These functions lead to an expected vote share that depends on distance and does not have discontinuities when agents' positions coincide. A scoring rule is somewhat simpler: it behaves like a step function that only depends on ordinal information (ranking) and not on relative distances. Moreover, the discontinuities associated with the deterministic model persist in our model. Other models include the stochastic model of Anderson et al. (1994) and Enelow et al. (1999). The former again involves a function depending on distance, and the latter involves a finite number of voters in a multidimensional space. De Palma et al. (1990) look at a model incorporating uncertainty and numerically calculate equilibria for up to six agents, comparing them with the deterministic model. Interestingly, some of the observed equilibria for four and five agents show similarities to those we find in our model (see the footnote following Theorem 5.3).

As for the economic interpretation, our results are related only to those models in which price competition and transport costs do not come into play, such as in, again, Eaton and Lipsey (1975) and Denzau et al. (1985). The most basic models incorporating price competition can be found in standard economics textbooks, such as Vega-Redondo (2003, pp. 171–176).

3 The model

There is a continuum of voters, assumed to have ideal positions uniformly distributedon the interval $[0,1]$, the issue space, on which candidates adopt positions. Since the two- and three-candidate cases are well-known—in the former case we have the classical median voter result, and in the latter no NCNE exist (see the end of this section)—we assume there are $m \geq 4$ candidates. Candidate i's position is denoted x_i. A strategy profile $x = (x_1, \ldots, x_m) \in [0,1]^m$ specifies the positions adopted by all the candidates. For a given strategy profile x, denote by $x^1, \ldots, x^q$ the *distinct* positions that appear in x, labelled so that $x^1 < \cdots < x^q$. In a NCNE, we will always have $q \geq 2$. Let $n_i \in \mathbb{N}$ denote the number of candidates adopting position x^i. Then we have $\sum_{i=1}^{q} n_i = m$. For this strategy profile x, we will often use the equivalent notation $x = ((x^1, n_1), \ldots, (x^q, n_q))$.

We use the notation $[n] = \{1, \ldots, n\}$. If $I = [a,b] \subseteq [0,1]$ is an interval with endpoints a and b ($a \leq b$) then the length of that interval will be denoted $l(I) = b - a$; since voters' ideal positions are uniformly distributed, $l(I)$ will be the measure of voters with ideal positions in I.

Voters are not strategic and have single-peaked, symmetric utility functions. Hence, a voter ranks the candidates according to the distance between her own personal ideal position and the positions adopted by the candidates. If $n_i \geq 2$, that is, two or more candidates adopt position x^i, then all voters will be indifferent between these candidates. In this case the voter will randomly choose a strict ordering by way of a fair lottery.[3]

Candidate i's score, $v_i(x)$, is the total number of points received on integrating across all voters.[4] The candidates maximise $v_i(x)$—that is, they are score (share) maximisers. Information is complete and candidates adopt positions simultaneously.

The following example shows how the scores are calculated.

Example 3.1. Consider a three-candidate election contested between candidates i, j and k and held under a scoring rule $s = (s_1, s_2, s_3)$. Consider the profile $x = ((x^1, 2), (x^2, 1))$, where $x_i = x_j = x^1$ and $x_k = x^2$, illustrated in Figure 1. Voters with ideal positions in the interval $I_1 = [0, (x^1 + x^2)/2]$ are indifferent between candidates i and j, but prefer either of them to k. The voters in the interval $I_2 = [(x^1 + x^2)/2, 1]$ have k as their unique favourite candidate, and are indifferent between i and j. In this case, candidates i and j both receive the score $v_i(x) = v_j(x) = \left(\frac{s_1+s_2}{2}\right) \ell(I_1) + \left(\frac{s_2+s_3}{2}\right) \ell(I_2)$ while candidate k receives $v_k(x) = s_1 \ell(I_2) + s_3 \ell(I_1)$.

Given a scoring rule $s = (s_1, \ldots, s_m)$, sometimes we will consider *subrules* of s. A subrule of the rule s will be a vector $s' = (s_i, s_{i+1}, \ldots, s_{i+j})$ where $i, j \geq 1$ and $i + j \leq m$. Thus, if $s_i > s_{i+j}$, a subrule is itself a scoring rule corresponding to an election with $j+1$ candidates. If $s_i = \cdots = s_{i+j}$, then s' does not define a scoring rule and we say s' is a *constant subrule* of s.

[3] Alternatively we could allow indifferences on voters' ballots and modify the scoring rule accordingly.

[4] This function is deterministic by the tie-breaking assumption.

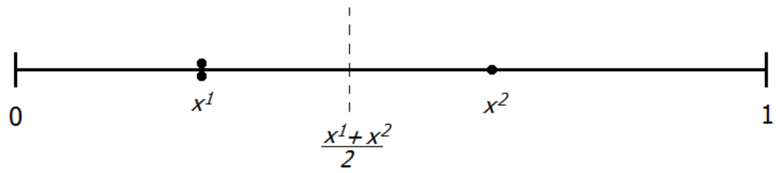

Figure 1: The situation in Example 3.1

Our equilibrium concept is the pure strategy Nash equilibrium (NE). A strategy profile $x^* = (x^*_1, \ldots, x^*_m)$ is in NE if, for all $i \in [m]$ and $x_i \in [0, 1]$, we have $v_i(x^*) \geq v_i(x_i, x^*_{-i})$, where $(x_i, x^*_{-i}) = (x^*_1, \ldots, x^*_{i-1}, x_i, x^*_{i+1}, \ldots, x^*_m)$. A NE x is said to be a *convergent Nash equilibrium* (CNE) if all candidates adopt the same position, i.e., $x = ((x^1, m))$. If, in a NE, at least two candidates adopt distinct positions, it is a *nonconvergent Nash equilibrium* (NCNE). If a strategy profile $x = ((x^1, n_1), \ldots, (x^q, n_q))$ is an NE then we will say that $(n_1, \ldots, n_q)$ is its *type*.

Let us restate Cox's (1987) characterisation of CNE for arbitrary scoring rules.

Theorem 3.2 (Cox 1987)**.** *Given a scoring rule s, the profile $x = ((x^1, m))$ is a CNE if and only if*

$$c(s, m) \leq x^1 \leq 1 - c(s, m),$$

where $c(s, m) = \dfrac{s_1 - \bar{s}}{s_1 - s_m}$ *and* $\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$ *is an average score.*

For the inequality in Theorem 3.2 to be satisfied for some x^1, it must be that $c(s, m) \leq 1/2$. The number $c(s, m)$ encodes important information about the competing incentives characteristic of a given scoring rule: in particular, it measures the first-to-average drop in the value of the points relative to the first-to-last drop. Motivated by the above theorem, Cox defined rules with $c(s, m) > 1/2$, $c(s, m) < 1/2$ and $c(s, m) = 1/2$ as *best-rewarding*, *worst-punishing* and *intermediate*, respectively.[5] Hence, CNE exist if and only if the rule is worst-punishing or intermediate.

Cox also observed that in any NCNE the most extreme positions, x^1 and x^q, must be occupied by at least two candidates and, hence, in the case of a three-candidate election, no NCNE exist.

4 Preliminaries

We begin by deriving a number of key lemmas. We first investigate how candidate i's score changes on deviating—that is, we are interested in the function $v_i(t, x_{-i})$.

Proposition 4.1. *Let x be a profile, i be a candidate, and suppose the other $m - 1$ candidates besides i are located at positions $x^1, \ldots, x^q$. Then*

(a) In the intervals $(0, x^1)$ and $(x^q, 1)$ the function $v_i(t, x_{-i})$ is linear with slopes $(s_1 - s_m)/2$ and $-(s_1 - s_m)/2$, respectively.

[5] Terminology of R. Myerson (1999).

(b) Suppose that, apart from candidate i, for some ℓ there are j candidates in positions $x^1, \ldots, x^{\ell-1}$ and k candidates in positions $x^\ell, \ldots, x^q$, where $j + k = m - 1$. Then in the interval $(x^{\ell-1}, x^\ell)$ the function $v_i(t, x_{-i})$ is linear with slope $(s_{j+1} - s_{k+1})/2$.

We note that at $x^1, \ldots, x^q$ the function $v_i(t, x_{-i})$ is in general discontinuous and all three values $v_i(x^{j-}, x_{-i}) = \lim_{t \to x^{j-}} v(t, x_{-i})$, $v_i(x^j, x_{-i})$, $v_i(x^{j+}, x_{-i}) = \lim_{t \to x^{j+}} v(t, x_{-i})$, where the limits are one-sided, may be different. Continuity, even linearity of this function in the intervals between candidates' positions, proved in Proposition 4.1, will be an important tool in identifying Nash equilibria. It allows us to conclude that unpaired candidates are actually quite rare.

