University of Auckland

Doctoral Thesis

A Robin Function for Algebraic Varieties and Applications to Pluripotential Theory

By Jesse Hart

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Abstract

The Robin function associated to a compact set K captures information about the asymptotic growth of the logarithmic extremal function associated to K and has found numerous applications within pluripotential theory in $\mathbb{C}^N$. Despite its use, an analogous function for affine algebraic varieties has not been described in the literature. The work presented here shows how such a function can be constructed and, supposing only mild geometric conditions, a wealth of classical pluripotential theoretic results can be recovered on an affine algebraic variety. Moreover this thesis defines special coordinates for an algebraic variety ('Noether presentation') which are particularly suited to studying the class $\mathcal{L}^+(\mathcal{V})$.

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– J.J.E.H.

Contents

Table of Symbols

Symbol	Reference	Meaning
$\vert . \vert$		Absolute value (of a number in $\mathbb{C}$)
$\Vert . \Vert$		Euclidean norm in $\mathbb{C}^N$, $\Vert z\Vert^2 = z\overline{z}$
$\Vert f\Vert_K$, $\Vert f\Vert_{L^\infty(K)}$		Supremum norm of f on the set K
$\Vert f\Vert_\mu$, $\Vert f\Vert_{L^2(\mu)}$		L^2-norm with respect to μ, equal to $\langle f, f\rangle_\mu^{1/2}$
$\langle f, g\rangle_\mu$		Inner product of f and g, equal to $\int f\overline{g}\, d\mu$
$\langle T, \alpha\rangle$	Definition 1.55	The duality pairing between a current T and smooth form α.
$\varnothing$		The empty set
$\lightning$		Contradiction
$A \Subset B$		A is a relatively compact subset of B
$A \subset B$		A is a subset of B (with possible equality)
$int(A)$		Interior of the set A
$\overline{A}$		The closure of the set A (in $\mathbb{C}^N$)
$\overline{A}_{\mathbb{P}}$	Notation 1.132	The projective closure of the set A (in $\mathbb{P}^N$)
Atlas $\{(U_\alpha, \phi_\alpha)\}$		$\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{C}^N$, ϕ_α biholomoprhic
∂A		The boundary of the set A
B_π	Definition 1.38	Branch locus of π
$B_r(a)$		Complex ball of radius r center $a \in \mathbb{C}^N$
$\mathbb{C}[z]$		The set of polynomials in $z_1, ..., z_N$ with coefficients in $\mathbb{C}$
$\mathbb{C}[\mathcal{V}]$	Definition 1.114	$\mathbb{C}[z]/\mathbf{I}(\mathcal{V})$
∂f	Definition 1.58	$\sum_{j=1}^{N} \frac{df}{dz_j}\, dz_j$
$\bar{\partial} f$	Definition 1.58	$\sum_{j=1}^{N} \frac{df}{d\overline{z}_j}\, d\overline{z}_j$
df	Definition 1.58	Exterior derivative of f, $(\partial + \bar{\partial})f$
$d^c f$	Definition 1.58	$i(\bar{\partial} - \partial)f$
$dd^c f$	Definition 1.58	$2i\partial\bar{\partial} f$
$\delta(K)$	Definition 1.143	Transfinite diameter of K
$f\vert_K$		f with domain restricted to K
$f^+(z)$		$\max\{f(z), 0\}$
f^*	Definition 1.4	Upper semicontinuous regularisation of f
$\mathcal{L}(\Omega)$	Definition 1.80	The set of psh functions on Ω of at most logarithmic growth
$\mathcal{L}^+(\Omega)$	Definition 1.80	The set of psh functions on Ω of logarithmic growth
L^∞_{loc}		The set of locally bounded functions

Symbol	Reference	Meaning
LHS		Left hand side
$\phi^* u$		Pullback of u by ϕ, equal to $u \circ \phi^{-1}$
$\phi_* u$		Pushforward of u by ϕ, equal to $u \circ \phi$
psh	Definition 1.8	Plurisubharmonic
$PSH(\Omega)$	Definition 1.8	The set of plurisubharmonic functions on Ω
$o_\pi(z)$	Definition 1.38	Branching order of π at z
$f = O(g)$	[37] §5.1	$\lvert f(z)\rvert \leq M\lvert g(z)\rvert$ for some $M > 0$
$f \leq O(g)$	[37] §5.1	$f(z) \leq h(z)$ for some $h(z) = O(g)$ (f, h real valued)
$f = o(g)$	[37] §5.1	$\frac{f(z)}{g(z)} \to 0$, $\lvert z\rvert \to \infty$
q.e.	Definition 1.12	Quasi-everywhere
RHS		Right hand side
$\tau(K)$	Definition 1.139	Chebyshev constant for K
$\tau(K, \lambda_i)$	Definition 1.139	Chebyshev constant for K in the direction λ_i
$T(K, \alpha, \lambda_i)$	Definition 1.139	α-Chebyshev constant for K in the direction λ_i
usc	Definition 1.1	Upper semicontinuous
$u_j \downarrow u$, $u_j \uparrow u$		The sequence u_j decreases (resp. increases) to u (pointwise)
ω		Kähler form, $\frac{1}{2} dd^c \log(1 + \lVert z\rVert^2)$
$\bigwedge_{j=1}^N T_j$		$T_1 \wedge T_2 \wedge ... \wedge T_N$
$VDM_{\mathcal{B}}(\zeta_1, ..., \zeta_s)$	Definition 1.142	Vandermonde determinant over the basis $\mathcal{B}$
$[z] = [z_0 : z_1 : ... : z_N]$	Definition 1.88	A point in projective space $\mathbb{P}^N$.
$\overline{z}$		(Component wise) complex conjugate of $z \in \mathbb{C}^N$
$\mathbb{Z}^N_{\geq 0}$		$\{(a_1, ..., a_N) : a_i \in \mathbb{Z}, a_i \geq 0\}$

Introduction

Affine algebraic varieties have largely been a sideline to the main study of pluripotential theory. Investigations are usually conducted in either the generality of a complex manifold or the concreteness of $\mathbb{C}^N$. Algebraic varieties have traditionally been seen as a special case of pluripotential theory on complex manifolds, since all smooth algebraic varieties are complex manifolds, and the literature reflects this point of view. The main results of this thesis exploit the fact that affine algebraic varieties are algebraic subvarieties of $\mathbb{C}^N$ and are relatively finite over $\mathbb{C}^M$ to show that a pluripotential theory can be developed on varieties which closely resembles the $\mathbb{C}^N$ case. From this point of view, algebraic varieties can be seen as an intermediate situation between pluripotential theory on $\mathbb{C}^N$ and pluripotential theory on complex manifolds.

Brief history of pluripotential theory on algebraic varieties

The study of pluripotential theory on algebraic varieties can broadly be split into two time periods; 1983-1995 and contemporary. One of the first major papers in 1983 was due to A. Sadullaev [50] which showed that if the logarithmic extremal function (with pole at infinity) associated to a compact subset K of an analytic variety $\mathcal{V}$ is finite at each point then $\mathcal{V}$ is algebraic. In particular, a logarithmic extremal function for an analytic variety which is not algebraic must have some points where it is not finite. This distinguished algebraic varieties from analytic varieties as spaces where classical pluripotential notions were likely to carry over.

The main thrust of technical development for pluripotential theory on algebraic varieties was due to Demailly and later Zeriahi. Demailly's point of view can be largely summarised as being motivated by the complex geometric consequences of pluripotential theoretic notions. For instance, the geometric concept of intersection theory is studied via the pluripotential theoretic notion of the Lelong number. His study culminated in the books 'Complex analytic and differential geometry' [25] and 'Applications of pluripotential theory to algebraic geometry' [26] – both have a distinctly differential geometric flavour to them. As far as we are aware, there is little overlap beyond basic definitions between his work and this thesis.

Zeriahi's contribution established a number of classical pluripotential theoretic results in the setting of a Stein space with parabolic potentials – of which algebraic varieties are an important example. Between his PhD thesis [55] and '91 [56] & '96 papers [57] he developed tools to show that many aspects of classical pluripotential theory (i.e. the ideas presented in Klimek [37]) could be recovered on an algebraic variety.

Contemporary results usually take the form of treating algebraic varieties as a special case of complex manifolds. There are a few exceptions. Firstly, Dihn-Sibony [28] have done work invoking pluripotential theory to study complex dynamics on algebraic varieties. Bloom-Levenberg

[14] studied the distribution of nodes on an algebraic curve for polynomial interpolation and as a special case obtained the convergence of an equilibrium measure in this setting. Finally, Coman-Guedj-Zeriahi [21] have investigated extensions of functions in the Lelong class on an algebraic variety to functions in the Lelong class on $\mathbb{C}^N$. We note that we don't use their results in this thesis, but it would be interesting to see if the ideas here can be streamlined using their theory.

We must also mention the pioneering work of Berman-Boucksom [6–8] who developed deep machinery to study the equidistribution of Fekete points on complex manifolds. While not specifically on algebraic varieties, the impact of their work on pluripotential theory is significant. We will use some of their results in Section 3.5.

The other studies conducted on algebraic varieties are those done by Ma'u. Ma'u's work focuses on studying the transfinite diameter and Chebyshev constants on an affine algebraic variety with emphasis placed on computation using the underlying geometry. These constants were first defined on an algebraic curve in $\mathbb{C}^2$ in [44]. Working with Baleikorocau, the extension to algebraic curves in $\mathbb{C}^N$ was established in [2]. The generalisation of these concepts to an algebraic variety was work done in conjunction with Cox in [23]. Development of generalisations of Robin constants to affine algebraic curves were considered in conjunction with the present author in [33].

Motivation

Our aim was to define and study a Robin function for affine algebraic varieties in $\mathbb{C}^N$ along classical lines. The results in [23, 33] suggest that with the right geometric assumptions that a 'classical' generalisation is possible. With this in mind, our main goal became generalising the results from the paper by Bloom-Levenberg [15] to an affine algebraic variety. Their main results make correspondences between the transfinite diameter and Chebyshev constants in different settings and make extensive use of weighted potential theory and the Robin function to do so.

The work of Berman-Boucksom [6–8] gave rise to a 'Robin function' realised as the restriction of a positive metric. It is not a priori given that the Cox-Ma'u formulation of the transfinite diameter is compatible with the Berman-Boucksom approach. Without knowing this, a 'classical' approach to the problem of defining the Robin function seemed like a more direct way of generalising the results in [15] hence our decision to pursue this route. In Section 3.5 we show precisely how the Cox-Ma'u formulation of a transfinite diameter is compatible. It would hence be interesting to see the results of this thesis recast in the formulation of Berman-Boucksom.

Overview of this thesis

In Chapter 1 we recall basic facts in the areas of pluripotential theory, algebraic geometry and the notion of Chebyshev constants and transfinite diameter on an algebraic variety (from the point of view of Cox-Ma'u [23]) which will be used in subsequent sections. Within this section we develop a higher dimensional analogue of a branch cut (Definition 1.40). This is a set $C \subset \mathbb{C}^M$ containing the branching set B_π of an affine algebraic variety $\mathcal{V} \subset \mathbb{C}^N$ of dimension M with respect to a projection π for which $\mathbb{C}^M \backslash C$ is simply connected. Theorem 1.44 shows that C can be chosen to be a real $2M-1$ dimensional set and an explicit construction is given by utilising classical branch cuts along 1 dimensional slices. The main use of this result is that it allows us to define a biholomorphic projection $\pi_i : V_i \to \mathbb{C}^M \backslash C$ from the ith branch of $\mathcal{V}$ to $\mathbb{C}^M \backslash C$.

Our main geometric condition used throughout the thesis is the notion of an algebraic variety having distinct intersections with infinity, introduced in Definition 1.133. A variety $\mathcal{V}$ has distinct intersections with infinity if

(i) $\mathbb{C}[z_1, ..., z_M] \subset \mathbb{C}[\mathcal{V}]$ is a Noether normalisation for $\mathcal{V}$.

(ii) Let $P = \{\mathbf{V}(\{z_0, ..., z_{M-1}\})\} \subset \mathbb{P}^M$. The set $\overline{\mathcal{V}}_{\mathbb{P}} \cap P$ consists of d distinct points.*

(iii) Let $\overline{\mathcal{V}}_{\mathbb{P}} \cap P = \{\lambda_1, ..., \lambda_d\}$ with $\lambda_i = [0 : ... : \lambda_{iM} : ... : \lambda_{iN}]$. Then for each i, $\lambda_{iM} \neq 0$.

We verify in Theorem 2.51 that a geometric interpretation of this condition is that the sheets of the variety intersect the hyperplane at infinity in a way which preserves the number of sheets i.e. that the homogeneous variety $\mathcal{V}^h$ has the same number of sheets as $\mathcal{V}$.

In Chapter 2 we study pluripotential theory on an affine algebraic variety with the aim of generalising a result of Bedford-Taylor. A key observation is that it is convenient to do calculations using coordinates $z = (x, y), x \in \mathbb{C}^M, y \in \mathbb{C}^{N-M}$ where M is the dimension of $\mathcal{V}$ which satisfy the following two properties:

(i) $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$.

(ii) For all $(x, y) \in \mathcal{V}$ we have the growth estimate $\|y\| \leq A(1 + \|x\|)$ for some $A > 0$.

We call coordinates satisfying these properties a 'Noether presentation' for $\mathcal{V}$. Theorem 2.21 gives an algorithm to compute a Noether presentation given arbitrary coordinates, consequently such coordinates can always be obtained via a linear change of coordinates. As an immediate application of these coordinates we show that

Theorem 0.1 (Theorem 2.27). *Suppose that $\mathcal{V}$ is an affine smooth algebraic variety with a Noether presentation (x, y). Let $u \in \mathcal{L}^+(\mathcal{V})$. Then*

$$\int_{\mathcal{V}} (dd^c u)^M = d(2\pi)^M,$$

*$\overline{\mathcal{V}_{\mathbb{P}}}$ is the projective closure of $\mathcal{V}$, see the notation table on page vi.

where d is the number of branches of $\mathcal{V}$ (over $\mathbb{C}^M$).

For an affine algebraic variety $\mathcal{V}$ with Noether presentation (x, y) which has distinct intersections with infinity, given $u \in \mathcal{L}^+$ we define the Robin function ρ_u pointwise in the following way.

Definition 0.2 (Definition 2.52)**.** *Given a point $(x, \tilde{y}_i(x)) \in \mathcal{V}^{h,reg} \backslash B_\pi^h$ we can choose a suitable branch cut C such that*

$$\pi_i^{-1}(x) = (x, y_i(x)),$$
$$\lim_{\substack{(t,x)\to(0,x)\\ x/t\in\mathbb{C}^M\backslash C}} y_i(x/t) = \tilde{y}_i(x),$$

i.e. we can find a path of points in $(x/t, y_i(x/t)) \in \mathcal{V}$ tending to $(x, \tilde{y}_i(x)) \in \mathcal{V}^h$ as $t \to 0$. Then we define the value of the Robin function ρ_u at $(x, \tilde{y}_i(x))$ to be

$$\rho_u(x, \tilde{y}(x)) = \limsup_{\substack{t\to 0\\ x/t\in\mathbb{C}^M\backslash C}} u(x/t, y(x/t)) + \log|t|.$$

This defines the Robin function pointwise on $\mathcal{V}^h \backslash B_\pi^h$. We extend the Robin function to be defined everywhere on $\mathcal{V}^h$ by

$$\rho_u(x, y) = \limsup_{\mathcal{V}^{h,reg}\backslash B_\pi^h \ni (\zeta,\eta)\to(x,y)} \rho_u(\zeta, \eta).$$

This is a direct generalisation of the $\mathbb{C}^N$ formulation of the Robin function. Throughout the rest of Chapter 2 and Chapter 3 we show that our Robin function satisfies generalised versions of important classical results.

The first of these is a Bedford-Taylor formula.

Theorem 0.3 (Theorem 2.67)**.** *Let $\mathcal{V}$ be a smooth irreducible algebraic variety with Noether presentation (x, y) which has distinct intersections at infinity. Let $u, v, w_2, ..., w_M \in \mathcal{L}^+(\mathcal{V})$ Then*

$$\int_{\mathcal{V}} (u\, dd^c v - v\, dd^c u) \wedge T = 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_u^* - \tilde{\rho}_v^*) \wedge \tilde{T}$$

where $T = dd^c w_2 \wedge ... \wedge dd^c w_M$, $\tilde{\rho}_u^*$ is usc regularisation of the projective Robin function, and $\tilde{T} = (dd^c \tilde{\rho}_{w_2}^* + \omega) \wedge ... \wedge (dd^c \tilde{\rho}_{w_M}^* + \omega)$, where ω is the usual Kähler form (see Notation, page viii).

