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Finite Element Solution of an Eikonal Equation for Excitation Wavefront Propagation in Ventricular Myocardium

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A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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Abstract

An efficient finite element method is developed to model the spreading of excitation in ventricular myocardium by treating the thin region of rapidly depolarizing tissue as a propagating wavefront. The model is used to investigate the excitation sequence in the full canine ventricular myocardium.

The solution to an eikonal–curvature equation for excitation time is shown to satisfy a reaction–diffusion equation for the bidomain myocardial model at the wavefront, while the solution to an eikonal–diffusion equation approximately satisfies the reaction–diffusion equation in the vicinity of the wavefront. The features of these two eikonal equations are discussed.

A Petrov–Galerkin finite element method with cubic Hermite elements is developed to solve the eikonal–diffusion equation. The oscillatory errors seen when using the Galerkin weighted residual method with high mesh Péclet numbers are avoided by supplementing the Galerkin weights with $C^0$ continuous functions based on derivatives of the interpolation functions. The ratio of the Galerkin and supplementary weights is a function of the Péclet number such that, for one-dimensional propagation, the error in the solution is within a small constant factor of the optimal error achievable in the trial space. An additional no-inflow boundary term is developed to prevent spurious excitation initiating on the boundary. The need for discretization in time is avoided by using a continuation method to gradually introduce the non-linear term of the governing equation. A small amount of artificial diffusion is sometimes necessary.

Simulations of excitation are performed using a model of the anisotropic canine ventricular myocardium with 2355 degrees of freedom for the dependent variable, and results are compared with reported experimental observations. When it was assumed that Purkinje fibres influence propagation only on the endocardial surface, excitation of the entire myocardium was completed in 56 ms. Altering material parameters to represent penetration of the Purkinje fibres beneath the left endocardial surface reduced the completion time to 48 ms. Modelling the effects of the laminar structure of myocardium by reducing the propagation speed by 40% in the direction normal to the layers delayed completion of excitation by only 4%.
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Glossary of Symbols

A  scalar constant.
\( A \)  set of possible vectors \( \mathbf{a} \) for the spread of excitation.
\( A_0 \)  coefficient of \( w_0 \) in \( w_k \).
\( A_\infty \)  coefficient of \( w_\infty \) in \( w_k \).
\( A_b \)  multiplier in no-inflow boundary term.
\( \mathbf{a} \)  vector for the spread of excitation or velocity field for advection.
\( \mathbf{a}_k \)  error transport eigenvector.
\( \mathbf{a}_l \)  fibre (or longitudinal) direction.
\( \mathbf{a}_n \)  direction normal to sheets.
\( \mathbf{a}_t \)  direction transverse to fibres within sheets.

B  scalar constant.
\( B(v, w) \)  form representing the left hand side of the weighted residual equations.
\( B_S(v, w) \)  symmetric part of \( B(v, w) \).

\( C_m \)  membrane capacitance per unit area.
\( c_m \)  membrane capacitance per unit volume.
\( c_0 \)  dimensionless propagation speed for a planar wavefront in homogeneous tissue.

\( D \)  set of numbers for parameters that are known from Dirichlet boundary conditions.

\( e_p^r \)  coefficient of \( h_p \) in the power series expansion of error in \( U_i^r \).

\( e_p^{(n)} \)  \( n \)th derivative of \( e_p^r \).

\( e_p \)  coefficient of \( h_p \) in the power series expansion of error in \( U_i \).

\( f_{\|} \)  fraction of cross-sectional area normal to \( \mathbf{a}_l \) occupied by intracellular space.
\( f_{1}, f_{2} \)  unknown scalar functions.

\( f_{\text{IIN}} \)  ionic activity function (dimension of voltage).
\( G^e \) extracellular effective conductivity tensor.
\( G^i \) intracellular effective conductivity tensor.
\( G^o \) conductivity tensor outside the myocardium.
\( g^e_{01}, g^e_{02}, g^e_{03} \) principal extracellular effective conductivities for directions \( \alpha_1, \alpha_2, \) and \( \alpha_3. \)
\( g^i_{01}, g^i_{02}, g^i_{03} \) principal intracellular effective conductivities for directions \( \alpha_1, \alpha_2, \) and \( \alpha_3. \)

\( H^p(\Omega) \) space of functions for which the squares of derivatives up to order \( p \) may be integrated over \( \Omega. \)

\( H^1_D \) space of functions in \( H^1(\Omega) \) satisfying the Dirichlet boundary conditions.

\( H^1_{0D} \) space of functions in \( H^1(\Omega) \) equal to zero on \( \Gamma_D. \)

\( h \) element length.