Corollary 4.2. *If m is even and $s_{m/2} \neq s_{m/2+1}$, there can be no unpaired candidate. If m is odd and $s_{(m-1)/2} \neq s_{(m+3)/2}$ then the only candidate that could possibly be unpaired is the median candidate.*

We continue investigating the behaviour of the function $v_i(t, x_{-i})$, this time in the neighborhood of a pair of candidates occupying the same position. We prove that in this case there can be no discontinuity as mentioned above.

Lemma 4.3. *Suppose at profile x candidate i is at x^ℓ and $n_\ell = 2$. Then $v_i(x^{\ell-}, x_{-i}) + v_i(x^{\ell+}, x_{-i}) = 2v_i(x)$. In particular, when x is a NCNE, $v_i(x^{\ell-}, x_{-i}) = v_i(x^{\ell+}, x_{-i}) = v_i(x)$.*

The following powerful fact relates to NCNE where n_1 or n_q is equal to two.

Lemma 4.4. *If $n_1 = 2$ or $n_q = 2$ then a necessary condition for NCNE is $s_2 = \ldots = s_{m-1}$.*

Lemmas 4.3 and 4.4 tell us the only rules that allow paired candidates at the end positions are the rules of the form $s = (a, b, \ldots, b, c)$. The following lemma places a lower bound on the number of candidates at extreme positions. This is a generalisation of Cox's (1987, p. 93) observation that there can be no less than two candidates at any of the two extreme positions.

Lemma 4.5. *Given a scoring rule s, let $1 \leq k \leq m - 1$ be such that $s_1 = \cdots = s_k > s_{k+1}$. Then a necessary condition for a profile x to be a NCNE is $\min(n_1, n_q) > k$.*

This already allows us to rules out NCNE for a large class of scoring rules.

Corollary 4.6. *Any scoring rule s with $s_1 = \cdots = s_k > s_{k+1}$ for some $k \geq \lfloor m/2 \rfloor$ does not allows an NCNE.*

Proof. By Lemma 4.5, $\min(n_1, n_q) > k \geq \lfloor m/2 \rfloor$. Hence, $n_1 + n_q > m$, a contradiction. $\square$

The rules specified in Corollary 4.6 are usually worst-punishing. However, there are some that are slightly best-rewarding ones, such as k-approval[6] with m

[6] That is, $s = (1, \ldots, 1, 0, \ldots, 0)$, where the first k components are ones.

odd and $k = (m-1)/2$. This shows that there exist best-rewarding rules which, unlike plurality, do not allow NCNE (and, hence, no Nash equilibria at all).

The next lemma says that in a NCNE no candidates may occupy the most extreme positions of the issue space, 0 and 1.

Lemma 4.7. *Let s be a scoring rule. In a NCNE x no candidate may adopt the most extreme positions. That is, $0 < x^1$ and $x^q < 1$.*

Finally, for a given scoring rule, we place an upper bound on the length of the interval between two occupied positions, or between the boundary of the issue space and the nearest occupied position. It also gives a lower bound on the number of occupied positions.

Lemma 4.8. *Given a scoring rule s, if x is a NCNE, then the following conditions must be satisfied:*

(a) $x^1 \le 1 - c(s,m)$ and $x^q \ge c(s,m)$;

(b) $x^i - x^{i-1} \le 2(1-c(s,m))$ for any i such that $2 \le i \le q$.

(c) the number of occupied positions q is at least $\left\lceil \frac{1}{2(1-c(s,m))} \right\rceil$.

Note that condition (a) of the this lemma generalises Cox's (1987, p. 88) argument that in a plurality election the most extreme candidates on either side are located outside the interval $(1/m, 1-1/m)$. We note that the interval $(1-c(s,m), c(s,m))$ reaches its maximal width under plurality rule.

5 The four-, five- and six-candidate cases

This section is devoted to the analysis of special cases where the number of candidates is small. When $m = 4$ and $m = 5$ we will provide a complete characterisation of the rules allowing NCNE. For $m = 6$ we will identify all types of equilibria that might exist.

Theorem 5.1. *Given $m = 4$ and scoring rule $s = (s_1, s_2, s_3, s_4)$, NCNE exist if and only if both the following conditions are satisfied: (a) $c(s,4) > 1/2$; (b) $s_1 > s_2 = s_3$. Moreover, the NCNE is unique and symmetric with equilibrium profile $x = ((x^1,2),(x^2,2))$, where*

$$x^1 = \frac{1}{4}\left(\frac{s_1 - s_4}{s_1 - s_2}\right) \quad \textit{and} \quad x^2 = 1 - x^1. \tag{1}$$

Proof. By Lemma 4.5, a NCNE with $m = 4$ must have exactly two distinct positions, $x^1 < x^2$, with $n_1 = n_2 = 2$ (agents 1 and 2 at x^1, 3 and 4 at x^2). Hence, by Lemma 4.4, it is necessary that $s_2 = s_3$. By Lemma 4.5, we also need $s_1 > s_2$. Hence, (b) is necessary.

By Lemma 4.7, we have $0 < x^1$ and $x^2 < 1$. By Lemma 4.3, in any NCNE we have $v_1(x^{2+}, x_{-1}) \le v_1(x^{1-}, x_{-1}) = v_1(x)$. That is, $v_1(x^{2+}, x_{-1}) = s_1(1-x^2) + s_2\left(\frac{x^2-x^1}{2}\right) + s_4\left(\frac{x^1+x^2}{2}\right) \le s_1 x^1 + s_2\left(\frac{x^2-x^1}{2}\right) + s_4\left(1 - \frac{x^1+x^2}{2}\right) = v_1(x^{1-}, x_{-1})$, which implies $s_1(1-x^1-x^2) \leq s_4(1-x^1-x^2)$. This is only possible if $x^1+x^2 \geq 1$. Considering the symmetric moves by candidate 4 gives $(1-x^1)+(1-x^2) \geq 1$ or $x^1+x^2 \leq 1$. Hence, $x^2 = 1-x^1$.

Then, since $v_1(x^{1-}, x_{-1}) = v_1(x)$, we have $s_1 x^1 + s_2\left(\frac{x^2-x^1}{2}\right) + \frac{1}{2}s_4 = \frac{1}{4}s_1 + \frac{1}{2}s_2 + \frac{1}{4}s_4$, from which, after substituting for x^2, equation (1) follows. For this to be a valid position, we need $x^1 < 1/2$, from which it follows that $2s_2 < s_1 + s_4$. This is equivalent to $c(s,4) > 1/2$, so (a) is also necessary.

For sufficiency, it is easy to check that (1) is actually a NCNE. However this is not necessary since, as we will see later, a rule satisfying conditions (a) and (b) also satisfies the conditions of Theorem 7.6, from which we conclude that the profile given by (1) is a NCNE. □

For the five-candidate case, we could either have: $q = 2$ and $(n_1, n_2) = (2,3)$ or $(n_1, n_2) = (3,2)$; or, $q = 3$ with $(n_1, n_2, n_3) = (2,1,2)$. In fact, $q = 2$ is not possible. To show this we need the following lemma. It will be also used later so we formulate it in a more general setting.

Lemma 5.2. *If $s = (s_1, \ldots, s_m) = (a, b, \ldots, b, 0)$, where $a > 2b$, then in any NCNE we have $n_1 = n_q = 2$.*

We are ready to describe all the rules that have NCNE for $m = 5$.

Theorem 5.3. *Given $m = 5$ and scoring rule $s = (s_1, s_2, s_3, s_4, s_5)$, NCNE exist if and only if both the following conditions are satisfied: (a) $c(s,5) > 1/2$; (b) $s_1 > s_2 = s_3 = s_4$. Moreover, the NCNE is unique and symmetric with equilibrium profile $x = ((x^1, 2), (1/2, 1), (x^3, 2))$, where*

$$x^1 = \frac{1}{6}\left(\frac{s_1+s_2}{s_1-s_2}\right) \quad \textit{and} \quad x^3 = 1-x^1. \tag{2}$$

Equation (2) shows that, when s_2 grows towards $s_1/2$, the positions of candidates become less extreme, converging at the median voter position when $s_2 = s_1/2$. As s_2 increases beyond this point, by Theorem 3.2 we know that infinitely many CNE are possible in an interval that becomes increasingly wide.[7]

Since for $m > 5$ the equilibria are no longer unique even for plurality (1975), it makes sense to describe only their types.

