CUBIC ARC-TRANSITIVE k-MULTICIRCULANTS

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Abstract. For an integer $k \geqslant 1$, a graph is called a *k-multicirculant* if its automorphism group contains a cyclic semiregular subgroup with k orbits on the vertices. If k is even, there exist infinitely many cubic arc-transitive k-multicirculants. We conjecture that, if k is odd, then a cubic arc-transitive k-multicirculant has order at most $6k^2$. Our main result is a proof of this conjecture when k is squarefree and coprime to 6.

1. Introduction

All graphs in this paper are finite, simple and connected. A permutation group is called *semiregular* if its only element fixing a point is the identity. For an integer $k \geqslant 1$, we say that a graph is a *k-multicirculant* if its automorphism group contains a cyclic semiregular subgroup with k orbits on the vertices. (Note that the terminology surrounding this topic varies.)

Clearly, every graph is a k-multicirculant for some k, for example if k is the order of the graph. Moving beyond this trivial observation is often quite difficult: whether every vertex-transitive graph is a k-multicirculant for some other k is a famous open problem (see [3, 28]). This question has been settled in the affirmative for graphs of valency at most four [12, 30]. The question has also been settled positively for vertex-quasiprimitive graphs [18], locally-quasiprimitive graphs (and hence arc-transitive graphs of prime valency and 2-arc-transitive graphs) [19], distance-transitive graphs [25], graphs of square-free order [13] and arc-transitive graphs of valency 8 [33]. Finally, [4] proves that vertex-transitive cubic graphs admit a semiregular automorphism of order at least three.

On the other hand, studying k-multicirculants for fixed k often yields interesting results. For example, 1-multicirculants, usually called simply *circulants*, are exactly Cayley graphs on cyclic groups. These graphs have been intensively studied. The family of 2-multicirculants (sometimes called *bicirculants*) has also attracted some attention.

In many cases, additional symmetry conditions are imposed on the graphs. In particular, cubic arc-transitive k-multicirculants have been the focus of some recent investigation. (A graph is called *arc-transitive* if its automorphism group acts transitively on ordered pairs of adjacent vertices. A graph is *cubic* if each of its vertices has degree 3.) It is a rather easy exercise to show that a cubic arc-transitive circulant is isomorphic to either K_4 or $\mathrm{K}_{3,3}$. The classification of cubic arc-transitive bicirculants can be deduced from [17, 29, 31], while cubic arc-transitive k-multicirculants for $k \in \{3, 4, 5\}$ are classified in [16, 23]. Rather than describe these classifications in detail, we would simply like to point out one striking feature: for $k = 2$ or $k = 4$, there exist infinitely many cubic arc-transitive k-multicirculants, whereas for

2010 *Mathematics Subject Classification.* 20B25, 05E18.

Key words and phrases. Cubic graph, arc-transitive, circulant.

This research was supported by the Australian Research Council grants DE130101001 and DP150101066. The second author was supported in part by the Slovenian Research Agency (research program P1-0285 and research projects N1-0032, N1-0038, J1-5433, and J1-6720) and would also like to thank The University of Western Australia for its hospitality.

$k \in \{1, 3, 5\}$, there are only finitely many. This immediately suggests the following question.

Question 1.1. Given a positive integer k, does there exist infinitely many cubic arc-transitive k-multicirculants?

Investigating this question is the main topic of this paper. It follows by [24, Theorem 1.2, Corollary 5.8] that, if k is an even positive integer, then there exist infinitely many cubic arc-transitive k-multicirculants. We thus focus on the case when k is odd. We are unable to settle Question 1.1 in full generality in this case, but we prove the following, which is our main result.

Theorem 1.2. *If k is a squarefree positive integer coprime to* 6*, then a cubic arc-transitive k-multicirculant has order at most* $6k^2$*.*

We would like to note that our proof of Theorem 1.2 relies on the Classification of the Finite Simple Groups. We would also like to note that the methods used in the proof of Theorem 1.2 can, with some more effort, yield a complete classification of cubic arc-transitive k-multicirculants, for k squarefree and coprime to 6.

If k is an odd positive integer, then there exists a cubic arc-transitive k-multicirculant of order $6k^2$. (See for example [24, Theorem 1.1], where such a graph is called a $I_k^{6k}(3k)$-path.) In particular, the bound in Theorem 1.2 is best possible. In view of this and computational evidence gathered from the census of cubic arc-transitive graphs of order at most 10000 [6, 7], we would like to propose the following conjecture which would completely settle Question 1.1.

Conjecture 1.3. *If k is an odd positive integer, then a cubic arc-transitive k-multicirculant has order at most* $6k^2$*.*

Question 1.1 has the following obvious but interesting generalisation.

Question 1.4. Given a pair of positive integers d and k, with $d \geqslant 3$, does there exist infinitely many d-valent arc-transitive k-multicirculants?

Note that arc-transitivity plays an important role here. Indeed, for all $d \geqslant 3$ and all $n \geqslant d + 1$, not both odd, there exists a Cayley graph of valency d on a cyclic group of order n. It easily follows that, if we relax from arc-transitivity to vertex-transitivity in Question 1.4, then the answer is positive for all d and k.

2. Preliminaries

We start with some notation and definitions. Let G be a group of automorphisms of a graph Γ. We denote by G_v the stabiliser in G of the vertex v, by $\Gamma(v)$ the neighbourhood of v, and by $G_v^{\Gamma(v)}$ the permutation group induced by the action of G_v on $\Gamma(v)$. We say that Γ is *G-vertex-transitive* (*G-arc-transitive*, respectively) if G is transitive on the set of vertices (arcs, respectively) of Γ, and that it is *G-locally-transitive* if $G_v^{\Gamma(v)}$ is transitive for every vertex v. We define *locally-primitive*, etc. similarly.

A *t-arc* of Γ is a sequence of $t + 1$ vertices such that any two consecutive vertices in the sequence are adjacent, and with any repeated vertices being more than 2 steps apart. We say that Γ is *(G, t)-arc-transitive* if G is transitive on the set of t-arcs of Γ.

