APPENDIX

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Extensions with Human Capital in R & D

The model described in this paper investigates the effects of barriers to entry on the growth rate of technology. The general setup is machines producing machines in the capital goods producing sector and the cost associated with research and development fixed, so that a set amount is paid for each innovation.[1] This setup is sufficient to investigate the effects of barriers to entry namely patents in the capital goods producing sector and the growth rate of technology. However it leaves the relationship between human capital and research and development unexplored. Taking the approach suggested by Boldrin and Levine (2004) the fixed sunk cost of an innovation can be interpreted as the set amount of human capital required for one innovation which is then paid the rental price of human capital.

7.1 The Boldrin and Levine approach

The basic set up is the same as the benchmark case except the fixed sunk cost associated with an innovation has been replaced by the fixed quantity of human capital required for an innovation to take place which is paid the same rental rate as human capital in the final goods sector. An innovation requires $\bar{h}$ units of human capital per effective labour where $\bar{h}$ is a parameter. The key

[1] This does not differ drastically from recent models in the literature. As mentioned from about 1998 onwards there has been a shift away from using human capital in these types of model because this results in the perverse result that physical capital accumulation results in a lower growth rate. Instead the trend has been to set marginal revenue equal to marginal cost in the capital goods producing sector implying that machines are produced by machines and to have a fraction of the final good allocated to research and development, or a monetary payment for research and development some what unrelated to human capital or labour. For example see Aghion and Howitt (1998a) and Howitt (1998).

difference now is the price of this “fixed sunk cost” can change. Using this new specification equation (39) can be rewritten as

$$V_{0,t} = \int_0^{x_t} \pi_{\upsilon,t} e^{-r_t \upsilon} d\upsilon - r_t^H \overline{h} \tag{1}$$

The present discounted stream of profits a firm earns during a fixed time period is equal to

$$\Phi_{0,t} = \int_0^{\frac{1}{\chi_t}} e^{-r_t \chi_t \tau} \left[\int_0^{x_t} \left(\frac{1-\alpha}{\alpha} \right) \theta^{\upsilon} e^{-r_t \upsilon} s_K B k_{t-1}^{\alpha} h_{t-1}^{\beta} d\upsilon - r_t^H \overline{h} \right] d\tau \tag{2}$$

The firm chooses the length of an innovation that will maximise their profits. The firm does not choose the amount of human capital that is required to undertake research and development; this is because the amount required for an innovation to take place is fixed. The length of an innovation that maximises the innovators profits is obtained by differentiating equation (60) with respect to χ_t

$$\begin{aligned} \frac{\partial \Phi_{0,t}}{\partial \chi_t} = \left(\frac{e^{-r_t} - 1}{\chi_t^2 r_t} \right) \left[\left(\frac{1-\alpha}{\alpha} \right) \left(\frac{1 - \theta^{\chi_t} e^{-r_t \chi_t}}{r_t - \ln \theta} \right) s_K B k_{t-1}^{\alpha} h_{t-1}^{\beta} - r_t^H \overline{h} \right] \\ + \left(\frac{1 - e^{-r_t}}{\chi_t r_t} \right) \left(\frac{1-\alpha}{\alpha} \right) \theta^{\chi_t} e^{-r_t \chi_t} s_K B k_{t-1}^{\alpha} h_{t-1}^{\beta} = 0 \end{aligned} \tag{3}$$

Rearranging gives

$$\frac{r_t^H \overline{h}}{\left(\frac{1-\alpha}{\alpha}\right) B k_{t-1}^{\alpha} h_{t-1}^{\beta}} = \left[\left(\frac{1-\theta^{\chi_t} e^{-\chi_t r_t}}{r_t - \ln\theta} \right) - \chi_t \theta^{\chi_t} e^{-\chi_t r_t} \right] \tag{4}$$

The same equilibrium conditions apply as with the benchmark case.

8. Extensions with Intertemporal Utility Maximisation

This section provides an outline for extending the model to include intertemporal utility maximisation. The model described previously uses Solow's (1956) fixed saving rate assumption for simplicity. The method used to endogenise the consumption saving decision follows Cooley and Prescott (1995) and Greenwood, Hercowitz and Krusell (1997).

Lifetime utility *U* is equal to the discounted sum of each period's utility pay off multiplied by the labour force where ρ is the discount factor, c_t is consumption per effective labour and n is the growth rate of labour. Life time utility is given by[2]

$$U = \sum_{t=0}^{\infty} \rho^t (1+n)^t \ln c_t \tag{5}$$

where $0 < \rho(1 + n) < 1$

[2] Note that log utility is assumed for simplicity, following Cooley and Prescott (1995) and Greenwood, Hercowitz and Krusell (1997).

The production of output per effective labour y_t requires capital per effective labour k_t and human capital per effective labour h_t according to

$$y_t = k_t^{\alpha} h_t^{\beta} \tag{6}$$

Final output is used for investment in physical capital x_{t+1}, investment in human capital i_t^H or it is consumed, so that

$$y_t = c_t + x_{t+1} + i_t^H \tag{7}$$

Physical capital per effective labour evolves according to

$$\left[(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta_{t+1}})^{\frac{1-\beta}{1-\alpha-\beta}}(1+n)\right]k_{t+1} = (1-\delta)k_t + \gamma^{\eta_{t+1}} x_{t+1} \tag{8}$$

Human capital per effective labour evolves according to

$$\left[(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}}(1+n)\right]h_{t+1} = (1-\delta)h_t + i_t^H \tag{9}$$

Dividing equation (50) through by $B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t$ gives the national income accounting identity in per effective labour.

$$y_t = P_t^K k_t q_t + r_t^H h_t + w_t (B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}})^{-1} \tag{10}$$

Competitive equilibrium is characterised by the optimisation problems of the household and the final goods producing firm and the economy wide resource constraint. The firm's problem is the same one described in the body of this paper. The next part of this section formulates the household's problem.

The representative household maximises lifetime utility in this model economy

$$\max_{\{c_t, k_{t+1}, h_{t+1}\}} \sum_{t=0}^{\infty} \rho^t (1+n)^t \ln c_t \tag{11}$$

Subject to

$$c_t + \left[(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}} (1+n) \right] \left(k_{t+1} + h_{t+1} \right) = P_t^K k_t q_t + r_t^H h_t + w_t (B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}})^{-1} + \frac{(1-\delta)k_t}{\gamma^{\eta_{t+1}}} + (1-\delta)h_t \tag{12}$$

The resource constraint holds each period so that

$$c_t + x_{t+1} + i_t^H = k_t^{\alpha} h_t^{\beta} \tag{13}$$

where

$$x_{t+1} = \left[(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}} (1+n) \right] k_{t+1} - \frac{(1-\delta)k_t}{\gamma^{\eta_{t+1}}} \tag{14}$$

and

$$i_t^H = \left[(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}} (1+n) \right] h_{t+1} - (1-\delta)h_t \tag{15}$$

The Lagrangean is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \rho^t (1+n)^t \ln c_t - \sum_{t=0}^{\infty} \rho^t (1+n)^t \lambda_t \begin{bmatrix} c_t + \left[(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}} (1+n) \right] (k_{t+1} + h_{t+1}) - P_t^K k_t q_t \\ - r_t^H h_t - w_t (B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}})^{-1} - \frac{(1-\delta) k_t}{\gamma^{\eta_{t+1}}} - (1-\delta) h_t \end{bmatrix} \quad (16)$$

First Order Conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \quad (17)$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\rho^t (1+n)^t \lambda_t \left[(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}} (1+n) \right] + \rho^{t+1} (1+n)^{t+1} \lambda_{t+1} \left[P_{t+1}^K q_{t+1} + \frac{(1-\delta)}{\gamma^{\eta_{t+1}}} \right] = 0 \quad (18)$$

$$\frac{\partial \mathcal{L}}{\partial h_{t+1}} = -\rho^t (1+n)^t \lambda_t \left[(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}} (1+n) \right] + \rho^{t+1} (1+n)^{t+1} \lambda_{t+1} \left[1 + r_t^H - \delta \right] = 0 \quad (19)$$

From equations (75) and (76) the Euler equation for physical capital is given by[3]

$$\frac{c_{t+1}}{c_t} = \rho \left[\frac{\alpha k_{t+1}^{\alpha-1} h_{t+1}^{\beta} + \frac{(1-\delta)}{\gamma^{\eta_{t+1}}}}{(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}}} \right] \tag{20}$$

From equation (75) and (77) the Euler equation for human capital is given by[4]

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[3] The model could be rewritten so that $\frac{c_{t+1}}{c_t} = \rho[R_K(t)]$ where $R_K(t)$ denotes the gross rate of return on saving which turns out to be $R_K(t) = \frac{\alpha k_{t+1}^{\alpha-1} h_{t+1}^{\beta} + \frac{(1-\delta)}{\gamma^{\eta_{t+1}}}}{(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}}}$. If growth in embodied and disembodied technology does not take place, the denominator becomes 1, so that $R_K(t) = 1 + P_{t+1}^K q_{t+1} - \delta$. In the standard model $R_K(t) = 1 + r_{t+1} - \delta$.