Theorem 5.4. *Given $m = 6$ and scoring rule $s = (s_1, s_2, s_3, s_4, s_5, s_6)$. Then there are four possible types of Nash equilibria that may occur; they can be split in two groups:*

$$\{(2,2,2),\ (2,1,1,2)\} \ \textit{and} \ \{(3,3),(6)\}.$$

[7] We note that this equilibrium behaviour shows certain similarities to some of the equilibria numerically calculated by De Palma et al. (1990) using a probabilistic model. When the level of uncertainty is low (which corresponds roughly to when $c(s,m)$ is large), they observe NCNE where the candidates are configured as in our NCNE. As the level of uncertainty increases from zero (the value of $c(s,m)$ decreases), they also observe the candidates' positions becoming less extreme. Beyond a certain point, only convergent equilibria are observed. However, in addition to these, they also observe other kinds of equilibria that do not arise in our model.

The equilibria of the first group occur for rules s that satisfy: (a) $c(s,6) > 1/2$; (b) $s_1 > s_2 = s_3 = s_4 = s_5$. The equilibria within each group can coexist. No equilibrium of the first group can coexist with an equilibrium of the second group.

Proof. We have to show that equilibria of types $(2,4)$, $(2,1,3)$, $(3,1,2)$ and $(4,2)$ do not exist. What they all have in common is that they have two candidates at one of the extreme positions and more than two the other. Without loss of generality, we can assume $s_6 = 0$. By Lemma 4.4, for NCNE, we must have $s_2 = s_3 = s_4 = s_5$ and by Lemma 4.5 we have $s_1 > s_2$. Hence, our rule is one of those studied in Lemma 5.2. Then there cannot be three or more candidates at any given position, which rules out all equilibrium types above and shows that equilibria of the first group are incompatible with those of the second.

Example 7.8(i) demonstrates that equilibria of the type $(3,3)$ exist and may coexist with CNE as the rule in that example is worst punishing. The two types of equilibria of the first group are shown to exist for plurality in Eaton and Lipsey (1975). □

6 Nonexistence of NCNE

In this section we will identify two broad classes of scoring rules for which NCNE do not exist or do not exist with a few well-defined exceptions.

The first class is all scoring rules with convex scores. These are best-rewarding rules and hence do not allow CNE. We show that such rules do not have NCNE (in fact, they have no NE whatsoever) with the exception of some derivatives of Borda rule. The second class consists of rules with a certain symmetry condition. These rules are intermediate, hence, allow a unique CNE at the median voter's ideal position by Theorem 3.2. However, we show that they do not allow NCNE.

Convex rules

We say that the rule $s = (s_1, \ldots, s_m)$ is *convex* if

$$s_1 - s_2 \geq s_2 - s_3 \geq \ldots \geq s_{m-1} - s_m. \tag{3}$$

We note that as soon as $s_i = s_{i+1}$ for some i, all the subsequent scores must also be equal for convexity to be satisfied. We aim to show that such rules, with one class of possible exceptions, have no NCNE; moreover they have no Nash equilibria at all. Firstly we show that a convex scoring rule s is either best-rewarding or intermediate. In fact, we show a bit more.

Proposition 6.1. *Let s be a scoring rule. Then s is convex if and only if every nonconstant m'-candidate subrule s', where $2 \leq m' \leq m$, has $c(s', m') \geq 1/2$.*

Proof. Suppose s satisfies (3). It suffices to show the rule itself is best-rewarding or intermediate, since any nonconstant subrule also satisfies (3). We have, for any $1 \leq i \leq \lfloor m/2 \rfloor$, $s_i - s_{i+1} \geq s_{i+1} - s_{i+2} \geq \ldots \geq s_{m-i} - s_{m-i+1}$. In particular,

$$s_i - s_{i+1} \geq s_{m-i} - s_{m-i+1}. \tag{4}$$

Suppose m is even. Equation (4) implies $s_1 + s_m \geq s_2 + s_{m-1} \geq \cdots \geq s_{m/2} + s_{m/2+1}$. Then

$$\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i = \frac{(s_1 + s_m) + (s_2 + s_{m-1}) + \cdots + (s_{m/2} + s_{m/2+1})}{m}$$
$$\leq \frac{m/2}{m}(s_1 + s_m) = \frac{1}{2}(s_1 + s_m).$$

A similar calculation shows that $2\bar{s} \leq s_1 + s_m$ if m is odd. So in both cases $s_1 + s_m \geq 2\bar{s}$, which is equivalent to $c(s, m) \geq 1/2$.

Conversely, suppose every nonconstant subrule s' is best-rewarding or intermediate. Then any 3-candidate subrule $s' = (s_i, s_{i+1}, s_{i+2})$ has $c(s', 3) \geq 1/2$, which is equivalent to $s_i - s_{i+1} \geq s_{i+1} - s_{i+2}$, so (3) is satisfied. □

The following lemma is crucial for the proof of Theorem 6.3, the main result of this section.

Lemma 6.2. *Let s be a convex scoring rule. Then the two conditions:*

(a) all inequalities in (3) *are equalities,*

(b) s satisfies $s_1 + s_m = \frac{2}{m}\sum_{i=1}^{m} s_i$,

are jointly equivalent to s being a Borda rule.[8]

Proof. (a) Let d be the common value of all the differences in (3). Then $s_i = (m-i)d + s_m$. Subtracting s_m from all scores does not change the rule. Dividing all scores by d after that does not change it either. But then we get the canonical Borda score vector with $s_i = m - i$.

(b) The condition (3) implies $s_1 + s_m \geq s_2 + s_{m-1} \geq s_3 + s_{m-2} \geq \ldots$ from which $s_1 + s_m \geq \frac{2}{m}\sum_{i=1}^{m} s_i$. An equality in the latter inequality is possible only if we had all equalities in the former, and this is possible only if we had equalities in (3). Now the result follows from (a). □

Now we can prove the main theorem of this section.

Theorem 6.3. *Let s be a scoring rule with convex scores and let $1 \leq n < m$ be such that $s_n > s_{n+1} = \cdots = s_m$. Then there are no NCNE, unless the subrule $s' = (s_1, \cdots, s_n, s_{n+1})$ is Borda and $n+1 \leq \lfloor m/2 \rfloor$ (i.e., more than half the scores are constant).*

Proof. Let x be a profile. Consider candidate 1 at x^1. Without loss of generality, assume $n_1 \leq \lfloor m/2 \rfloor$, since at least one of the two end positions has less than half the candidates. Let $I_1 = [0, x^1]$ and $I_2 = [x^1, (x^1 + x^2)/2]$. The rest of the issue space to the right of $(x^1 + x^2)/2$ can be partitioned into subintervals $J_1 = \left[\frac{x^1+x^2}{2}, \frac{x^1+x^3}{2}\right), \ldots, J_j = \left[\frac{x^1+x^{j+1}}{2}, \frac{x^1+x^{j+2}}{2}\right), \ldots, J_{q-1} = \left[\frac{x^q+x^q}{2}, 1\right]$, where

[8] Borda rule was defined in the introduction.

voters in each of these intervals rank candidate 1 in the same way. More specifically, candidate 1 shares k_i-th through to $(k_i + n_1 - 1)$-th place in the rankings of all voters in J_i, for some $k_i \geq 1$ such that $k_i + n_1 - 1 \leq m$. Then 1's score is

$$v_1(x) = \left(\frac{1}{n_1} \sum_{i=1}^{n_1} s_i \right) (\ell(I_1) + \ell(I_2)) + \sum_{j=1}^{q-1} \left(\frac{1}{n_1} \sum_{i=k_j}^{k_j+n_1-1} s_i \right) \ell(J_j).$$

If candidate 1 moves infinitesimally to the left, then $v_1(x^{1-}, x_{-1}) = s_1\ell(I_1) + s_{n_1}\ell(I_2) + \sum_{j=1}^{q-1} s_{k_j+n_1-1}\ell(J_j)$. Similarly, if she moves infinitesimally to the right, then $v_1(x^{1+}, x_{-1}) = s_1\ell(I_2) + s_{n_1}\ell(I_1) + \sum_{j=1}^{q-1} s_{k_j}\ell(J_j)$. Let x be a NCNE. Then $v_1(x^{1-}, x_{-1}) \leq v_1(x)$ and $v_1(x^{1+}, x_{-1}) \leq v_1(x)$. This implies that $v_1(x^{1-}, x_{-1}) + v_1(x^{1+}, x_{-1}) \leq 2v_1(x)$. That is,

$$\begin{aligned} &(s_1 + s_{n_1})(\ell(I_1) + \ell(I_2)) + \sum_{j=1}^{q-1}(s_{k_j} + s_{k_j+n_1-1})\ell(J_j) \\ &\qquad \leq \left(\frac{2}{n_1} \sum_{i=1}^{n_1} s_i \right) (\ell(I_1) + \ell(I_2)) + 2\sum_{j=1}^{q-1} \left(\frac{1}{n_1} \sum_{i=k_j}^{k_j+n_1-1} s_i \right) \ell(J_j), \end{aligned}$$

which implies

$$\begin{aligned} &\left(s_1 + s_{n_1} - \frac{2}{n_1} \sum_{i=1}^{n_1} s_i \right) (\ell(I_1) + \ell(I_2)) \\ &\qquad + \sum_{j=1}^{q-1} \left(s_{k_j} + s_{k_j+n_1-1} - \frac{2}{n_1} \sum_{i=k_j}^{k_j+n_1-1} s_i \right) \ell(J_j) \leq 0. \qquad (5) \end{aligned}$$