Given an integer n and a prime p, we will sometimes denote by n_p the p-part of n (that is, the largest power of p dividing n) and by $n_{p'}$ the p' part (that is, n/n_p).

Given a graph Γ and a group of automorphisms N of Γ, the *quotient graph* Γ/N is the graph whose vertices are the N-orbits, and with two such N-orbits v^N and u^N adjacent whenever there is a pair of vertices $v' \in v^N$ and $u' \in u^N$ that are adjacent in Γ. If the natural projection $\pi : \Gamma \to \Gamma/N$ is a local bijection (that is, if $\pi_{|\Gamma(v)} : \Gamma(v) \to (\Gamma/N)(v^N)$ is a bijection for every vertex v of Γ) then Γ is called a *regular N-cover* of Γ/N. Such covers have many important properties that will be used repeatedly, most of which are folklore. (See [32, Lemma 3.2] for example.)

Given a group G with subgroups H and K, we define$[H,K] = \langle h^{-1}k^{-1}hk \mid h \in H, k \in K\rangle$. Letting $G^{(0)} = G$, we define $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$. Then

$$G = G^{(0)} \geqslant G^{(1)} \geqslant G^{(2)} \geqslant G^{(3)} \cdots .$$

Since G is finite, there is some integer n such that $G^{(k)} = G^{(n)}$ for all $k \geqslant n$, and we write $G^{\infty} = G^{(n)}$. We say that G is *perfect* if $G = [G,G]$. Note that G^{∞} is perfect. We say that G is *soluble* if $G^{\infty} = 1$, and *insoluble* otherwise. The *soluble radical* S of G is the largest normal soluble subgroup of G. Note that G/S has trivial soluble radical. We denote the centre of G by $\mathrm{Z}(G)$. Finally, G is called *quasisimple* if it is perfect and $G/\mathrm{Z}(G)$ is simple.

The *socle* of a group is the subgroup generated by its minimal normal subgroups (see [10, Section 4.3] for properties of the socle). A minimal normal subgroup is the direct product of isomorphic simple groups hence the socle is a direct product of simple groups.

A group G is called *almost simple* if it has a unique minimal normal subgroup T and T is a nonabelian simple group. Identifying T with its group of inner automorphisms, we have $T \leqslant G \leqslant \mathrm{Aut}(T)$.

We now collect a few results that will be useful in the proof.

Lemma 2.1. *Let p be an odd prime, let P be a p-group with a cyclic maximal subgroup and let X be the group generated by elements of order p in P.*

(1) *If $P = X$, then P is elementary abelian.*
(2) *If X is cyclic, then so is P.*
(3) *If P is cyclic of order at least p^2, then an automorphism of P of order 2 cannot centralise the maximal subgroup of P.*

Proof. It is obvious that (1) and (2) hold if P is abelian, hence assume that P is nonabelian. Since p is odd, it is well-known that $P \cong \langle a, b \mid a^{p^n} = b^p = 1, a^b = a^{p^{n-1}+1}\rangle$, where $n \geqslant 2$ (see, e.g., [1, 23.4]). One can easily check that $X = \langle a^{p^{n-1}}, b\rangle \cong \mathbb{Z}_p^2$. This completes the proof of (1) and (2). For (3), recall that the automorphism group of a cyclic group of order the power of an odd prime is itself cyclic, and that its unique involution acts by inversion (see, e.g., [1, 23.3]). $\square$

Lemma 2.2. *Let G be a group with a normal subgroup N and let T be a perfect group acting on G and centralising N. If T acts trivially on G/N, then it acts trivially on G.*

Proof. Since T acts trivially on G/N, we have $[G,T] \leqslant N$ and thus $[G,T,T] \leqslant [N,T] = 1$. Similarly, $[T,G,T] = 1$. By the three subgroups lemma (see, e.g., [1, 8.7]), it follows that $[T,T,G] = 1$ and, since T is perfect, $[T,G] = 1$. $\square$

Lemma 2.3. *Let Γ be a graph with every vertex having odd valency and let C be a semiregular cyclic group of automorphisms of Γ. If C has an odd number of orbits, then C has even order and the unique involution of C reverses some edge of Γ.*

Proof. Let $(C_1,\ldots,C_k)$ be an ordering of the orbits of C and let $A = \{a_{ij}\}$ be the $k \times k$ matrix such that a_{ij} is the number of vertices of C_j adjacent to a given vertex of C_i. It is not hard to see that this is independent of the choice of the vertex, hence A is well-defined and, moreover, $a_{ij} = a_{ji}$ hence A is symmetric.

By hypothesis, k is odd and the sum of every row and column is odd. In particular, the sum of all the entries of A is odd. On the other hand, A is symmetric and thus the sum of the non-diagonal entries is even. This shows that at least one diagonal entry of A, say a_{nn}, must be odd.

Let X be the graph induced on C_n. Since C is semiregular, it acts regularly on X and we can view X as a Cayley graph $\mathrm{Cay}(C,S)$. Since X has odd valency, $|S|$ is odd, $|C|$ is even and S contains the unique involution of C. The result follows. $\square$

Lemma 2.4. *Let Γ be a G-arc-transitive group and let N be a normal subgroup of G. If N contains an element reversing some edge of Γ, then Γ is N-vertex-transitive.*

Proof. Let e be an edge of Γ. Since N is normal in the arc-transitive group G, N must contain an element reversing e. In particular, the endpoints of e are in the same N-orbit. By connectedness, N is vertex-transitive. $\square$

Lemma 2.5. *Let G be a transitive permutation group on the set Ω, let N be a normal subgroup of G and let C be a semiregular subgroup of G with k orbits. If $|N|$ is coprime to $|G_v|$, then the induced action of C on the N-orbits is semiregular with k' orbits, where k' divides k.*

Proof. Since $|N|$ is coprime to $|G_v|$, $N_v = 1$ and thus $|N| = |v^N|$ for every point v. It follows that $|(G/N)_{v^N}| = \frac{|G/N|}{|\Omega|/|v^N|} = \frac{|G|}{|\Omega|} = |G_v|$.