[4] The model could be rewritten so that $\frac{c_{t+1}}{c_t} = \rho[R_H(t)]$ where $R_H(t)$ denotes the gross rate of return on saving which turns out to be $R_H(t) = \frac{1 + \beta k_{t+1}^{\alpha} h_{t+1}^{\beta-1} - \delta}{(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_t})^{\frac{\alpha}{1-\alpha-\beta}}}$. If growth in embodied and disembodied technology does not take place, the denominator becomes 1, so that $R_H(t) = 1 + r_{t+1}^H - \delta$.

$$\frac{c_{t+1}}{c_t} = \rho \left[\frac{1 + \beta k_{t+1}^{\alpha} h_{t+1}^{\beta-1} - \delta}{(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta_{t+1}})^{\frac{\alpha}{1-\alpha-\beta}}} \right] \tag{21}$$

From equation (78) along a balanced growth path

$$1 = \left[\frac{\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}} \right] \left[\frac{\alpha y}{k} + \frac{(1-\delta)}{\gamma^{\eta}} \right] \tag{22}$$

From equation (79) along a balanced growth path

$$1 = \left[\frac{\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}} (\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}} \right] \left[\frac{\beta y}{h} + 1 - \delta \right] \tag{23}$$

Cooley and Prescott (1995) and Greenwood, Hercowitz and Krusell (1997) use their equations that correspond with (80) and (81) for calibrating parameters of their models.

Solving equation (80) and (81) on a balanced growth path for capital per effective labour gives

$$k=\left[\frac{\alpha\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}-\frac{\rho(1-\delta)}{\gamma^{\eta}}}\right]^{\frac{1-\beta}{1-\alpha-\beta}}\left[\frac{\beta\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}-\rho(1-\delta)}\right]^{\frac{\beta}{1-\alpha-\beta}} \tag{24}$$

Solving equation (80) and (81) on a balanced growth path for human capital per effective labour gives

$$h=\left[\frac{\beta\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}-\rho(1-\delta)}\right]^{\frac{1-\alpha}{1-\alpha-\beta}}\left[\frac{\alpha\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}-\frac{\rho(1-\delta)}{\gamma^{\eta}}}\right]^{\frac{\alpha}{1-\alpha-\beta}} \tag{25}$$

Substituting equations (82) and (83) into (64) gives steady state output per effective labour

$$y=\left[\frac{\alpha\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}-\frac{\rho(1-\delta)}{\gamma^{\eta}}}\right]^{\frac{\alpha}{1-\alpha-\beta}}\left[\frac{\beta\rho}{(1+g_B)^{\frac{1}{1-\alpha-\beta}}(\gamma^{\eta})^{\frac{\alpha}{1-\alpha-\beta}}-\rho(1-\delta)}\right]^{\frac{\beta}{1-\alpha-\beta}} \tag{26}$$

Appendix I

Demand for New Machinery

The representative firm chooses allocations of labour and human capital to use with each vintage of machinery such that the marginal products of labour and human capital are the same for labour and human capital used with all machinery regardless of vintage and the surviving quantity. Any differential is eliminated through arbitrage.

The demand for embodied capital goods is determined using Jorgenson's (1963) neoclassical theory of investment. The price of a capital good reflects the discounted future stream of earnings it is expected to generate over its useful life. Jorgenson (1963) was the first to propose this method of pricing capital goods in a Cobb Douglas production function. The method and solutions in this paper mirror those of Jorgenson (1963). Constant returns in the production function imply that all factors are paid their marginal products. This suggests that a capital good's earnings at any given point in time are the marginal product of its surviving stock. Hence the price of a new capital good reflects the sum of its discounted future stream of marginal products.

Let $Y_{v,t}$ denote supply of the final good produced by machinery of vintage v at time t, B a scaling parameter, q_v the quality of technology embodied in vintage v capital at date v, X_v the quantity of vintage v capital produced at date v, δ the depreciation rate, $L_{v,t}$ and $H_{v,t}$ respectively the labour and human capital allocated to production with vintage *v* capital at date *t* and α and β the shares of physical and human capital respectively. Where $v \le t$.

Machinery of vintage *v* supply output according to the Cobb Douglas production function.[5]

$$Y_{v,t} = B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{1-\alpha-\beta} \tag{A.I.1}$$

The discounted lifetime profits a firm in the final goods producing sector makes using vintage *t* capital are given by

$$\Pi^F(t) = \sum_{u=t}^{\infty} \left[\frac{Y_{t,u} - w_u L_{t,u} - r_u^H H_{t,u}}{R_t^u} \right] - P_t X_t \tag{A.I.2}$$

Where $R_t^t = 1$ and $R_t^u = \prod_{\tau=t+1}^{u} (1+r_\tau)$ where $u \ge t$ and r_t is determined in Appendix 2.

Substituting in the production function for a firm using vintage *t* capital (equation (A.I.2)) gives

$$\Pi^F(t) = \sum_{u=t}^{\infty} \left[\frac{B(q_t X_t (1-\delta)^{u-t})^{\alpha} H_{t,u}^{\beta} L_{t,u}^{1-\alpha-\beta} - w_u L_{t,u} - r_u^H H_{t,u}}{R_t^u} \right] - P_t X_t \tag{A.I.3}$$

The representative firm chooses the quantity of new capital machinery to purchase, and allocations of human capital and labour to maximise its present discounted profits taking prices as given.

$$\max_{\{X_t, H_{t,u}, L_{t,u}\}} \Pi^F(t)$$

The firm's First Order Conditions are given by

[5] See Appendix III for the aggregation properties of this model.

$$\frac{\partial \Pi^F(t)}{\partial X_t} = \alpha B q_t^{\alpha} X_t^{\alpha-1} \sum_{u=t}^{\infty} \left[\frac{((1-\delta)^{u-t})^{\alpha}}{R_t^u} \right] H_{t,u}^{\beta} L_{t,u}^{1-\alpha-\beta} - P_t = 0 \,, \qquad \text{(A.I.4)}$$

$$\frac{\partial \Pi^F(t)}{\partial H_{t,u}} = \frac{\beta B q_t^{\alpha} X_t^{\alpha} (1-\delta)^{\alpha(u-t)} H_{t,u}^{\beta-1} L_{t,u}^{1-\alpha-\beta} - r_u^H}{R_t^u} = 0 \,, \qquad \text{(A.I.5)}$$

$$\frac{\partial \Pi^F(t)}{\partial L_{t,u}} = \frac{(1-\alpha-\beta) B q_t^{\alpha} X_t^{\alpha} (1-\delta)^{\alpha(u-t)} H_{t,u}^{\beta} L_{t,u}^{-\alpha-\beta} - w_u}{R_t^u} = 0 \,. \qquad \text{(A.I.6)}$$

From equation (A.I.5) the inverse demand for human capital to be used with vintage *t* capital at date *u* is given by

$$r_u^H = \beta B q_t^{\alpha} X_t^{\alpha} (1-\delta)^{\alpha(u-t)} H_{t,u}^{\beta-1} L_{t,u}^{1-\alpha-\beta} \qquad \text{(A.I.7)}$$

From equation (A.I.6) the inverse demand for labour used in conjunction with vintage *t* capital at time *u* is given by

$$w_u = (1-\alpha-\beta) B q_t^{\alpha} X_t^{\alpha} (1-\delta)^{\alpha(u-t)} H_{t,u}^{\beta} L_{t,u}^{-\alpha-\beta} \qquad \text{(A.I.8)}$$

The industry demand for capital equipment can be derived from equation (A.I.4) following Solow (1960) and Boucekkine, del Rio and Licandro (2004). This is given by

$$P_t = \alpha B q_t^{\alpha} X_t^{\alpha-1} \sum_{u=t}^{\infty} \left[\frac{((1-\delta)^{u-t})^{\alpha}}{R_t^u} \right] H_{t,u}^{\beta} L_{t,u}^{1-\alpha-\beta} \equiv P(X_t) \qquad \text{(A.I.9)}$$

Rearranging equation (A.I.7) the demand for human capital to be used with vintage *t* capital at time *u* is given by

$$H_{t,u} = \left[\frac{\beta B q_t^{\alpha} X_t^{\alpha} (1-\delta)^{\alpha(u-t)} L_{t,u}^{1-\alpha-\beta}}{r_u^H} \right]^{\frac{1}{1-\beta}} \tag{A.I.10}$$

Notice the demand for human capital to be allocated to vintage *t* machinery at date *u* is a function of the surviving stock of vintage *t* capital at date *u*. Because this stock declines over time, the demand for human capital to be used with this machinery also declines over time.

Rearranging equation (A.I.8) to get the demand for labour to be used with vintage *t* capital at time *u* gives

$$L_{t,u} = \left[\frac{(1-\alpha-\beta) B q_t^{\alpha} X_t^{\alpha} (1-\delta)^{\alpha(u-t)} H_{t,u}^{\beta}}{w_u} \right]^{\frac{1}{\alpha+\beta}} \tag{A.I.11}$$

Just like the demand for human capital to be used with vintage *t* capital at date *u* the demand for labour is an increasing function of the surviving stock of vintage *t* capital at date *u*. This stock declines over time so that demand for labour to be used in combination with this machinery also declines with time.

The demand for labour and human capital to be used with each vintage of capital declines with time. The labour and human capital withdrawn from production with each vintage of capital as its productive quantity declines is used to man the leading vintage of capital that has just been created.

Appendix II

The Real Interest Rate and the Price of Capital Services

This section describes the determination of the market interest rate in this model. The real interest rate is determined using the steps outlined in Boucekkine Del Rio and Licandro (2000) and Howitt (1998). The use of a fixed saving rate assumption as in Solow (1956) means the market interest rate is flexible ensuring the supply of investment is equal to its demand.