Our method of proof relies on using the fact that the projection is locally biholomoprhic to utilise the $\mathbb{C}^M$ version of the Bedford-Taylor formula to establish the formula on branches of $\mathcal{V}$, and then piecing everything together. Many consequences of this formula are derived in Sections 3.1-3.3.

In Section 3.4 we generalise results found in Hart-Ma'u [33] to an algebraic variety. Precisely, in Theorem 3.33 we show that for a compact, nonpluripolar, regular set K the projective capacity $\kappa(K,\zeta)$ in the direction $\zeta \in \mathcal{V}^h$ is equal to $e^{-\Psi_K(\zeta)} = e^{-\rho_K(\zeta)}$. This in turn generalises the well known equality between the classical projective capacity and the classical Robin function. In the case where $\mathcal{V}$ is an algebraic curve the directional projective capacity coincides with the directional Chebyshev constants and we obtain $e^{-\rho_K(\lambda_j)} = \tau(K,\lambda_j)$.

In Section 3.5 we establish a Rumely type formula relating the transfinite diameter to our Robin function. We first show that the Cox-Ma'u transfinite diameter is compatible with the Berman-Boucksom theory. The difference between these formulations is that the Cox-Ma'u formulation utilises the reduced monomials which span $\mathbb{C}[\mathcal{V}]$ to form the Vandermonde determinants, while the Berman-Boucksom theory utilises an $L^2(\mu)$-orthonormal basis (with respect to a probability measure μ) to form the Vandermonde determinants. The key to relating these is using Gram-Schmidt with respect to the probability measure $\mu = \frac{1}{d(2\pi)^M}(dd^cV_{T_\mathcal{V}})^M$ where $T_\mathcal{V} = \{|x_j| \leq 1\} \cap \mathcal{V}$. These calculations are done in Propositions 3.45 and 3.46. As a consequence we obtain

$$\lim_{k\to\infty} \frac{1}{kN_k} \log \| \det[D_k] \|_{L^\infty(K)} = \lim_{k\to\infty} \frac{1}{kN_k} \log \| \det[S_k] \|_{L^\infty(K)}$$

where D_k are the elements of $\mathbb{C}[\mathcal{V}]$ of degree at most k and S_k the elements of the μ-orthonormal basis of degree at most k. From this, we clarify the relationship between the Cox-Ma'u formulation and the Berman-Boucksom theory when $\mu = (dd^cV_{T_\mathcal{V}})^M$ in the following result

Theorem 0.4 (Theorem 3.51.). *Let $K \subset \mathcal{V}$ be a compact set, $\mu = (dd^cT_\mathcal{V})^M$ and S_k be an $L^2(\mu)$-orthonormal basis. Then*

$$\log(K) = \frac{M+1}{M} \lim_{k\to\infty} \frac{1}{kN_k} \log \| \det[S_k] \|_{L^\infty(K)}.$$

Where $\log(K)$ is the Cox-Ma'u transfinite diameter and $\lim_{k\to\infty} \frac{1}{kN_k} \log \| \det[S_k] \|_{L^\infty(K)}$ is the Berman-Boucksom transfinite diameter.

With this key technical result we are able to prove the following Rumely type formula by invoking the Berman-Boucksom theory.

Theorem 0.5 (Theorem 3.53). *Suppose that $K \subset \mathcal{V}$ is compact, regular and $K = \{V_K(z) \leq 0\}$.*

Then

$$-\log\delta(K) = \frac{1}{dM}\left[\frac{1}{(2\pi)^{M-1}}\int_{\mathcal{V}^h}\rho_K(1,z_2,...,z_N)(dd^c\rho_K(1,x_2,...,x_M,y))^{M-1}\right.$$
$$+\frac{1}{(2\pi)^{M-2}}\int_{\mathcal{V}^h}\rho_K(0,1,x_3,...,x_M,y)(dd^c\rho_K(0,1,x_3,...,x_M,y))^{M-2}$$
$$+...+\frac{1}{2\pi}\int_{\mathcal{V}^h}\rho_K(0,...,0,1,x_N,y)(dd^c\rho_K(0,...,0,1,x_N,y))$$
$$\left.+\sum_{j=1}^{d}\rho_K(0,...,0,1,y_i)\right].$$

Equivalently, we have an "energy version" of this formula

$$-\log\delta(K) = \frac{1}{dM(2\pi)^{M-1}}\int_{\mathcal{V}^h}[\tilde{\rho}_K - \tilde{\rho}_{T_\mathcal{V}}]\sum_{j=0}^{N-1}(dd^c\tilde{\rho}_K+\omega)^j\wedge(dd^c\tilde{\rho}_{T_\mathcal{V}}+\omega)^{M-j-1}.$$

Chapter 4 is dedicated to exploring weighted pluripotential theory on algebraic varieties and makes extensive use of the technical results derived in Chapter 3. One of the key ideas used in this section is that of relating the *weighted* pluripotential theory on $K\subset\mathcal{V}$ with the *unweighted* potential theory of $K^w_\uparrow\subset V_\uparrow$ where $K^w_\uparrow := \{(t,t\zeta)\in V_\uparrow : |t| = w(\zeta), \zeta\in K\}$ and $\mathcal{V}_\uparrow = \{(t,tz) : z\in\mathcal{V}, t\in\mathbb{C}\}$. The weighted $\mathcal{H}$-principle (Theorem 4.19) provides the maps to relate the potential theories.

Theorem 4.44 uses this connection to show that the limit of Vandermonde determinants defining the weighted transfinite diameter $\delta^w(K)$ exists by showing that it is related to the homogeneous transfinite diameter by $\delta^w(K) = d^H(K^w_\uparrow)^{(M+1)/M}$.

We extend this idea in Section 4.5 by relating the weighted potential theory of $K\subset\mathcal{V}$ to the unweighted potential theory on $K^w_\rho\subset\mathcal{V}^h$ where $K^w_\rho := \{z\in V^h : \rho_Q(z)\le 0\}$. This is an extension of an idea of Bloom-Levenberg [15] where in their case $K\subset\mathbb{C}^N$ and $K^w_\rho\subset\mathbb{C}^N$ i.e. the underlying space does not change. The changing of the underlying space from $\mathcal{V}$ to $\mathcal{V}^h$ in our work introduces a minor technicality. We deduce the following 'K^w_ρ lemmas' in this section.

(i) $\log\delta^w(K) = \log\delta(K^w_\rho) - \frac{1}{dM(2\pi)^M}\int_K Q(dd^cV_{K,Q})^M$. (Lemma 4.46).

(ii) $d^w(K) = d(K^w_\rho)$ (Lemma 4.51).

(iii) $d^w(K) = \delta^w(K)\exp\left(\frac{1}{dM(2\pi)^M}\int_K Q(dd^cV_{K,Q})^M\right)$ (Corollary 4.53).

(iv) $\tau^w(K,\theta,\lambda) = \tau(K^w_\rho,\theta,\lambda)$ (Lemma 4.55).

Using the K^w_ρ lemmas in conjunction with the technique of turning weighted problems into unweighted problems allows us to derive the convergence of Fekete polynomials to the extremal function.

Theorem 0.6 (Theorem 4.58). *Let $K \subset \mathcal{V}$ be compact, regular and polynomially convex. Let w be a continuous admissible weight function on K. For each i, let $\{p_{j,i}\}_{j\in\mathbb{N}}$ be a sequence of polynomials such that for all θ there exists a subsequence $Y_{\theta,i} \in \mathbb{Z}_{\geq 0}$ with $p_{j,i} \in \{p : \textsc{lt}(p) = \boldsymbol{v}_{\lambda_i} x^{\alpha_j}\}$, $j \in Y_{\theta,i}$ and*

$$\lim_{j\in Y_{\theta,i}} \|w^{\deg p_{j,i}} p_{j,i}\|_K^{1/\deg p_{j,i}} = \tau^w(K,\theta,\lambda_i).$$

Then for all $z \notin K$,

$$\max_{1\leq i\leq d}\left[\limsup_{j\to\infty} \frac{1}{\deg p_{j,i}} \log \frac{|p_{j,i}(z)|}{\|w^{\deg p_{j,i}} p_{j,i}\|_K}\right]^* = V_{K,Q}(z).$$

Where we are treating $\boldsymbol{v}_{\lambda,i}x^\alpha$ as a monomial as in Lemma 1.134 and Definition 1.138.

In Chapter 5 we pose some unanswered questions. Notably, in Section 5.3 we consider the following extension problem. The work of Coman-Geudj-Zeirahi [21] guarantees an extension of functions with logarithmic growth on an affine algebraic variety to all of $\mathbb{C}^N$ under certain circumstances, however an explicit extension is not given in their work. By utilising a generalisation of the theory of uniform algebra (i.e. the restriction of a uniform algebra to a subspace) we give some results to partially answer this question.

1 Preliminaries

1.1 Pluripotential Theory

There are many exceptional resources explaining classical potential theory. The classic resource is Klimek [37] but many of those in the field have notes which cover the concepts important to the modern study of the subject. See Błocki [9], Levenberg [42], [41] or Bracci-Trapani [17] for such modern renditions of the classical material. For pluripotential theory on complex manifolds Demailly's book [25] is the main reference, however his set of notes [24] is more than sufficient for what we will cover and is more accessible.

Our main focus here is to establish definitions and key results that will be used in subsequent sections. Proofs will largely be omitted in favour of references.

1.1.1 Plurisubharmonic Functions

Definition 1.1 (§2.3, [37]). *Let X be a metric space and $u : X \to [-\infty, \infty)$ a function. We say u is upper semicontinuous (usc) if for each $c \in \mathbb{R}$ the set $\{x \in X : u(x) < c\}$ is open. A function u is lower semicontinuous (lsc) if $-u$ is usc.*

Lemma 1.2 (§2.3, [37]). *Let X be a metric space with norm $|.|$ and $u : X \to [-\infty, \infty)$ a function. u is usc if and only if for each $a \in X$,*

$$\limsup_{x \to a} u(x) := \inf_{\varepsilon > 0}(\sup\{u(y) : y \in |y - a| \leq \varepsilon\}) = u(a).$$

Lemma 1.3 (Lemma 2.3.2, [37]). *Suppose that $\{u_\alpha\}_{\alpha \in A}$ is a collection of usc functions uniformly bounded above. The pointwise infimum $v(z) = \inf_{\alpha \in A} u_\alpha(z)$ is usc and A can be replaced by a countable subset B.*

Definition 1.4 (§2.3, [37]). *Let X be a metric space and $U \subset X$. Suppose $u : U \to [-\infty, \infty)$ is a function which is locally bounded above near each point of $\overline{U}$. Then the upper semicontinuous regularisation u^* of u is defined by the formula*

$$u^*(x) = \limsup_{\substack{y \to x \in \overline{X} \\ y \in U}} u(y).$$

Moreover, $u^* : \overline{U} \to [-\infty, \infty)$ is upper semicontinuous with $u^* \geq u$ in U.

Lemma 1.5 (Choquet's Lemma, Lemma 2.3.4, [37]). *Suppose that $\{u_\alpha\}_{\alpha \in A}$ is a collection of real valued functions locally bounded above. Then there exists a countable subset B of A such that*

$$\left(\sup_{\alpha \in A} u_\alpha\right)^* = \left(\sup_{\beta \in B} u_\beta\right)^*.$$

Definition 1.6 (Definition 1.2.3, [9]). *Let $\Omega \subset \mathbb{R}^N$ be an open set. An usc function $u : \Omega \to [-\infty, \infty)$ is called subharmonic if $u \not\equiv -\infty$ on any connected component of Ω and for every ball $B_r = B_r(x_0) \Subset \Omega$*

$$u(x_0) \leq \frac{1}{\sigma(\partial B_r)} \int_{\partial B_r} u(x)\, d\sigma$$

where σ is 'surface area' measure of the ball. (i.e. the spherical measure on the N sphere)

Theorem 1.7 (Hartogs' Lemma, Theorem 2.6.4, [37]). *Suppose that $\Omega \subset \mathbb{R}^N$ is an open set. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of subharmonic functions on Ω uniformly bounded above in Ω. Suppose that*

$$\limsup_{j\to\infty} u_j(x) \leq M$$

for each $x \in \Omega$ and some constant M. Then, for each $\varepsilon > 0$ and each compact set $K \subset \Omega$, there exists a natural number j_0 such that for $j \geq j_0$

$$\sup_{x\in K} u_j(x) \leq M + \varepsilon.$$

Definition 1.8 (§2.9, [37]). *Let $\Omega \subset \mathbb{C}^N$ be an open set and $u : \Omega \to [-\infty, \infty)$ a usc function which is not identically $-\infty$ on any connected component of Ω. We say u is plurisubharmonic (psh) if for each $a \in \Omega$ and $b \in \mathbb{C}^N$, the function $\lambda \mapsto u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in \Omega\}$. In this case we write $u \in PSH(\Omega)$.*

Theorem 1.9 (Theorem 2.9.1, [37]). *Plurisubharmonicity is a local property. That is, if $\Omega \subset \mathbb{C}^N$ is an open set then a function $u : \Omega \to [-\infty, \infty)$ is psh on Ω if and only if it is psh in a neighbourhood of each point of Ω.*

Definition 1.10 (Corollary 2.9.10, [37]). *A set $E \subset \mathbb{C}^N$ is pluripolar if for each point $a \in E$ there is a neighbourhood $\mathcal{N}$ of a and a function $u \in PSH(\mathcal{N})$ such that $E \cap \mathcal{N} \subset \{z \in \mathcal{N} : u(z) = -\infty\}$.*

Definition 1.11 (Theorem 0.5, [24]). *We say a set $E \subset \mathbb{C}^N$ is complete pluripolar if there exists an open covering $\{\Omega_j\}$ of E and psh functions $u_j \in PSH(\Omega_j)$ with $E \cap \Omega_j = u_j^{-1}(\{-\infty\})$.*

Definition 1.12. *We say a property P holds on a set $\Omega \subset \mathbb{C}^N$ quasi-everywhere (q.e.) if it holds on $\Omega \backslash E$ where E is a pluripolar set.*

Proposition 1.13 (Corollary 2.9.5, [37]). *Let Ω and Ω' be open sets in $\mathbb{C}^N$ and $\mathbb{C}^M$ respectively. If $u \in PSH(\Omega)$ and $f : \Omega' \to \Omega$ is a holomorphic mapping, then $u \circ f$ is plurisubharmonic in Ω'.*

Proposition 1.14 (Theorem 2.9.14, [37]). *Let $\Omega \subset \mathbb{C}^N$ be open.*

(i) The family $PSH(\Omega)$ is a convex cone, i.e. if α, β are non-negative real numbers and $u, v \in PSH(\Omega)$ then $\alpha u + \beta v \in PSH(\Omega)$.

(ii) If Ω is connected and $u_j \subset PSH(\Omega), j \in \mathbb{N}$ is a decreasing sequence, then $u = \lim_{j\to\infty} u_j \in PSH(\Omega)$ or $u \equiv -\infty$.

(iii) If $u : \Omega \to \mathbb{R}$ and if $\{u_j\}_{j\in\mathbb{N}} \in PSH(\Omega)$ converges to u uniformly on compact subsets of Ω then $u \in PSH(\Omega)$.

Proposition 1.15 (Proposition 2.9.17, [37])**.** *Let $\Omega \subset \mathbb{C}^N$ be an open set. Suppose that $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of psh functions on Ω which are locally uniformly bounded above. Define $u(z) = \limsup_{j\to\infty} u_j(z)$ for $z \in \Omega$. Then u^* is psh.*

Proposition 1.16 (Corollary 2.9.8, [37])**.** *Let $\Omega \subset \mathbb{C}^N$ be an open set. If $u, v \in PSH(\Omega)$ and $u = v$ almost everywhere in Ω then $u \equiv v$.*

Proposition 1.17 (Corollary 2.9.15, [37])**.** *Let $\Omega \subset \mathbb{C}^N$ be an open set and let ω be a non-empty proper open subset of Ω. If $u \in PSH(\Omega)$, $v \in PSH(\omega)$ and $\limsup_{x\to y} v(x) \le u(y)$ for each $y \in \partial\omega \cap \Omega$ then the formula*

$$w = \begin{cases} \max\{u, v\} & \text{in } \omega \\ u & \text{in } \Omega\backslash\omega \end{cases}$$

defines a psh function in Ω.