Let $c \in C$ such that Nc (viewed throughout this proof as an element in the quotient group G/N) fixes some v^N. For the first part, it suffices to show that Nc is the identity element of G/N. Note that, by the previous paragraph, the order of Nc divides $|G_v|$. On the other hand, since v^N is fixed by Nc, v^N can be partitioned in $\langle c\rangle$-orbits, but these all have the same size, namely $|c|$, and thus $|c|$ divides $|N|$. It follows that the order of Nc divides both $|G_v|$ and $|N|$ but these are coprime and thus Nc is trivial.

As for the second claim, $k = \frac{|\Omega|}{|C|}$ while $k' = \frac{|\Omega|}{|v^N|}\frac{|C\cap N|}{|C|}$ thus $\frac{k}{k'} = \frac{|v^N|}{|C\cap N|}$. Recall that $|N| = |v^N|$ and thus $\frac{k}{k'} = \frac{|N|}{|C\cap N|}$ which is an integer. $\square$

Lemma 2.6. *Let Γ be a G-arc-transitive graph. If G has a normal semiregular subgroup with at most two orbits on vertices, then the subgroup of G fixing a vertex and all its neighbours is trivial.*

Proof. If G has a normal regular group, then the result follows by [20, Lemma 2.1]. Otherwise, it is not hard to see that Γ must be bipartite and the result follows by applying [26, Lemma 2.4] with $X = G$ and N the bipartition-preserving subgroup of G. $\square$

The structure of the vertex-stabiliser in a cubic G-arc-transitive graph has been known for a long time (see [11] for example).

Theorem 2.7. *Let Γ be a cubic graph. If Γ is G-arc-transitive, then it is $(G, t+1)$-arc-regular for some $0 \leqslant t \leqslant 4$. Moreover, the structure of G_v is uniquely determined by t and is as in Table 1.*

t	0	1	2	3	4
G_v	$\mathbb{Z}_3$	$\mathrm{Sym}(3)$	$\mathrm{Sym}(3) \times \mathbb{Z}_2$	$\mathrm{Sym}(4)$	$\mathrm{Sym}(4) \times \mathbb{Z}_2$

TABLE 1. Vertex-stabilisers in cubic $(t+1)$-arc-regular graphs

Proposition 2.8. [27, Corollary 4.6] *Let Γ be a cubic $(G, t+1)$-arc-transitive graph. If G is soluble, then $t \leqslant 2$. Moreover, if $t = 2$, then Γ is a regular cover of* $\mathrm{K}_{3,3}$.

Lemma 2.9. *Let Γ be a cubic G-arc-transitive graph and let N be a normal subgroup of G that is locally-transitive on Γ. If $|N_v| \leqslant 12$, then $|G_v| \leqslant 12$.*

Proof. Suppose, by contradiction, that $|G_v| > 12$. By Theorem 2.7, G_v is isomorphic to either $\mathrm{Sym}(4)$ or $\mathrm{Sym}(4) \times \mathbb{Z}_2$. Now, N_v is a normal subgroup of G_v of order divisible by 3. Since $|N_v| \leqslant 12$, it is not hard to check that this implies $N_v \cong \mathrm{Alt}(4)$. Since $N_v^{\Gamma(v)}$ is a quotient of N_v with order divisible by 3, we have that $N_v^{\Gamma(v)}$ is regular of order 3. As N is normal in a vertex-transitive group, this holds for every vertex, but this implies that N_v itself has order 3, a contradiction. □

Lemma 2.10. *Let Γ be a G-locally-primitive cubic graph. If N is a normal subgroup of G such that G/N is insoluble, then N has at least three orbits and is semiregular on the vertices of Γ. In particular, Γ is a regular cover of Γ/N.*

Proof. If N has at most two orbits on vertices, then $|G : N|$ divides $2|G_v|$. Since $|G_v|$ is a $\{2, 3\}$-group, so is G/N and thus G/N is soluble, a contradiction. If follows that N has at least three orbits on vertices.

Suppose that $N_v^{\Gamma(v)} \neq 1$. Since G is locally-primitive, it follows that $N_v^{\Gamma(v)}$ is transitive. In particular, vertices at distance 2 from each other are in the same N-orbit. By connectedness, N has at most 2 orbits, which is a contradiction. We conclude that $N_v^{\Gamma(v)} = 1$, thus $N_v = 1$ and N is semiregular. □

Lemma 2.11. *Let C be a cyclic subgroup of a group G. If N is a normal subgroup of G, then $N \cap C$ is a cyclic subgroup of N and NC/N is a cyclic subgroup of G/N. Moreover, $|G : C| = |G/N : NC/N||N : N \cap C|$.*

Proof. The first part follows from the fact that $NC/N \cong C/(N \cap C)$. As for the second part, note that $|G : C| = |G : NC||NC : C| = |G/N : NC/N||N : N \cap C|$. □

Lemma 2.12. *Let Γ be a cubic $(G, t+1)$-arc-regular graph such that G is insoluble and let S be the soluble radical of G.*

(1) *If C is a semiregular cyclic subgroup of G with an odd number of orbits, then $|C \cap S|$ is odd and $|G/S : CS/S|_2|S|_2 = 2^t$.*
(2) *If a Sylow 2-subgroup of G has a cyclic subgroup of index at most 2^t, then G/S is almost simple.*

Proof. We first prove (1). Suppose, by contradiction, that $|C \cap S|$ is even. This implies that S contains the unique involution of C. By Lemma 2.3, this involution reverses an edge of G and it follows by Lemma 2.4 that S is vertex-transitive, contradicting Lemma 2.10. We conclude that $|C \cap S|$ is odd. Note that $|G/S : CS/S| = |G|/|CS| = |G||C \cap S|/|S||C| = 3 \cdot 2^t k|C \cap S|/|S|$, where k is the number of orbits of C. Since $|C \cap S|$ is odd it follows that $|G/S : CS/S|_2|S|_2 = 2^t$. This concludes the proof of (1).

