Dimension functions for T_0 digital spaces

Petra Wiederhold and Richard G. Wilson[1]

Universidad Autónoma Metropolitana

Unidad Iztapalapa, Depto. de Matemáticas

México 09340 D.F., México

Abstract: Alexandroff T_0-spaces have been studied as topological models of the supports of digital images and as discrete models of continuous spaces in theoretical physics. In this paper we discuss three different dimension functions for this class of spaces, namely the *Alexandroff dimension*, the *Order dimension* and the *Krull dimension* and we outline a proof of the equality of these dimension functions in this class. The first of these is essentially the small inductive dimension well-known in topology, the second has been studied in the theory of posets while the third has been studied extensively as a dimension function for lattices and rings and was first applied to topological spaces by Vinokurov in 1966. Since the category of Alexandroff T_0-spaces is known to be isomorphic to the category of posets, these results could be formulated in this latter category as well.

1. Introduction

In digital image processing and computer graphics, it is necessary to describe topological properties of n-dimensional digital image arrays, hence the search for models of the supports of such images. Recently, topological models have been constructed. Based on the Khalimsky topology τ on the integers, given by the subbase $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$, *digital n-space* was defined as a product of n copies of $(\mathbb{Z}, \tau)$ and Jordan curve (surface) theorems were proved for digital 2-space [8] and digital 3-space [10]. A more general construction of a digital space was proposed in [12] where a collection of locally finite disjoint open subsets of $\mathbb{R}^n$, whose union is dense in $\mathbb{R}^n$, is used to define a partition of $\mathbb{R}^n$ giving a digital image which is a locally finite T_0-space, that is, a space in which each point x has

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a finite, and hence also a minimal, neighbourhood which we will denote by $U(x)$. Spaces in which every point has a minimal neighbourhood are called *Alexandroff*; such spaces were first considered by Alexandroff [1] under the name of *discrete spaces* and are the objects of study in this article; thus our definition of a digital space generalizes those considered previously. Other approaches to digital topology, which lead to locally finite spaces are for example, the models based on cellular complexes developed by Kovalevsky [11] and Lee and Rosenfeld [13] or the model of molecular spaces developed by Ivashchenko [5,6]. These latter papers apply discrete topological models in theoretical physics as well.

A problem on which research has been focused recently, is that of the dimension of a digital space. A digital image which is obtained by discretization of an image defined on $\mathbb{R}^n$, should be modeled by an n-dimensional topological space. A dimension function for Alexandroff spaces called *Alexandroff dimension*, which is essentially the small inductive dimension of [14], was studied in [17] and later by Ivashchenko et al. in [6]. An Alexandroff T_0-space (X, τ) is completely determined by a poset $(X, \leq)$, where $\leq$ is defined by $x \leq y \Leftrightarrow x \in cl(\{y\}) \Leftrightarrow y \in U(x)$. The partial order $\leq$ was first introduced in [1] and has been called the *specialization order* by [7], [9] and others. The *partial order dimension* or *poset dimension* of a poset $(X, \leq)$ is defined as the supremum of all lengths of chains in $(X, \leq)$ (see [2]). It was shown in [17] that for any Alexandroff T_0-space, its Alexandroff dimension equals its poset dimension. This implies immediately that digital n-space has Alexandroff dimension n. In another context the same poset dimension was defined in [6] for transitive graphs, and the same equality was proved. In [6] it was also proved, that both dimensions coincide with a third dimension function, which is defined inductively for directed graphs, and which was studied previously by Ivashchenko [5] in relation to his model of molecular space.

In this article we consider not only the two previously studied dimension functions for digital spaces, but also a third, the *Krull dimension*, first applied to topological spaces in [16], but previously known in algebra as a dimension function for rings and lattices. In general spaces, this dimension is difficult to calculate, but it is interesting to note that the

Krull dimension agrees with other dimension functions in one well-known class of space: In [16] it was proved that the Krull dimension is equal to the small inductive and to the covering dimensions for separable metrizable spaces. Here we outline the proofs of a series of theorems which show that for any Alexandroff T_0-space, the Krull dimension coincides with its Alexandroff dimension and its poset dimension. Our results generalize those of [4] and [15] for finite spaces, but we note that in [4], Krull dimension is defined differently in terms of chains of irreducible closed sets and coincides with the definition used here in the case of finite spaces.

2. Preliminaries

Recall that if $A(X)$ is the set of all closed sets of a topological space X, then $(A(X), \cap, \cup)$ is a distributive lattice with greatest and least elements X and $\emptyset$, respectively. It is clear that in the case of a discrete space this lattice is a Boolean algebra.

Definition 1. Let $(L, \wedge, \vee)$ be a lattice; a non-empty subset $F \subset L$ is said to be a *filter* if $a, b \in F, c \in L$ implies that $a \wedge b, a \vee c \in F$ and $F \neq L$. A filter F is said to be *prime* if whenever $a, b \in L$ and $a \vee b \in F$ then $a \in F$ or $b \in F$.