Definition 1.18. *We say $u \in PSH \cap L^\infty_{loc}(\mathbb{C}^N)$ is maximal on a nonpluripolar compact set E if $\int_E (dd^c u)^N = 0$.*

Lemma 1.19 (Lemma 3.7.5, [37])**.** *If $u \in PSH \cap L^\infty_{loc}(\mathbb{C}^N)$ is maximal on a nonpluripolar compact set E then any $v \in PSH(\mathbb{C}^N)$ with $v \le u$ on ∂E satisfies $v \le u$ on E.*

Definition 1.20. *Suppose that X is a complex manifold and $u : X \to [-\infty, \infty)$ is a function. We say $u \in PSH(X)$ if for any atlas $\{(U_\alpha, \phi_\alpha)\}$ the function $\phi^*_\alpha u = u \circ \phi^{-1}_\alpha \in PSH(\phi_\alpha(U_\alpha))$ for every α (note $\phi_\alpha(U_\alpha) \subset \mathbb{C}^N$).*

Definition 1.21. *Let $\Omega \subset \mathbb{C}^N$ be an open set. We say that a psh function u is strongly psh on Ω if for every open $U \Subset \Omega$ there exists $\lambda \in [0, \infty)$ such that $u(z) - \lambda|z|^2$ is psh in U. If instead $\Omega \subset X$ is an open set where X is a complex manifold, then we say that u is strongly psh if on every chart (U_α, ϕ_α) the function $\phi^*_\alpha u$ is strongly psh on $\phi_\alpha(U_\alpha)$.*

Remark 1.22. Definition 1.20 and Proposition 1.13 allows us to prove any local result from $\mathbb{C}^N$ pluripotential theory on a complex manifold. We will encounter this idea frequently in Sections 2.1-2.2. For instance, Proposition 1.17 is a local result so the result for Ω' in a complex manifold holds by Proposition 1.13.

1.1.2 Plurisubharmonic Functions on Algebraic Varieties

Definition 1.23. *Suppose that $\{p_i(z) : i \in A\}$ is a collection of holomorphic polynomials in N complex variables with coefficients in $\mathbb{C}$. Then we say that their common zero set is an affine* *algebraic variety i.e.*

$$\mathcal{V} = \{z \in \mathbb{C}^N : p_i(z) = 0, i \in A\}.$$

The collection $\{p_i(z) : i \in A\}$ are called defining polynomials for $\mathcal{V}$.

Recall the Hilbert basis theorem which says that there is always a finite collection of polynomials that define an algebraic variety $\mathcal{V}$. Our usual situation will consider algebraic varieties formed from a finite collection of polynomials.

Definition 1.24. *We say an algebraic variety $\mathcal{V}$ is reducible if it can be written as the union of two non-empty proper subvarieties. Otherwise, we say $\mathcal{V}$ is irreducible.*

Definition 1.25. *We define the dimension of $\mathcal{V}$ at ζ (denoted $dim_\zeta(\mathcal{V})$) to be the nullity* of the Jacobian of the defining polynomials at ζ. In other words, $Jac_\zeta(p_1, ..., p_{N-M})$ has rank $N - dim_\zeta(\mathcal{V})$ where*

$$Jac_\zeta(p_1, ..., p_{N-M}) = \begin{pmatrix} \frac{\partial p_1}{\partial z_1}(\zeta) & ... & \frac{\partial p_1}{\partial z_N}(\zeta) \\ \vdots & \ddots & \vdots \\ \frac{\partial p_{N-M}}{\partial z_1}(\zeta) & ... & \frac{\partial p_{N-M}}{\partial z_N}(\zeta) \end{pmatrix}.$$

We say the dimension of $\mathcal{V}$ is $dim(\mathcal{V}) = \min\{dim_\zeta(\mathcal{V}) : \zeta \in \mathcal{V}\}$.

Example 1.26. Consider the algebraic variety $\mathcal{V} = \{(x, y) \in \mathbb{C}^2 : xy = 0\}$. Then

$$Jac(xy) = \begin{pmatrix} y & x \end{pmatrix}.$$

There are three kinds of points on $\mathcal{V}$, $\zeta_1 = (\zeta_1', 0), \zeta_1' \neq 0$, $\zeta_2 = (0, \zeta_2'), \zeta_2' \neq 0$ and $\zeta_3 = (0, 0)$. We calculate

$$\begin{aligned} Nullity(Jac_{\zeta_1}) &= Nullity \begin{pmatrix} 0 & \zeta_1' \end{pmatrix} = 1, \\ Nullity(Jac_{\zeta_2}) &= Nullity \begin{pmatrix} \zeta_2' & 0 \end{pmatrix} = 1, Nullity(Jac_{\zeta_3}) &= Nullity \begin{pmatrix} 0 & 0 \end{pmatrix} = 2. \end{aligned}$$

Hence $dim(\mathcal{V}) = 1$.

Definition 1.27. *If $dim_\zeta(\mathcal{V}) = dim(\mathcal{V})$ then we say ζ is a regular point of $\mathcal{V}$. The set of all regular points on $\mathcal{V}$ defines the regular part of $\mathcal{V}$ which we denote $\mathcal{V}^{reg}$. If $dim_\zeta(\mathcal{V}) > dim(\mathcal{V})$ then we say that ζ is a singular point of $\mathcal{V}$. The set of all singular points on $\mathcal{V}$ defines the singular part of $\mathcal{V}$ which we denote $\mathcal{V}^{sing}$. Evidently, $\mathcal{V} \backslash \mathcal{V}^{sing} = \mathcal{V}^{reg}$. If $\mathcal{V}^{reg} = \mathcal{V}$ then we say that $\mathcal{V}$ is smooth.*

The following is a consequence of the implicit function theorem.

*Dimension of the null space

Lemma 1.28. *$\mathcal{V}^{reg}$ is a complex manifold.*

Definition 1.29. *We say $u \in PSH(\mathcal{V})$ if for every point $z \in \mathcal{V}$ there exists a neighbourhood $\Omega(z) \subset \mathbb{C}^N$ and a function $v \in PSH(\Omega(z))$ with $v|_{\mathcal{V}} = u$.*

Definition 1.30. *We say that u is weakly psh (denoted $wPSH$) if for all $z \in \mathcal{V}^{reg}$ there exists a neighbourhood $\Omega(z) \subset \mathbb{C}^N$ and a function $v \in PSH(\Omega(z))$ with $v|_{\mathcal{V}} = u$. Moreover for $z \in \mathcal{V}^{sing}$ we require that u satisfies $\limsup_{\zeta \to z} u(\zeta) = u(z)$.*

Theorem 1.31 (Fornæss-Narasimhan, Theorem 5.3.1, [29])**.** *When $\mathcal{V}$ is smooth, definitions 1.20 and 1.29 are equivalent.*

Definition 1.32 (Definition C.1, [32])**.** *A mapping $\pi : V \to W$ between two second-countable Hausdorff spaces is a finite branched covering if*

(i) π is a continuous finite proper surjective mapping.

(ii) There are dense open subsets $V_0 \subset V$, $W_0 \subset W$ such that $V_0 = \pi^{-1}(W_0)$ and the restriction $\pi|_{V_0} : V_0 \to W_0$ is a covering mapping. That is, for any $x \in W_0$ and $\pi^{-1}(x) = \{z_j\} \subset V_0$ then there is an open neighbourhood W_x of x in W_0 and disjoint open neighbourhoods V_j (called sheets) of the distinct points z_j in V_0 such that $\pi^{-1}(W_x) = \cup_j V_j$ and each restriction $\pi|_{V_j} : V_j \to W_x$ a homeomorphism.

We call the space W the base space and the space V the covering space in this instance. The restriction $\pi|_{V_0} : V_0 \to W_0$ will be called the generic part of the finite branched covering π. Points in W_0 will be called generic points.[†]

Definition 1.33. *We say a function $f : A \to B$ is locally biholomorphic if for each point $x \in A$ there is an open neighbourhood U of x such that $f : U \to f(U)$ is a biholomorphic function (i.e. holomorphic with holomorphic inverse).*

Definition 1.34 (Definition C.3, [32])**.** *A mapping $\pi : V \to W$ between two algebraic varieties is a finite branched holomorphic covering if*

(i) π is a finite branched covering.

(ii) There is a generic part $\pi|_{V_0} : V_0 \to W_0$ (where V_0 and W_0 are as in Definition 1.32) of this finite branched covering for which $W - W_0$ is a algebraic subvariety of W and $\pi|_{V_0}$ is a locally biholomorphic mapping.

Theorem 1.35 (Theorem D.1, [32])**.** *For any smooth algebraic variety of dimension M, the projection $\pi : \mathcal{V} \to \mathbb{C}^M$ is a finite branched holomorphic covering.*

Definition 1.36 (Lemma C.3, [32])**.** *Using the notation of Definition 1.34, the order of a branched covering $\pi : V \to W$ is the maximum number of sheets over W_0. The order will be denoted o_π.*

[†]Note that Gunning calls this part the regular part of the finite branched covering. Due to regular meaning non-singular for algebraic varieties in our case we avoid this terminology.

Lemma 1.37 (Lemma C.3, [32]). *Using the notation of Definition 1.34, for each $z \in V$ there is an arbitrarily small connected open neighbourhood $V_z \subset V$ such that the restriction $\pi|_{V_z} : V_z \to \pi(V_z)$ is a finite branched covering. The order $o_{(\pi|V_z)}$ is a positive integer and decreases as V_z shrinks and is eventually constant for all sufficiently small neighbourhoods V_z.*

Definition 1.38 (Lemma C.3, [32]). *The common order from the conclusion of Lemma 1.37 will be called the branching order of the mapping π at the point z and denoted $o_\pi(z)$. For any point $z \in V_0$ we have $o_\pi(z) = 1$. The set*

$$B_\pi = \{z \in V : o_\pi(z) > 1\}$$

will be called the branch locus of the mapping π.

Lemma 1.39 (Lemma C.14, [32]). *If $\pi : V \to W$ is a finite branched homomorphic covering then B_π is a proper holomorphic subvariety of V and $\pi(B_\pi)$ is a proper holomorphic subvariety of W.*

1.1.3 Branch Cuts

Recall that an algebraic variety V of dimension M is a finite branched holomorphic covering over $\mathbb{C}^M$ with projection map $\pi : (x, y) \in V \setminus W \mapsto x \in \mathbb{C}^M$ locally biholomorphic where $V \setminus W$ is a dense open set in V. Our goal in this section is to show that we can choose W so that the projection restricted to each sheet of $V \setminus W$ is biholomorphic. This is a multi-dimensional analogue of the notion of branch cuts in complex analysis of one variable. While a number of related notions are discussed in the literature we are unsure if branch cuts have been explicitly constructed.

Definition 1.40. *Let V be an algebraic variety with projection π. A branch cut $C \subset \mathbb{C}^M$ is a set satisfying*

(i) $\mathbb{C}^M \setminus C$ *is simply connected;*

(ii) $\mathbb{C}^M \setminus C$ *is dense in* $\mathbb{C}^M$*.*

In this case we say that C is a branch cut of V over $\mathbb{C}^M$ for π.

We will often say that C is a branch cut for V, in this context the base space $\mathbb{C}^M$ and projection π are implied.

Theorem 1.41 (Monodromy Theorem, 1.6 [53]). *Let $U \subset \mathbb{C}^N$. Suppose that a function $f : U \to \mathbb{C}$ is holomorphic in some neighbourhood of a point z_0 and is analytically continued outside this neighbourhood along every path lying entirely in some domain G. Then, the result of continuing $f(z)$ to an arbitrary point $z_0' \in G$ along all homotopic paths in G connecting the points z_0 and z_0' will be the same. In particular, if the domain G is simply connected, $f(z)$ will be single-valued in G.*

Corollary 1.42. *Let V be an M-dimensional algebraic variety. If C is a branch cut for V over $\mathbb{C}^M$ then every point of $\mathbb{C}^M \backslash C$ is generic and $V \backslash \pi^{-1}(C)$ is a finite holomorphic covering over $\mathbb{C}^M \backslash C$.*

We need to understand how 'large' the branch locus is in order to understand how to form a branch cut C. The Zariski-Nagata purity theorem does this.

Theorem 1.43 (Zariski-Nagata Purity Theorem, Theorem G.17, [32])**.** *Let V be an M-dimensional affine algebraic variety. If $\pi : V \to U$ is a finite branched holomorphic covering over an open subset $U \subset \mathbb{C}^M$ with branch locus $B_\pi \subset V$ and if V is (locally) irreducible at a point $z \in B_\pi$ then $dim_a(B_\pi) = dim_{\pi(z)}\pi(B_\pi) = M - 1$.*

Our standard hypothesis on V will include the fact that it is irreducible, so we will always be in this situation unless explicitly stated otherwise.

Theorem 1.44. *Suppose that V is an irreducible M-dimensional algebraic variety, $\pi : V \to \mathbb{C}^M$ the projection and that $\pi : V \backslash B_\pi \to \mathbb{C}^M \backslash \pi(B_\pi)$ is a finite branched holomorphic covering. Then a branch cut C for V over $\mathbb{C}^M$ can be chosen to be a real $2M - 1$ dimensional set (noting $\mathbb{C}^M \cong \mathbb{R}^{2M}$).*

Proof. If B_π is empty then we are done, so suppose that B_π is not empty. By the Zariski-Nagata purity theorem we know that B_π is, at most, an $M - 1$ (complex) dimensional set and hence $\pi(B_\pi)$ is at most $M - 1$ dimensional. Without loss of generality, assume that $\pi(B_\pi)$ has finite intersection with any line parallel to the z_1 axis (else we can rotate coordinates to ensure this). If $z' = (z_2, ..., z_M)$, define

$$S_{z'} := \{(t, z_2, ..., z_M) : t \in \mathbb{C}\}$$

Then $S_{z'}$ is a complex line in $\mathbb{C}^M$ and, by hypothesis and the Zariski-Nagata purity theorem, intersects $\pi(B_\pi)$ at most finitely many points. If the intersection is empty then we do nothing. If the intersection is nonempty then enumerate the points in the intersection by $(t_1, z'), ..., (t_d, z') \in S_{z'} \cap \pi(B_\pi)$. Fix some $\theta \in [0, 2\pi)$ and define the ray

$$r_\theta(t) := \{t + re^{i\theta} : r \in [0, \infty)\}.$$

We can identify the line $S_{z'}$ with the complex plane by projection to the first coordinate (i.e. projection to t). With this identification $S_{z'} \cap \pi(B_\pi)$ is identified with $\mathbb{C}\backslash\{t_1, ..., t_d\}$. Observe that $\mathbb{C}\backslash\{t_1, ..., t_d\}$ is not simply connected. Let $C_{z'} := \cup_{1 \le i \le d}\{r_\theta(t_i)\}$. Then $\mathbb{C}\backslash C_{z'}$ is simply connected and $C_{z'}$ has real dimension 1. This consideration can be done for any $z' \in \mathbb{C}^{M-1}$ (note that for every z' we choose the same θ). Now define

$$C = \{(t, z_2, ..., z_M) \in \mathbb{C}^M : t \in C_{(z_2,...,z_M)}\} = \bigcup_{z' \in \mathbb{C}^{M-1}} C_{z'} \times \{z'\}.$$

Clearly C is at most real $2M - 1$ dimensional, we claim that $\mathbb{C}^M \backslash C$ is simply connected. To this end let γ be any closed curve in $\mathbb{C}^M \backslash C$. Let z' be such that $S_{z'}$ intersects γ. By construction

C is a closed set. Since $\mathbb{C}^M$ is a metric space, we can find disjoint open neighbourhoods containing C and γ respectively. It follows that we can find a homotopy that sends γ to a curve that is contained in $S_{z'}$ while remaining outside of an open neighbourhood of C. Taking the projection to the first coordinate, we have a closed loop γ' contained in the set $\mathbb{C}\backslash C_{z'}$. By construction, $\mathbb{C}\backslash C_{z'}$ is simply connected and hence γ' can be contracted to a point. Since this process is arbitrary, it follows that any closed curve is null-homotopic and hence $\mathbb{C}^M\backslash C$ is simply connected. By definition C is a branch cut and fulfils the conclusion of the theorem. □

Corollary 1.45. *With the setup as in Theorem 1.44, $V\backslash\pi^{-1}(C)$ is not connected. Moreover if π has d fibers then $V\backslash\pi^{-1}(C)$ consists of d simply connected sets.*

Definition 1.46. *Let V be an irreducible M-dimensional algebraic variety. Suppose that C is a branch cut for V over $\mathbb{C}^M$ for the projection π. Enumerate the d simply connected sets from Corollary 1.45 as $V_1, ..., V_d$. We call $V_1, ..., V_d$ the branches of V (with repsect to the branch cut C).*

Remark 1.47. There are two equivalent formulations of Definition 1.46 that utilise the idea of analytic continuation.