We now prove (2). By Theorem 2.7, we have $0 \leqslant t \leqslant 4$. By Lemma 2.10, Γ is a regular cover of Γ/S and $G_v \cong (G/S)_{v^S}$. Let N be the socle of G/S. Write $N = T_1 \times \cdots \times T_m$, such that the T_i's are nonabelian simple and ordered such that the exponent of their Sylow 2-subgroups is non-increasing. We suppose that $m \geqslant 2$ and will obtain a contradiction.

Let N_2 be a Sylow 2-subgroup of N. Recall that the Sylow 2-subgroup of a nonabelian simple group is never cyclic (see [1, 39.2]) and, in particular, has order at least 4. Thus, any cyclic subgroup of N_2 has index at least $2|T_2|_2 \cdots |T_m|_2$. By Lemma 2.11, N_2 has a cyclic subgroup of index at most 2^t. It follows that $2^t \geqslant 2|T_2|_2 \cdots |T_m|_2 \geqslant 2 \cdot 4^{m-1}$. Since $t \leqslant 4$, we have $m = 2$, $|T_2|_2 \leqslant 8$ and $t \geqslant 3$.

Since G/S has trivial soluble radical, G/S can be embedded in the automorphism group of N, and $(G/S)/N$ can be embedded in the outer automorphism group of N.

Since $N = T_1 \times T_2$, the Schreier Conjecture implies that the outer automorphism of N is soluble, and thus so is $(G/S)/N$. If N has at least three orbits on the vertices of Γ/S, then Γ/S is a regular cover of $(\Gamma/S)/N$ and thus $t \leqslant 2$ by Proposition 2.8, a contradiction. It follows that N has at most two orbits. If N is semiregular, then it follows by Lemma 2.6 that $t \leqslant 1$. We may thus assume that N is locally-transitive.

We apply Lemma 2.10 to conclude that Γ/S is a regular cover of $(\Gamma/S)/T_1$. In particular, N/T_1 is locally-transitive. Since $N/T_1 \cong T_2$, $|T_2|_2 \leqslant 8$ and $(\Gamma/S)/T_1$ has even order, we find that $|(N/T_1)_{\overline{v}}|_2 \leqslant 4$, where $\overline{v}$ is a vertex in $(\Gamma/S)/T_1$, and thus $|N_{v^S}| = |(N/T_1)_{\overline{v}}| \leqslant 12$. By Lemma 2.9, this implies $|G_v| = |(G/S)_{v^S}| \leqslant 12$ and thus $t \leqslant 2$, a contradiction. □

Proposition 2.13. *Let t be an integer with $0 \leqslant t \leqslant 4$, let k be a squarefree positive integer coprime to 6 and let $\overline{G}$ be an almost simple group with order divisible by 3. If $\overline{G}$ has a cyclic subgroup $\overline{C}$ of even order and index dividing $3 \cdot 2^t k$, then $\overline{G}$, $|\overline{C}|$ and $\log_2 |\overline{G} : \overline{C}|_2$ are given in Table 2. (The meaning of the last two columns of Table 2 will be explained in Section 3.2.)*

	$\overline{G}$	$\lvert\overline{C}\rvert$	$\log_2 \lvert\overline{G} : \overline{C}\rvert_2$	Upper bound on t	Upper bound on $\log_2 \lvert S\rvert_2$
(1)	Alt(5)	2	1	1	0
(2)	Sym(5)	2, 4 or 6	1 or 2	2	1
(3)	Sym(6)	6	3	3	0
(4)	Aut(Sym(6))	6	4	4	0
(5)	Alt(7)	6	2	3	1
(6)	Sym(7)	6 or 12	2 or 3	4	2
(7)	M_{11}	6	3	3	0
(8)	J_1	2, 6 or 10	2	2	0
(9)	$\mathrm{Aut}(^2B_2(8))$	4 or 12	4	0	–
(10)	$\mathrm{PSL}(2,2^4)$	2	3	1	–
(11)	$\mathrm{PSL}(2,2^4).2$	2, 4, 6 or 10	3 or 4	2	–
(12)	$\mathrm{P\Gamma L}(2,2^4)$	4, 8 or 12	3 or 4	2	–
(13)	$\mathrm{PSL}(2,2^5)$	2	4	1	–
(14)	$\mathrm{P\Gamma L}(2,2^5)$	2 or 10	4	1	–
(15)	$\mathrm{PSL}(2,r)$, $r \geqslant 7$	$\leqslant (r+1)/2$	$\geqslant 1$	3	2
(16)	$\mathrm{PGL}(2,r)$, $r \geqslant 7$	$\leqslant r+1$	$\geqslant 1$	3	2
(17)	$\mathrm{P\Sigma L}(2,r^2)$, $r \geqslant 5$	$2r$	$\geqslant 3$	4	1
(18)	$\mathrm{P\Gamma L}(2,r^2)$, $r \geqslant 5$	$2r$	$\geqslant 4$	4	0

TABLE 2.

Proof. Let T be the socle of $\overline{G}$. By Lemma 2.11, $|T : T \cap \overline{C}|$ divides $3 \cdot 2^t k$; this will play a crucial role.

If $T \cong \mathrm{Alt}(n)$, then $n < 9$ since the Sylow 3-subgroup of $\overline{G}$ contains a cyclic subgroup of index dividing 3. The cases $n \in \{5, 6, 7, 8\}$ yield rows $(1-6)$ of Table 2.