It is clear that if F is a filter in $A(X)$ then $X \in F$ and $\emptyset \notin F$.

Definition 2. The *Krull dimension* of a non-empty lattice $(L, \wedge, \vee)$ is defined as

$$kdim\, L = \sup\{n \in \omega : \exists \;\; a \; chain \; F_0 \subset F_1 \subset \cdots \subset F_n \text{ of distinct prime filters in } L\}.$$

It is known that a lattice with a greatest and a least element has Krull dimension zero if and only if it is a Boolean algebra [16]. Consequently, $kdim\, A(X) = 0$ if X is a discrete space. In general the function $kdim$ is not monotone with respect to sublattices, as the following example shows.

Example 1. Let X be an infinite discrete space. Clearly $A(X)$ itself is a sublattice of $A(X)$, and $kdim\, A(X) = 0$. Now let $\mathcal{B}$ be the sublattice generated by $\{X \setminus \{x\},\ x \in X\} \cup \{\emptyset, X\}$. Then

$$\mathcal{B} = \{X \setminus F,\ F \subset X,\ F \textit{ finite or empty}\} \cup \{\emptyset\}.$$

Since $\mathcal{B} \setminus \{\emptyset\}$ is closed under finite intersections, it is a prime filter in $\mathcal{B}$. Moreover, for any $x \in X$,

$$F_x = \{M \in \mathcal{B} :\ x \in M\} = \{X \setminus F,\ F \subset X,\ F \textit{ finite or empty},\ x \notin F\}$$

is a prime filter in $\mathcal{B}$, and $F_x \subset (\mathcal{B} \setminus \{\emptyset\})$ implies $kdim\, \mathcal{B} \geq 1$.

Definition 2 was first applied to the lattice of closed sets of a topological space by Vinokurov [16]; the same definition was used in [15], although the term *ideal* was used instead of filter.

The *Krull dimension* of a space Y, will be defined in terms of the Krull dimensions of bases of the lattice $(A(Y), \cap, \cup)$, where again, $A(Y)$ is the set of all closed subsets of Y. The following definitions and results are taken from [17].

Definition 3. The *Alexandroff dimension* (*adim*) of an Alexandroff space (X, τ) is defined inductively in terms of a local dimension *adil* determined by the minimal neighbourhoods:

(i) $adim\,(X) = -1 \iff X = \emptyset.$

(ii) *If* $X \neq \emptyset$ *then define*

$$adim\,(X) = sup\,\{adil\,(x), x \in X\}, \textit{ where for } x \in X \textit{ and } n \in \omega \textit{ we define}$$

$$adil\,(x) \leq n \iff adim(FrU(x)) \leq n - 1,$$

$$adil\,(x) = n \iff adil\,(x) \leq n \textit{ and } adil\,(x) \not\leq n - 1,$$

$$adil\,(x) = \infty \iff adil\,(x) \leq n \textit{ is false } \forall n.$$

Full details can be found in [17]. It is easy to see that a discrete space has Alexandroff dimension 0, and for an Alexandroff T_0-space the converse is also true. As mentioned in the introduction, the function *adim* is essentially the small inductive dimension *ind* of [14].

Definition 4. The *poset dimension* (*odim*) of a poset $(X, \leq)$ is defined in terms of a local poset dimension function *odil* as follows:

$$odil\,(x) = \sup\{l : \exists a_1, a_2, \cdots, a_l \in X \ \textit{such that}\ a_l < a_{l-1} < \cdots < a_0 = x\}$$

$$odim\,(X) = \sup\{odil\,(x),\ x \in X\}$$

Again, full details may be found in [17] where the following two propositions were proved:

Proposition 1. *If (X, τ) is an Alexandroff T_0-space and $x \in X$ is such that $odil\,(y) \leq n$ for all $y \in U(x)$, then $adil\,(x) \leq n$.*

Proposition 2. *If (X, τ) is an Alexandroff T_0-space, then*

(i) If $odil\,(x) \geq n$ and $odil\,(y) \leq n$ for all $y \in U(x)$, then $adil\,(x) = n$.

(ii) If $adil\,(x) = n$ then $odil\,(x) \leq n$ and there is some $y \in U(x)$ such that $odil\,(y) \geq n$.

As an immediate corollary to these results we obtain:

Theorem 1. *If (X, τ) is an Alexandroff T_0-space with specialization order $\leq$, then*

$$adim\,(X, \tau) = odim\,(X, \leq).$$

Suppose now that (Y, τ) is a topological space. In order to define the Krull dimension of the space (Y, τ), we will need the concept of a base for the lattice $A(Y)$:

Definition 5. A subset $\mathcal{B}$ of $A(Y)$ is said to be a *lattice base* of $A(Y)$ if it is a (topological) base for the closed sets and additionally, is a sublattice of $(A(Y), \cap, \cup)$ which contains $\emptyset$ and Y.