(i) Let $z = (x, y)$ where $x \in \mathbb{C}^M$ and $y \in \mathbb{C}^{N-M}$. Suppose moreover that $x \in \mathbb{C}^M\backslash C$ so that $\pi^{-1}(x) = \{(x, y_1), ..., (x, y_d)\} \subset V$. Define V_i to be the set obtained from the analytic continuation of π^{-1} to $\mathbb{C}^M\backslash C$ which sends $x \mapsto (x, y_i)$.

(ii) Each y_i coordinate on $V\backslash\pi^{-1}(C)$ can be considered as an analytic function (given by the projection) over $\mathbb{C}^M\backslash C$. We can then define $V_i = \{(x, y_i(x)) : x \in \mathbb{C}^M\backslash C\}$.

Corollary 1.48. *With the setup as in Definition 1.46, if $\overline{V}_i$ is the closure of V_i (in V or $\mathbb{C}^N$) then $V = \bigcup_{1\leq i\leq d} \overline{V}_i$.*

Proof. Observe that by definition $\bigcup_{1\leq i\leq d} V_i$ is dense in V. □

1.1.4 The Complex Monge-Ampère Operator

Definition 1.49 (§1.1, [9])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set. On the vector space $C^k(\Omega)$, $k = 0, 1, ..., \infty$ of k-differentiable functions with compact support define the following topology: a sequence $\{\phi_j\}$ is convergent to ϕ if and only if*

(i) there exists $K \Subset \Omega$ such that $\operatorname{supp}\phi_j \subset K$ for all j;

(ii) for all multi-indices $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq k$ we have $D^\alpha\phi_j \to D^\alpha\phi$ uniformly (if $k = \infty$ then for all α) where D^α is the α-partial derivative.

We call $\mathcal{D}(\Omega) = C^\infty(\Omega)$ with this topology the space of test functions on Ω. By $\mathcal{D}'(\Omega)$ we denote the set of all (complex) continuous linear functionals on $\mathcal{D}(\Omega)$ and call functionals in this class distributions on Ω. We say that $u \in \mathcal{D}'(\Omega)$ is a distribution of order k if it can be continuously extended to a linear functional on $C^k(\Omega)$.

A sufficient treatment of distributions (insofar as they are used in pluripotential theory) can be found in Błocki [9] §1.1.

Remark 1.50. While we will make definitions where the underlying space is $\mathbb{C}^N$, these definitions can be extended to the case of a complex manifold X in the natural way. That is, ensuring the definition is valid on each chart (U_α, ϕ_α) of an atlas for X via pullback.

Definition 1.51 (§1.3, [9]). *A current Θ of bidegree (p,q) is a differential (p,q)-form with distribution coefficients. That is, if $\Omega \subset \mathbb{C}^N$ is an open set then*

$$\Theta = \sum_{I,J} \Theta_{IJ}\, dz_I \wedge d\overline{z}_J,$$

where the sum is taken over increasing multi-indices I, J with $|I| = p, |J| = q$ and each $\Theta_{IJ} \in \mathcal{D}'(\Omega)$.

Definition 1.52 (§1, [24]). *Suppose that $\Theta \in D_{(p,q)}(\Omega)$ has measure coefficients. Let $K \subset \Omega$ be a compact set. We define a mass semi-norm*

$$\|\Theta\|_K = \sum_j \int_{K_j} \sum_{I,J} |\Omega_{I,J}|$$

by taking a partition $K = \cup K_j$ where each $\overline{K}_j$ is contained in a coordinate patch and where $\Theta_{I,J}$ are the corresponding measure coefficients.

Remark 1.53. The semi-norm $\|\Theta\|_K$ does not depend on the choice of coordinate system.

Corollary 1.54 (§3.3, Page 107, [37]). *Let $\Omega \subset \mathbb{C}^N$ be an open set. Write $\mathcal{D}'_{(p,q)}(\Omega)$ to denote the currents of bidegree (p,q). Then $\left(\mathcal{D}'_{(p,q)}(\Omega)\right)' \cong \mathcal{D}_{(N-p,N-q)}(\Omega)$ where $\mathcal{D}_{(p,q)}(\Omega)$ is the space of smooth $(N-p, N-q)$-forms (differential forms with C^∞ coefficients with compact support in Ω). We say that a current of bidegree (p,q) is a (p,q)-current.*

Definition 1.55 (Equation 0.2, [24]). *Let $\Omega \subset \mathbb{C}^N$ be an open set. The duality pairing between a current Θ of bidegree (p,q) and a smooth differential form with compact support ϕ of bidegree $(N-p, N-q)$ is given by*

$$\langle \Theta, \phi \rangle = \sum \langle \Theta_{IJ}, \phi_{I'J'} \rangle = \int_\Omega \Theta \wedge \phi$$

where the sum is taken over all I, I', J, J' such that $I + I' = J + J' = N$. The integration symbol and wedge product is used to represent the pairing. These symbols are chosen because when everything is smooth the situation reduces to classical integration.

Definition 1.56. *Let $\Omega \subset \mathbb{C}^N$, a form $\omega \in D_{(p,p)}(\Omega)$ is said to be elementary strongly positive if there are linearly independent $\eta_j \in D_{(1,0)}(\Omega)$, $j = 1, ..., p$ such that*

$$\omega = \frac{i}{2}\eta_1 \wedge \overline{\eta_1} \wedge ... \wedge \frac{i}{2}\eta_p \wedge \overline{\eta_p}.$$

Definition 1.57. *Let $\Omega \subset \mathbb{C}^N$ be an open set and let $\Theta \in D'_{(p,p)}(\Omega)$. Then we say that T is a (weakly) positive current if $T \wedge \alpha \geq 0$ for all elementary positive forms $\alpha \in D_{(p,p)}(\Omega)$.*

Definition 1.58. *Let $\Omega \subset \mathbb{C}^N$ be an open set. Let $(z_1, ..., z_N)$ be local coordinates for Ω. We define $\partial : \mathcal{D}_{(p,q)}(\Omega) \to \mathcal{D}_{(p+1,q)}(\Omega)$ to be the operator*

$$\partial\phi = \sum_{j=1}^{N} \frac{\partial \phi}{\partial z_j} dz_j$$

where $\phi \in \mathcal{D}_{(p,q)}(\Omega)$. Similarly we define $\bar{\partial} : \mathcal{D}_{(p,q)}(\Omega) \to \mathcal{D}_{(p,q+1)}(\Omega)$ to be the operator

$$\bar{\partial}\phi = \sum_{j=1}^{N} \frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j.$$

We define the exterior derivative to be $d = \partial + \bar{\partial}$ and operator $d^c = i(\bar{\partial} - \partial)$. The operator $dd^c : \mathcal{D}_{(p,q)}(\Omega) \to \mathcal{D}_{(p+1,q+1)}(\Omega)$ is the operator $2i\partial\bar{\partial}$. Each of these operators can be extended to be defined for currents in the obvious way using the duality pairing.

Definition 1.59. *If T is a (p,q)-current and α an (m,n)-form then $T \wedge \alpha$ is defined for a test $(N-p-m, N-q-n)$-form φ by*

$$\langle T \wedge \alpha, \varphi \rangle = \langle T, \alpha \wedge \varphi \rangle.$$

Moreover, $T \wedge \alpha = (-1)^{m+n+p+q} \alpha \wedge T$.

Theorem 1.60 (Stokes Theorem). *Let Ω be an open set (in $\mathbb{C}^N$ or a complex manifold X) with oriented boundary $\partial\Omega$ and let $\alpha \in \mathcal{D}_{(N,N-1)}(\Omega) \oplus \mathcal{D}_{(N-1,N)}(\Omega)$ be a differential form on Ω (so that $d\alpha \in \mathcal{D}_{(N,N)}(\Omega)$). Then*

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha.$$

Where the integral on the RHS is the integral of the restriction of α to $\partial\Omega$.

Theorem 1.61 (Integration by Parts). *Let Ω be an open set (in $\mathbb{C}^N$ or a complex manifold X). Let α, β be differential forms on Ω. Suppose that Θ is a current such that $\alpha \wedge \beta \wedge \Theta \in \mathcal{D}'_{(N,N-1)} \oplus \mathcal{D}'_{(N-1,N)}$ and $d\Theta = 0$. Then*

$$\int_{\partial\Omega} \alpha \wedge \beta \wedge \Theta = \int_{\Omega} d\alpha \wedge \beta \wedge \Theta + \int_{\Omega} \alpha \wedge d\beta \wedge \Theta.$$

Definition 1.62 (§0, [24]). *We say that a (p,p)-current Θ is closed if $d\Theta = 0$.*

Lemma 1.63 (Proposition 3.3.4, [37]). *The coefficients of a positive current are complex measures.*

Theorem 1.64 (Stokes' Theorem for Currents, Theorem 1.3.4, [9]). *Let Ω be a bounded domain in $\mathbb{C}^N$ with C^1 boundry. Assume that T is a current in Ω such that T is C^1 on $\overline{\Omega}\backslash U$ where $U \Subset \Omega$. If dT is an (N,N)-current then*

$$\int_{\partial\Omega} T = \int_{\Omega} dT.$$

Theorem 1.65 (Skoda-El Mir, Theorem 0.5, [24]). *Let X be a complex manifold. Suppose that E is a closed complete pluripolar set in X and let Θ be a closed positive current on $X\backslash E$ such that the coefficients Θ_{IJ} of Θ are measures with locally finite mass near E. Then the trivial extension $\tilde{\Theta}$ obtained by extending the measures Θ_{IJ} by 0 on E is still closed.*

Definition 1.66. *Let X be a complex manifold. The current of integration $[S]$ over an oriented submanifold $S \subset X$ is given by*

$$\langle [S], \alpha \rangle = \int_S \alpha.$$

Theorem 1.67 (Equation 0.4, [24]). *Let X be a complex manifold. Suppose that $A \subset X$ is closed analytic set of pure dimension p. Then*

$$\langle [A], \alpha \rangle = \int_{A^{reg}} \alpha, \qquad \alpha \in \mathcal{D}_{(p,p)}(X),$$

is a positive closed current with locally finite mass near A^{sing}.

Theorem 1.68 (Proposition 3.3.5, [37]). *Let X be a complex manifold. If $u \in PSH(X)$ then*

$$T = 2i\partial\bar{\partial}u = 2i \sum_{1\leq i,j\leq N} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} dz_i \wedge d\overline{z}_j$$

is a closed positive current of bidgree $(1,1)$.

It is common in the literature to use a normalised dd^c operator, precisely $\frac{i}{\pi}\partial\bar{\partial}$. We will avoid this, but we alert the reader because the convention is popular in modern papers. Our decision to not normalise is motivated by our calculation of the mass of functions in $\mathcal{L}^+(V)$ (Theorem 2.27). We now define some commonly used wedge products.

Definition 1.69. *Suppose that $[X_j]$ is the current of integration on an algebraic variety X_j for $1 \leq j \leq q$. We define*

$$[X_1] \wedge ... \wedge [X_q] = [X_1 \cap .... \cap X_q].$$

In other words, for any test $(N-q, N-q)$-form α, we have

$$\langle [X_1] \wedge ... \wedge [X_q], \alpha \rangle = \int_{X_1 \cap ... \cap X_j} \alpha = \langle [X_1 \cap .... \cap X_q], \alpha \rangle.$$

Remark 1.70. The previous definition is made in the spirit of the Poincaré-Lelong formula [39].

Definition 1.71 (§1, [24]). *Let $\Omega \subset \mathbb{C}^N$ be an open set. Suppose that $u \in PSH \cap L^\infty_{loc}(\Omega)$ and T is a positive current of bidegree $(N-p, N-p)$. We define*

$$dd^c u \wedge T := dd^c(uT).$$

Theorem 1.72 (Theorem 1.2, [24]). *With the same hypothesis as Definition 1.71, the wedge product $dd^c u \wedge T$ is a closed positive current.*

The following is the primary operator of study in this thesis.

Definition 1.73 (Theorem 1.2, [24]). *We define inductively the N-fold exterior product*

$$\underbrace{dd^c u \wedge ... \wedge dd^c u}_{N \text{ terms}} = (dd^c u)^N := dd^c(u(dd^c u)^{N-1}).$$

We will call the operator $(dd^c)^N$ the Monge-Ampère operator. $(dd^c u)^N$ is a positive current of bidegree (N, N).

Definition 1.74. *Suppose that $u, v \in PSH \cap L^\infty_{loc}(\Omega)$ and T a positive closed current. Then we define*

$$\begin{aligned} du \wedge d^c u \wedge T &:= dd^c u^2 \wedge T - 2u\, dd^c u \wedge T \\ du \wedge d^c v \wedge T &:= \frac{1}{2}\big(d(u+v) \wedge d^c(u+v) \wedge T - du \wedge d^c u \wedge T - dv \wedge d^c v \wedge T\big). \end{aligned}$$

Theorem 1.75 (First Integration by Parts Formula for dd^c, Formula 1.1, [24]). *Let $\Omega \Subset X$ be a smoothly bounded open set in a complex manifold X and let $f \in C^2 \cap \mathcal{D}_{(p,p)}(\overline{\Omega})$, $g \in C^2 \cap \mathcal{D}_{(q,q)}(\overline{\Omega})$ with $p + q = N - 1$. Then*

$$\int_\Omega f \wedge dd^c g - dd^c f \wedge g = \int_{\partial\Omega} f \wedge d^c g - d^c f \wedge g.$$

Theorem 1.76 (Second Integration by Parts Formula for dd^c, Proposition 2.1, [27]). *Let X be a complex manifold and Ω an open set in X. Suppose that $u, v \in PSH \cap L^\infty_{loc}(\Omega)$ and that u, v are negative. Let T be a positive closed $(N-1, N-1)$ current. If $\lim_{z \to w \in \partial\Omega} u(z) = 0$ then*

$$\int_\Omega v\, dd^c u \wedge T \leq \int_\Omega u\, dd^c v \wedge T$$

with equality if $\lim_{z \to w \in \partial\Omega} v(z) = 0$.

Lemma 1.77 (Corollary 1.10, [24]). *Let X be a complex manifold and Ω an open set in X. Let $u_1, ..., u_q \in PSH \cap L^\infty_{loc}(\Omega)$. The wedge product $dd^c u_1 \wedge ... \wedge dd^c u_q$ is symmetric with respect to the order of $u_1, ..., u_q$. That is, interchanging $dd^c u_j$ with $dd^c u_i$ leaves the wedge product unchanged for all j, i.*

Theorem 1.78 (Chern-Levine-Nirenberg (CLN) Inequalities, 1.3, [24]). *Let X be a complex manifold. For all compact subsets K, L of X with $L \subset int(K)$ there exists a constant $C_{K,L} > 0$*

such that

$$\|dd^c u_1 \wedge ... \wedge dd^c u_q \wedge T\|_L \leq C_{K,L} \|u_1\|_{L^\infty(K)} ... \|u_q\|_{L^\infty(K)} \|T\|_K.$$

Theorem 1.79 (Proposition 1.11, [24]). *Let X be a complex manifold. Let K, L be compact subsets of X such that $L \subset int(K)$. If $v \in PSH(X)$ and $u_1, ..., u_q \in PSH \cap L^\infty_{loc}(X)$ then there is an inequality*

$$\|v dd^c u_1 \wedge ... \wedge dd^c u_q\|_L \leq C_{K,L} \|v\|_{L^1(K)} \|u_1\|_{L^\infty(K)} ... \|u_q\|_{L^\infty(K)}.$$

1.1.5 Logarithmic Extremal Functions

Definition 1.80. *Define the Lelong class of psh functions to be*

$$\mathcal{L}(\mathbb{C}^N) := \{u \in PSH(\mathbb{C}^N) : u(z) - \log \|z\| \leq \alpha, \|z\| \to \infty \text{ for some } \alpha \in \mathbb{R}\}.$$

This is the class of psh functions which have at most logarithmic growth. We will also need the following class of functions which have exactly logarithmic growth;

$$\mathcal{L}^+(\mathbb{C}^N) := \{u \in PSH(\mathbb{C}^N) : \alpha \leq u(z) - \log \|z\| \leq \beta, \|z\| \to \infty, \text{ for some } \alpha, \beta \in \mathbb{R}\}.$$

Define the class of log homogeneous psh functions to be

$$\mathcal{H}(\mathbb{C}^N) := \{u \in PSH(\mathbb{C}^N) : u(\lambda z) = u(z) + \log |\lambda|\}.$$

Definition 1.81. *The (logarithmic) extremal function for a compact set $K \subset \mathbb{C}^N$ is the function*

$$V_K(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^N), u \leq 0 \text{ on } K\}.$$

A complete study of $V_K(z)$ in $\mathbb{C}^N$ can be found in Section 5 of Klimek [37]. We will recall the important properties needed for our study here.