We know that the convexity of the scores implies $s_l + s_{l+n_1-1} \geq \frac{2}{n_1} \sum_{i=l}^{l+n_1-1} s_i$ for all $l \geq 1$ such that $l + n_1 - 1 \leq m$. Thus, each term on the left-hand side of (5) is nonnegative. If one or more of these terms is positive, then we have a contradiction and hence no NCNE exist. The only other possibility is that all these terms are equal to zero, which by Proposition 6.2 implies each of the n_1-candidate subrules appearing in the expression is equal to Borda or is constant (in particular, the rule $s' = (s_1, \ldots, s_{n_1})$ must be Borda, since $s_1 > s_{n_1}$ by Lemma 4.5). In particular we get $n_1 \leq n + 1$.

If this is the case then $v_1(x^{1-}, x_{-1}) + v_1(x^{1+}, x_{-1}) = 2v_1(x)$, so for x to be a NCNE we must have $v_1(x^{1-}, x_{-1}) = v_1(x^{1+}, x_{-1}) = v_1(x)$. Then, for x to be a NCNE we must have $v_1(t, x_{-1}) \leq v_1(x) = v_1(x^{1+}, x_{-1})$ for any $t \in (x^1, x^2)$, that is, the score cannot increase as 1 moves to the right from x^1. By Proposition 4.1(b), the slope of the linear function $v_1(t, x_{-1})$ for $t \in (x^1, x^2)$ is $\frac{1}{2}s_{n_1} - \frac{1}{2}s_{m-n_1+1}$ and since it is nonincreasing we have $\frac{1}{2}s_{n_1} - \frac{1}{2}s_{m-n_1+1} \leq 0$. On the other hand, $\frac{1}{2}s_{n_1} - \frac{1}{2}s_{m-n_1+1} \geq 0$ since $n_1 < m - n_1 + 1$. We conclude therefore that $s_{n_1} = s_{m-n_1+1}$. This means that the scores have stabilised on or earlier than s_{n_1}, whence $n_1 \geq n + 1$. As $n_1 \leq n + 1$, we must now conclude that $n_1 = n + 1$. Hence, there are no NCNE unless the subrule $s' = (s_1, \ldots, s_n, s_{n+1})$ is Borda and $n + 1 \leq \lfloor m/2 \rfloor$. □

For any rule s for which the subrule $s' = (s_1, \ldots, s_{n+1})$ is Borda and the scores from s_{n+1} through s_m are constant, Theorem 6.3 says nothing and for good reason since here NCNE can actually exist. This will follow from Theorem 7.1 and Example 7.4. Rules that do satisfy the conditions of Theorem 6.3 include Borda as well as the following example.

Example 6.4. The rule $s = (1, s_2, 0, \ldots, 0)$, for any $0 < s_2 < 1/2$, has convex scores and, hence, no NCNE. Thus, even a slight deviation from plurality destroys the NCNE which plurality is known to possess.

Symmetric scores

We say that the rule $s = (s_1, \ldots, s_m)$ is *symmetric* if

$$s_i - s_{i+1} = s_{m-i} - s_{m-i+1}, \tag{6}$$

for all $1 \leq i \leq \lfloor m/2 \rfloor$. That is, for every drop between consecutive scores at the top end there is an equal drop at the symmetric position at the bottom end.

Proposition 6.5. *A symmetric rule is intermediate.*

Proof. Note that (6) is condition (4) with equalities instead of inequalities. Hence, replacing all the inequalities in Proposition 6.1 with equalities, we obtain $c(s, m) = 1/2$. $\square$

We now show a symmetric rule has no NCNE. Note that a symmetric rule satisfies, for any valid n_1,

$$\frac{1}{n_1} \sum_{i=m-n_1+1}^{m} s_i + \frac{1}{n_1} \sum_{i=1}^{n_1} s_i = \frac{1}{n_1} \sum_{i=1}^{n_1} (s_i + s_{m-i+1}) = s_{n_1} + s_{m-n_1+1}. \tag{7}$$

Theorem 6.6. *A symmetric rule does not allow NCNE.*

Proof. Consider candidate 1 at position x^1, which is occupied by n_1 candidates. Consider intervals $I_1 = [0, x^1]$ and $I_2 = [(x^1 + x^q)/2, 1]$. If 1 makes an infinitesimal move to the right of x^1, then in the rankings of voters in I_1 she falls behind the other $n_1 - 1$ candidates originally at x^1. On the other hand, 1 rises ahead of these $n_1 - 1$ candidates in the rankings of all other voters. Then the score candidate 1 loses by making this move is $s_{lost} = \left(\frac{1}{n_1} \sum_{i=1}^{n_1} s_i - s_{n_1} \right) \ell(I_1)$. On the other hand, 1's gain from this move, s_{gained}, is *at least* the gain from I_2:

$$s_{gained} \geq \left(s_{m-n_1+1} - \frac{1}{n_1} \sum_{i=m-n_1+1}^{m} s_i \right) \ell(I_2) = \left(\frac{1}{n_1} \sum_{i=1}^{n_1} s_i - s_{n_1} \right) \ell(I_2), \tag{8}$$

where we have used (7). For this profile to be a NCNE, we need this move not be beneficial for candidate 1. That is, we need $s_{lost} \geq s_{gained}$, which, since $s_1 > s_{n_1}$, implies $\ell(I_1) \geq \ell(I_2)$ or, equivalently, $x_1 \geq 1 - (x^1 + x^q)/2$.

Similar considerations for candidate q give that $\ell([x^q, 1]) \geq \ell([0, (x^1 + x^q)/2])$. That is, $1 - x^q \geq (x^1 + x^q)/2$. Hence, $x^1 \geq x^q$, which is impossible for a NCNE. $\square$

Some examples of rules with symmetric scores are single-positive and single-negative voting, given by $s = (2, 1, \ldots, 1, 0)$, and the rule $s = (4, 3, 2, \ldots, 2, 1, 0)$.

7 Existence of NCNE

We will now turn our attention to rules for which NCNE do exist in general. First, we look at a class of best-rewarding rules for which we can find NCNE in which candidates cluster at positions spread across the issue space. Then, we characterise NCNE with two symmetric clusters when m is even. These results have implications for the Principle of Local Clustering, demonstrating how candidates (firms) are led to cluster at different points on the issue space (linear city).

Clustered NCNE

Here we provide a method of constructing rules for which multipositional NCNE exist. We introduce some additional notation. Let $I_1 = [0, (x^1 + x^2)/2]$, $I_i = [(x^{i-1} + x^i)/2, (x^i + x^{i+1})/2]$ for $2 \leq i \leq q-1$, and $I_q = [(x^{q-1} + x^q)/2, 1]$ be the "full-electorates" corresponding to each occupied position. For each $i \in [q]$ let $I_i^L = \{y \in I_i : y \leq x^i\}$ and $I_i^R = \{y \in I_i : y \geq x^i\}$ be the left and right "half-electorates" whose union is the full-electorate I_i, that is $I_i = I_i^L \cup I_i^R$. We note that $\ell(I_i^R) = \ell(I_{i+1}^L)$ for $i \in [q-1]$.

Theorem 7.1. *Let $m = qr$ be a composite number with $q \geq 2$. Consider an m-candidate scoring rule $s = (s_1, \ldots, s_{r-1}, 0, 0, \ldots, 0)$, where only the first $r-1$ scores can be non-zero. Then the profile given by $x = ((x^1, r), \ldots, (x^q, r))$ is a NCNE if and only if the following two conditions hold:*

(a) $\max_{i\in[q]} \max\{\ell(I_i^L), \ell(I_i^R)\} \leq (1 - c(s', r)) \min_{i\in[q]}\{\ell(I_i)\}$,

(b) $\max_{i\in[q]}\{\ell(I_i)\} \leq \left(1 + \frac{1}{r}\right) \min_{i\in[q]}\{\ell(I_i)\}$,

where $s' = (s_1, \ldots, s_{r-1}, 0)$.