The firm chooses allocations of human capital and labour to use with each vintage of capital at each date *t*. From equation (A.I.8), the inverse demand for labour to be used with vintage *v* machinery is given by

$$w_t = (1-\alpha-\beta)B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{-\alpha-\beta} \tag{A.II.1}$$

Solving for labour gives the demand for labour used with vintage *v* capital

$$L_{v,t} = \left[\frac{(1-\alpha-\beta)Bq_v^{\alpha} X_v^{\alpha} (1-\delta)^{\alpha(t-v)} H_{v,t}^{\beta}}{w_t} \right]^{\frac{1}{\alpha+\beta}} \tag{A.II.2}$$

Equation (11) gives the inverse demand for labour used with all vintages of capital, this is reproduced below

$$w_t = (1-\alpha-\beta)BK_t^{\alpha} H_t^{\beta} L_t^{-\alpha-\beta} \tag{A.II.3}$$

As mentioned firms choose allocations of labour and human capital such that the marginal products of labour and human capital are the same for all labour and human capital regardless of the vintage of machinery they are employed with.

Substituting (A.II.3) into (A.II.2) gives

$$L_{v,t} = \left[\frac{(1-\alpha-\beta) B q_v^{\alpha} X_v^{\alpha} (1-\delta)^{\alpha(t-v)} H_{v,t}^{\beta}}{(1-\alpha-\beta) B K_t^{\alpha} H_t^{\beta} L_t^{-\alpha-\beta}} \right]^{\frac{1}{\alpha+\beta}}$$

Cancelling terms gives

$$L_{v,t} = \left[\frac{q_v^{\alpha} X_v^{\alpha} (1-\delta)^{\alpha(t-v)} H_{v,t}^{\beta}}{K_t^{\alpha} H_t^{\beta} L_t^{-\alpha-\beta}} \right]^{\frac{1}{\alpha+\beta}} \qquad \text{(A.II.4)}$$

Expanding equation (A.II.4) gives

$$L_{v,t} = \left(\frac{L_t}{K_t} \right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{L_t}{H_t} \right)^{\frac{\beta}{\alpha+\beta}} q_v^{\frac{\alpha}{\alpha+\beta}} X_v^{\frac{\alpha}{\alpha+\beta}} (1-\delta)^{(t-v)\frac{\alpha}{\alpha+\beta}} H_{v,t}^{\frac{\beta}{\alpha+\beta}} \qquad \text{(A.II.5)}$$

Solving for labour gives

$$L_{v,t}^{1-\alpha-\beta} = \left(\frac{L_t}{K_t} \right)^{\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{L_t}{H_t} \right)^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}} q_v^{\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} X_v^{\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} (1-\delta)^{(t-v)\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} H_{v,t}^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}} \qquad \text{(A.II.6)}$$

The inverse demand for human capital used with vintage *v* capital from equation (A.I.7) is given by

$$r_t^H = \beta B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta-1} L_{v,t}^{1-\alpha-\beta}$$

Solving for human capital gives

$$H_{v,t}^{1-\beta} = \frac{\beta B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} L_{v,t}^{1-\alpha-\beta}}{r_t^H}$$

The demand for human capital used in conjunction with vintage *v* capital at time *t* is given by

$$H_{v,t} = \left[\frac{\beta B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} L_{v,t}^{1-\alpha-\beta}}{r_t^H}\right]^{\frac{1}{1-\beta}} \qquad \text{(A.II.7)}$$

From equation (10) the inverse demand for all human capital is given by

$$r_t^H = \beta B K_t^{\alpha} H_t^{\beta-1} L_t^{1-\alpha-\beta} \qquad \text{(A.II.8)}$$

Substituting this into the demand for human capital gives

$$H_{v,t} = \left[\frac{(q_v X_v (1-\delta)^{(t-v)})^{\alpha} L_{v,t}^{1-\alpha-\beta}}{K_t^{\alpha} H_t^{\beta-1} L_t^{1-\alpha-\beta}}\right]^{\frac{1}{1-\beta}} \qquad \text{(A.II.9)}$$

Expanding equation (A.II.9) gives

$$H_{v,t} = \left(\frac{L_t}{K_t}\right)^{\frac{\alpha}{1-\beta}} \left(\frac{H_t}{L_t}\right) q_v^{\frac{\alpha}{1-\beta}} X_v^{\frac{\alpha}{1-\beta}} (1-\delta)^{(t-v)\frac{\alpha}{1-\beta}} L_{v,t}^{\frac{1-\alpha-\beta}{1-\beta}}$$

$$H_{v,t}^{\beta} = \left(\frac{L_t}{K_t}\right)^{\frac{\alpha\beta}{1-\beta}} \left(\frac{H_t}{L_t}\right)^{\beta} q_v^{\frac{\alpha\beta}{1-\beta}} X_v^{\frac{\alpha\beta}{1-\beta}} (1-\delta)^{(t-v)\frac{\alpha\beta}{1-\beta}} L_{v,t}^{\frac{(1-\alpha-\beta)\beta}{1-\beta}} \qquad \text{(A.II.10)}$$

From equation (A.II.6)

$$L_{v,t}^{1-\alpha-\beta} = \left(\frac{L_t}{K_t}\right)^{\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{L_t}{H_t}\right)^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}} q_v^{\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} X_v^{\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} (1-\delta)^{(t-v)\frac{\alpha(1-\alpha-\beta)}{\alpha+\beta}} H_{v,t}^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}}$$

Substituting this into equation (A.II.10) gives

$$H_{v,t}^{\beta} = \left(\frac{L_t}{K_t}\right)^{\frac{\alpha\beta}{1-\beta}} \left(\frac{H_t}{L_{Y,t}}\right)^{\beta} q_v^{\frac{\alpha\beta}{1-\beta}} X_v^{\frac{\alpha\beta}{1-\beta}} (1-\delta)^{\frac{\alpha\beta}{1-\beta}} \left(\frac{L_t}{K_t}\right)^{\frac{\alpha\beta(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}} \left(\frac{L_t}{H_t}\right)^{\frac{\beta^2(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}} q_v^{\frac{\alpha\beta(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}}$$
$$\times X_v^{\frac{\alpha\beta(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}} (1-\delta)^{(t-v)\frac{\alpha\beta(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}} H_{v,t}^{\frac{\beta^2(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}}$$

Simplifying

$$H_{v,t}^{\beta} = \left(\frac{L_t}{K_t}\right)^{\frac{\alpha\beta}{(1-\beta)(\alpha+\beta)}} \left(\frac{H_t}{L_t}\right)^{\frac{\beta^2(1-\alpha-\beta)-\beta(\alpha+\beta)(1-\beta)}{(\alpha+\beta)(1-\beta)}} q_v^{\frac{\alpha\beta}{(1-\beta)(\alpha+\beta)}} X_v^{(t-v)\frac{\alpha\beta}{(1-\beta)(\alpha+\beta)}} (1-\delta)^{\frac{\alpha\beta}{(1-\beta)(\alpha+\beta)}}$$
$$\times H_{v,t}^{\frac{\beta^2(1-\alpha-\beta)}{(\alpha+\beta)(1-\beta)}}$$

$$H_{v,t}^{\beta(\alpha+\beta)(1-\beta)} = \left(\frac{L_t}{K_t}\right)^{\alpha\beta} \left(\frac{H_t}{L_t}\right)^{\beta^2(1-\alpha-\beta)-\beta(\alpha+\beta)(1-\beta)} q_v^{\alpha\beta} X_v^{\alpha\beta} (1-\delta)^{(t-v)\alpha\beta} H_{v,t}^{\beta^2(1-\alpha-\beta)}$$

$$H_{v,t}^{\alpha\beta} = \left(\frac{L_t}{K_t}\right)^{\alpha\beta} \left(\frac{H_t}{L_t}\right)^{\alpha\beta} q_v^{\alpha\beta} X_v^{\alpha\beta} (1-\delta)^{(t-v)\alpha\beta}$$

Demand for human capital used with vintage *v* capital at date *t* is given by

$$H_{v,t} = \left(\frac{L_t}{K_t}\right)\left(\frac{H_t}{L_t}\right) q_v X_v (1-\delta)^{(t-v)} \qquad \text{(A.II.11)}$$

From equation (A.II.5)

$$L_{v,t} = \left(\frac{L_t}{K_t}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{L_t}{H_t}\right)^{\frac{\beta}{\alpha+\beta}} q_v^{\frac{\alpha}{\alpha+\beta}} X_v^{\frac{\alpha}{\alpha+\beta}} (1-\delta)^{(t-v)\frac{\alpha}{\alpha+\beta}} H_{v,t}^{\frac{\beta}{\alpha+\beta}} \qquad \text{(A.II.12)}$$

Substituting (A.II.11) into (A.II.12) gives

$$L_{v,t} = \left(\frac{L_t}{K_t}\right) q_v X_v (1-\delta)^{(t-v)} \qquad \text{(A.II.13)}$$

Simplifying equation (A.II.11) gives

$$H_{v,t} = \left(\frac{H_t}{K_t}\right) q_v X_v (1-\delta)^{(t-v)} \qquad \text{(A.II.14)}$$

Equations (A.II.13) and (A.II.14) imply for all vintages of capital that

$$\frac{q_v X_v (1-\delta)^{(t-v)}}{K_t} = \frac{H_{v,t}}{H_t} = \frac{L_{v,t}}{L_t} \qquad \text{(A.II.15)}$$

That is the ratio of the surviving stock of vintage *v* capital to the aggregate capital stock at date *t* is equal to the ratio of human capital allocated to vintage *v* capital to the aggregate stock of human capital which is also equal to the ratio of labour allocated to vintage *v* machinery to the aggregate stock of labour at date *t*.