Notation 1.82. We will commonly use the notation $\log^+ |z|$ to denote the function $\max\{\log |z|, 0\}$.

Example 1.83 (Example 5.1.1, [37]). If $B_r(a) = \{z :\in \mathbb{C}^N : |z - a| \leq r\}$ then $V_{B_r(a)}(z) = \log^+ \frac{|z-a|}{r}$.

Definition 1.84. *If $K \subset \mathbb{C}^N$ is such that V_K is continuous then we say K is regular.*

Proposition 1.85 (Basic Properties of V_K).

(i) If $K_1 \supset K_2 \supset ...$ is a sequence of compact sets in $\mathbb{C}^N$ and $K = \cap K_j$ then $\lim_{j\to\infty} V_{K_j} = V_K$ at each point of $\mathbb{C}^N$. (Corollary 5.1.2, [37])

(ii) If $K \subset \mathbb{C}^N$ is compact then V_K is lower semicontinuous. (Corollary 5.1.3, [37])

(iii) If $K \subset \mathbb{C}^N$ is compact and $V_K|_K \equiv 0$ then V_K is continuous. (Corollary 5.1.4, [37])

(iv) If $K \subset \mathbb{C}^N$ is compact and $\varepsilon > 0$ then V_{K_ε} is continuous (equivalently, K_ε is regular) where $K_\varepsilon := \{z \in \mathbb{C}^N : \mathrm{dist}(z, K) < \varepsilon\}$. Moreover $\lim_{\varepsilon \to 0} V_{K_\varepsilon} = V_K$ at each point of $\mathbb{C}^N$. (Corollary 5.1.5, [37])

Theorem 1.86 (Theorem 5.1.7, [37])**.** *If $K \subset \mathbb{C}^N$ is compact define*

$$\Phi_K(z) := \sup\{|p(z)|^{1/\deg p} : \|p\|_K \leq 1, p \text{ a polynomial}\}.$$

Then $V_K = \log \Phi_K(z)$.

Theorem 1.87 (Theorem 5.5.4, [37])**.** *If $K \subset \mathbb{C}^N$ is a bounded non-pluripolar set, then $V_K^*(z) \in \mathcal{L}^+(\mathbb{C}^N)$.*‡ *Moreover,*

$$\int_{\mathbb{C}^N} (dd^c V_K^*(z))^N = \int_K (dd^c V_K^*(z))^N = (2\pi)^N.$$

In particular,

$$\int_{\mathbb{C}^N \setminus K} (dd^c V_K^*(z))^N = 0.$$

One of the most important objects of our study is the Robin function. Here we define this function and refer the reader to Bedford-Taylor [4] for a more complete study.

Definition 1.88. *For $u \in \mathcal{L}^+(\mathbb{C}^N)$ the Robin function of u is the function $\rho_u : \mathbb{C}^N \to [-\infty, \infty)$ given by*

$$\rho_u(z) = \limsup_{t \to 0} u(z/t) + \log|t|.$$

We also define the projective Robin function to be the function $\tilde{\rho}_u : \mathbb{P}^{N-1} \to \mathbb{R}$ given by

$$\tilde{\rho}_u([z]) = \limsup_{t \to 0} u(z/t) - \log\|z/t\|.$$

Where $[z] = [z_1 : ... : z_N] \in \mathbb{P}^{N-1}$ is identified with $z = (z_1, ..., z_N) \in \mathbb{C}^N$.

Lemma 1.89 (Section 1 & Section 3, [4])**.** *The Robin and projective Robin functions have the following properties.*

(i) The function ρ_u is psh for all $u \in \mathcal{L}^+(\mathbb{C}^N)$, and $\rho_u \in \mathcal{L}^+(\mathbb{C}^N)$.

(ii) For $|z| = 1$ we have $\rho_u(z) = \tilde{\rho}([z])$.

(iii) $\rho_u(z)$ is logarithmically homogeneous, i.e. $\rho_u(\lambda z) = \rho_u(z) + \log|\lambda|$ for all $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^N$.

1.2 Algebraic Preliminaries

There are three main things we need from algebra. The first is the basics of elimination theory for finite sets of polynomials which we will need for the proof of Theorem 2.21. This is one of

‡Recall that V_K^* is the usc regularisation of V_K, see Definition 1.4.

the major technical results of this thesis. The second is the basics of algebraic computations in quotients of polynomial rings in order to lay the foundations necessary for developing analogues of the Chebyshev constants on an algebraic variety in Section 1.3. And finally, we need the relative finiteness theorem and associated results. The main reference for this section is Cox-Little-O'Shea [22].

1.2.1 Basic Concepts

Definition 1.90. *A monomial in $z_1, ..., z_N$ is a product of the form*

$$z_1^{\alpha_1} \cdot z_1^{\alpha_2} ... \cdot z_N^{\alpha_N},$$

where each exponent $\alpha_1, ..., \alpha_N$ is a nonnegative integer. The degree of the monomial is the multiindex $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}^N_{\geq 0}$ and the total degree of the monomial is the sum $\alpha_1 + ... + \alpha_N$. We write $|\alpha| = \alpha_1 + ... + \alpha_N$. We will often simplify notation so that $z^\alpha = z_1^{\alpha_1} ... z_N^{\alpha_N}$, where $z = (z_1, ..., z_N)$.

Notation 1.91. $\mathbb{C}[z]$ will denote the set of polynomials in z with coefficients in $\mathbb{C}$. The elements of $\mathbb{C}[z]$ take the form

$$\sum_\alpha c_\alpha z^\alpha$$

where the sum is over a finite set of multiindices α and each $c_\alpha \in \mathbb{C}$.

Definition 1.92. *A subset $I \subset \mathbb{C}[z]$ is an ideal if it satisfies:*

(i) $0 \in I$.

(ii) If $f, g \in I$ then $f + g \in I$.

(iii) If $f \in I$ and $h \in \mathbb{C}[z]$ then $hf \in I$.

If $f_1, ..., f_s \in \mathbb{C}[z]$ then the ideal generated by $f_1, ..., f_s$ is

$$\langle f_1, ..., f_s \rangle = \left\{ \sum_{i=1}^{s} h_i f_i : h_1, ..., h_s \in \mathbb{C}[z] \right\}.$$

If I is representable by $\langle f_1, ..., f_s \rangle$ for some $f_1, ..., f_s \in I$ then we say that I is finitely generated. In this case, such a subset $\{f_1, ..., f_s\}$ is called a basis for I.

Remark 1.93. The Hilbert basis theorem guarantees that any ideal in $\mathbb{C}[z]$ is finitely generated.

Definition 1.94. *An ideal I is radical if $f^m \in I$ for any integer $m \geq 1$ implies that $f \in I$. Given an arbitrary ideal I the radical of I, denoted $\sqrt{I}$ is the set*

$$\sqrt{I} = \{f : f^m \in I \textit{ for some integer } m \geq 1\}.$$

Definition 1.95. *Let $I \subset \mathbb{C}[z]$ be an ideal and V be an algebraic variety. Write*

$$\boldsymbol{V}(I) = \{z \in \mathbb{C}^N : f(z) = 0, \forall f \in I\},$$
$$\boldsymbol{I}(V) = \{f \in \mathbb{C}[z] : f(z) = 0, \forall z \in V\}.$$

The map $\boldsymbol{V}$ is a map from affine varieties to ideals and the map $\boldsymbol{I}$ is a map from ideals to algebraic varieties. Points in $\boldsymbol{V}(I)$ are called solutions to the system of equations $f(z) = 0$ for all $f \in I$.

Theorem 1.96 (§4.2 Theorem 7, [22])**.** *The maps $\boldsymbol{V}$ and $\boldsymbol{I}$ (by definition) are inclusion reversing, that is for ideals $I_1 \subset I_2$ we have $\boldsymbol{V}(I_1) \supset \boldsymbol{V}(I_2)$ and for varieties $V_1 \subset V_2$ we have $\boldsymbol{I}(V_1) \supset \boldsymbol{I}(V_2)$. Moreover we have $\boldsymbol{V}(\boldsymbol{I}(V)) = V$ for all varieties V (i.e. $\boldsymbol{I}$ is one-to-one.). If we restrict the domain of $\boldsymbol{I}$ and range of $\boldsymbol{V}$ to radical ideals then the correspondences are inclusion-reversing bijections which are inverses of each other.*

Definition 1.97 (Monomial Ordering, §1.2 Definition 1, [22])**.** *A monomial ordering on $\mathbb{C}[z_1, ..., z_N]$ is any relation $>$ on the nonnegative integers $\mathbb{Z}^N_{\geq 0}$, or requivalently, any relation on the set of monomials z^α, $\alpha \in \mathbb{Z}^N_{\geq 0}$, satisfying:*

(i) $>$ is a total ordering on $\mathbb{Z}^N_{\geq 0}$.

(ii) If $\alpha > \beta$ and $\gamma \in \mathbb{Z}^N_{\geq 0}$, then $\alpha + \gamma > \beta + \gamma$.

(iii) $>$ is a well-ordering on $\mathbb{Z}^N_{\geq 0}$.

There are many ways one can order the monomials in $\mathbb{C}[z]$. We will only ever use two: 'graded reverse lexicographic' or 'grevlex' and elimination orderings of l-elimination type.

Definition 1.98 (Graded reverse lexicographic ordering)**.** *Let $\alpha, \beta \in \mathbb{Z}^N_{\geq 0}$. We say $\alpha >_{grevlex} \beta$ (or simply, $\alpha > \beta$) if either*

(a) $|\alpha| = \sum_{i=1}^N \alpha_i > |\beta| = \sum_{i=1}^N \beta_i$,

(b) $|\alpha| = |\beta|$ and in $\alpha - \beta$ the right-most nonzero entry is positive.

Definition 1.99. *Fix an integer $1 \leq l \leq N$ and define the order $>_l$ as follows; if $\alpha, \beta \in \mathbb{Z}^N_{\geq 0}$ then $\alpha >_l \beta$ if either*

(a) $\alpha_N + ... + \alpha_{N-l+1} > \beta_N + ... + \beta_{N-l+1}$ or

(b) $\alpha_N + ... + \alpha_{N-l+1} = \beta_N + ... + \beta_{N-l+1}$ and $\alpha >_{grevlex} \beta$.

We call the order $>_l$ the grevlex order of l-elimination type.

Remark 1.100. Our definition of an elimination ordering counts down from N while in [22] an elimination ordering counts up. The reason we invert is because the treatment in [22] uses an ordering where $z_1 > z_2 > ... > z_N$ and the elimination theory eliminates the variable z_1. Grevlex orders $z_N > ... > z_1$ so we wish to eliminate z_N necessitating the inversion.

Definition 1.101. *Let $I \subset \mathbb{C}[z]$ be an ideal other than $\{0\}$, $f = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}[z]$ and fix a monomial ordering $>$ on $\mathbb{C}[z]$.*

(i) We define $\operatorname{multideg}(f) = \max_>\{\alpha \in \mathbb{Z}^N_{\geq 0} : c_\alpha \neq 0\}$ *(where the max is with respect to grevlex ordering).*

(ii) We define $\deg(f) = |\operatorname{multideg}(f)|$.

(ii) We define $\mathrm{LT}(f) = c_{\operatorname{multideg}(f)} z^{\operatorname{multideg}(f)}$.

(iii) We define $\mathrm{LT}(I) = \{cz^\alpha : \textit{there exists } f \in I \textit{ with } \mathrm{LT}(f) = cz^\alpha\}$.

(iv) We define $\langle \mathrm{LT}(I) \rangle$ *to be the ideal generated by the elements of* $\mathrm{LT}(I)$.

From now on we will always assume we are using grevlex ordering on $\mathbb{C}[z]$ unless otherwise specified.

Lemma 1.102 (§2.5 Proposition 3, [22])**.** *Let $I \subset \mathbb{C}[z]$ be an ideal. Then $\langle \mathrm{LT}(I) \rangle$ is a finitely generated monomial ideal. That is, $\langle \mathrm{LT}(I) \rangle$ is generated by finitely many monomials of the form z^α.*

Definition 1.103. *A finite subset $G = \{g_1, ..., g_s\}$ of an ideal I is a Gröbner basis if $\langle \mathrm{LT}(g_1, ..., \mathrm{LT}(g_s)) \rangle = \langle \mathrm{LT}(I) \rangle$.*

Corollary 1.104 (§2.5 Corollary 6, [22])**.** *Every ideal $I \subset \mathbb{C}[z]$ other than $\{0\}$ has a Gröbner basis. Furthermore, any Gröbner basis for an ideal I is a basis of I.*

Suppose that $f_1, ..., f_s$ are polynomials in $\mathbb{C}[z]$. To 'divide' $f \in \mathbb{C}[z]$ by $f_1, ..., f_s$ is to find polynomials $a_1, ..., a_s \in \mathbb{C}[z]$ and a remainder $r \in \mathbb{C}[z]$ such that

$$f = a_1 f_1 + ... + f_s g_s + r.$$

In contrast to the one variable case, there is no division algorithm which returns unique $a_1, ..., a_s, r$ in general. The main utility of Gröbner bases is that they allow for a division algorithm which returns a unique remainder.

Theorem 1.105 (Division Algorithm, §2.6 Proposition 1, [22])**.** *Let $G = \{g_1, ..., g_s\}$ be a Gröbner basis for an ideal $I \subset \mathbb{C}[z]$ and let $f \in \mathbb{C}[z]$. Then there is a unique $r \in \mathbb{C}[z]$ with the following properties.*

(i) No term of r is divisible by any of $\mathrm{LT}(g_1), ..., \mathrm{LT}(g_s)$.

(ii) There is $g \in I$ such that $f = g + r$.

In particular, r is the remainder of division of f by G no matter how the elements of G are listed when using a division algorithm.

1.2.2 Elimination Theory

Definition 1.106. *Let $1 \leq l \leq N$. Given $I = \langle f_1, .., f_s \rangle \subset \mathbb{C}[z]$, the lth elimination ideal I_l is the ideal of $\mathbb{C}[z_1, ..., z_{N-l}]$ defined by*

$$I_l = I \cap \mathbb{C}[z_1, ..., z_{N-l}].$$

As with the grevlex monomial order of l-elimination type, this is the opposite of what is defined in Cox-Little-O'Shea [22] (c.f. Remark 1.100).

Theorem 1.107 (Elimination Theorem, Theorem 2 §3.1, [22])**.** *Let $I \subset \mathbb{C}[z]$ be an ideal and let G be a Gröbner basis of I with respect to grevlex order of l-elimination type. Then for every $0 \leq l \leq N$ the set*

$$G_l = G \cap \mathbb{C}[z_1, ..., z_{N-l}]$$

is a Gröbner basis of the lth elimination ideal I_l.