The idea is that for this kind of scoring rule, each occupied position is "isolated" from the rest of the issue space, since a candidate at this position receives nothing from voters who rank her rth or worse. So the candidates have to compete "locally". Note that condition (a) can only be satisfied if $c(s', r) \leq 1/2$, since it implies

$$\begin{aligned}\max_{i\in[q]} \max\{\ell(I_i^L), \ell(I_i^R)\} &\leq (1 - c(s', r)) \min_{i\in[q]}\{\ell(I_i)\} \\ &\leq 2(1 - c(s', r)) \max_{i\in[q]} \max\{\ell(I_i^L), \ell(I_i^R)\},\end{aligned}$$

from which $c(s', r) \leq 1/2$ follows. That is, though the scoring rule is best-rewarding, the subrule s', for x to be a NCNE, must be worst-punishing or intermediate. Hence, comparing this with Theorem 3.2, we see that locally each occupied position behaves with respect to the rule s' in a similar way to a CNE on the whole issue space.

Proof. Consider candidate i at position x^k. Since all of i's score is garnered from the immediate full-electorate I_k, i's score is $v_i(x) = \left(\frac{1}{r}\sum_{j=1}^{r} s_j\right) \ell(I_k)$. Suppose that i moves to some position t between two occupied positions or between an occupied position and the boundary of the issue space. In the latter case, i is now ranked first by, at best, all voters in the intervals I_1^L or I_q^R. In the former case, when $x^l < t < x^{l+1}$ for some l, candidate i is ranked first by voters in the interval $[(x^l+t)/2, (t+x^{l+1})/2]$, which is equal in length to $\ell(I_l^R) = \ell(I_{l+1}^L)$. From the rest of the issue space, i is ranked at best rth, so receives nothing. In each case, i's score is now $v_i(t, x_{-i}) = s_1\ell(J)$, for the half-electorate J that i moves into. For NCNE we need this move not be beneficial, that is, $v_i(t, x_{-i}) \leq v_i(x)$. Thus, for NCNE we must have $s_1\ell(J) \leq \left(\frac{1}{r}\sum_{j=1}^r s_j\right)\ell(I_k)$, which occurs if and only if $\ell(J) \leq (1 - c(s', r))\ell(I_k)$. This must hold when i moves into any of the half-electorates, and for any candidate at any initial position. This yields the necessity of condition (a).

Also, there is a possibility that i moves to some position x^l that is already occupied. In this case her score becomes $v_i(x^l, x_{-i}) = \left(\frac{1}{r+1}\sum_{j=1}^{r+1} s_j\right)\ell(I_l) = \left(\frac{1}{r+1}\sum_{j=1}^{r} s_j\right)\ell(I_l)$, which again must not exceed $v_i(x)$. Hence, we must have $\left(\frac{1}{r+1}\sum_{j=1}^{r} s_j\right)\ell(I_l) \leq \left(\frac{1}{r}\sum_{j=1}^{r} s_j\right)\ell(I_k)$ or $\ell(I_l) \leq \left(1 + \frac{1}{r}\right)\ell(I_k)$. This must hold for any pair of full-electorates, which implies that condition (b) is necessary. With no other possible moves, so (a) and (b) are sufficient for NCNE. $\square$

We note that the degree to which the positions can be nonsymmetric depends on how small $c(s', r)$ is. If $c(s', r) = 1/2$, for example, then by condition (a) of Theorem 7.1 we must have that all the electorates are the same size and the occupied positions are at the halfway point of each one. If the profile is symmetric, with the candidates positioned so as to divide the issue space into equally sized full-electorates, Theorem 7.1 simplifies.

Corollary 7.2. *Let there be $m = qr$ candidates, $q \geq 2$, and consider the scoring rule $s = (s_1, \ldots, s_{r-1}, 0, 0, \ldots, 0)$, where only the first $r - 1$ scores can be non-zero. Then the profile given by $x = ((x^1, r), \ldots, (x^q, r))$ such that $\ell(I_i) = 1/q$ for all $i \in [q]$, is a NCNE if and only if $c(s', r) \leq 1/2$, where $s' = (s_1, \ldots, s_{r-1}, 0)$.*

Proof. Since each I_i is the same length, condition (b) of Theorem 7.1 is satisfied. Condition (a) reduces to $1/2q \leq (1 - c(s', r))/q$ or $c(s', r) \leq 1/2$. $\square$

Example 7.3. Consider r-candidate k-approval $s' = (1, \ldots, 1, 0, \ldots, 0)$ with $r \geq k + 1$. The condition $c(s', r) \leq 1/2$ holds if and only if $r \leq 2k$, suppose this is true. By appending zeros to the end of s', we can extend s' to k-approval with $m = qr$ candidates for any $q \geq 2$. Then Theorem 7.1 implies there exist NCNE in which r candidates position themselves at each of the q distinct locations.

As a special case, consider plurality: $s = (1, 0, \ldots, 0)$. For any even m, set $r = 2$ to obtain $s' = (1, 0)$ with $c(s', 2) = 1/2$. So the profile where two candidates locate at each position so as to divide the space into equally sized intervals is a NCNE, the only one in which there are two candidates at each position. We cannot have $r > 2$, as then we would have $c(s', r) > 1/2$. So plurality has no equilibria in which more than two candidates locate at each position, as is well-known by Eaton and Lipsey (1975) and Denzau et al. (1985).

Example 7.4. Let $s' = (r-1, r-2, \dots, 2, 1, 0)$, that is, s' is Borda. Let s, of length $m = qr$, $q \geq 2$, be the rule resulting from appending $(q-1)r$ zeros to s'. Then $c(s', r) = 1/2$, so there exists a NCNE in which r candidates position themselves at the q halfway points of q equally sized full-electorates.

Recall that Theorem 6.3 stated that a rule with convex scores has no NCNE, unless the nonconstant part of the scoring rule is exactly Borda and is shorter than the constant part. The rule s in Example 7.4 is precisely such a rule. Hence, the exception in Theorem 6.3 does indeed need to be made.

Example 7.5. For a given scoring rule, NCNE with different partitions of the candidates can exist simultaneously. Consider 3-approval, $s = (1, 1, 1, 0, \dots, 0)$, with $m = 20$. It can be verified that there exist equilibria with five distinct positions with four candidates apiece, as well as equilibria with four positions and five candidates apiece.

Bipositional clustered NCNE

A CNE is the simplest kind of Nash equilibrium that may exist. The next simplest would be a NCNE in which there are only two occupied positions. To keep things straightforward, we restrict to the case where m is even and the equilibrium positions are symmetric.

Theorem 7.6. *Suppose m is even. Then the profile $x = ((x^1, m/2), (x^2, m/2))$, with $0 < x^1 < 1/2$ and $x^2 = 1 - x^1$, is a NCNE if and only if both*

$$\frac{s_{m/2} + s_{m/2+1}}{2} < \bar{s} \tag{9}$$

and

$$\frac{s_1 + s_{m/2} - 2\bar{s}}{2(s_1 - s_{m/2+1})} \leq x^1 \leq \frac{2\bar{s} - s_m - s_{m/2}}{2(s_1 - s_{m/2})}. \tag{10}$$

If in addition $c(s, m) \leq 1/2$, then the profile x is a NCNE whenever

$$\frac{s_1 + s_{m/2} - 2\bar{s}}{2(s_1 - s_{m/2+1})} \leq x^1 < \frac{1}{2}. \tag{11}$$

Moreover, (11) *can always be satisfied.*

Proof. By the symmetry of the positions, $(x^1 + x^2)/2 = 1/2$ and $(x^2 - x^1)/2 = 1/2 - x^1$. At x, all candidates receive $1/m$-th of the points, so $v_i(x) = \bar{s}$ for all $i = 1, \dots, m$. Note that it is necessary that $s_1 > s_{m/2}$, since otherwise there would need to be more than $m/2$ candidates at each position by Lemma 4.5.

By symmetry, for NCNE it is enough to require candidate 1 not be able to deviate profitably, and there are only three moves to consider: a move to x^{1-}, which is always better than a move to x^{2+}, since 1 is ranked one place higher for half of voter in the middle interval; a move to x^{2-}, which is the best move out of any into the middle interval since the slope of $v_1(t, x_{-1})$ in that interval is nonnegative by Proposition 4.1(b); and, finally, a move to x^2.