From Equation (A.I.9) the inverse demand for new capital goods at date *t* is given by

$$P = \alpha B q_t^{\alpha} X_t^{\alpha-1} \sum_{u=t}^{\infty} \left[\frac{((1-\delta)^{u-t})^{\alpha}}{R_t^u} \right] H_{t,u}^{\beta} L_{t,u}^{1-\alpha-\beta} \equiv P(X_t) \qquad \text{(A.II.16)}$$

Note, in equilibrium *P* is constant because price is just a constant mark-up over marginal cost so that the time subscript can be dropped. (See the capital goods producer's equilibrium condition, equation (21))

Substituting equations (A.II.13) and (A.II.14) into equation (A.II.16) gives

$$P = \alpha B q_t^{\alpha} X_t^{\alpha-1} \sum_{u=t}^{\infty} \left[\frac{((1-\delta)^{u-t})^{\alpha}}{R_t^u} \right] \left(\left(\frac{H_u}{K_u} \right) q_t X_t (1-\delta)^{u-t} \right)^{\beta} \left(\left(\frac{L_u}{K_u} \right) q_t X_t (1-\delta)^{u-t} \right)^{(1-\alpha-\beta)}$$

Simplifying

$$P = \alpha B q_t \sum_{u=t}^{\infty} \left[\frac{(1-\delta)^{u-t}}{R_t^u} K_u^{\alpha-1} H_u^{\beta} L_u^{1-\alpha-\beta} \right] \qquad \text{(A.II.17)}$$

Dividing equation (A.II.17) by q_t gives the quality adjusted price of capital

$$\frac{P}{q_t} = \tilde{P}_t = \alpha B \sum_{u=t}^{\infty} \left[\frac{(1-\delta)^{u-t}}{R_t^u} K_u^{\alpha-1} H_u^{\beta} L_u^{1-\alpha-\beta} \right] \tag{A.II.18}$$

Note the quality adjusted price of capital declines over time, a feature that is consistent with the data. Quality adjusted capital goods prices have fallen in the United States since the Second World War a stylised fact this model is able to replicate.[6]

Combining equation (A.II.18) with equation (25) gives

$$\tilde{P}_t = \frac{1}{q_t \left[1 - \phi \left[1 - \alpha\right]\right]}$$

If $\phi = 1$, there is monopolistic competition and the quality adjusted price of capital is given by

$$\tilde{P}_t = \frac{1}{\alpha q_t}$$

This is the same result as in Boucekkine, Del Rio and Licandro (2000) and (2004).

If ϕ equals 0 there is perfect competition and the quality adjusted price of capital is given by

$$\tilde{P}_t = \frac{1}{q_t}$$

This is the same result found in Greenwood, Hercowitz and Krusell (1997). Any values of ϕ between 0 and 1 will give quality adjusted capital prices somewhere in between the monopoly prices

[6] This is according to the Gordon (1990) and Greenwood, Hercowitz and Krusell (1997) data series.

suggested in Boucekkine, Del Rio and Licandro (2000) and (2004) and the perfectly competitive prices determined in Greenwood, Hercowitz and Krusell (1997).

Substituting equation (9) into (A.II.18) gives

$$\widetilde{P}_t = \sum_{u=t}^{\infty}\left[\frac{(1-\delta)^{u-t}}{R_t^u} P_u^K\right] \qquad \text{(A.II.19)}$$

This says the quality adjusted price of a new capital good is a function of the real interest rate, the rate of depreciation and the price of capital services. This is the same result Solow (1960) specifies for the price of an asset, the only difference being he assumes $\widetilde{P}_t$ is constant over time, normalising it to unity. This relationship relates the imputed rental on capital of all qualities determined in equation (9) with Jorgenson's (1963) neoclassical theory of investment.

Differentiating quality adjusted capital prices with respect to time gives the growth rate of quality adjusted capital prices, rearranging this will give the real interest rate.

Differentiating Equation (A.II.18) with respect to time gives

$$\frac{\partial \widetilde{P}_t}{\partial t} = \widetilde{P}_t(r_t + \delta) - \alpha B K_t^{\alpha-1} H_t^{\beta} L_t^{1-\alpha-\beta}$$

Rearranging

$$\alpha BK_t^{\alpha-1}H_t^{\beta}L_t^{1-\alpha-\beta} = \tilde{P}_t\left[r_t + \delta - \frac{\partial \tilde{P}_t}{\partial t}\right] \tag{A.II.20}$$

Quality adjusted prices decline at the rate of technology growth, where g_{qt} is the growth rate of embodied technology at time *t* so that equation (A.II.20) can be rewritten as

$$\alpha BK_t^{\alpha-1}H_t^{\beta}L_t^{1-\alpha-\beta} = \tilde{P}_t\left[r_t + \delta + g_{qt}\right] \tag{A.II.21}$$

Notice that the left side of equation (A.II.21) is the marginal product of quality capital determined in equation (9). This is equal to the price of capital services in equilibrium. The right side is the user cost of capital so that the marginal product of capital is equal to its user cost in equilibrium. Note if there were no embodied technical change, and the marginal cost of producing capital were set to unity, under perfect competition equation (A.II.21) would revert to the standard result of the marginal product of capital equal to the interest rate plus the rate of depreciation. For example

$$\alpha BK_t^{\alpha-1}H_t^{\beta}L_t^{1-\alpha-\beta} = r_t + \delta$$

This is the same result derived in Mankiw, Romer and Weil (1992).

Substituting in the price of capital services from equation (9) gives

$$P_t^K = \tilde{P}_t\left[r_t + \delta + g_{qt}\right] \tag{A.II.22}$$

Equation (A.II.22) measures the user cost of capital and is similar to Jorgenson and Griliches' (1967) price of capital services. The key difference between equation (A.II.22) and Jorgenson and Griliches (1967) is the third term in parentheses. This term measures changes in capital prices i.e. capital gains or losses. In this case quality adjusted capital prices decline at the rate technology grows so that the growth rate of technology enters in place of the change in prices. Another important feature of equation (A.II.22) is the real interest rate; the depreciation rate and the growth rate are all constant over time, while the quality adjusted price of capital goods declines at the same rate technology grows. This means the rental price of capital P_t^K must also decline with time at the same rate technology is growing. This will be useful for the national accounting section to show capital is paid the same share of income in a vintage capital model as it is paid in more conventional models that consist of heterogeneous capital stocks.

Multiplying equation (A.II.21) through by q_t gives.

$$q_t \alpha B K_t^{\alpha-1} H_t^{\beta} L_t^{1-\alpha-\beta} = P_t \left[r_t + \delta + g_{qt} \right]$$

This can be rewritten as

$$q_t MPK = P_t \left[r_t + \delta + g_{qt} \right]$$

Which is the same result obtained in Greenwood and Jovanovic (2000). Rewriting the left side in per effective labour terms gives

$$\alpha B k_t^{\alpha-1} h_t^{\beta} = P_t \left[r_t + \delta + g_{qt} \right] \qquad \text{(A.II.23)}$$

Substituting in the equilibrium price of capital goods from equation (25) gives

$$\alpha B k_t^{\alpha-1} h_t^{\beta} = \left(\frac{1}{1-\phi[1-\alpha]} \right) [r_t + \delta + g_{qt}]$$

$$\alpha(1-\phi[1-\alpha]) B k_t^{\alpha-1} h_t^{\beta} = [r_t + \delta + g_{qt}] \tag{A.II.24}$$

Solving for the real interest rate gives

$$r_t = \alpha(1-\phi[1-\alpha]) B k_t^{\alpha-1} h_t^{\beta} - \delta - g_{qt} \tag{A.II.25}$$

Under perfect competition the real interest rate is given by

$$r_t = \alpha B k_t^{\alpha-1} h_t^{\beta} - \delta - g_{qt}$$

Under monopoly the real interest rate is given by

$$r_t = \alpha^2 B k_t^{\alpha-1} h_t^{\beta} - \delta - g_{qt}$$

Appendix III

Aggregation

This section shows how output produced by each vintage of capital can be aggregated to obtain gross domestic product. It follows closely the method outlined in Solow (1960) but differs with the inclusion of human capital. As mentioned this is a vintage capital model so that there are many different vintages of capital goods in use at any particular time. Each vintage of capital is associated with its own production function, which can be aggregated to obtain GDP.

Output produced by vintage *v* capital denoted by $Y_{v,t}$, is produced using the surviving quantity of vintage *v* capital represented by $q_v X_v (1-\delta)^{t-v}$ and $H_{v,t}$ and $L_{v,t}$ the quantity of human capital and labour allocated to machinery of vintage *v* respectively.

From equation (A.I.1), machinery of vintage *v* supply final output according to the Cobb Douglas production function.

$$Y_{v,t} = B(q_v X_v (1-\delta)^{t-v})^{\alpha} H_{v,t}^{\beta} L_{v.t}^{1-\alpha-\beta} \qquad \text{(A.III.1)}$$

The following conditions hold in equilibrium

Gross Domestic Product is equal to the sum of the outputs produced by machinery of each vintage in existence at date *t* so that

$$Y_t = \sum_{v=-\infty}^{t} Y_{v,t} \qquad \text{(A.III.2)}$$

In equilibrium the supply of human capital is equal to the sum of the demands for human capital to be used with each vintage of capital

$$H_t = \sum_{v=-\infty}^{t} H_{v,t} \qquad \text{(A.III.3)}$$

In equilibrium the supply of labour is equal to the sum of the demands for labour used with each vintage of capital

$$L_t = \sum_{v=-\infty}^{t} L_{v,t} \qquad \text{(A.III.4)}$$

The effective stock of physical capital at date *t* is the sum of the surviving stock of each vintage of equipment in existence at date *t*

$$K_t = \sum_{v=-\infty}^{t} q_v X_v (1-\delta)^{t-v} = q_t X_t + (1-\delta) K_{t-1} \qquad \text{(A.III.5)}$$

Define $X_{v,t} \equiv X_v (1-\delta)^{(t-v)}$ as the surviving stock of vintage *v* capital at date *t*.