Suppose now that T is a sporadic simple group (including the Tits group). By considering the order of elements in T (see [8]), one can check that T does not have a cyclic subgroup of index dividing $3 \cdot 2^t k$ unless T is isomorphic to the Matthieu group M_{11} or the Janko group J_1. Both of these have trivial outer automorphism group, hence $\overline{G} = T$ and it is easy to check that $\overline{C}$ must be as in rows (7) and (8) of Table 2.

From now on, we may thus assume that T is a simple group of Lie type, of characteristic r, say. We record the order and a crude upper bound on the exponent

T	$\lvert T\rvert_r$	Upper bound on r-exponent	Condition
$\mathrm{PSL}(n,r^f)$	$r^{fn(n-1)/2}$	$r^{(n+1)/2}$	$n\geqslant 2$
$\mathrm{PSU}(n,r^f)$	$r^{fn(n-1)/2}$	$r^{(n+1)/2}$	$n\geqslant 3$
$\mathrm{PSp}(n,r^f)$	$r^{fn^2/4}$	$r^{(n+1)/2}$	$n\geqslant 4$, even
$\mathrm{P\Omega}(n,r^f)$	$r^{f(n-1)^2/4}$	$r^{(n+1)/2}$	$n\geqslant 7$, nr odd
$\mathrm{P\Omega}^{\epsilon}(n,r^f)$	$r^{fn(n-2)/4}$	$r^{(n+1)/2}$	$n\geqslant 8$, n even
$E_8(r^f)$	r^{120f}	r^8	
$E_7(r^f)$	r^{63f}	r^6	
$E_6(r^f)$	r^{36f}	r^5	
${}^2E_6(r^f)$	r^{36f}	r^5	
$F_4(r^f)$	r^{24f}	r^5	
${}^2F_4(2^{2m+1})$	2^{12f}	2^5	$m\geqslant 1$
$G_2(r^f)$	r^{6f}	r^2,	r odd
$G_2(r^f)$	r^{6f}	r^3,	$r=2$, $f\geqslant 2$
${}^2G_2(3^{2m+1})$	$3^{3(2m+1)}$	3^2	$m\geqslant 1$
${}^2B_2(2^{2m+1})$	$2^{2(2m+1)}$	2^2	$m\geqslant 1$
${}^3D_4(r^f)$	r^{12f}	r^3	

TABLE 3. Orders and exponents of Sylow r-subgroups of simple groups of Lie type of characteristic r

of a Sylow r-subgroup of T in Table 3. The orders can be found in [8, p xvi]. Note that an r-element in $\mathrm{GL}(n,r^f)$ has order at most r^e where $e=\lceil \log_r n\rceil(n+1)/2$ (see [21, §6.5] for example). We then use [22, Table 5.4C] to find the smallest value n such that T (or some central extension of T by a cyclic subgroup of order coprime to r) embeds as a subgroup of $\mathrm{GL}(n,F)$ for F a field of characteristic r. The maximum exponent of an r-element in $\mathrm{GL}(n,F)$ then gives a crude upper bound on the maximum exponent of an r-element in T.

Recall that $|T:T\cap\overline{C}|$ divides $3\cdot 2^t k$. In particular, a Sylow r-subgroup of T must contain a cyclic subgroup of index at most r if r is odd and at most 16 if $r=2$. Using this fact and Table 3, we deduce that T is isomorphic to one of $\mathrm{PSp}(4,2)$, $\mathrm{PSU}(4,2)$, $\mathrm{PSL}(4,2)$, ${}^2B_2(8)$, $\mathrm{PSU}(3,r^f)$, or $\mathrm{PSL}(n,r^f)$ with $n\leqslant 3$.

It can be checked that $\mathrm{PSU}(4,2)$ and $\mathrm{PSL}(4,2)$ do not contain a cyclic subgroup of index dividing $3\cdot 2^t k$, whereas the case $T\cong\mathrm{PSp}(4,2)\cong\mathrm{Sym}(6)$ has already been dealt with. The group ${}^2B_2(8)$ has order coprime to 3 but its automorphism group yields row (9) of Table 2.

Suppose now that T is isomorphic to $\mathrm{PSL}(3,r^f)$ or $\mathrm{PSU}(3,r^f)$. A Sylow r-subgroup of T has order r^{3f} and exponent 2^2 if $r=2$, and r otherwise. It follows that $r=2$ and $f\leqslant 2$. It can be checked that no example arise when $f=2$, while $\mathrm{PSU}(3,2)$ is soluble. Finally, we will deal with $T\cong\mathrm{PSL}(3,2)\cong\mathrm{PSL}(2,7)$ as part of our next and last case.

It remains to deal with the case $T\cong\mathrm{PSL}(2,r^f)$. Since $\mathrm{PSL}(2,2)$ and $\mathrm{PSL}(2,3)$ are soluble, $\mathrm{PSL}(2,4)\cong\mathrm{PSL}(2,5)\cong\mathrm{Alt}(5)$ and $\mathrm{PSL}(2,9)\cong\mathrm{Alt}(6)$, we may assume that $r^f\geqslant 7$ and $r^f\neq 9$. The Sylow r-subgroup of T has order r^f and exponent r. In particular, $f\leqslant 2$ unless $r=2$ in which case $f\leqslant 5$.

It can be checked that when $r=2$ and $f\in\{3,4,5\}$, the examples that arise are in rows $(10-14)$ of Table 2.

Suppose now that $f=1$. In particular, r is odd and $\overline{G}=\mathrm{PSL}(2,r)$ or $\overline{G}=\mathrm{PGL}(2,r)$. The orders of maximal cyclic subgroups of $\mathrm{PSL}(2,r)$ are $(r+1)/2$,

$(r-1)/2$ and r, while the orders of maximal cyclic subgroups of $\mathrm{PGL}(2,r)$ are $(r+1)$, $(r-1)$ and r [9]. Since $|\overline{C}|$ is even, we get rows (15) and (16) of Table 2.