Clearly, $A(Y)$ itself is a lattice base of $A(Y)$ and since a lattice base of $A(Y)$ is closed under finite intersections, $A(Y)$ is the unique lattice base in the case that Y is a finite set.

Definition 6. The *Krull dimension* of a topological space (Y, τ) is defined by

$$kdim\,(Y) = \min\{kdim\,\mathcal{B} :\ \mathcal{B}\ \textit{is a lattice base of}\ A(Y)\}.$$

It is a consequence of the remarks following Definition 2, that if Y is a discrete space, then $kdim\,(Y) = 0$, since we may take $A(Y) = \mathcal{P}(Y)$ which is a Boolean algebra. In [15] and [16], many properties of $kdim$ were obtained; specifically, it was shown in [16] that

$kdim\,(Y) = ind\,Y = dim\,Y$ for any separable metrizable space Y (where ind, dim are the small inductive and the covering dimension, respectively). However, in general spaces, $kdim$ is frequently difficult to calculate.

Lemma 1. *If L is an arbitrary sublattice of $A(Y)$ such that $\cap L = \emptyset$, then*

$$F_{y,L} = \{M \in L : \; cl(\{y\}) \subset M\}$$

is a prime filter in L.

Proof: We leave the routine verifications to the reader and note only that if $\cap L = \emptyset$, then $F_{y,L} \neq L$. □

3. The Krull dimension of an Alexandroff T_0-space

In this section we outline a proof of the fact that if (X, τ) is an Alexandroff T_0-space, then the Krull dimension of X coincides with the Alexandroff dimension of X, thus generalizing a result of [15] for finite spaces.

Throughout, $\leq$ will denote the specialization order of the space (X, τ). Note that $odil\,(x) = 0$ if and only if $\{x\}$ is closed, and x is maximal with respect to $\leq$ if and only if $\{x\}$ is open.

The main theorem will be a consequence of the following two propositions.

Proposition 3. *If (X, τ) is an Alexandroff T_0-space, and L is a topological base for the closed sets which is also a sublattice of $(A(X), \cap, \cup)$, then $kdim\,L \geq adim\,(X)$.*

Proof: From Theorem 1, it suffices to show that $kdim\,L \geq odim\,(X)$. Since L is a base for the closed sets, it follows that $\cap L = \emptyset$. Hence, by Lemma 1 we have that

$$F_a = \{M \in L : \; cl(\{a\}) \subset M\}$$

is a prime filter in L for any $a \in X$.

We claim that if $a_1 < a_2$ for $a_1, a_2 \in X$, then $F_{a_2} \subset F_{a_1}$, and $F_{a_2} \neq F_{a_1}$. For, if $M \in F_{a_2}$ then $cl(\{a_2\}) \subset M$; but clearly, $cl(\{a_1\}) \subset cl(\{a_2\}) \subset M$, and so $M \in F_{a_1}$. Now, since (X, τ) is a T_0-space, $cl(\{a_1\}) \neq cl(\{a_2\})$ and so, since L is a base of $A(X)$, there is some closed set $M \in L$ such that $cl(\{a_1\}) \subset M$ and $a_2 \notin M$. Thus $M \in F_{a_1} \setminus F_{a_2}$ and the claim is proved.

Consequently, for any $a \in X$ with $odil\,(a) = k$ there exists a chain of prime filters $F_{a_k} \subset F_{a_{k-1}} \subset \cdots \subset F_{a_0}$ (with $a_k = a$). Therefore, if $adim\,(X) = odim\,(X) \geq n$ then there is $x \in X$ with $odil\,(x) \geq n$, implying $kdim\,L \geq n$. The result follows. □

Corollary 1. *If (X, τ) is an Alexandroff T_0-space, then $kdim\,(X) \geq adim\,(X)$.*

Proposition 4. *If (X, τ) is an Alexandroff T_0-space, then $kdim\,A(X) = adim\,(X)$.*

From Proposition 3, we have $kdim\,A(X) \geq adim\,(X)$; hence if $adim\,(X)$ is infinite, then so is $kdim\,A(X)$ and we are done. Thus we suppose that $adim\,(X) = odim\,(X) = n$ is finite and in order to complete the proof it is necessary to show that $kdim\,A(X) \leq n$. We omit the technical details of this proof which can be found in [18].

Combining this result with Corollary 1, we obtain:

Theorem 2. *If X is an Alexandroff T_0-space, then*

$$kdim\,(X) = adim\,(X) = odim\,(X).$$

It is well known that the category of Alexandroff T_0-spaces is isomorphic to the category of partially ordered sets under the functor which takes the space (X, τ) to the poset $(X, \leq)$, where $\leq$ is the specialization order of (X, τ) (see [3]). Thus if we define the Krull dimension of a poset to be the Krull dimension of the corresponding Alexandroff space, Theorem 1 can be stated in the following order-theoretic form:

Theorem 3. *If $(X, \leq)$ is a poset, then*

$$kdim\,(X, \leq) = odim\,(X, \leq).$$

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