The firm maximises profits by choosing quantities of vintage *v* capital, and quantities of labour and human capital to use with vintage *v* capital. That is they solve the following

$$\max_{\{H_{v,t}, L_{v,t}, X_{v,t}\}} \Pi_{v,t} \qquad \text{(A.III.6)}$$

Where the profit the firm makes from using vintage *v* capital at time *t* is given by

$$\Pi_{v,t} = Y_{v,t} - r_t^H H_{v,t} - w_t L_{v,t} - P_t^K X_{v,t} \qquad \text{(A.III.7)}$$

Substituting in the production function for a firm using vintage *v* capital at time *t* gives

$$\Pi_{v,t} = B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{1-\alpha-\beta} - r_t^H H_{v,t} - w_t L_{v,t} - P_t^K X_v (1-\delta)^{(t-v)}$$

$$\Pi_{v,t} = B(q_v X_{v,t})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{1-\alpha-\beta} - r_t^H H_{v,t} - w_t L_{v,t} - P_t^K X_{v,t} \qquad \text{(A.III.8)}$$

The First Order Conditions are given by

$$\frac{\partial \Pi_{v,t}}{\partial X_{v,t}} = \alpha B q_v^{\alpha} X_{v,t}^{\alpha-1} H_{v,t}^{\beta} L_{v,t}^{1-\alpha-\beta} - P_t^K = 0, \qquad \text{(A.III.9)}$$

$$\frac{\partial \Pi_t}{\partial L_{v,t}} = (1-\alpha-\beta) B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{-\alpha-\beta} - w_t = 0, \qquad \text{(A.III.10)}$$

$$\frac{\partial \Pi_t}{\partial H_{v,t}} = \beta B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{1-\alpha-\beta} - r_t^H = 0. \qquad \text{(A.III.11)}$$

From (A.III.9) the inverse demand for physical capital of vintage *v* at date *t* is given by

$$P_t^K = \alpha B q_v^{\alpha} (X_v (1-\delta)^{(t-v)})^{\alpha-1} H_{v,t}^{\beta} L_{v,t}^{1-\alpha-\beta} \qquad \text{(A.III.12)}$$

Remembering that firms own their capital so that equation (A.III.12) represents an imputed rental the firm pays to itself.

From equation (A.III.10) the inverse demand for labour is given by

$$w_t = (1-\alpha-\beta)B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta} L_{v,t}^{-\alpha-\beta} \qquad \text{(A.III.13)}$$

This is the same as equation (A.I.8) determined in Appendix I above.

From equation (A.III.11) the inverse demand for human capital to be used with vintage *v* capital at time *t* is given by

$$r_t^H = \beta B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta-1} L_{v,t}^{1-\alpha-\beta} \qquad \text{(A.III.14)}$$

Rearranging equation (A.III.13) gives the demand for labour to be used with vintage *v* capital at time *t*, which is given by

$$L_{v,t} = \left[\frac{(1-\alpha-\beta)B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta}}{w_t} \right]^{\frac{1}{\alpha+\beta}} \qquad \text{(A.III.15)}$$

Rearranging equation (A.III.14) gives the demand for human capital used with vintage *v* capital at time *t*, which is given by

$$H_{v,t} = \left[\frac{\beta B(q_v X_v (1-\delta)^{(t-v)})^{\alpha} L_{v,t}^{1-\alpha-\beta}}{r_t^H} \right]^{\frac{1}{1-\beta}} \qquad \text{(A.III.16)}$$

Substituting equation (A.III.15) into (A.III.16) gives

$$H_{v,t} = \left[\frac{\beta B (q_v X_v (1-\delta)^{(t-v)})^{\alpha} \left[\frac{(1-\alpha-\beta) B (q_v X_v (1-\delta)^{(t-v)})^{\alpha} H_{v,t}^{\beta}}{w_t} \right]^{\frac{1-\alpha-\beta}{\alpha+\beta}}}{r_t^H} \right]^{\frac{1}{1-\beta}}$$

Simplifying

$$H_{v,t} = \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \beta B^{\frac{1}{\alpha+\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{\alpha+\beta}} H_{v,t}^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}}}{r_t^H (w_t)^{\frac{1-\alpha-\beta}{\alpha+\beta}}} \right]^{\frac{1}{1-\beta}}$$

Solving for human capital gives

$$H_{v,t}^{1-\beta} = \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \beta B^{\frac{1}{\alpha+\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{\alpha+\beta}} H_{v,t}^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}}}{r_t^H (w_t)^{\frac{1-\alpha-\beta}{\alpha+\beta}}} \right]$$

$$H_{v,t}^{1-\beta} = \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \beta B^{\frac{1}{\alpha+\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{\alpha+\beta}} H_{v,t}^{\frac{\beta(1-\alpha-\beta)}{\alpha+\beta}}}{r_t^H (w_t)^{\frac{1-\alpha-\beta}{\alpha+\beta}}} \right]$$

$$H_{v,t}^{\frac{\alpha}{\alpha+\beta}} = \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \beta B^{\frac{1}{\alpha+\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{\alpha+\beta}}}{r_t^H (w_t)^{\frac{1-\alpha-\beta}{\alpha+\beta}}} \right]$$

The demand for human capital to be used in conjunction with vintage *v* capital is given by

$$H_{v,t} = \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right] \qquad \text{(A.III.17)}$$

Substituting equation (A.III.16) into (A.III.15) gives

$$L_{v,t} = \left[\frac{(1-\alpha-\beta) B (q_v X_v (1-\delta)^{(t-v)})^{\alpha} \left[\frac{\beta B (q_v X_v (1-\delta)^{(t-v)})^{\alpha} L_{v,t}^{1-\alpha-\beta}}{r_t^H} \right]^{\frac{\beta}{1-\beta}}}{w_t} \right]^{\frac{1}{\alpha+\beta}}$$

Simplifying

$$L_{v,t} = \left[\frac{(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{1-\beta}} L_{v,t}^{\frac{(1-\alpha-\beta)\beta}{1-\beta}}}{w_t \left(r_t^H\right)^{\frac{\beta}{1-\beta}}} \right]^{\frac{1}{\alpha+\beta}}$$

Solving for labour

$$L_{v,t}^{\alpha+\beta} = \left[\frac{(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{1-\beta}} L_{v,t}^{\frac{(1-\alpha-\beta)\beta}{1-\beta}}}{w_t \left(r_t^H\right)^{\frac{\beta}{1-\beta}}} \right]$$

$$L_{v,t}^{\frac{\alpha}{1-\beta}} = \left[\frac{(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} B^{\frac{1}{1-\beta}} (q_v X_v (1-\delta)^{(t-v)})^{\frac{\alpha}{1-\beta}}}{w_t \left(r_t^H\right)^{\frac{\beta}{1-\beta}}} \right]$$

The demand for labour used with vintage *v* capital at time *t* is given by

$$L_{v,t} = \frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \qquad \text{(A.III.18)}$$

The demand for human capital used with vintage *v* capital at time *t* is given by

$$H_{v,t} = \frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}}$$

From equation (A.III.4) the sum of the demands for labour to be used with each vintage of capital *v* equals the supply of labour so that

$$L_t = \sum_{v=-\infty}^{t} L_{v,t} \qquad \text{(A.III.19)}$$

Substituting equation (A.III.18) into (A.III.4) gives

$$L_t = \sum_{v=0}^{t} \left[\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \right]$$

Rearranging gives

$$L_t = \left[\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}}}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \right] \sum_{v=0}^{t} q_v X_v (1-\delta)^{(t-v)} \qquad \text{(A.III.20)}$$

From equation (A.III.3) the demand for human capital to be used with each vintage of capital equals the supply of human capital.

$$H_t = \sum_{v=-\infty}^{t} H_{v,t} \qquad \text{(A.III.21)}$$

Substituting equation (A.III.17) into (A.III.3) gives

$$H_t = \sum_{v=0}^{t} \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right]$$

Simplifying gives

$$H_t = \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}}}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right] \sum_{v=0}^{t} (q_v X_v (1-\delta)^{(t-v)}) \qquad \text{(A.III.22)}$$

From equation (A.III.1) the production function for a firm using capital of vintage *v* is given by

$$Y_{v,t} = B(q_v X_v (1-\delta)^{t-v})^{\alpha} H_{v,t}^{\beta} L_{v.t}^{1-\alpha-\beta}$$

Substituting equations (A.III.18) and (A.III.17) into (A.III.1) gives

$$Y_{v,t} = B(q_v X_v (1-\delta)^{t-v})^{\alpha} \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right]^{\beta} \times \left[\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}} (q_v X_v (1-\delta)^{(t-v)})}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \right]^{1-\alpha-\beta}$$

Simplifying

$$Y_{v,t} = B(q_v X_v (1-\delta)^{t-v}) \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}}}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right]^{\beta} \left[\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}}}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \right]^{1-\alpha-\beta} \qquad \text{(A.III.23)}$$

Substituting equation (A.III.23) into (A.III.2) gives

$$Y_t = \sum_{v=0}^{t} \left(B(q_v X_v (1-\delta)^{t-v}) \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}}}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right]^{\beta} \left[\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}}}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \right]^{1-\alpha-\beta} \right)$$

Simplifying

$$Y_t = B \left[\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}} \beta^{\frac{\alpha+\beta}{\alpha}} B^{\frac{1}{\alpha}}}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}} \left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}} \right]^{\beta} \left[\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}} \beta^{\frac{\beta}{\alpha}} B^{\frac{1}{\alpha}}}{w_t^{\frac{1-\beta}{\alpha}} \left(r_t^H\right)^{\frac{\beta}{\alpha}}} \right]^{1-\alpha-\beta} \sum_{v=0}^{t} (q_v X_v (1-\delta)^{t-v})$$