Finally, suppose that $f=2$ and $r \geqslant 5$. Since k is squarefree and r^2 divides $|\mathrm{PSL}(2,r^2)|$, r must divide $|\mathrm{PSL}(2,r^2) \cap \overline{C}|$. On the other hand, a Sylow r-subgroup S of $\mathrm{PSL}(2,r^2)$ is elementary abelian hence $|\mathrm{PSL}(2,r^2) \cap \overline{C}| = r$. Moreover, for each element c of order r in S, the centraliser of c in $\mathrm{PGL}(2,r^2)$ is S [9]. Since $|\overline{C}|$ is even, it follows that $\mathrm{P\Sigma L}(2,r^2) \leqslant \overline{G}$ and $|\overline{C}| = 2r$. Note that $|\mathrm{P\Sigma L}(2,r^2)|_2 \geqslant 2^4$ and $|\mathrm{P\Gamma L}(2,r^2)|_2 \geqslant 2^5$. This gives rows (17) and (18) of Table 2. $\square$

3. Proof of Theorem 1.2

In view of the statement of Theorem 1.2, we will consider the following hypothesis.

Hypothesis 3.1. Let $k \geqslant 5$ be a squarefree integer coprime to 6 and let Γ be a cubic $(G, t+1)$-arc-regular graph such that C is semiregular with k orbits.

Our goal is to show that Γ has order at most $6k^2$. We introduce the following notation which we will use whenever we assume Hypothesis 3.1.

Notation. For a prime p dividing $|G|$, we denote by C_p a Sylow p-subgroup of C, and let P_p denote a Sylow p-subgroup of G containing C_p. (Note that we may have $C_p = 1$.) Let c be the unique involution in C. (C has even order since Γ does but k is odd.) We denote by S the soluble radical of G and write $\overline{G} = G/S$ and $\overline{C} = CS/S$. Let T be the socle of $\overline{G}$.

We note a few obvious facts about G and C that will be very useful.

Lemma 3.2. *Assuming Hypothesis 3.1, the following holds.*

(1) $|G| = 3 \cdot 2^t k |C|$.
(2) $|P_2 : C_2| = 2^t$.
(3) *For every odd prime p, we have that $|P_p : C_p|$ divides p.*

Proof. Recall that $|G_v| = 3 \cdot 2^t$. (1) then follows by the Orbit-Stabiliser Theorem. We get (2) and (3) by considering the 2-part and the p-part of the equation in (1), respectively. $\square$

3.1. G soluble.

We first focus on the case when G is soluble.

Lemma 3.3. *Assume Hypothesis 3.1. If G is soluble, then $t \leqslant 1$ and, for every prime p, we have $|P_p : C_p| \leqslant p$.*

Proof. By Lemma 3.2, it suffices to show that $t \leqslant 1$. Suppose that $t \geqslant 2$. By Proposition 2.8, $t = 2$ and Γ is a regular cover of $\mathrm{K}_{3,3}$.

In particular, $\Gamma/N \cong K_{3,3}$ for some normal subgroup $N \trianglelefteq G$. There is a chief series for G through N. Let M be the member of this series immediately preceding N (that is, $M \trianglelefteq N$). Then, N/M is a minimal normal subgroup of G/M and, since G is soluble, $N/M \cong \mathbb{Z}_q^a$ for some prime q and some integer $a \geqslant 1$. Let $\Gamma^* = \Gamma/M$. Note that Γ is a regular cover of Γ^* which is itself a regular $\mathbb{Z}_q^a$-cover of $K_{3,3}$. Since $t=2$, it follows by [27, Proposition 3.3] (see also [14, Theorem 4.1] and [15, Theorem 1.1]) that $a \geqslant 4$, or $q = 3$ and $a \neq 2$.

By Lemma 3.2, P_q has a cyclic subgroup of index dividing q or 4. In particular, every elementary abelian section of P_q has rank at most 2, unless $q=2$, in which case it has rank at most 3. By the previous paragraph, we get that $q=3$ and $a=1$ and, by [15, Theorem 1.1], Γ^* is isomorphic to the Pappus graph. Since $t=2$ and the Pappus graph is 3-arc-regular, $\mathrm{Aut}(\Gamma^*)$ is a quotient of G. This is a contradiction because the Sylow 3-subgroup of $\mathrm{Aut}(\Gamma^*)$ does not have a cyclic maximal subgroup. $\square$

Lemma 3.4. *Assume Hypothesis 3.1, and let $p \geqslant 5$ be a prime dividing the order of Γ. If P_p is normal in G then*

(1) *c does not centralise P_p, and*
(2) *$|P_p : C_p| = p$ and $|C_p| \leqslant p$.*

Proof. (Recall that, by definition, c is the unique involution of C.) We first prove (1). Suppose, by contradiction, that c centralises P_p. Let Z be the centraliser of P_p in G. This is a normal subgroup of G. By the Schur-Zassenhaus Theorem, we can write $Z = \mathrm{Z}(P_p) \times Y$ where Y is a p'-group. Note that Y is characteristic in Z and thus normal in G. Since p is odd, we have $c \in Y$. By Lemmas 2.3 and 2.4, it follows that Y is transitive on the vertices of Γ, a contradiction, as p divides the order of Γ. This concludes the proof of (1).

We now prove (2). By (1), $P_p \nleqslant C$ and thus Lemma 3.2 implies $|P_p : C_p| = p$. In particular, P_p is a p-group with a cyclic maximal subgroup.

Let X be the group generated by elements of order p in P_p. This is a characteristic subgroup of P_p and thus normal in G. Suppose first that $X = P_p$. By Lemma 2.1(1), P is elementary abelian. This implies immediately that $P \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$, and in either case, $|C_p| \leqslant p$, as required.

Suppose next that $C_pX < P_p$. Since C_p is maximal in P_p, this implies that $X \leqslant C_p$. It follows by Lemma 2.1(2) that P_p is cyclic. By (1), c centralises C_p but not P_p and thus $|P_p| = p$ by Lemma 2.1(3).