\( h_1, h_2, h_3 \) reference vectors for specification of \( \alpha_1, -\alpha_3, \) and \( \alpha_2. \)

\( I \) total applied current.

\( I_{\text{mem}} \) ionic membrane current per unit membrane area.

\( i_{\text{app}} \) applied extracellular current per unit tissue volume.

\( i_{\text{mem}} \) ionic membrane current per unit volume.

\( i_{\text{m}} \) transmembrane current per unit tissue volume.

\( j_e \) extracellular current density.

\( j_i \) intracellular current density.

\( j_o \) current density outside the myocardium.

\( K_{\text{Galer}} \) constant in Galerkin error bound.

\( k_{\text{Poincare}} \) constant in Poincaré bound.

\( k_{\text{c}} \) constant for \( \zeta. \)

\( L \) domain length or characteristic distance.

\( N \) number of nodes.

\( N \) set of parameter numbers not in \( D. \)

\( n \) normal to the boundary.

\( n^\nu \) normal to the boundary in the natural coordinate system.

\( O(h^p) \) terms involving powers of \( h \) order \( p \) and higher.

\( P_e \) mesh Pécel number.

\( P^b_e \) mesh Pécel number at the boundary.
GLOSSARY OF SYMBOLS

$P_g$ global Péclet number.

$p_S$ piecewise polynomial function for modification of the diffusion term near a singularity.

$p_\zeta$ piecewise polynomial function for $\zeta$.

$p$ direction of propagation (normal to the wavefront).

$R_m$ reciprocal of membrane conductance per unit area.

$R_S$ Riesz representer.

$r_{el}, r_{in}, r_m$ principal extracellular effective resistivities for directions $a_1$, $a_4$, and $a_6$.

$r_{II}, r_{I}, r_m$ principal intracellular effective resistivities for directions $a_2$, $a_4$, and $a_{III}$.

$r$, $r_{I}, r_{II}$ bulk tissue resistivities for directions $a_1$, $a_4$, and $a_{III}$.

$r_m$ reciprocal of membrane conductance per unit volume.

$\bar{r}_m$ spatially averaged value of $r_m$.

$r_b$ residual for no-inflow boundary term.

$S \equiv \left( \frac{\partial y}{\partial \xi} \right)^{-1}.$

$S_D^h$ trial space.

$S_D^h$ space of possible variation within $S_D^h$.

$s \equiv S(\xi, 0)$.

$T$ coordinate transformation function, $T(\xi, \tau) = t$.

$T^h$ test space.

$T^{h*}$ optimal test space.

$t$ time.

$U$ numerical approximation of $u$.

$U^*$ optimal approximation to $u$ in $S_D^h$.

$U_i$ $i$th parameter determining $U$.

$U_{ij}^{pqr}$ parameter at node $j$ corresponding to $U$ differentiated $p, q,$ and $r$ times with respect to $\xi_1, \xi_2,$ and $\xi_3$, respectively.

$U_i$ vector of parameters for $U$ at node $i$.

$\hat{U}_D$ sum of terms for $U$ that are determined by Dirichlet boundary conditions.

$u$ excitation time.

$\bar{u}_D$ Dirichlet boundary condition value.

$V_{el}$ jump in extracellular potential across a wavefront propagating along fibres.

$V_{el}$ jump in extracellular potential across a wavefront propagating transverse to fibres.
GLOSSARY OF SYMBOLS

$V_p$ plateau transmembrane potential.
$V_r$ resting transmembrane potential.
$V_{m}$ transmembrane potential.
$v_{in}$ transmembrane potential $v_{in}(r)$.

$w_i$ finite element weighting function corresponding to parameter $i$.
$\hat{w}_i^*$ optimal weighting function corresponding to parameter $i$.
$w_{\infty}$ weight terms based on $w_i^*$ when $P_c$ approaches 0 and $\infty$.
$w_{\hat{c}}^p$ finite element weighting function corresponding to derivative $p$ at node $i$.
$\hat{w}_i^p$ optimal weighting function corresponding to derivative $p$ at node $i$.
$\hat{w}_i^{p*}$ localized optimal weighting function corresponding to derivative $p$ at node $i$.
$w_{\infty}^p, w_{\hat{c}}^p$ weight terms based on $\hat{w}_i^{p*}$ when $P_c$ approaches 0 and $\infty$.
$\hat{w}_i^p$ Petrov–Galerkin supplementary weighting function corresponding to parameter $i$.
$\hat{w}_i^p$ supplementary weighting function corresponding to derivative $p$ at node $i$.

$\hat{r}$ position in space.