Partitioning the aggregate capital stock into the different factor shares gives

$$Y_t = B\left[\sum_{v=0}^{t}(q_v X_v (1-\delta)^{t-v})\right]^{\alpha}\left[\left(\frac{(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha}}\beta^{\frac{\alpha+\beta}{\alpha}}B^{\frac{1}{\alpha}}}{\left(r_t^H\right)^{\frac{\alpha+\beta}{\alpha}}\left(w_t\right)^{\frac{1-\alpha-\beta}{\alpha}}}\right)\sum_{v=0}^{t}(q_v X_v (1-\delta)^{t-v})\right]^{\beta} \times \times\left[\left(\frac{(1-\alpha-\beta)^{\frac{1-\beta}{\alpha}}\beta^{\frac{\beta}{\alpha}}B^{\frac{1}{\alpha}}}{w_t^{\frac{1-\beta}{\alpha}}\left(r_t^H\right)^{\frac{\beta}{\alpha}}}\right)\sum_{v=0}^{t}(q_v X_v (1-\delta)^{t-v})\right]^{1-\alpha-\beta} \quad \text{(A.III.24)}$$

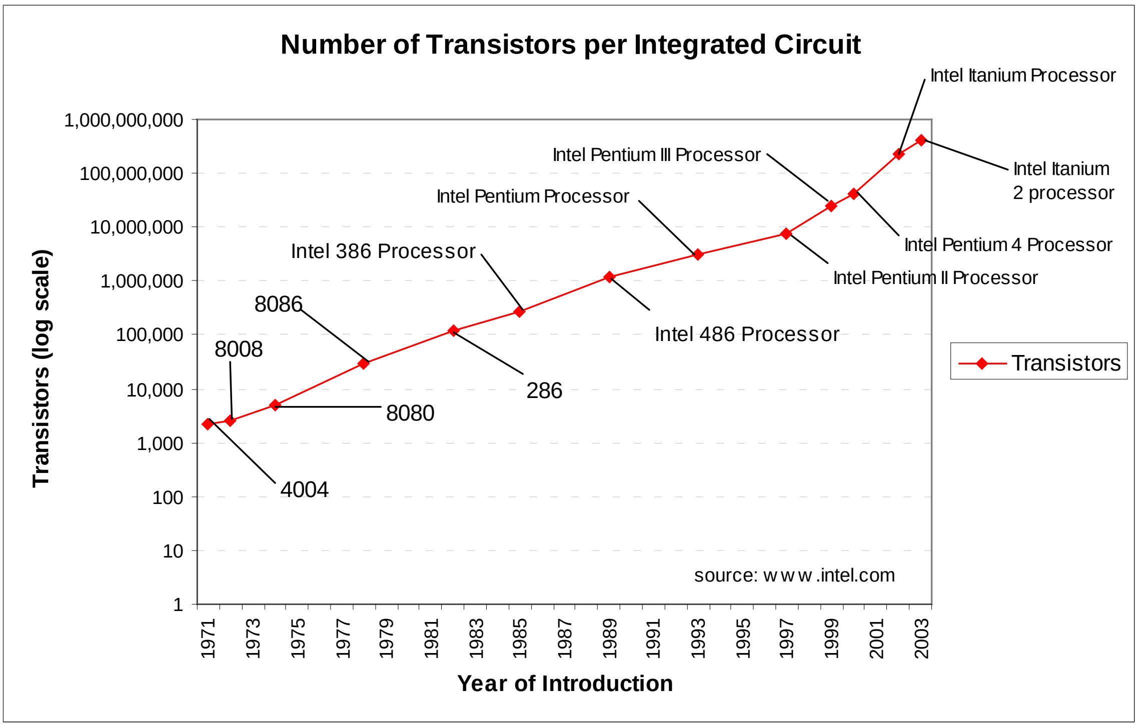
Substituting equations (A.III.5), (A.III.22) and (A.III.20) into equation (A.III.24), the aggregate production function in equation (1) can be obtained.

$$Y_t = BK_t^{\alpha} H_t^{\beta} L_t^{1-\alpha-\beta} \quad (1)$$

Appendix IV

Moore’s Law

Figure 1.



Appendix V

Balanced Growth and the Steady State

In this section the growth rates of real GDP, physical capital and human capital are defined for a balanced growth path. From the definition of a balanced growth path the growth rates of GDP, physical capital and human capital are determined in terms of the growth rate of embodied technology and labour. From these relationships expressions for output, physical capital and human capital per effective labour can be determined such that these measures are stationary on a balanced growth path. Subsequently it will be shown that the model exhibits similar features to Solow (1956) and Mankiw, Romer and Weil (1992).

Characterising a Balanced Growth Path

On a balanced growth path, the growth rates of quality capital, aggregate output and human capital are all constant. Subsequently the growth rates of GDP, physical capital and human capital can all be written in terms of the growth rate of embodied technology and labour.

Let K_t denote the stock of quality capital, H_t the stock of human capital, L_t the stock of labour, Y_t aggregate output, X_t investment in physical capital and I_t^H investment in human capital. Also define δ to be the rate of depreciation, s_K the saving rate for physical capital and s_H the saving rate for human capital. Let the growth rate of aggregate output be represented by g_Y, let n denote the growth rate of labour, g_K the growth rate of quality capital, g_H the growth rate of human capital and g_q the growth rate or embodied technology.

From equation (4) the law of motion of physical capital is given by

$$K_{t+1} = q_{t+1} X_{t+1} + (1-\delta) K_t \qquad \text{(A.V.1)}$$

Backdating by one period gives

$$K_t = q_t X_t + (1-\delta) K_{t-1} \qquad \text{(A.V.2)}$$

Dividing equation (A.V.1) by (A.V.2) gives

$$\frac{K_{t+1}}{K_t} = \frac{q_{t+1} X_{t+1} + (1-\delta) K_t}{q_t X_t + (1-\delta) K_{t-1}} \qquad \text{(A.V.3)}$$

Substituting equation (16) into (A.V.3) gives

$$\frac{K_{t+1}}{K_t} = \frac{q_{t+1} s_K Y_t + (1-\delta)K_t}{q_t s_K Y_{t-1} + (1-\delta)K_{t-1}}$$

On the balanced growth path physical capital grows at a constant rate

$$\frac{K_t(1+g_K)}{K_t} = \frac{q_t(1+g_q)s_K Y_{t-1}(1+g_Y) + (1-\delta)K_{t-1}(1+g_K)}{q_t s_K Y_{t-1} + (1-\delta)K_{t-1}}$$

Solving for the growth rate of physical capital gives

$$(1+g_K) = \frac{q_t(1+g_q)s_K Y_{t-1}(1+g_Y) + (1-\delta)K_{t-1}(1+g_K)}{q_t s_K Y_{t-1} + (1-\delta)K_{t-1}}$$

$$q_t s_K Y_{t-1}(1+g_K) + (1-\delta)K_{t-1}(1+g_K) = q_t(1+g_q)s_K Y_{t-1}(1+g_Y) + (1-\delta)K_{t-1}(1+g_K)$$

$$q_t s_K Y_{t-1}(1+g_K) = q_t s_K Y_{t-1}(1+g_q)(1+g_Y)$$

On a balanced growth path physical capital grows at the rate.

$$(1+g_K) = (1+g_q)(1+g_Y) \qquad \text{(A.V.4)}$$

Conducting a similar exercise for human capital

From equation (3) the law of motion for human capital is given by

$$H_{t+1} = I_t^H + (1-\delta)H_t \tag{A.V.5}$$

Back dating equation (A.V.5) by one period gives

$$H_t = I_{t-1}^H + (1-\delta)H_{t-1} \tag{A.V.6}$$

Dividing equation (A.V.5) by (A.V.6) gives

$$\frac{H_{t+1}}{H_t} = \frac{I_t^H + (1-\delta)H_t}{I_{t-1}^H + (1-\delta)H_{t-1}} \tag{A.V.7}$$

Substituting equation (17) into (A.V.7) gives

$$\frac{H_{t+1}}{H_t} = \frac{s_H Y_t + (1-\delta)H_t}{s_H Y_{t-1} + (1-\delta)H_{t-1}}$$

On a balanced growth path human capital grows at a constant rate so that

$$\frac{H_t(1+g_H)}{H_t} = \frac{s_H Y_{t-1}(1+g_Y) + (1-\delta)H_{t-1}(1+g_H)}{s_H Y_{t-1} + (1-\delta)H_{t-1}}$$

Solving for the growth rate of human capital

$$(1+g_H) = \frac{s_H Y_{t-1}(1+g_Y) + (1-\delta)H_{t-1}(1+g_H)}{s_H Y_{t-1} + (1-\delta)H_{t-1}}$$

$$s_H Y_{t-1}(1+g_H) + (1-\delta)H_{t-1}(1+g_H) = s_H Y_{t-1}(1+g_Y) + (1-\delta)H_{t-1}(1+g_H)$$

$$s_H Y_{t-1}(1+g_H) = s_H Y_{t-1}(1+g_Y)$$

Human capital grows at the rate

$$(1+g_H) = (1+g_Y) \qquad \text{(A.V.8)}$$

Let B_t denote an index of neutral disembodied technology, where g_B is the growth rate of this technology and B_0 is the initial value at time zero. So that

$$B_t = (1+g_B)^t B_0 \qquad \text{(A.V.9)}$$

From equation (1) the production function is given by

$$Y_t = B_t K_t^{\alpha} H_t^{\beta} L_t^{1-\alpha-\beta} \qquad (1)$$

From equation (A.V.4) the growth rate of physical capital is given by

$$(1+g_K) = (1+g_q)(1+g_Y)$$

Combining equations (1) and (A.V.4) gives the growth rate of aggregate output

$$(1+g_Y) = (1+g_B)(1+g_K)^{\alpha}(1+g_H)^{\beta}(1+n)^{1-\alpha-\beta} \qquad \text{(A.V.10)}$$

Substituting in the values for the growth rates of human and physical capital from equations (A.V.8) and (A.V.4) gives

$$(1+g_Y)=(1+g_B)(1+g_Y)^{\alpha}(1+g_q)^{\alpha}(1+g_Y)^{\beta}(1+n)^{1-\alpha-\beta}$$

$$(1+g_Y)^{1-\alpha-\beta}=(1+g_B)(1+g_q)^{\alpha}(1+n)^{1-\alpha-\beta}$$

So that on a balanced growth path aggregate output grows at the rate

$$(1+g_Y)=(1+g_B)^{\frac{1}{1-\alpha-\beta}}(1+g_q)^{\frac{\alpha}{1-\alpha-\beta}}(1+n) \qquad \text{(A.V.11)}$$

Letting y_t denote output per effective labour, equation (A.V.11) implies that output per effective labour can be written as

$$y_t=\frac{Y_t}{B_t^{\frac{1}{1-\alpha-\beta}}q_t^{\frac{\alpha}{1-\alpha-\beta}}L_t} \qquad \text{(A.V.12)}$$

Output per effective labour will be stationary on a balanced growth path.