Definition 1.108. *Let $I = \langle f_1, ..., f_s \rangle \subset \mathbb{C}[z]$ be an ideal. Suppose $1 \leq l \leq N$ with l fixed. We will call a solution $(a_1, ..., a_{N-l}) \in \boldsymbol{V}(I_l)$ a partial solution of the original system $\boldsymbol{V}(I)$.*

Theorem 1.109 (Extension Theorem, Theorem 3 §3.1, [22])**.** *Let $I = \langle f_1, ..., f_s \rangle \subset \mathbb{C}[z]$ be an ideal and let I_1 be the first elimination ideal of I. For each $1 \leq i \leq s$, write f_i in the form*

$$f_i = g_i(z_1, ..., z_{N-1}) z_N^{m_i} + \text{ terms in which } z_N \text{ has degree } < m_i,$$

where $m_i \geq 0$ and $g_i \in \mathbb{C}[z_1, ..., z_{N-1}]$ is nonzero. Suppose that we have a partial solution $(a_1, ..., a_{N-1}) \in \boldsymbol{V}(I_1)$. If $(a_1, ..., a_{N-1}) \notin \boldsymbol{V}(g_1, ..., g_s)$ then there exists $a_1 \in \mathbb{C}$ such that $(a_1, ..., a_{N-1}, a_N) \in \boldsymbol{V}(I)$.

Our key technical result (Theorem 2.21) relies on the following special case of the Extension Theorem.

Corollary 1.110 (§3.1 Corollary 4, [22])**.** *Let $I = \langle f_1, ..., f_s \rangle \subset \mathbb{C}[z]$ and assume that for some i, f_i is of the form*

$$f_i = c z_N^m + \text{ terms in which } z_N \text{ has degree } < m,$$

where $c \in \mathbb{C}$ is nonzero and $m > 0$. If I_1 is the first elimination ideal of I and $(a_1, ..., a_{N-1}) \in \boldsymbol{V}(I_1)$, then there is $a_1 \in \mathbb{C}$ so that $(a_1, ..., a_{N-1}, a_N) \in \boldsymbol{V}(I)$.

1.2.3 Quotients of Polynomial Rings

Definition 1.111. *Let $I \subset \mathbb{C}[z]$ be an ideal and let $f, g \in \mathbb{C}[z]$. We say that f and g are congruent modulo I, written*

$$f \equiv g \mod I,$$

if $f - g \in I$. 'Congruence modulo I' is an equivalence relation on $\mathbb{C}[z]$.

Definition 1.112. *The quotient of $\mathbb{C}[z]$ modulo I, written $\mathbb{C}[z]/I$ is the set of equivalence classes for congruence modulo I:*

$$\mathbb{C}[z]/I = \{[f] : f \in \mathbb{C}[z]\}.$$

Theorem 1.113 (§5.2 Proposition 5, [22])**.** *Suppose I is an ideal and $[f], [g] \in \mathbb{C}[z]/I$. The operations*

$$\begin{aligned} [f] + [g] &= [f + g] \quad \textit{(sum in } \mathbb{C}[z]\textit{)} \\ [f] \cdot [g] &= [f \cdot g] \quad \textit{(product in } \mathbb{C}[z]\textit{)} \end{aligned}$$

are well defined.

The following definition will play a significant role in this thesis.

Definition 1.114. *Suppose that V is an affine algebraic variety. We define $\mathbb{C}[V] := \mathbb{C}[z]/\boldsymbol{I}(V)$.*

Theorem 1.115 (§5.2 Proposition 3, Theorem 7, [22])**.** *Suppose V is an algebraic variety. There is a one to one correspondence between the non-zero polynomials in $\mathbb{C}[z]$ restricted to V and the equivalences classes of $\mathbb{C}[V]$. Moreover, this correspondence preserves the sum and product operations from Theorem 1.113.*

Motivated by this, we call the elements of $\mathbb{C}[V]$ polynomials on V.

Theorem 1.116 (§5.3 Proposition 1, [22])**.** *Fix a monomial ordering on $\mathbb{C}[z]$ and let $I \subset \mathbb{C}[z]$ be an ideal. Let $\langle \mathrm{LT}(I) \rangle$ denote the ideal generated by the leading terms of the elements of I.*

(i) Every $f \in \mathbb{C}[z]$ is congruent modulo I to a unique polynomial r which is a $\mathbb{C}$-linear combination of the monomials in the complement of $\langle \mathrm{LT}(I) \rangle$.

(ii) The elements of $\{z^\alpha : z^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$ are 'linearly independent modulo I'. That is, if

$$\sum_\alpha c_\alpha z^\alpha \equiv 0 \mod I$$

where the z^α appearing are all in the complement of $\langle \mathrm{LT}(I) \rangle$, then $c_\alpha = 0$ for all α.

Theorem 1.117 (§5.3 Proposition 4, [22])**.** *Let $I \subset \mathbb{C}[z]$ be an ideal. Then $\mathbb{C}[z]/I$ is isomorphic as a $\mathbb{C}$-vector space to $S = span\{z^\alpha : z^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$.*

Theorem 1.116 and 1.117 allow us to choose canonical representatives for equivalence classes in $\mathbb{C}[V]$, precisely, the representative for $[f] \in \mathbb{C}[V]$ is chosen to be $f' \in S$ such that $f' \in [f]$ with S given in Theorem 1.117. For this reason we will represent the elements of $\mathbb{C}[V]$ using elements of $\text{span}\{z^\alpha : z^\alpha \notin \langle \text{LT}(\mathbf{I}(V)) \rangle\}$ rather than the equivalence class notation. Theorem 1.115 gives a multiplication and addition operation on $\mathbb{C}[V]$. In particular, for $f, g \in \mathbb{C}[V]$ we have $f \cdot g = [f \cdot g] = f'$ where f' is the canonical representative of the equivalence class $[f \cdot g]$ in $\mathbb{C}[z]/\mathbf{I}(V)$. Recall that Theorem 1.105 ensures that the canonical representatives (i.e. remainders after division) resulting from these calculations is unique. This relationship is formalised in the following Lemma.

Lemma 1.118 (§5.3 Proposition 5, [22]). *If $G = \{g_1, ..., g_s\}$ is a Gröbner basis for an ideal I and $\overline{f}^G$ denotes the remainder of $f \in \mathbb{C}[z]$ under division by the elements in G then the canonical representative in Theorem 1.117 for $[f]$ is $\overline{f}^G$. Moreover if $g \in \mathbb{C}[z]$ we have $[f] + [g] = \overline{f}^G + \overline{g}^G$ and $[f] \cdot [g] = \overline{\overline{f}^G \cdot \overline{g}^G}^G$.*

Definition 1.119. *Let G be a Groöbner basis for an ideal $I \subset \mathbb{C}[z]$. If $f \in \mathbb{C}[z]$ we say that $\overline{f}^G$ is f written in normal form.*

1.2.4 Finiteness

Definition 1.120. *A $\mathbb{C}$-algebra is a ring which contains $\mathbb{C}$ as a subring. We say a $\mathbb{C}$-algebra is finitely generated if it contains finitely many elements such that every element can be expressed as a polynomial (with coefficients in $\mathbb{C}$) in these finitely many elements.*

Definition 1.121. *Given a commutative ring S and a subring $R \subset S$, we say that S is finite over R if there are finitely many elements $s_1, ..., s_l \in S$ such that every $s \in S$ can be written in the form $s = a_1 s_1 + ... + a_l s_l$ where $a_1, ..., a_l \in R$.*

Theorem 1.122 (Relative Finiteness Theorem, §5.6 Theorem 4, [22]). *Let $z = (x_1, ..., x_M, y_1, ..., y_{N-M}) = (x, y)$. Let $I \subset \mathbb{C}[x, y]$ be such that $I \cap \mathbb{C}[x] = \{0\}$ and order monomials by $(N-M)$-elimination type. Then the following statements are equivalent.*

(i) For each i, $1 \le i \le N - M$ there is some $m_i \ge 0$ such that $y_i^{m_i} \in \langle \text{LT}(I) \rangle$.

(ii) Let G be a Gröbner basis for I. Then for each i, $1 \le i \le N - M$ there is some $m_i \ge 0$ such that $y_i^{m_i} = \text{LM}(g)$ for some $g \in G$.

(iii) The set $\{y^\alpha :$ there is $\beta \in Z^m_{\ge 0}$ such that $x^\beta y^\alpha \notin \langle \text{LT}(I) \rangle\}$ is finite.

(iv) The ring $\mathbb{C}[x, y]/I$ is finite over the subring $\mathbb{C}[x]$.

When this condition is satisfied we say that $\mathbb{C}[\mathcal{V}] = \mathbb{C}[x, y]/I$ is finite over $\mathbb{C}[x]$.

Lemma 1.123 (§5.6, Pg 281, [22]). *Suppose that $(a, b) = (a_1, ..., a_M, b_1, .., b_{N-M}) \in \boldsymbol{V}(I)$. Then the inclusion $\mathbb{C}[x] \subset \mathbb{C}[x, y]$ corresponds to the projection $\pi : \mathbb{C}^N \to \mathbb{C}^M$ sending $(a, b) \mapsto a$. Moreover, the point a determines* *(i) the ideal $I_a \subset C[y]$ which is obtained by setting $x_i = a_i$ in all elements of the ideal I,*

(ii) the fiber $\pi^{-1}(a) = \boldsymbol{V}(I) \cap (\{a\} \times \mathbb{C}^{N-M})$, which consists of all points of $\boldsymbol{V}(I)$ whose first M coordinates are given by a.

We also have the relation $\pi^{-1}(a) = \{a\} \times \boldsymbol{V}(I_a)$ and

$$\bigcup_{a \in \mathbb{C}^M} \{a\} \times \boldsymbol{V}(I_a) = \bigcup_{a \in \mathbb{C}^M} \pi^{-1}(a) = \boldsymbol{V}(I) \subset \mathbb{C}^N.$$

Theorem 1.124 (Geometric Relative Finiteness Theorem, §5.6 Theorem 5, [22])**.** *Suppose that $I \subset \mathbb{C}[z] = \mathbb{C}[x, y]$ is an ideal such that $I \cap \mathbb{C}[x] = \{0\}$. If, in addition, $\mathbb{C}[x, y]/I$ is finite over $\mathbb{C}[x]$ then*

(i) The projection map $\pi : \boldsymbol{V}(I) \to \mathbb{C}^M, (x, y) \mapsto x$ is onto and has finite fibers.

(ii) For each $a \in \mathbb{C}^M$ the variety $\boldsymbol{V}(I_a) \subset \mathbb{C}^{N-M}$ is finite and non-empty.

Theorem 1.125 (Noether Normalisation, §5.6 Theorem 6, [22])**.** *If $A \subset \mathbb{C}[z]$ is a finitely generated algebra then there are algebraically independent elements $u_1, ..., u_M \in A$ such that:*

(i) A is finite over $\mathbb{C}[u_1, ..., u_M]$.

(ii) If additionally A is generated by $s_1, ..., s_l$ as a $\mathbb{C}$-algebra, then $M \leq l$ and $u_1, ..., u_M$ is a $\mathbb{C}$-linear combination of $s_1, ..., s_l$.

In this case we say that $\mathbb{C}[u] \subset A$ is a Noether normalisation for A.

Theorem 1.126 (Geometric Noether Normalisation, §5.6 Theorem 8, [22])**.** *Let $V \subset \mathbb{C}^N$ be a variety. Then a Noether normalisation $\mathbb{C}[x] = \mathbb{C}[x_1, ..., x_M]$ of $\mathbb{C}[V]$ can be chosen (i.e. after a suitable linear change of coordinates) so that the projection map $\pi : V \to \mathbb{C}^M, (x, y) \mapsto x$ has the following properties*

(i) π is the composition of the inclusion $V \subset \mathbb{C}^N$ with a linear map $\mathbb{C}^N \to \mathbb{C}^M$.

(ii) π is onto with finite fibers.

Remark 1.127. It is worth clarifying the relationship between the Geometric Noether Normalisation Theorem and the notion that V is a finite branch holomorphic covering over $\mathbb{C}^M$. If V is a finite branched holomorphic covering then $\pi : V \to W \subset \mathbb{C}^M$ is a surjective mapping to W which can be a strict subset of $\mathbb{C}^M$. For instance, the projection to x or y for $V = \{yx = 1\}$ is onto $\mathbb{C}\backslash\{0\}$. The Geometric Noether Normalisation theorem says that we can always find a linear change of variables so that $W = \mathbb{C}^M$. For instance, the linear change of variables $u = x + y$ and $v = x - y$ yields $V = \{(u + v)(u - v) = 1\}$ and the projection to u is onto $\mathbb{C}$.

1.3 Transfinite Diameter and Chebyshev Constants on Algebraic Varieties

We will assume for the majority of this thesis that certain geometric conditions on our algebraic varieties are satisfied, which allow us to discuss natural analogues of the transfinite diameter and Chebyshev constants on an affine variety. This section closely follows that of Cox-Ma'u [23] in order to establish those conditions and prove the existence of the generalised transfinite diameter and Chebyshev constants. First let us recall the classical situation due to Zakharjuta [54].

Notation 1.128. Fix a monomial order on $\mathbb{C}[z_1, ..., z_N] = \mathbb{C}[z]$. We make the following notational conventions:

(i) $m^{(N)}(i) =$ the number of monomials of degree at most i in $\mathbb{C}[z_1, ..., z_N]$;

(ii) $h^{(N)}(i) = m^{(N)}(i) - m^{(N)}(i-1) =$ the number of monomials degree exactly i in $\mathbb{C}[z_1, ..., z_N]$;

(iii) $l^{(N)}(i) = \sum_{j=1}^{i} jh^{(N)}(j) =$ the sum of degree of the monomials of at most degree i in $\mathbb{C}[z]$.

Definition 1.129. *Fix a monomial order (e.g. grevlex) on $\mathbb{C}[z]$ and enumerate the monomials as $\{\mathbf{e}_1, \mathbf{e}_2, ...\}$. Given a positive integer s and points $\zeta_1, ..., \zeta_s \in \mathbb{C}^N$, we define the Vandermonde determinant as*

$$VDM_{\mathbb{C}[z]}(\zeta_1, ..., \zeta_s) = VDM(\zeta_1, ..., \zeta_s) = \det \begin{pmatrix} \mathbf{e}_1(\zeta_1) & \mathbf{e}_1(\zeta_2) & ... & \mathbf{e}_1(\zeta_s) \\ \mathbf{e}_2(\zeta_1) & \mathbf{e}_2(\zeta_2) & ... & \mathbf{e}_2(\zeta_s) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{e}_s(\zeta_1) & \mathbf{e}_s(\zeta_2) & ... & \mathbf{e}_s(\zeta_s) \end{pmatrix}.$$

Let $K \subset \mathbb{C}^N$ be a compact set. We define the transfinite diameter for K to be

$$\delta(K) = \limsup_{s\to\infty} \max_{\zeta_1,...,\zeta_s \in K} \left| VDM(\zeta_1, ..., \zeta_s)^{1/l^{(N)}(s)} \right|.$$

Definition 1.130. *Fix a monomial order of $\mathbb{C}[z]$ and suppose $K \subset \mathbb{C}^N$ is compact. Let $\alpha \in \mathbb{Z}^N_{\geq 0}$. The α-Chebyshev constant is defined to be*

$$T(K, \alpha) = \inf\{\|p\|_K : \mathrm{LT}(p) = z^\alpha\}.$$

Let Σ be the N-dimensional simplex. That is,

$$\Sigma = \left\{ (\theta_1, ..., \theta_N) \in \mathbb{R}^N : \sum_{i=1}^{N} \theta_i = 1, \theta_i \geq 0 \right\}.$$

We also define $\Sigma_0 = \{\theta \in \Sigma : \theta_i > 0, \forall i\}$. Let $\theta \in \Sigma_0$. We define the θ-partial Chebyshev constant to be

$$\tau(K, \theta) = \limsup_{\frac{\alpha}{|\alpha|}\to\theta} T(K, \alpha)^{1/|\alpha|}. \tag{1}$$

We define the principal Chebyshev constant to be

$$\tau(K) = \exp\left(\frac{1}{vol(\Sigma)}\int_{\Sigma_0} \log \tau(K,\theta)\, d\theta\right).$$

Theorem 1.131 (Zakharjuta, [54])**.** *The* lim sup *defining the transfinite diameter in Definition 1.129 and θ-partial Chebyshev constants in Definition 1.130 can be replaced by a limit. Moreover, $\delta(K) = \tau(K)$.*

Bloom-Levenberg [15] developed a very general system to prove that the lim sup can be replaced by a limit in equation (1), which depends on so-called sub-multiplicative functions, and allowed for a number of analogous concepts to be proven in the same manner. The equality $\delta(K) = \tau(K)$ was proven by Zakharjuta [54] and also originally showed that the lim sup in Definition 1.129 can be replaced by a limit.