For the first one we have $v_1(x^{1-}, x_{-1}) = s_1 x^1 + s_{m/2}\left(\frac{1}{2} - x^1\right) + \frac{1}{2}s_m$. For NCNE, it must be that $v_1(x^{1-}, x_{-1}) \leq v_1(x)$, which yields the requirement

$$x^1 \leq \frac{2\bar{s} - s_m - s_{m/2}}{2(s_1 - s_{m/2})}. \tag{12}$$

For the second move we have $v_1(x^{2-}, x_{-1}) = s_1\left(\frac{1}{2} - x^1\right) + \frac{1}{2}s_{m/2} + s_{m/2+1}x^1$. The fact that $v_1(x^{2-}, x_{-1}) \leq v_1(x)$ yields

$$x^1 \geq \frac{s_1 + s_{m/2} - 2\bar{s}}{2(s_1 - s_{m/2+1})}. \tag{13}$$

Finally,

$$v_1(x^2, x_{-1}) = \frac{1}{m/2+1}\left(\sum_{i=1}^{m/2+1} s_i + \sum_{i=m/2}^{m} s_i\right)\frac{1}{2} = \frac{1}{m+2}\left(m\bar{s} + s_{m/2} + s_{m/2+1}\right).$$

Since in a NCNE $v_1(x^2, x_{-1}) \leq v_1(x)$, it must be that $s_{m/2} + s_{m/2+1} \leq 2\bar{s}$.

There are no other moves to consider, so if the position x^1 is valid, that is, satisfies (12), (13) and is in the range $0 < x^1 < 1/2$, then we have a NCNE. The condition $x^1 < 1/2$ combined with (13) implies the strict inequality in (9). The condition $x^1 > 0$ means that we need the right-hand side of (12) to be strictly greater than zero, which is always true.

Finally, note that the requirement that $s_1 > s_{m/2}$ is implied by (9), since if $s_{m/2} = s_1$ then we have $s_{m/2} + s_{m/2+1} = s_1 + s_{m/2+1} < 2\bar{s} = s_1 + \frac{2}{m}\sum_{i=m/2+1}^{m} s_i \leq s_1 + s_{m/2+1}$, a contradiction.

To prove the second statement, suppose a scoring rule satisfies both $c(s, m) \leq 1/2$ and (9). Since $c(s, m) \leq 1/2$ is equivalent to $s_1 + s_m \leq 2\bar{s}$, the right-hand side of (10) always satisfies

$$\frac{1}{2} \leq \frac{2\bar{s} - s_m - s_{m/2}}{2(s_1 - s_{m/2})}. \tag{14}$$

Similarly, the left-hand side satisfies

$$\frac{s_1 + s_{m/2} - 2\bar{s}}{2(s_1 - s_{m/2+1})} < \frac{1}{2}. \tag{15}$$

Putting together (14) and (15), we see that it will always be possible to find valid values of x^1 in the desired range. □

Example 7.7. Again consider k-approval with $k < m/2$. Clearly, (9) is satisfied. Then we have symmetric bipositional NCNE whenever $1/2 - k/m \leq x^1 \leq k/m$, which is valid whenever $k \geq m/4$.

Theorem 7.6 allows us to conclude that bipositional NCNE may exist for both best-rewarding and worst-punishing rules, as we will see in the examples below.

Example 7.8. Let $m = 6$. The rules $s = (2, 2, 1, 1, 1, 0)$, $s = (10, 10, 4, 3, 3, 0)$ and $s = (4, 3, 1, 1, 0, 0)$ have bipositional NCNE of the form $x = ((x^1, 3), (1 - x^1, 3))$ when, respectively: $1/3 \leq x^1 < 1/2$; $2/7 \leq x^1 < 1/2$; and, $x^1 = 1/3$. These rules are worst-punishing, intermediate and best-rewarding, respectively. In particular, we see CNE and NCNE can coexist for the same rule.

Though Theorem 7.6 restricts to even m, it is possible to have bipositional equilibrium in which the number of candidates is different at the two positions. Here is an example.

Example 7.9. Let $m = 7$ and consider the rule $s = (10, 10, 4, 3, 3, 1, 0)$. It can be verified that the profile $((x^1, 4), (x^2, 3))$ with $x^1 = 1/3$ and $x^2 = 2/3$, is a NCNE.

8 Conclusion

We have investigated how the particular scoring rule in use influences the candidates' position-taking behaviour. We have looked at the equilibrium properties of a number of different classes of scoring rules. We were able to identify several broad classes of scoring rules under which NCNE are impossible. For other large classes, we found that NCNE can exist and we calculated a number of them. Many of the rules most frequently appearing in the literature—Borda, k-approval, plurality and so on—fall nicely into the cases considered.

A strong argument in favour of using scoring rules more general than plurality is that some of them give us a more realistic set of nonconvergent Nash equilibria. Indeed, plurality does not allow more than two candidates in a Nash equilibrium to cluster together while some of the rules we discovered do allow this behaviour, providing an explanation for Eaton and Lipsey's (1975) Principle of Local Clustering.

A number of questions remain open. A particularly interesting class for further investigation is the class of concave rules which are defined similarly to convex rules but with inequalities in (3) reversed. Also, there are a number of simplifying assumptions that we would like to relax. One is the assumption that the voters are uniformly distributed along the issue space[9] and another is the unidimensionality of the issue space[10]. Whether any of our results can be extended in some form is not known.

References

Anderson, S., Kats, A. and J.-F Thisse (1994). Probabilistic voting and platform selection in multi-party elections. *Social Choice and Welfare 11*, 305–322.

Aragonès, E. and D. Xefteris (2012). Candidate quality in a downsian model with a continuous policy space. *Games and Economic Behavior 75*, 464–480.

[9] Cox's (1987) characterisation of CNE holds for an arbitrary nonatomic distribution of voter ideal points. The existence of NCNE, however, is much more vulnerable to changes in the distribution as is shown by Osborne (1993).

[10] Cox (1987) provides a version of Theorem 3.2 for a multidimensional space

Brusco, S., Dziubinski, M. and J. Roy (2012). The hotelling-downs model with runoff voting. *Games and Economic Behavior 74*, 447–469.

Chamberlin, E. (1933). *The Theory of Monopolistic Competition*. Cambridge: Harvard University Press.

Coughlin, P. J. (1992). *Probabilistic Voting Theory*. New York: Cambridge University Press.

Cox, G. W. (1987). Electoral equilibrium under alternative voting institutions. *American Journal of Political Science 31*, 82–108.

Cox, G. W. (1990). Multicandidate Spatial Competition, in *Advances in the Spatial Theory of Voting*, Enelow J.M. and M.J. Hinich, eds., pp. 179–198. Cambridge: Cambridge University Press.

De Palma, A., Hong, G. and J.-F Thisse (1990). Equilibria in multi-party competition under uncertainty. *Social Choice and Welfare 7*, 247–259.

Denzau, A., Kats, A. and S. Slutsky (1985). Multi-agent equilibria with market share and ranking objectives. *Social Choice and Welfare 2*, 96–117.

Downs, A. (1957). An economic theory of political action in a democracy. *The Journal of Political Economy 65*, 135–150.

Duggan, J. (2005). A survey of equilibrium analysis in spatial models of elections. Unpublished manuscript available at http://www.rochester.edu/College/PSC/duggan/papers/existsurvey4.pdf.

Eaton, C. B. and R. G. Lipsey (1975). The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition. *The Review of Economic Studies 42*, 27–49.

Enelow, J.M., Lin, T. and H. Dorrusen (1999). Equilibrium in multicandidate probabilistic spatial voting. *Public Choice 98*, 59–82.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica 41*, 587–601.

Hotelling, H. (1929). Stability in competition. *Economic Journal 39*, 41–59.

Laslier, J.-F. and F. Maniquet (2010). Classical Electoral Competition under Approval Voting, in *Handbook on Approval Voting*, Laslier, J.-F and M. Remzi Sanver, eds., pp. 415-430. New York: Springer.

Myerson, R. and R. J. Weber (1993). A theory of voting equilibria. *American Political Science Review 87*, 102–114.

Myerson, R. B. (1993). Incentives to cultivate favored minorities under alternative electoral systems. *American Political Science Review 87*, 856–869.

Myerson, R. B. (1999). Theoretical comparisons of electoral systems. *European Economic Review 43*, 671–697.

Myerson, R. B. (2002). Comparison of scoring rules in poisson voting games. *Journal of Economic Theory 103*, 219–251.

Osborne, M. (1993). Candidate positioning and entry in a political competition. *Games and Economic Behavior 5*, 133–151.

Osborne, M. (1995). Spatial models of political competition under plurality rule: A survey of some explanations of the number of candidates and the positions they take. *Canadian Journal of Economics 28*, 261–301.

Satterthwaite, M. A. (1975). Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory 10*, 187–217.

Sengupta, A. and K. Sengupta (2008). A hotelling-downs model of electoral competition with the option to quit. *Games and Economic Behavior 62*, 661–674.