$\alpha$ a positive scalar.
$\alpha_{\infty}$ constant for smoothing $w_{\infty}$.
$\alpha_{\hat{r}}$ constant for smoothing $r_{\hat{r}}$.
$\alpha_c$ continuation variable.
$\alpha_e$ ratio of $M$ to $M^e$ for equal anisotropy.
$\alpha_q$ ratio of $M$ to $M^q$ for equal anisotropy.
$\alpha_i$ multiplier in Petrov–Galerkin supplementary weighting functions.
$\alpha_{\hat{c}}^p$ multiplier in supplementary weighting functions corresponding to derivative $p$.

$\beta$ coefficient of artificial diffusion or stabilizing term.
$\beta_p$ coefficient of artificial diffusion term corresponding to derivative $p$.

$\Gamma_D$ portion of the boundary on which Dirichlet boundary conditions are applied.
$\Gamma_N$ $\partial\Omega = \Gamma_D$.
$\gamma$ ratio of first to second derivative coefficients in the one-dimensional eikonal equation.

$\Delta_S$ constant determining test space performance.
$\epsilon_k$ coefficients of error transport eigenfunctions.
\( \zeta \) multiplier in \( w_{\infty} \) to ensure that \( w_i \in H_{\Omega}^1 \).

\( \theta \) propagation speed.

\( \theta, \theta_l, \theta_n \) propagation speeds in directions \( a_l, a_e, \) and \( a_n \).

\( \lambda \) space constant.

\( \lambda_k \) error transport eigenvalue.

\( \lambda_{\xi l}, \lambda_{\xi e}, \lambda_{\xi n} \) singular values of \( M^\xi \) corresponding to eigenvectors \( a_l, a_e, \) and \( a_n \).

\( \lambda_{\xi l}, \lambda_{\xi e}, \lambda_{\xi n} \) singular values of \( M^i \) corresponding to eigenvectors \( a_l, a_e, \) and \( a_n \).

\( \lambda_{\xi l}, \lambda_{\xi e}, \lambda_{\xi n} \) space constants for directions \( a_l, a_e, \) and \( a_n \).

\( M \) effective coupling tensor.

\( M^e \) extracellular coupling tensor.

\( M^i \) intracellular coupling tensor.

\( M^\xi \) \( \xi \)-coordinate based coupling tensor.

\( \mu_{ij} \) component of \( M \).

\( \mu_{iir} \) component of \( M^\xi \).

\( \bar{\mu}^\xi \) average of the diagonal components of \( M^\xi \).

\( \nu \) function mapping element-local parameter numbers to global parameter numbers.

\( \xi \) (Chapter 2) position in space. \( \xi = x \) but \( \frac{\partial}{\partial \xi_a} \) is a derivative with \( \tau \) held constant.

\( \xi \) (Chapters 3,5,6) element-local coordinate system.

\( \xi_a \) component of \( \xi \).

\( \sigma_{ll} \) intracellular specific conductivity for the fibre direction.

\( \tau \) scalar function \( \tau(x,t) \) defined such that \( V_{\text{int}}(x,t) = v_{\text{int}}(\tau) \).

\( \tau_f \) time constant of the early exponential rise in the foot of an action potential.

\( \tau_{\text{in}} \) membrane time constant.

\( \nu \) local natural coordinate system.

\( \nu_p \) component of \( \nu \).

\( \phi_e \) extracellular potential.

\( \phi_l \) intracellular potential.

\( \phi_{\text{ПМД}} \) potential in a monodomain.

\( \phi_o \) potential outside the myocardium.

\( \varphi_e \) extracellular potential \( \varphi_e(\tau) \).
GLOSSARY OF SYMBOLS

χ  ratio of membrane surface area to volume of tissue.

Ψ  element-local basis functions.

ψi  interpolation function corresponding to parameter i.

ψi  interpolation function corresponding to derivative p at node i.

Ω  domain of interest.

∂Ω  boundary of Ω.

Ωe  domain of element e.

∂Ωe  boundary of element e.
Notation

- The \( \equiv \) symbol denotes definition.
- Repeated indices imply summation over all values of the index unless explicitly stated:
  \[ a_i \phi_i \equiv \sum_i a_i \phi_i. \]
- A following prime denotes differentiation: for \( u \coloneqq u(x) \), \( u' \equiv \frac{du}{dx} \).
  Higher derivatives are denoted by multiple following primes or by a superscript in parentheses:
  \[ u^{(4)} \equiv \frac{d^4u}{dx^4}. \]
- A subscript on a differentiation operator indicates the coordinate system in which the
  differentiation is performed: \( \nabla_\nu u \) is the gradient of \( u \) with respect to \( \nu \) coordinates.
- Angle brackets indicate an inner product over the domain \( \Omega \): \( \langle u, v \rangle \) is the inner product of \( u \) and \( v \).