Taking logs of (A.V.11) and subtracting n from both sides gives

$$g_Y-n=\left(\frac{1}{1-\alpha-\beta}\right)g_B+\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q \qquad \text{(A.V.13)}$$

This is the growth rate of GDP per labour; this will be used in the calibration section for disaggregating technology into its different components.

From equation (A.V.4) the growth rate of physical capital is given by

$$(1+g_K)=(1+g_q)(1+g_Y)$$

Substituting in the growth rate of aggregate output from equation (A.V.11) gives

$$(1+g_K)=(1+g_B)^{\frac{1}{1-\alpha-\beta}}(1+g_q)^{\frac{1-\beta}{1-\alpha-\beta}}(1+n) \qquad \text{(A.V.14)}$$

Using equation (A.V.14), physical capital per effective labour can be written as

$$k_t=\frac{K_t}{B_t^{\frac{1}{1-\alpha-\beta}}q_t^{\frac{1-\beta}{1-\alpha-\beta}}L_t} \qquad \text{(A.V.15)}$$

On a balanced growth path capital per effective labour will be stationary. Differentiating capital per effective labour with respect to time, setting this equal to zero and then solving for *k* gives the steady state value of capital per effective labour.

Differentiating equation (A.V.15) with respect to time gives

$$\dot{k}_t = \frac{\dot{K}_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{1-\beta}{1-\alpha-\beta}} L_t} - \frac{\left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)\dot{q}_t B_t^{\frac{1}{1-\alpha-\beta}} L_t + \dot{L}_t B q_t^{\frac{1-\beta}{1-\alpha-\beta}} + \left(\frac{1}{1-\alpha-\beta}\right)\dot{B}_t q_t^{\frac{1}{1-\alpha-\beta}} L_t\right] K_t}{\left[B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{1-\beta}{1-\alpha-\beta}} L_t\right]^2}$$

Simplifying

$$\dot{k}_t = \frac{q_{t+1} X_{t+1} - \delta K_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{1-\beta}{1-\alpha-\beta}} L_t} - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \left(\frac{1}{1-\alpha-\beta}\right) g_B\right] k_t$$

$$\dot{k}_t = \frac{q_{t+1} X_{t+1}}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{1-\beta}{1-\alpha-\beta}} L_t} - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B\right] k_t$$

$$\dot{k}_t = \frac{q_{t+1} s_K Y_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{1-\beta}{1-\alpha-\beta}} L_t} - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B\right] k_t$$

$$\dot{k}_t = \frac{(1+g_q) q_t s_K Y_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{1-\beta}{1-\alpha-\beta}} L_t} - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B\right] k_t$$

$$\dot{k}_t = \frac{(1+g_q) s_K Y_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t} - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B\right] k_t$$

$$\dot{k}_t = (1+g_q) s_K y_t - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B\right] k_t$$

$$\dot{k}_t = (1+g_q)s_K k_t^{\alpha} h_t^{\beta} - \left[\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right)g_B\right]k_t$$

Because human capital and aggregate output grow at the same rate, human capital per effective labour can be written as

$$h_t = \frac{H_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t} \qquad \text{(A.V.16)}$$

Differentiating equation (A.V.16) with respect to time gives

$$\dot{h}_t = \frac{\dot{H}_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t} - \frac{\left[\left(\frac{\alpha}{1-\alpha-\beta}\right)\dot{q}_t B_t^{\frac{1}{1-\alpha-\beta}} L_t + \dot{L}_t B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} + \left(\frac{1}{1-\alpha-\beta}\right)\dot{B}_t q_t^{\frac{\alpha}{1-\alpha-\beta}} L\right]H_t}{\left[B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t\right]^2}$$

$$\dot{h}_t = \frac{I_t^H - \delta H_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t} - \left[\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q + n + \left(\frac{1}{1-\alpha-\beta}\right)g_B\right]h_t$$

$$\dot{h}_t = \frac{s_H Y_t - \delta H_t}{B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} L_t} - \left[\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q + n + \left(\frac{1}{1-\alpha-\beta}\right)g_B\right]h_t$$

$$\dot{h}_t = s_H k_t^{\alpha} h_t^{\beta} - \left[\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q + n + \delta\left(\frac{1}{1-\alpha-\beta}\right)g_B\right]h_t$$

Setting the change in physical capital per labour to zero and solving for the steady state value gives

$$k = \left[\frac{(1+g_q)s_K h^{\beta}}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1}{1-\alpha}} \quad \text{(A.V.17)}$$

Setting the change in human capital per worker to zero and solving for the steady state value gives

$$h = \left[\frac{s_H k^{\alpha}}{\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q + n + \delta\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1}{1-\beta}} \quad \text{(A.V.18)}$$

Solving for the steady state physical capital per labour in terms of parameters

$$k = h^{\frac{\beta}{1-\alpha}}\left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q + n + \delta\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1}{1-\alpha}}$$

$$k = \left[\frac{s_H k^{\alpha}}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\beta}{(1-\beta)(1-\alpha)}} \times \left[\frac{(1+g_q) s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{1}{1-\alpha}}$$

$$k = k^{\frac{\alpha\beta}{(1-\beta)(1-\alpha)}} \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\beta}{(1-\beta)(1-\alpha)}} \times \left[\frac{(1+g_q) s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{1}{1-\alpha}}$$

$$k^{\frac{1-\alpha-\beta}{(1-\beta)(1-\alpha)}} = \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\beta}{(1-\beta)(1-\alpha)}} \times \left[\frac{(1+g_q) s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{1}{1-\alpha}}$$

$$k=\left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\beta}{1-\alpha-\beta}} \times\left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1-\beta}{1-\alpha-\beta}}$$

The steady state capital per effective labour is given by

$$k=\left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\beta}{1-\alpha-\beta}} \times\left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1-\beta}{1-\alpha-\beta}} \qquad \text{(A.V.19)}$$

Solving the steady state human capital per effective labour gives

$$h=k^{\frac{\alpha}{1-\beta}}\left[\frac{s_H B}{\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1}{1-\beta}}$$

$$k^{\frac{\alpha}{1-\beta}} = \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha\beta}{(1-\alpha-\beta)(1-\beta)}} \times \left[\frac{(1+g_q) s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha}{(1-\alpha-\beta)}}$$

$$h = \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha\beta}{(1-\alpha-\beta)(1-\beta)}} \times \left[\frac{(1+g_q) s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{s_H B}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta} \right]^{\frac{1}{1-\beta}}$$

Steady state human capital per effective labour is given by

$$h = \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{1-\alpha}{1-\alpha-\beta}} \times \left[\frac{(1+g_q) s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha}{1-\alpha-\beta}} \quad \text{(A.V.20)}$$

Equations (1), (A.V.20), (A.V.19) and (A.V.12) imply that aggregate output per effective labour is given by

$$y_t = k_t^{\alpha} h_t^{\beta} \tag{A.V.21}$$

This is stationary on a balanced growth path.

Substituting the steady state values for human capital and physical capital from equations (A.V.19) and (A.V.20)

$$k^{\alpha} = \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha\beta}{1-\alpha-\beta}} \times \left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha(1-\beta)}{1-\alpha-\beta}}$$

$$h^{\beta} = \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\beta(1-\alpha)}{1-\alpha-\beta}} \times \left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right) g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha\beta}{1-\alpha-\beta}}$$

Aggregate output per effective labour is given by

$$y=\left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\alpha}{1-\alpha-\beta}} \times\left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q+n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\beta}{1-\alpha-\beta}} \qquad \text{(A.V.22)}$$

Note if g_q = 0, that is there is no embodied technological progress then the model collapses to Mankiw, Romer and Weil (1992) with Hicks neutral technology.