The work of Berman-Boucksom [7] proved that an analog of Theorem 1.131 for a transfinite diameter defined using an $L^2(\mu)$-orthonormal basis with respect to a probability measure μ on V. The paper of Cox-Ma'u [23] shows that under reasonable geometric conditions on an algebraic variety V, one can formulate the transfinite diameter on V in a purely algebraic way. We will relate these two different conceptions of the transfinite diameter in Section 3.5. The construction given by Cox-Ma'u provides a natural setting to study Chebyshev constants. Our work focuses primarily on this construction.

1.3.1 Distinguished Basis for $\mathbb{C}[V]$

Notation 1.132. Suppose that V is an affine algebraic variety. Let $\overline{V}_{\mathbb{P}}$ denote the projective closure of V. Precisely, if $\tilde{p}(t,z) = t^{\deg p} p(z/t)$ is the homogenisation of p and $I = \mathbf{I}(V)$ then

$$\begin{aligned} I^h &:= \{\tilde{p} \in \mathbb{C}[z_0, z_1, ..., z_N] : p \in I\} \\ \overline{V}_{\mathbb{P}} &= \mathbf{V}(I^h) = \{z = [z_0 : ... : z_N] \in \mathbb{P}^N : p(z) = 0 \text{ for all } p \in I^h\}. \end{aligned}$$

Definition 1.133. *We say a d-sheeted affine algebraic variety V of dimension M has distinct intersections with infinity if it satisfies the following properties.*

(i) $\mathbb{C}[z_1, ..., z_M] \subset \mathbb{C}[V]$ *is a Noether normalisation for V.*

(ii) If $P = \mathbf{V}(\{z_0, ..., z_{M-1}\}) \subset \mathbb{P}^M$ the set $\overline{V}_{\mathbb{P}} \cap P$ consists of d distinct points.

(iii) Label the point of $\overline{V}_{\mathbb{P}} \cap P$ as $\lambda_1, ..., \lambda_d$ where each $\lambda_i = [0 : ... : 0 : \lambda_{iM} : ... : \lambda_{iN}]$. Then for each i, $\lambda_{iM} \neq 0$.

The name 'distinct intersections with infinity' is motivated by the fact that the sheets of V intersect the hyperplane at infinity in a way which preserves the number of branches and is singular at a subvariety of H_∞. The justification for these claims will be proven in Section 2.5.4. The following lemma distinguishes certain polynomials which will be used in our study.

Lemma 1.134 (Corollary 2.6, Lemma 2.7-10, Proposition 2.11, [23]). *Suppose that V is an M-dimensional affine algebraic variety with distinct intersections with infinity. For some $t \in \mathbb{N}$ sufficiently large there are polynomials $v_1, ..., v_d \in \mathbb{C}[V]$ of degree t satisfying the following properties*

(i) $v_i^2 = z_M^t v_i + \sum_{k=1}^{M-1} z_k h_k + h_0$ *with* $\deg(h_k) \leq 2t - 1$ *for each* $k = 0, ..., m - 1$.

(ii) $v_i v_j = \sum_{k=1}^{M-1} z_k q_k + q + 0$ *if* $i \neq j$ *with* $\deg(q_k) \leq 2t - 1$ *for each* k.

(iii) There is $\tilde{v}_i \in \mathbb{C}[z_0, ..., z_N]/(I^h + \langle z_0 \rangle)$ *such that* $\tilde{v}_i(\lambda_j) = \delta_{ij}$ *where* δ_{ij} *is the usual Kronecker delta function and the canonical representation of* $\tilde{v}_i(0, z_1, ..., z_N)$ *in* $\mathbb{C}[V]$ *is* v_i.

(iv) ${\tilde{v}_i}^2 = z_M^t \tilde{v}_i + \sum_{k=1}^{M-1} z_k H_k(z_1, ..., z_N)$ *and* $\tilde{v}_i \tilde{v}_j = \sum_{k=1}^{M-1} z_k Q_k(z_1, ..., z_N)$ *where* H_k *and* Q_k *are homogeneous polynomials of degree* $2t - 1$.

(v) $\mathbb{C}[V]$ *is spanned by*

$$\begin{aligned} (*) \quad & z^\alpha z_M^l z^\beta : \quad \alpha \in \mathbb{Z}_{\geq}^{M-1} 0,\ z_M^l z^\beta \in \mathcal{B} \\ (**) \quad & \left[z^\alpha z_M^l v_i \right] : \quad \alpha \in \mathbb{Z}_{\geq}^{M-1} 0,\ l \geq 0,\ i = 1, ..., d. \end{aligned}$$

where $\mathcal{B} = \{ z_M^l z^\beta \notin \langle \mathrm{LT}(\mathbf{I}(V)) \rangle,\ l + |\beta| \leq t - 1 \}$.

Remark 1.135. It is possible that $z^\alpha z_M^l v_i$ is not written in normal form (c.f. Lemma 1.118). However for simple examples (such as the hypersurface) $z^\alpha z_M^l v_i$ is usually in normal form. Including the normal form of $[z^\alpha z_M^l v_i]$ in the definition is largely just a technicality and won't impact our work.

Notation 1.136. Basis elements of the form $(*)$ will be called type-1 monomials. The multiplicative behavior of the terms $\left[z^\alpha z_M^l v_i \right]$ resembles that of monomials and they will often be treated as such. Elements of the form $(**)$ will be called type-2 monomials. See Cox-Ma'u [23] for further details.

Theorem 1.137 (Theorem 2.13, [23]). *All type-2 monomials are in* $\mathcal{C}$, *i.e.* $\left[z^\alpha z_M^l v_i \right]$ *and* $\left[z^{\alpha'} z_M^{l'} v_j \right]$ *are linearly independent unless* $\alpha = \alpha'$, $l = l'$, $i = j$.

Definition 1.138. *Suppose that V is an M-dimensional affine algebraic variety with distinct intersections with infinity. The following construction defines the distinguished basis $\mathcal{C}$ for $\mathbb{C}[V]$. Define an ordering $\prec$ on the type-1 and type-2 monomials as follows:*

(i) Let type-2 monomials precede type-1 monomials.

(ii) Let $z^\alpha z_M^l v_i \prec z^{\alpha'} z_M^{l'}$ *if* $z^\alpha z_M^l <_{grevlex} z^{\alpha'} z_M^{l'}$.

(iii) Let $z^\alpha z_M^l v_i \prec z^\alpha z_M^l v_j$ *if* $i < j$.

(iv) Order type-1 monomials by grevlex.

We inductively define the set $\mathcal{C}$ by going through this collection with respect to the order $\prec$ and adding elements which are linearly independent to those already in the set. (This removes any monomial which is not linearly independent with a $z^\alpha z_M^l v_i$ term). With this process completed, we redefine the ordering on $\mathcal{C}$ so that type-1 monomials precede type-2 monomials, leaving the other conditions unchanged. We will write $\mathbf{e}_i$ to be the ith element in $\mathcal{C}$ (with the ordering $\prec$).

1.3.2 Transfinite Diameter and Chebyshev Constants

Cox-Ma'u [23] extended the convergence properties of submultiplicative functions due to Bloom-Levenberg (pp. 10-12, [15]) to that of weakly submultiplicative functions of subexponential growth in order to exploit 'classical' arguments to show the convergence of the Chebyshev constants on an algebraic variety. The following definition is made in the same spirit as Definition 1.130.

Definition 1.139. *Suppose that V is an M-dimensional affine algebraic variety with distinct intersections with infinity and let $\prec$ be monomial ordering from Definition 1.138. Let $K \subset V$ be a compact set. For $1 \le i \le d$ let $\mathcal{C}(\alpha, i) := \{p(z) \in \mathbb{C}[V] : p(z) = z^\alpha v_i + g(z),\, g(z) < z^\alpha v_i\}$. Define*

$$T(K, \alpha, \lambda_i) = \inf\{\|p\|_K : p \in \mathcal{C}(\alpha, i)\}.$$

$T(K, \alpha, \lambda_i)$ will be called the α-Chebyshev constant in the direction λ_i and a polynomial $p \in \mathcal{C}(\alpha, i)$ which reaches the infimum an α-Chebyshev polynomial in the direction λ_i. The limit

$$\tau(K, \theta, \lambda_i) = \limsup_{\substack{|\alpha| \to \infty \\ \frac{\alpha}{|\alpha|} \to \theta}} T(K, \alpha, \lambda_i)^{1/|\alpha|}$$

will be called the θ-partial Chebyshev constant in the direction λ_i. We define the Chebyshev constant in the direction λ to be

$$\tau(K, \lambda_i) = \exp\left(\frac{1}{vol(\Sigma)} \int_{\Sigma_0} \log \tau(K, \theta, \lambda_i)\, d\theta\right).$$

We define the principal Chebyshev constant for K to be the geometric average $\tau(K) = \left(\prod_{i=1}^d \tau(K, \lambda_i)\right)^{1/d}$.

Lemma 1.140 (Lemma 3.4, [23])**.** *Suppose that we are in the situation of Definition 1.139. Then $\tau(K, \theta, \lambda_i)$ is log-convex with respect to θ, i.e. for any $\theta_1, \theta_2 \in \Sigma_0$ and $t \in [0, 1]$,*

$$\log \tau(K, t\theta_1 + (1-t)\theta_2, \lambda_i) \le t \log \tau(K, \theta_1, \lambda_i) + (1-t) \log \tau(K, \theta_2, \lambda_i).$$

In the spirit of the notation introduced in 1.128 we make the following conventions.

Notation 1.141. Suppose that V is an algebraic variety that has distinct intersections with infinity and let $\mathcal{C}$ be the distinguished polynomial basis for $\mathbb{C}[V]$. We make the following notational conventions:

(i) $m^{(V)}(i) =$ the number of monomials of degree at most i in $\mathcal{C}$;

(ii) $h^{(V)}(i) = m^{(V)}(i) - m^{(V)}(i-1) =$ the number of monomials degree exactly i in $\mathcal{C}$;

(iii) $l^{(V)}(i) = \sum_{j=1}^{i} jh^{(V)}(j) =$ the sum of degree of the monomials of at most degree i in $\mathcal{C}$.

Definition 1.142. *Suppose that V is an affine algebraic variety. We define the Vandermonde determinant for a finite set $\{\zeta_1, ..., \zeta_s\} \subset V$ with respect to the basis $\mathcal{C}$ to be*

$$VDM_{\mathcal{C}}(\zeta_1, ..., \zeta_s) := \det \begin{pmatrix} \mathbf{e}_1(\zeta_1) & \mathbf{e}_1(\zeta_2) & ... & \mathbf{e}_1(\zeta_s) \\ \mathbf{e}_2(\zeta_1) & \mathbf{e}_2(\zeta_2) & ... & \mathbf{e}_2(\zeta_s) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{e}_s(\zeta_1) & \mathbf{e}_s(\zeta_2) & ... & \mathbf{e}_s(\zeta_s) \end{pmatrix}.$$

where $\{\mathbf{e}_i\}_{i=1}^{\infty}$ is the enumeration of $\mathcal{C}$ with the ordering $\prec$.

Definition 1.143. *Suppose that V is an affine algebraic variety and that $K \subset V$ is a compact set. Let*

$$\delta(K) := \limsup_{s\to\infty} \left| \left(\max_{\zeta_1,...,\zeta_{m^{(V)}(s)} \in K} VDM_{\mathcal{C}}(\zeta_1, ..., \zeta_{m^{(V)}(s)}) \right)^{1/l^{(V)}(s)} \right|.$$

We say that $\delta(K)$ is the transfinite diameter of the set K.

The following is the main result of Cox-Ma'u [23].

Theorem 1.144. *Suppose we are in the situation of Definition 1.143. The lim sup in Definition 1.143 can be replaced by a limit, moreover*

$$\delta(K) = \tau(K) = \left(\prod_{i=1}^{d} \tau(K, \lambda_i) \right)^{1/d}.$$

We now want to relate the transfinite diameter from the $\mathcal{C}$ basis to the transfinite diameter using the monomial basis for $\mathbb{C}[\mathcal{V}]$. Write $\mathcal{B}$ for the monomial basis, then $\delta_{\mathcal{C}}(K)$ is the transfinite diameter using $VDM_{\mathcal{C}}$ and $\delta_{\mathcal{B}}(K)$ is the transfinite diameter using $VDM_{\mathcal{B}}$.

Theorem 1.145. *Suppose that V is an affine algebraic variety which has distinct intersections with infinity and K a compact subset of $\mathcal{V}$. Write $\mathcal{B}$ to be usual monomial basis for $\mathbb{C}[V]$. Let $v_1, ..., v_d$ be the polynomials guaranteed by Lemma 1.134 and $\mathcal{C}$ the corresponding distinguished basis. Then $\delta_{\mathcal{B}}(K) = \delta_{\mathcal{C}}(K)$.*

We note a preliminary Lemma.

Lemma 1.146. *With the hypothesis as in Theorem 1.145, if $\mathcal{C}'$ is the distinguished basis but with the normal form $[z^\alpha z_M^l v_i]$ replaced with the monomial form $z^\alpha z_M^l v_i$ then $\delta_{\mathcal{C}'}(K) = \delta_{\mathcal{C}}(K)$.*

Proof. Observe that $[z^\alpha z_M^l v_i]\big|_{\mathcal{V}} = z^\alpha z_M^l v_i\big|_{\mathcal{V}}$ by definition of the normal form (Lemma 1.118). The result now follows immediately. $\square$

Proof of Theorem 1.145. Write $z = (x', x_M, y)$ where $(x', x_M) \in \mathbb{C}^M$ and $y \in \mathbb{C}^{N-M}$ and choose t so that type-1 monomials in $\mathcal{C}$ have the form $x^\alpha x_M^l y^\beta$ where $|\beta| + l \leq t - 1$ for some $t \in \mathbb{N}$ and by Lemma 1.146 we may choose our type-2 monomials to have the form $x^\alpha x_M^l v_i$ without changing the transfinite diameter. Observe that every type-1 monomial in $\mathcal{C}$ is an element of $\mathcal{B}$, so the monomials which are not type-1 in the set $\mathcal{B}$ belong to the set $\mathcal{B}' = \{x_M^l y^\beta : |\beta| + l \geq t\}$.

By the Noether normalisation assumption, there exists $m_i \in \mathbb{N}$ such that $y_i^{m_i+1} \notin \mathbb{C}[V]$ while $y_i^{m_i} \in \mathbb{C}[V]$. It follows then that $|\beta| \leq \sum_{i=1}^{N-M} m_i =: m_Y$. Then elements of $\mathcal{B}'$ have the form of $x_M^s f_j, s \in \mathbb{N}$ where f_j is an element of the finite set $F = \{x_M^l y^\beta : m_Y \geq |\beta| \geq t - l\}$.

Let $f_j \in F$. By Lemma 1.134 property (v) there exists a linear combination of elements $e_k \in \mathcal{C}$ such that $f_j = \sum_{k=1}^{s_j} c_{jk} e_k$. It is apparent by linear independence that $\deg(e_k) \leq \deg(f_j)$. The observation about the form of elements in $\mathcal{B}'$ means that every element can be represented as $x_M^s \left(\sum_{k=1}^{s_j} c_{jk} e_k\right)$ for some j, again from the form of the elements in $\mathcal{B}'$, this sum is a linear combination of type-2 monomials.

Now choose $r \in \mathbb{N}$ sufficiently large $F \subset \mathcal{B}_r$ where $\mathcal{B}_r := \{b \in \mathcal{B} : \deg(b) \leq r\}$ (define $\mathcal{C}_r$ similarly). It follows by doing elementary row operations that

$$\det VDM_{\mathcal{B}}(\zeta_1, ..., \zeta_{m^{(V)}(r)}) = C \det VDM_{\mathcal{C}}(\zeta_1, ..., \zeta_{m^{(V)}(r)})$$

where C is a normalisation constant depending on the c_{jk} terms arising from the elementary row operations calculated previously.[§] Since the same linear operations are used to transform $\mathcal{C}_{2r} \backslash \mathcal{C}_r$ into $\mathcal{B}_{2r} \backslash \mathcal{B}_r$ (which is a consequence of the form of the elements of $\mathcal{B}'$ observed above) it follows that

$$\det VDM_{\mathcal{B}}(\zeta_1, ..., \zeta_{m^{(V)}(2r)}) = C^2 VDM_{\mathcal{C}}(\zeta_1, ..., \zeta_{m^{(V)}(2r)})$$

and so on. We have the following estimate and limit calculation,

$$\frac{2}{nr(nr+1)m^{(V)}(nr)} \leq \frac{1}{l^{(V)}(nr)} \leq \frac{1}{nrm^{(V)}(nr)},$$

$$1 = \lim_{n\to\infty} C^{2/r(nr+1)m^{(V)}(nr)} \leq \lim_{n\to\infty} C^{n/l^{(V)}(nr)} \leq \lim_{n\to\infty} C^{1/rm^{(V)}(nr)} = 1.$$

[§]Despite not using e_k written in normal form, the elementary row operations preserve the value of the determinant since the elementary row operations only involve the addition of rows, and the normal form of a sum is the sum of the normal forms (Lemma 1.118).