From now, we assume that $X < P_p = C_pX$. This implies that $1 \neq P_p/X \leqslant CX/X$ and thus P_p/X is a non-trivial normal Sylow p-subgroup of G/X. By Lemma 2.5, CX/X is a semiregular cyclic subgroup with k' orbits for some divisor k' of k. Since p divides the order of CX/X, it follows that p divides also the order of Γ/X, and Γ/X is a (G/X)-arc-transitive cubic graph. If $k' = 1$, then Γ/X is an arc-transitive cubic circulant hence its order divides 6, a contradiction. Since k' is coprime to 6, it follows that $k' \geqslant 5$ and Hypothesis 3.1 is satisfied with (k, Γ, G, C) replaced by $(k', \Gamma/X, G/X, CX/X)$. In particular, we may apply (1) to conclude that cX does not centralise P_p/X, contradicting the fact that $P_p/X \leqslant CX/X$. □

Theorem 3.5. *Assume Hypothesis 3.1. If G is soluble, then Γ has order at most $6k^2$.*

Proof. By Lemma 3.3, every Sylow p-subgroup of G is metacyclic. It follows by [5, Theorem 1] that $G = N \rtimes A$, where A is a Hall $\{2,3\}$-subgroup of G and N has a normal series

$$1 = N_0 \trianglelefteqslant N_1 \trianglelefteqslant \cdots \trianglelefteqslant N_n = N$$

where $N_{i+1}/N_i \cong P_{p_i}$. For every $i \in \{0, \ldots, n\}$, $|N_i|$ is coprime to 6 and thus semiregular.

Suppose that $i < n$. Then p_{i+1} divides the order of Γ but not $|N_i|$ hence N_i has at least 3 orbits and Γ is a regular cover of Γ/N_i. By Lemma 2.5, CN_i/N_i is semiregular and has k_i orbits for some divisor k_i of k. Note that $k_i = 1$ is impossible, for otherwise, Γ/N_i would be a cubic arc-transitive circulant and its order would not be divisible by p_{i+1}. Since k_i is coprime to 6, it follows that $k_i \geqslant 5$ and that $(\Gamma/N_i, G/N_i)$ satisfies Hypothesis 3.1 with (k, Γ, G, t, C) replaced by $(k_i, \Gamma/N_i, G/N_i, t, CN_i/N_i)$. Note that N_{i+1}/N_i is a normal Sylow p_i-subgroup of G/N_i and we may thus apply Lemma 3.4 to conclude that $|C_{p_i}| \leqslant |P_{p_i} : C_{p_i}| = k_{p_i}$. Finally, k is coprime to 6 but $G/N_n = G/N \cong A$ is a $\{2,3\}$-group and thus $k_n = 1$. It follows that Γ/N has order dividing 6 and that $|C_2||C_3| \leqslant 6$ hence $|C| \leqslant 6k$, which concludes the proof. □

3.2. G **not soluble.** We now consider the remaining case, namely when G is not soluble.

Lemma 3.6. *Assume Hypothesis 3.1. If G is insoluble, then $\overline{G}$, $|\overline{C}|$, $\log_2|\overline{G}:\overline{C}|_2$ and upper bounds for t and $\log_2|S|_2$ are as in rows $(1-8)$ or $(15-18)$ of Table 2.*

Proof. By Lemma 2.10, Γ/S is a cubic $(\overline{G}, t+1)$-arc-regular graph. In particular, 3 divides $|\overline{G}|$. By Lemma 2.12, $|C\cap S|$ is odd, $\overline{C}$ is a cyclic group of even order and $\overline{G}$ is an almost simple group. Recall that $|G:C| = 3\cdot 2^t k$ hence by Lemma 2.11 $|\overline{G}:\overline{C}|$ divides $3\cdot 2^t k$. By Proposition 2.13, $\overline{G}$, $|\overline{C}|$ and $\log_2|\overline{G}:\overline{S}|$ are as in one of the rows of Table 2.

We now compute upper bounds on t and record them in Table 2. We do this by using the fact that the isomorphism type of the vertex-stabiliser $\overline{G}_{v^S}$ is uniquely determined by t (see Theorem 2.7). For example, $\mathrm{Alt}(7)$ does not contain a subgroup isomorphic to $\mathrm{Sym}(4)\times\mathbb{Z}_2$ and thus $t\leqslant 3$ when $\overline{G}\cong\mathrm{Alt}(7)$. The fact that $\mathrm{PSL}(2,r)$ does not contain a subgroup isomorphic to $\mathrm{Sym}(4)\times\mathrm{Sym}(2)$ follows from Dickson's classification of the subgroups of $\mathrm{PSL}(2,r)$ [9].

We then combine this upper bound on t with Lemma 2.12(1) to obtain an upper bound on $\log_2|S|_2$, which we also record in Table 2. (When $\log_2|\overline{G}:\overline{C}|_2 > t$, we obtain a contradiction and record this as a –.) □

Theorem 3.7. *Assume Hypothesis 3.1. If G is insoluble, then Γ has order at most $6k^2$.*

Proof. By Lemma 3.6, $\overline{G}$, $|\overline{C}|$, $\log_2|\overline{G}:\overline{C}|_2$ and upper bounds for t and $|S|_2$ are as in Table 2. Write $G = S.T.A$. Note that $\overline{G}\cong T.A$ and we can read off A from Table 2. In fact, $|A|\leqslant 2$, unless $\overline{G}\cong\mathrm{P\Gamma L}(2,r^2)$, in which case $|A| = 4$. Since A is soluble, $G^\infty\cong Y.T$ for some normal subgroup Y of S. Let

$$1 = S_0 \trianglelefteq S_1 \trianglelefteq \cdots \trianglelefteq S_n = S$$

be a characteristic series for S maximal subject to its length. For every $i\in\{0,\dots,n-1\}$, let $\phi_i : G\to\mathrm{Aut}(S_{i+1}/S_i)$ be the homomorphism induced by the action of G on S_{i+1}/S_i by conjugation and let $K_i = \ker\phi_i\cap G^\infty$.