Capital per effective labour is given by

$$k=\left[\frac{s_H^{\beta}s_K^{1-\beta}}{n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1}{1-\alpha-\beta}}$$

Human capital per effective labour is given by

$$h=\left[\frac{s_H^{1-\alpha}s_K^{\alpha}}{n+\delta+\left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{1}{1-\alpha-\beta}}$$

Output per effective labour is given by

$$y = s_H^{\frac{\beta}{1-\alpha-\beta}} s_K^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{1}{n+\delta+\left(\frac{1}{1-\alpha-\beta}\right) g_B} \right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

Let $\tilde{y}_t$ denote output per labour, that is

$$\tilde{y}_t = \frac{Y_t}{L_t} \tag{A.V.23}$$

On a balanced growth path output per labour is given by

$$\tilde{y}_t = B_t^{\frac{1}{1-\alpha-\beta}} q_t^{\frac{\alpha}{1-\alpha-\beta}} y \tag{A.V.24}$$

From equation (A.V.9) disembodied neutral technology is given by

$$B_t^{\frac{1}{1-\alpha-\beta}} = \left[(1+g_B)^t B_0 \right]^{\frac{1}{1-\alpha-\beta}} \tag{A.V.25}$$

From equation (29) embodied technology is given by

$$q_t^{\frac{\alpha}{1-\alpha-\beta}} = \left(\gamma^{Zt}\right)^{\frac{\alpha}{1-\alpha-\beta}} \tag{A.V.26}$$

Which on a balanced growth path is equal to

$$q_t^{\frac{\alpha}{1-\alpha-\beta}} = \left(\gamma^{\eta t}\right)^{\frac{\alpha}{1-\alpha-\beta}} \qquad \text{(A.V.27)}$$

That is $Z_t = \eta t$

Substituting equations (A.V.25) and (A.V.27) into (A.V.22) gives output per effective labour on a balanced growth path which is equal to.

$$\tilde{y}_t = \left[(1+g_B)^t B_0\right]^{\frac{1}{1-\alpha-\beta}} \left(\gamma^{\eta t}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{(1+g_q)s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\alpha}{1-\alpha-\beta}} \times \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right)g_q + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\beta}{1-\alpha-\beta}} \qquad \text{(A.V.28)}$$

Substituting equation (32) into (A.V.28) gives

$$\tilde{y}_t = \left[(1+g_B)^t B_0\right]^{\frac{1}{1-\alpha-\beta}} \left(\gamma^{\eta(t+1)}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{s_K}{\left(\frac{1-\beta}{1-\alpha-\beta}\right)\eta \ln\gamma + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\alpha}{1-\alpha-\beta}} \times \left[\frac{s_H}{\left(\frac{\alpha}{1-\alpha-\beta}\right)\eta \ln\gamma + n + \delta + \left(\frac{1}{1-\alpha-\beta}\right)g_B}\right]^{\frac{\beta}{1-\alpha-\beta}} \qquad \text{(A.V.29)}$$

This equation is very similar to Mankiw, Romer and Weil (1992). This equation will form the basis for calibrating the model used in this paper.

Appendix VI

Participation Constraint

This section describes how to derive the length of time it takes for the first copying firm to enter the capital goods producing sector. This is necessary for determining the participation constraint which ensures the innovator wants to remain an innovator.

Let Θ denote the fixed sunk cost associated with copying a technology, ω the time taken for the first copying firm to copy the technology and $V_{0,t}^{1C}$ the value of an innovation at age 0 to the first copying firm to enter the market at age ω so that the following participation constraint must be satisfied for the innovating firm to remain an innovator

$$V_{0,t} = \int_0^{\chi_t} \pi_{\upsilon,t} e^{-r_t \upsilon} d\upsilon - F \geq \int_\omega^{\chi_t} \pi_{\upsilon,t} e^{-r_t \upsilon} d\upsilon - \Theta = V_{0,t}^{1C} \qquad \text{(A.VI.1)}$$

Where

$$\omega = \frac{\ln\left[\dfrac{\alpha}{2(1+\alpha)}\right]}{\ln\theta} \qquad \text{(A.VI.2)}$$

From equation (38) the inverse of the number of the firms in the capital goods producing sector υ after an innovation is released is given by

$$\phi_{\upsilon} = \frac{-\theta^{\upsilon}(1-\alpha) + \sqrt{\left[\theta^{\upsilon}(1-\alpha)\right]^2 + 4\alpha\theta^{\upsilon}}}{2\alpha}$$

When there are two firms in the capital goods producing sector that is the innovating firm and the first copying firm, ϕ_{υ} = ½ so that

$$\frac{1}{2} = \frac{-\theta^{\upsilon}(1-\alpha) + \sqrt{\left[\theta^{\upsilon}(1-\alpha)\right]^2 + 4\alpha\theta^{\upsilon}}}{2\alpha}$$

Solving for the age of an innovation υ at which there are two firms in the capital goods producing sector, letting ω denote this time length.

$$\alpha = -\theta^{\omega}(1-\alpha) + \sqrt{\left[\theta^{\omega}(1-\alpha)\right]^2 + 4\alpha\theta^{\omega}}$$

$$\alpha + \theta^{\omega}(1-\alpha) = \sqrt{\left[\theta^{\omega}(1-\alpha)\right]^2 + 4\alpha\theta^{\omega}}$$

$$\left[\alpha + \theta^{\omega}(1-\alpha)\right]^2 = \left[\theta^{\omega}(1-\alpha)\right]^2 + 4\alpha\theta^{\omega}$$

$$\alpha^2 + 2\alpha\theta^{\omega}(1-\alpha) + \left[\theta^{\omega}(1-\alpha)\right]^2 = \left[\theta^{\omega}(1-\alpha)\right]^2 + 4\alpha\theta^{\omega}$$

$$\alpha^2 + 2\alpha\theta^{\omega}(1-\alpha) = 4\alpha\theta^{\omega}$$

$$\alpha^2 + 2\alpha\theta^{\omega} - 2\alpha^2\theta^{\omega} = 4\alpha\theta^{\omega}$$

$$\alpha^2 - 2\alpha^2\theta^{\omega} = 2\alpha\theta^{\omega}$$

$$\alpha - 2\alpha\theta^{\omega} = 2\theta^{\omega}$$

$$\alpha = 2\theta^{\omega}(1+\alpha)$$

$$\frac{\alpha}{2(1+\alpha)} = \theta^{\omega}$$

$$\ln\left[\frac{\alpha}{2(1+\alpha)}\right] = \omega \ln\theta$$

The length of time that it takes for the first copying firm to enter the industry is given by

$$\omega = \frac{\ln\left[\frac{\alpha}{2(1+\alpha)}\right]}{\ln\theta} \qquad \text{(A.VI.3)}$$

Note that this is only a function of parameters and is therefore time invariant.

Appendix VII

Determining γ from the Data

The size of an innovation can be determined from the data if two assumptions are made. First it must be assumed that the rate of innovation follows a Poisson distribution then γ can be determined directly from the data. Second it must be assumed that the economy is on a balanced growth path. The following describes how γ can be determined from the data.

The index for embodied technology q_t can be written as

$$q_t = \gamma^{Zt}$$

Along a balanced growth path this can be written as

$$q_t = \gamma^{t\eta}$$

The average growth rate of technology along a balanced growth path is given by

$$g_q = \eta \ln \gamma$$

From the properties of the Poisson distribution the variance of the growth rate along the balanced growth path is given by

$$\sigma_{gq}^2 = \eta (\ln \gamma)^2$$

Dividing the variance of the growth rate by the average growth rate gives a value for lnγ. Taking the exponential of this reveals γ from the data. Note that η can also be determined from the data although this is determined endogenously in the model.

Table of Parameters

Parameter	Description
α	Capital's Share of Income
β	Human Capital's Share of Income
δ	Depreciation Rate of both Physical and Human Capital
γ	Measures the size of an innovation
θ	Barriers to Entry
g_B	The growth rate of disembodied technology
B	A scaling parameter in the production function
n	The growth rate of the labour force
s_H	The saving rate for investment in human capital
s_K	The saving rate for investment in physical capital
F	Fixed Sunk Cost of an innovation
$\overline{h}$	Fixed amount of human capital used in Research and Development
ρ	Time discount factor

Table of Variables

Variable	Description
Y_t	Aggregate Output
K_t	The aggregate effective stock of physical capital
H_t	The aggregate stock of human capital
L_t	The total stock of labour at date *t*
I_t^H	Investment in human capital
q_t	An index of embodied technology
X_t	The amount of capital goods produced at date *t*
C_t	Aggregate consumption at date *t*
$H_{v,t}$	The amount of human capital used with vintage *v* machinery in production
$L_{v,t}$	The amount of labour used with vintage *v* machinery in production
P_t^K	The price of capital services
r_t^H	The real rental on human capital
w_t	Real wages paid to labour
r_t	The real interest rate
P_t	Price of a new capital good
C	The Marginal Cost of producing quality capital goods
$X_{j,t}$	Firm *j*'s share of demand for new capital goods
J	The number of firms in the capital goods producing sector
ϕ	Inverse of the number of firms in the capital goods producing sector
ε	The price elasticity of demand
Z_t	The cumulative number of innovations that have taken place up to date *t*
η_t	Measures the flow rate of innovations, the number of innovations in a discrete time period.

χ_t	The length of time an innovation remains in the lead
π_t	The capital goods producers profit per effective labour
y_t	Output per effective worker
h_t	Human capital per effective worker
k_t	Physical capital
y	Steady state output per effective worker
k	Steady state capital per effective worker
h	Steady state human capital per effective worker
υ	The age of an innovation
$V_{0,t}$	The value of an innovation at the date of its release
ω	The length of time it takes for the second firm to enter the capital goods producing sector
$\Phi_{0,t}$	The discounted stream of net profits earned in a discrete time period
B_t	An index of neutral disembodied technology
ζ_t	The period expenditure on innovations
κ	The time taken to pay of the fixed sunk cost of an innovation
$Y_{v,t}$	Output produced using machinery of vintage *v*
$\tilde{P}_t$	Quality adjusted price of capital
g_{qt}	The growth rate of embodied technology at date *t*
g_q	The growth rate of embodied technology on a balanced growth path
g_y	The growth rate of aggregate output on a balanced growth path
g_K	The growth rate of quality capital on a balanced growth path