Shepsle, K. A. (2001). *Models of Multiparty Electoral Competition.* London: Routledge.

Slinko, A. and S. White (2010). Proportional representation and strategic voters. *Journal of Theoretical Politics 22*, 301–332.

Stigler, G. J. (1972). Economic competition and political competition. *Public Choice 13*, 91–106.

Vega-Redondo, F. (2003). *Economics and the Theory of Games.* New York: Cambridge University Press.

Appendix

Proof of Proposition 4.1. (a) Let $t \in [0, x^1)$. We split the issue space into intervals $I_1 = \left[0, \frac{t+x^1}{2}\right), \ldots, I_k = \left[\frac{t+x^{k-1}}{2}, \frac{t+x^k}{2}\right), \ldots, I_{q+1} = \left[\frac{t+x^q}{2}, 1\right]$. The voters in I_k rank candidate i in the same position on their ballots. For $1 \neq k \neq q+1$ the length of I_k does not depend on t (and hence the contribution to i's score from voters in I_k). So when t changes, only the contributions to i's score from the voters in the two end intervals change. The sum of these is $s_1\left(\frac{t+x^1}{2}\right) + s_m\left(1 - \frac{t+x^q}{2}\right)$, so the result follows. The second statement follows from the first due to an obvious symmetry.

(b) Suppose candidate i is currently at t. We split the issue space into intervals $[0, t]$ and $[t, 1]$ (so that candidate i belongs to both of them) and apply the previous proposition to rules $(s_1, \ldots, s_{j+1})$ and $(s_1, \ldots, s_{k+1})$. Then the nonconstant contributions to the score $v_i(t, x_{-i})$ from both intervals will be $\left[s_1\left(\frac{x^l-t}{2}\right) + s_{k+1}\left(1 - \frac{t+x^q}{2}\right)\right] + \left[s_1\left(\frac{t-x^{l-1}}{2}\right) + s_{j+1}\left(\frac{t+x^1}{2}\right)\right] = \frac{(s_{j+1}-s_{k+1})}{2}t + \text{const}$, which proves the proposition. □

Proof of Lemma 4.3. Again, the issue space can be divided into subintervals of voters who all rank i in the same position. The immediate interval around x^l is $I_l = I_l^L \cup I_l^R$, where: $I_l^L = [(x^{l-1}+x^l)/2, x^l]$ if $l > 1$ or $I_l^L = [0, x^1]$ if $l = 1$; and, $I_l^R = [x^l, (x^l + x^{l+1})/2]$ if $l < q$ or $I_l^R = [x^l, 1]$ if $l = q$. The contribution to $v_i(x)$ from the interval I_l is $\frac{s_1+s_2}{2}(\ell(I_l^L) + \ell(I_l^R))$. The contribution to $v_i(x^{l-}, x_{-i})$ from this interval is then $s_1\ell(I_l^L) + s_2\ell(I_l^R)$ and the contribution to $v_i(x^{l+}, x_{-i})$ is $s_2\ell(I_l^L) + s_1\ell(I_l^R)$.

The contribution to $v_i(x)$ from any interval J to the left of I_l, consisting of voters who all rank i similarly, is $\frac{s_t+s_{t+1}}{2}\ell(J)$ for some $2 \leq t \leq m-1$. The contribution to $v_i(x^{l-}, x_{-i})$ is $s_t\ell(J)$ since when candidate i moves infinitesimally to the left she rises one place in the rankings of these voters. The contribution to $v_i(x^{l+}, x_{-i})$ is $s_{t+1}\ell(J)$, since this move causes i to fall one place in these voters' rankings.

In the same way, the contribution to $v_i(x)$ from any interval J' to the right of I_l, consisting of voters who all rank i identically, is $\frac{s_t+s_{t+1}}{2}\ell(J')$ for some $2 \leq t \leq m-1$ while the contribution to $v_i(x^{l-}, x_{-i})$ is $s_{t+1}\ell(J')$ and the contribution to $v_i(x^{l+}, x_{-i})$ is $s_t\ell(J')$.

Hence, $v_i(x^{l-}, x_{-i}) + v_i(x^{l+}, x_{-i}) = 2v_i(x)$ since for any subinterval I_l, J or J' the sum of the contributions to $v_i(x^{l-}, x_{-i})$ and $v_i(x^{l+}, x_{-i})$ is twice the contribution to $v_i(x)$ from the same subinterval. For x to be a NCNE we need both $v_i(x^{l-}, x_{-i}) \leq v_i(x)$ and $v_i(x^{l+}, x_{-i}) \leq v_i(x)$. This is only possible when $v_i(x^{l-}, x_{-i}) = v_i(x^{l+}, x_{-i}) = v_i(x)$. □

Proof of Lemma 4.4. Let $n_1 = 2$. By Lemma 4.3 we have $v_1(x^{1+}, x_{-1}) = v_1(x)$. Hence, if 1 moves to a position $t \in (x^1, x^2)$ then for NCNE we need $v_1(t, x_{-1}) \leq v_1(x) = v_1(x^{1+}, x_{-1})$. Hence, the slope of the linear function $v_1(t, x_{-1})$ is nonpositive. By Proposition 4.1(b) we then have $s_2 - s_{m-1} \leq 0$, which can happen only if $s_2 = s_{m-1}$. □

Proof of Lemma 4.5 . Suppose $n_1 \leq k$ and candidate 1 is located at x^1. We split the issue space into intervals $I_1 = \left[0, \frac{x^1+x^2}{2}\right), \ldots, I_j = \left[\frac{x^1+x^j}{2}, \frac{x^1+x^{j+1}}{2}\right), \ldots, I_q = \left[\frac{x^1+x^q}{2}, 1\right]$. At x the contribution to candidate 1's score $v_1(x)$ from the interval $I_1 = [0, (x^1 + x^2)/2]$ is $\left(\frac{1}{n_1}\sum_{i=1}^{n_1} s_i\right)\ell(I_1) = s_1\ell(I_1)$, and the contribution from the interval I_j is $\left(\frac{1}{n_1}\sum_{i=k_j}^{k_j+n_1-1} s_i\right)\ell(J_j)$, for some k_j, since candidate 1 is tied in the rankings of all voters in I_j. If 1 moves infinitesimally to the right, then these contributions to $v_1(x^{1+}, x_{-1})$ become $s_1\ell(I_1)$ and $s_{k_j}\ell(I_j)$, respectively. Indeed, 1 is still ranked at worst kth by voters in I_1 and hence loses nothing. Also, 1 rises in all other voters' rankings. If $s_{k_j} > s_{k_j+n_1-1}$ for at least one j, this move is strictly beneficial. This infinitesimal "move" is not a real move. However, if it is strictly beneficial, then by Proposition 4.1(b) we may conclude that a sufficiently small move to the right will also be beneficial. If $s_{k_j} = s_{k_j+n_1-1}$ for all j, then we consider the move by candidate 1 to the right, to a position $t \in (x^1, x^2)$. The intervals where voters rank candidate 1 similarly will then be $I_1' = \left[0, \frac{t+x^2}{2}\right), \ldots, I_j' = \left[\frac{t+x^j}{2}, \frac{t+x^{j+1}}{2}\right), \ldots, I_q' = \left[\frac{t+x^q}{2}, 1\right]$. We see that candidate 1 has increased the length of the first interval, from which she receives s_1, at the expense of the far-right interval, from which she receives $s_{m-n_1+1} = s_m < s_1$, while keeping the lengths of all other intervals unchanged. So this move is beneficial.

So for NCNE we must have $n_1 > k$. Similarly, $n_q > k$. □

Proof of Lemma 4.7. Suppose $x^1 = 0$. Let $1 \leq k \leq m-1$ be such that $s_1 = \cdots = s_k > s_{k+1}$. Then by Lemma 4.5 we have $n_1 > k$. Candidate 1's score is $v_1(x) = \left(\frac{1}{n_1}\sum_{i=1}^{n_1} s_i\right)\frac{x^2}{2} + S$, where S is the contribution to $v_1(x)$ from voters in the interval $I = [x^2/2, 1]$. Now suppose candidate 1 moves infinitesimally to the right. Then, in the limit, 1's score is $v_1(x^{1+}, x_{-1}) = s_1\frac{x^2}{2} + S'$, where S' is the new contribution to 1's score from the interval I. Since 1 has moved up in the ranking of all the voters in I, $S' \geq S$. Also, since $s_1 = s_k > s_{n_1}$ we have

$$v_1(x^{1+}, x_{-1}) = s_1 \frac{x^2}{2} + S' > \left(\frac{1}{n_1} \sum_{i=1}^{n_1} s_i \right) \frac{x^2}{2} + S' \geq \left(\frac{1}{n_1} \sum_{i=1}^{n_1} s_i \right) \frac{x^2}{2} + S = v_1(x).$$

Hence, candidate 1 benefits from moving to the right, and hence x is not a NCNE. Thus, $x^1 > 0$. Similarly, $x^q < 1$. □

Proof of Lemma 4.8. (a) Since $s_m < \bar{s} < s_1$, there exists a real number α with the property that $0 < \alpha < 1$ and $\alpha s_1 + (1-\alpha)s_m = \bar{s}$. Rearranging this equation, one verifies that $\alpha = 1 - c(s, m)$.