Suppose that $\phi_i(G^\infty)$ is insoluble for some i. Since S_{i+1}/S_i is characteristically simple, it is elementary abelian, say $S_{i+1}/S_i\cong\mathbb{Z}_p^a$. By Table 2, $|S|_2\leqslant 4$. Together with Lemma 3.2, this implies that $a\leqslant 2$. Since $\phi_i(G^\infty)$ is insoluble, $a=2$, $p\geqslant 5$ and $\mathrm{Aut}(S_{i+1}/S_i)\cong\mathrm{GL}(2,p)$. By Dickson's classification of subgroups of $\mathrm{PSL}(2,p)$ [9], either $\mathrm{SL}(2,p)\leqslant\phi_i(G^\infty)$ or $\mathrm{SL}(2,5)\leqslant\phi_i(G^\infty)\leqslant\phi_i(G)\leqslant\mathrm{SL}(2,5)\circ\mathbb{Z}_{p-1}$. In the latter case, $\overline{G}\cong\mathrm{Alt}(5)$ and $|S|$ is even, contradicting Table 2. Thus $\mathrm{SL}(2,p)\leqslant\phi_i(G^\infty)$ and $T = \mathrm{PSL}(2,p)$. By [2, Table I], an extension of $\mathbb{Z}_p^2$ by $\mathrm{SL}(2,p)$ splits hence G contains a group of order p^3 and exponent p as a section, contradicting Lemma 3.2.

It follows that $G^\infty/K_i\cong\phi_i(G^\infty)$ is soluble for every i. Since G^∞ is perfect, it follows that $G^\infty = K_i$, $\phi_i(G^\infty) = 1$ and G^∞ acts trivially on S_{i+1}/S_i. Using Lemma 2.2 and induction on i, we conclude that G^∞ acts trivially on S_n, that is, $G^\infty\leqslant C_G(S)$ and $G^\infty\cap S\leqslant\mathrm{Z}(G^\infty)$.

On the other hand, $\mathrm{Z}(G^\infty)$ is an abelian normal subgroup of G hence $\mathrm{Z}(G^\infty)\leqslant S$ and thus $G^\infty\cap S = \mathrm{Z}(G^\infty)$. In particular, $G^\infty/\mathrm{Z}(G^\infty) = G^\infty/(G^\infty\cap S)\cong G^\infty S/S = T$. Since T is simple, we conclude that G^∞ is quasisimple. In particular, $\mathrm{Z}(G^\infty)$ is equal to a homomorphic image of the Schur multiplier of T (see [1, 33.8]).

We want to show that the order of Γ is at most $6k^2$. This is equivalent to $|C|\leqslant 6k = \frac{6|G|}{|G_v||C|} = \frac{|G|}{2^{t-1}|C|}$ and thus to $|G|\geqslant 2^{t-1}|C|^2$. On the other hand, $|C| = |\overline{C}||C\cap S|$ but $|C\cap S|$ is odd by Lemma 2.12(1) hence $|C|\leqslant|\overline{C}||S|_{2'}$. Since $|G| = |\overline{G}||S|\geqslant|\overline{G}||S|_{2'}$, it thus suffices to show that

$$|\overline{G}|\geqslant 2^{t-1}|\overline{C}|^2|S|_{2'}. \qquad (1)$$

Suppose first that G^∞ is semiregular on the vertices of Γ. This implies that $|G/G^\infty|_2 \geqslant 2^t$. On the other hand, $G/G^\infty \cong (S/Y).A$ therefore $|G/G^\infty|_2 \leqslant |S|_2|A|$. Combining this with Lemma 2.12(1), we get $|G/G^\infty|_2|\overline{G} : \overline{C}|_2 \leqslant 2^t|A| \leqslant |G/G^\infty|_2|A|$ and thus $|\overline{G} : \overline{C}|_2 \leqslant |A|$. By running through Table 2, we find that $\overline{G} \cong \mathrm{PGL}(2, r)$ with $r \geqslant 5$ and $|A| = |\overline{G} : \overline{C}|_2 = 2$. (Note that this includes the case $\overline{G} \cong \mathrm{Sym}(5)$.) Using the previous inequalities, this implies that $|G/G^\infty|_2 = 2^t$, and thus $|G^\infty|_2 = |C|_2$. Since Γ has order $k|C|$ and G^∞ is semiregular, this further implies that G^∞ has an odd number of orbits. If G^∞ has at least three orbits, then Γ/G^∞ is a cubic graph on an odd number of vertices, a contradiction. It follows that G^∞ is transitive, hence $|(S/Y).A| = |G/G^\infty| = |G_v| = 3 \cdot 2^t$ and Lemma 2.6 implies $t \leqslant 1$. Since $|A| = 2$, we have $t = 1$ and $|S/Y| = 3$. On the other hand, since the Schur multiplier of $\mathrm{PSL}(2, r)$ has order 2, we see that $|Y|_{2'} = 1$ and thus $|S|_{2'} = 3$. Now, $|\overline{G}| = (r + 1)r(r - 1)$ while $|\overline{C}| \leqslant r + 1$ hence (1) is satisfied.

We may thus assume that G^∞ is not semiregular on the vertices of Γ. In particular, G^∞ is locally transitive and has at most two orbits on the vertices of Γ. It follows that G/G^∞ is a 2-group hence so is S/Y and thus $|S|_{2'} = |Y|_{2'}$. By considering the Schur multiplier of T, we find that $|Y|_{2'} = 1$ unless T is isomorphic to $\mathrm{Alt}(6)$ or $\mathrm{Alt}(7)$, when we may have $|Y|_{2'} = 3$. It is then a matter of routine to go through Table 2 and verify that (1) is satisfied. $\square$

Acknowledgements. We would like to thank Gordon Royle for his help with the computations mentioned near the end of the introduction, and Luke Morgan for pointing out Lemma 2.2. We would also like to thank the referees for their many helpful comments.

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