At any profile x, there will be at least one candidate j who garners a total score $v_j(x) \leq \bar{s}$. Hence, if I is an end interval, namely $[0, x^1]$ or $[x^q, 1]$, then we must have $\ell(I) \leq \alpha$. To show this we assume without loss of generality that $\ell(I) > \alpha$ for $I = [0, x^1]$ and we show that candidate j would be able to make a profitable move to $x^1 - \epsilon$ for some very small ϵ. By Proposition 4.1(a), this would be proved if we could show that $v_j(x^{1-}, x_{-j}) > v_j(x)$. The idea is that by locating incrementally to the left from x^1 candidate j captures s_1 from all the voters in I and at the very worst s_m from all other voters. So $v_j(x^{1-}, x_{-j}) \geq s_1 \ell(I) + s_m(1 - \ell(I)) > \alpha s_1 + (1-\alpha)s_m = \bar{s} \geq v_j(x)$.

(b) Similarly, suppose $x^i - x^{i-1} > 2\alpha$ for some $2 \leq i \leq q$. Note that we can assume candidate j is not an unpaired candidate located at x^i or x^{i-1}, since then j already receives a score greater than $\alpha s_1 + (1-\alpha)s_m$, which contradicts that j's score is no more than $\bar{s}$. Hence, we may assume that position x^i and x^{i-1} remains occupied when j moves. Now, if j moves to any point t in the interval $I = (x^{i-1}, x^i)$, we have

$$v_j(t, x_{-j}) \geq s_1 \left(\frac{x^i - x^{i-1}}{2} \right) + s_m \left(1 - \frac{x^i - x^{i-1}}{2} \right) > \alpha s_1 + (1-\alpha)s_m \geq v_j(x).$$

Hence, for x to be a NCNE it must be that $x^i - x^{i-1} \leq 2\alpha$.

(c) If $c(s, m) \leq 1/2$ it is clearly true as the right-hand side is equal to 1. If $c(s, m) > 1/2$, only NCNE can exist with $q \geq 2$ occupied positions. Then the issue space can be partitioned into two end intervals, $[0, x^1)$ and $[x^q, 1]$, together with $q-1$ intervals of the form $[x^{i-1}, x^i)$ for $2 \leq i \leq q$. So, using Lemma 4.8, we see that $1 = x^1 + (1 - x^q) + \sum_{i=2}^{q}(x^i - x^{i-1}) \leq 2q(1 - c(s, m))$, whence the result. We round up since q is an integer. □

Proof of Lemma 5.2. Define $\alpha = x^1$, $\beta = (x^2 - x^1)/2$ and $\gamma = 1 - (x^1 + x^q)/2$. Note that $v_1(x) = \frac{s_1}{n_1}(\alpha + \beta) + s_2 - \frac{s_2}{n_1}(\alpha + \beta + \gamma) = \frac{(s_1 - s_2)}{n_1}(\alpha + \beta) + s_2 \left(1 - \frac{\gamma}{n_1}\right)$. Consider if 1 moves to x^{1-}. Then $v_1(x^{1-}, x_{-1}) = s_1\alpha + s_2(1 - \alpha - \gamma) = (s_1 - s_2)\alpha + s_2(1 - \gamma)$. If 1 moves to x^{1+} then $v_1(x^{1+}, x_{-1}) = s_1\beta + s_2(1 - \beta) = (s_1 - s_2)\beta + s_2$. For NCNE we require that these moves not be beneficial to candidate 1. That is, $v_1(x^{1-}, x_{-1}) \leq v_1(x)$ which implies $(s_1 - s_2)\alpha + s_2(1 - \gamma) \leq \frac{(s_1 - s_2)}{n_1}(\alpha + \beta) + s_2 \left(1 - \frac{\gamma}{n_1}\right)$ or

$$(s_1 - s_2) \left(1 - \frac{1}{n_1}\right) \alpha \leq \frac{(s_1 - s_2)}{n_1} \beta + s_2 \left(1 - \frac{1}{n_1}\right) \gamma. \tag{16}$$

Similarly, for the other move we have $v_1(x^{1+}, x_{-1}) \leq v_1(x)$ whence $(s_1 - s_2)\left(1 - \frac{1}{n_1}\right)\beta \leq \frac{(s_1 - s_2)}{n_1}\alpha - s_2 \frac{\gamma}{n_1}$ or $(s_1 - s_2)\left[\left(1 - \frac{1}{n_1}\right)\beta - \frac{\alpha}{n_1}\right] \leq -s_2 \frac{\gamma}{n_1} \leq 0$. The latter implies that

$\beta(n_1-1) \leq \alpha$ since we know that $s_1 > s_2$. In addition, rearranging this inequality gives $s_2\gamma \leq (s_1-s_2)(\alpha+\beta-n_1\beta)$ and multiplying through by the positive number $1-1/n_1$

$$s_2\left(1-\frac{1}{n_1}\right)\gamma \leq \left(1-\frac{1}{n_1}\right)(s_1-s_2)(\alpha+\beta-n_1\beta). \tag{17}$$

Substituting (17) into (16) and dividing through by $s_1-s_2>0$ we get $\left(1-\frac{1}{n_1}\right)\alpha \leq \frac{\beta}{n_1}+\left(1-\frac{1}{n_1}\right)(\alpha+\beta-n_1\beta)$ or $0 \leq \beta(2-n_1)$. Since $\beta>0$, this equation requires that $n_1 \leq 2$. So $n_1=2$. A similar argument gives $n_q=2$. □

Proof of Theorem 5.3. First note that, without loss of generality, we can assume $s_5=0$, since it is easy to see that subtracting s_5 from each score does not change the rule. By Lemma 4.4 we have $s_2=s_3=s_4$ and by Lemma 4.5 we have $s_1>s_2$, so condition (b) is necessary. Hence our rule is one of those studied in Lemma 5.2, which tells us that there are no NCNE of the forms $x=((x^1,2),(x^2,3))$ and $x=((x^1,3),(x^2,2))$.

By Lemma 4.7, the end points of the issue space are not occupied. As in the proof of Theorem 5.1, considering moves by candidate 1 to x^{1-} and x^{3+}, together with Lemma 4.3, gives $x^1 \geq 1-x^3$. Similar considerations for candidate 5 give $x^1 \leq 1-x^3$, hence $x^1=1-x^3$.

Let $t \in (x^1,x^2)$ and $t' \in (x^2,x^3)$ (all positions in these intervals yield the same score by Proposition 4.1(b)). Again by Lemma 4.3, we need $v_1(t',x_{-1}) = s_1\left(\frac{x^3-x^2}{2}\right) + s_2\left(1-\frac{x^3-x^2}{2}\right) \leq v_1(x) = v_1(t,x_{-1}) = s_1\left(\frac{x^2-x^1}{2}\right) + s_2\left(1-\frac{x^2-x^1}{2}\right)$, which implies $x^3-x^2 \leq x^2-x^1$ and hence $\frac{1}{2}(x^1+x^3) \leq x^2$. The same considerations with respect to candidate 5 give that $x^2 \leq \frac{1}{2}(x^1+x^3)$. So we have equality and, consequently, $x^2=1/2$.

We know that $v_1(x^{1-},x_{-1}) = v_1(x) = v_1(x^{1+},x_{-1})$. This yields $s_1x^1+s_2\left(\frac{x^3-x^1}{2}\right) = s_1\left(\frac{x^2-x^1}{2}\right)+s_2\left(1-\frac{x^2-x^1}{2}\right)$, from which, after substituting $x^2=1/2$ and $x^3=1-x^1$, equation (2) follows. For this to be a valid position, we need $x^1<x^2=1/2$. This gives $s_1>2s_2$, which is equivalent to $c(s,5)>1/2$, so condition (a) is necessary.

The proof of sufficiency is straightforward and is omitted: we simply check that if (a) and (b) are satisfied, then $x=((x^1,2),(1/2,1),(x^3,2))$, calculated according to (2), is an equilibrium with no profitable deviation for any candidate. □