It follows that

$$\begin{aligned}\delta_{\mathcal{B}}(K) &= \lim_{n\to\infty}\left|\max_{\zeta_1,...,\zeta_{m^{(V)}(nr)}\in K}\left(\det VDM_{\mathcal{B}}(\zeta_1,...,\zeta_{m^{(V)}(nr)})\right)^{1/l^{(V)}(nr)}\right| \\ &= \lim_{n\to\infty}\left|\max_{\zeta_1,...,\zeta_{m^{(V)}(nr)}\in K}\left(C^n\det VDM_{\mathcal{C}}(\zeta_1,...,\zeta_{m^{(V)}(nr)})\right)^{1/l^{(V)}(nr)}\right| \\ &= \delta_{\mathcal{C}}(K).\end{aligned}$$

This concludes the proof. □

The point of this is that convergence to the transfinite diameter in the basis $\mathcal{C}$ is equivalent to convergence to the transfinite diameter in the monomial basis for $\mathbb{C}[\mathcal{V}]$. The speciality of $\mathcal{C}$ is that it allows a geometric interpretation of the transfinite diameter (or rather, Chebyshev polynomials).

1.3.3 Homogeneity and Circled Sets

We record a few basic facts concerning circled sets and homogeneous polynomials for later use.

Definition 1.147. *We say a set $K \subset \mathbb{C}^N$ is circled if $z \in K$ implies $e^{i\theta}z \in K$ for all $\theta \in [0, 2\pi]$.*

The following is a standard consequence of the Cauchy integral formula.

Lemma 1.148 (Pg 7, [16])**.** *Given a compact circled set $K \subset \mathbb{C}^N$ and a polynomial $p = h_d + h_{d-1} + ... + h_0$ of degree d written as a sum of homogeneous polynomials $h_0, ..., h_d$ of degree $0, ..., d$ respectively, we have*

$$\|h_j\|_K \le \|p\|_K, \quad \forall j \in \{1, ..., d\}.$$

Corollary 1.149. *Let V be an affine algebraic variety with distinct intersections with infinity. If $K \subset V$ is a compact circled set then α-Chebyshev polynomials for K for any α can be chosen to be homogeneous.*

Definition 1.150. *If p is a polynomial then we define the top degree homogeneous part of p (denoted $\hat{p}$) to be*

$$\hat{p}(z) = \lim_{t\to 0} t^{\deg p} p(z/t) = \textit{the sum of all terms of } p \textit{ with degree equal to } \deg p.$$

Remark 1.151. In the notation of Lemma 1.148, $\hat{p}(z) = h_d(z)$ where $d = \deg(p)$.

2 The Robin Function and First Results

The main goal of this section is to define a Robin function for algebraic varieties and show that it sastifies the following generalisation of a Bedford-Taylor result.

Theorem 2.1 (Theorem 2.67)**.** *Let $\mathcal{V}$ be a smooth irreducible algebraic variety with Noether presentation (x, y) which has distinct intersections at infinity. Let $u, v, w_2, ..., w_M \in \mathcal{L}^+(\mathcal{V})$ Then*

$$\int_{\mathcal{V}} (u\, dd^c v - v\, dd^c u) \wedge T = 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_u^* - \tilde{\rho}_v^*) \wedge \tilde{T}$$

where $T = dd^c w_2 \wedge ... \wedge dd^c w_M$.

This theorem is essential to obtaining the results from Bloom-Levenberg [15] and our main results in Section 3. The first three parts of this chapter are dedicated to building the preliminary material to prove this result. Section 2.1 explicitly proves a Quasicontinuity Theorem (Corollary 2.8) and Comparison Theorem (Theorem 2.3) on affine algebraic varieties – these results were first obtained on an algebraic variety by Zeriahi through using classical arguments. For our work it is sufficient to consider only smooth affine algebraic varieties and so these results are also valid on a complex manifold, and in that setting are well known. Section 2.2 builds on the perliminary material to prove a Mass Comparison Theorem (Theorem 2.16) for algebraic varieties. Section 2.3 utilises the Mass Comparison Theorem and 'good' coordinates (which we call a Noether presentation, Definition 2.22) to calculate the mass of functions in $\mathcal{L}^+(\mathcal{V})$ (Theorem 2.27) which as far as we can tell is an original result.

Section 2.4 examines the definitional problems for the Robin function on an algebraic variety and culminates in a definition for the Robin function in Section 2.5 (Definition 2.52). Some preliminary properties are proven. Section 2.6 contains the proof of Theorem 2.67. Section 2.7 contains a sample calculation to illustrate Theorem 2.67 while Section 2.8 justifies some of the hypotheses imposed through this section.

Unless explicitly stated otherwise, all algebraic varieties henceforth will be affine algebraic varieties.

2.1 Comparison and Quasicontinuity Theorems

Two important theoretical results due to Bedford-Taylor are the following theorems:

Theorem 2.2 (Quasicontinuity, [37] 3.5.5)**.** *Let $u \in PSH \cap L^\infty_{loc}(\Omega)$, where Ω is an open subset of $\mathbb{C}^N$. For each $\varepsilon > 0$ there exists an open subset E of Ω such that $cap_N(w, \Omega) < \varepsilon$ and the restriction of u to $\Omega \backslash E$ is continuous, where*

$$cap_N(E, \Omega) = \sup \left\{ \int_E (dd^c u)^N : u \in PSH(\Omega, (0,1)) \right\}.$$

Theorem 2.3 (Comparison Theorem, [37] 3.7.1)**.** *Let Ω be a bounded open subset of $\mathbb{C}^N$ and let $u, v \in PSH \cap L^\infty(\Omega)$ be such that for each $w \in \partial\Omega$*

$$\liminf_{\substack{z \to w \\ z \in \Omega}} (u(z) - v(z)) \geq 0. \tag{2}$$

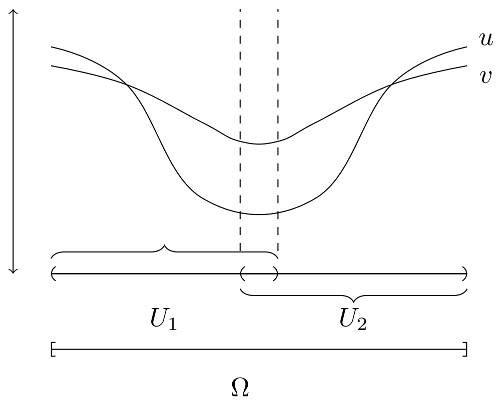
(i.e. $u \geq v$ on $\partial\Omega$) Then

$$\int_{\{u<v\}} (dd^c v)^N \leq \int_{\{u<v\}} (dd^c u)^N.$$

Generalisations of the above were considered by Zeriahi in [55] and [56] (Theorem 1.9), who claimed that they could be obtained using the methods of Bedford-Taylor (essentially by the arguments presented in Klimek [37]). We will simplify the proof of the Quasicontinuity theorem by utilisation a localisation argument and give the details to prove the Comparison theorem along the lines claimed by Zeriahi.

The Quasicontinuity Theorem is local so can be reduced to the classical case, provided the relative capacities behave appropriately. This means the result can be obtained directly from the $\mathbb{C}^N$ case without needing to re-develop the theory.

The Comparison Theorem depends on the boundary data from equation (2) so only reduces to the classical case provided there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$ for $\mathcal{V}$ such that $\{u < v\}$ is contained U_j for some j. The technicality is illustrated in the diagram below.



Suppose that $\mathcal{V}$ is M-dimensional. When U_1 is pushed forward via ϕ_1 to $\mathbb{C}^M$ the push forward of the boundary of $\{u < v\} \cap U_1$ does not satisfy the boundary data (2). As such the classical theory yields no information for us. Removing this obstruction is more difficult than building up the theorem from Bedford-Taylor methods. Thus in order to obtain a generalisation of Theorem 2.3 we pursue a direct proof. It is worth pointing out that Guedj-Zeriahi [31] obtained a Comparison Theorem for compact Kähler manifolds. One can probably reduce the affine algebraic variety version from this case, but a number of useful technical results are obtained in the pursuit of a classically inspired argument hence our choice to present the result in this way.

2.1.1 Quasicontinuity

Our strategy to prove a generalisation of Theorem 2.2 is to reduce the problem to the classical case by relating a version of the relative capacity on $\mathcal{V}$ to the relative capacity in $\mathbb{C}^M$. Recall that we are using the notation that an atlas $\{(X_j, \phi_j)\}$ of $\mathcal{V}$ is a covering of $\mathcal{V}$ by open sets X_j and for each j there is a biholomorphic mapping $\phi_j : X_j \to \phi(X_j) \subset \mathbb{C}^M$. Also recall that $\phi_j^* u = u \circ \phi^{-1}$. To begin note the following.

Lemma 2.4. *Suppose that $\mathcal{V}$ is an M-dimensional algebraic variety, and let $\Omega \subset \mathcal{V}$ be an open set, $\{(X_j, \phi_j) : 1 \leq j \leq n\}$ be a finite atlas for Ω,* and $u \in PSH \cap L^\infty_{loc}(\Omega)$. Then for all $\varepsilon_1, ..., \varepsilon_n > 0$ there exists an open set E of Ω such that u restricted to $\Omega \backslash E$ is continuous and*

$$cap_M(\phi_j(X_j \cap E), \phi_j(X_j)) < \varepsilon_j.$$

Proof. Let $u \in PSH \cap L^\infty_{loc}(\Omega)$ then $u_j := u|_{X_j} \in PSH \cap L^\infty_{loc}(X_j)$ by restriction. Since plurisubharmonicity is preserved under holomorphic maps, $\phi_j^* u_j \in PSH \cap L^\infty_{loc}(\phi_j(X_j))$. Since $\phi_j(X_j) \subset \mathbb{C}^M$ we may use the classic theory (i.e. Theorem 2.2) to deduce that we can find a (possibly empty) subset $E_j \subset \phi_j(X_j)$ such that $cap_M(E_j, \phi_j(X_j)) < \varepsilon$ and $\phi_j^* u_j$ is continuous on $\phi(X_j) \backslash E_j$. If we let $E = \bigcup_j \phi_j^{-1}(E_j)$ then the conclusion of the lemma is satisfied. $\square$

It is desirable to be able to formulate this result so that it says that the set E is 'small' in $\mathcal{V}$, rather than its image under ϕ_α in $\mathbb{C}^M$. To this end we define the relative capacity for the variety.

Definition 2.5. *Suppose that $\mathcal{V} \subset \mathbb{C}^N$ is a smooth M-dimensional algebraic variety and that $E \subset \Omega \subset \mathcal{V}$ where Ω is an open set and E is a Borel set. Then we set*

$$cap_{\mathcal{V}}(E, \Omega) := \sup \left\{ \int_E (dd^c u)^M : u \in PSH(\Omega), 0 \leq u \leq 1 \right\}.$$

We say that $cap_{\mathcal{V}}(E, \Omega)$ is the relative Monge-Ampère capacity (or simply, relative capacity).

Remark 2.6. This coincides with Zeriahi's definition (Equation (1.13), [56]) except Zeriahi makes no assumption of smoothness of $\mathcal{V}$, which leads to the supremum being over all weakly psh functions in $\Omega \subset \mathcal{V}$. We note that Zeriahi doesn't supply a quasicontinuity theorem with respect to this capacity in [56].

Using the generalised CLN inequality (Theorem 1.78) it is easily shown that $cap_{\mathcal{V}}(E, \Omega)$ is finite when E is compact.

Theorem 2.7. *Suppose that $\mathcal{V}$ is a smooth M-dimensional algebraic variety and let $\Omega \subset \mathcal{V}$ be*

*Recall that a projective algebraic variety is compact, hence there exists a finite atlas $\{(X_j, \phi_j) : 1 \leq j \leq n\}$ for any projective algebraic variety. Since every affine algebraic variety is the local restriction of some projective algebraic variety, we can take the local restriction of the finite atlas to induce a finite atlas for an affine algebraic variety.

an open set and $\{(X_j, \phi_j)\}$ a finite atlas for Ω. Then for any Borel subset $E \subset \Omega$,

$$cap_{\mathcal{V}}(E, \Omega) \leq \sum_j cap_M(\phi_j(E \cap X_j), \phi_j(X_j)).$$

Proof. Let $E_j = E \cap X_j$. Then clearly $\bigcup_j E_j = E$. It follows from the definition that

$$cap_{\mathcal{V}}\left(\bigcup_j E_j, \Omega\right) \leq \sup_u \sum_j \int_{E_j} (dd^c u)^M \leq \sum_j \sup_u \int_{E_j} (dd^c u)^M = \sum_j cap_{\mathcal{V}}(E_j, \Omega).$$

Given a function u which is a competitor for the relative capacity $cap_{\mathcal{V}}(E_j, \Omega)$ we observe that $\phi_j^* u$ is a competitor for the relative capacity $cap_M(\phi_j(E_j), \phi_j(X_j))$. It follows that

$$\int_{E_j} (dd^c u)^M = \int_{\phi_j(E_j)} (dd^c \phi_j^* u)^M \leq \sup_{v \in PSH(\phi_j(X_j))} \int_{\phi_j(E_j)} (dd^c v)^M = cap_M(\phi_j(E_j), \phi_j(X_j)).$$

By taking the supremum over the left hand side we conclude that $cap_{\mathcal{V}}(E_j, \Omega) \leq cap_M(\phi_j(E_j), \phi_j(X_j))$. Combining both parts of this argument yields

$$cap_{\mathcal{V}}(E, \Omega) \leq \sum_j cap_{\mathcal{V}}(E_j, \Omega) \leq \sum_j cap_M(\phi_j(E_j), \phi_j(X_j)).$$

$\square$

Corollary 2.8 (Quasicontinuity theorem for varieties)**.** *Let $\Omega \subset \mathcal{V}$ where $\mathcal{V}$ is a smooth M-dimensional algebraic variety and Ω is open. Suppose that $u \in PSH \cap L^\infty_{loc}(\Omega)$. Then for all $\varepsilon > 0$ there exists an open set E of Ω such that $cap_{\mathcal{V}}(E, \Omega) < \varepsilon$ and u restricted to $\Omega \backslash E$ is continuous.*

2.1.2 Comparison theorem

Essential for the following result is that given a branch cut C for $\mathcal{V}$ over $\mathbb{C}^M$ the projection π is biholomorphic when restricted to the branches of $\mathcal{V}$.

Theorem 2.9. *Let C be a branch cut for a smooth algebraic variety $\mathcal{V}$ over $\mathbb{C}^M$. Suppose Ω is a bounded open subset of $\mathcal{V} \backslash \pi^{-1}(C)$. Let $u, v \in PSH \cap L^\infty_{loc}(\Omega)$ be such that for each $w \in \partial\Omega$*

$$\liminf_{\substack{z \to w \\ z \in \Omega}} (u(z) - v(z)) \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^M \leq \int_{\{u<v\}} (dd^c u)^M.$$

Proof. Since every point of Ω is generic (by Corollary 1.42) it follows that Ω is a d-sheeted covering over $W \subset \mathbb{C}^M$ for some d. We can enumerate these sheets $\Omega_1, ..., \Omega_d$ and now the projection $\pi_i : \Omega_i \to W$ is biholomorphic for each i. The boundary data of Ω is preserved under the holomorphic map π_i since Ω is the disjoint union of all the sheets Ω_i. $\pi_i^* u$ and $\pi_i^* v$ are both psh functions in $\mathbb{C}^M$ and so using $\mathbb{C}^M$ theory we have

$$\int_{\{\pi_i^* u < \pi_i^* v\}} (dd^c \pi_i^* v)^M \leq \int_{\{\pi_i^* u < \pi_i^* v\}} (dd^c \pi_i^* u)^M.$$

Transferring this statement back to the variety we obtain

$$\int_{\{u<v\}\cap\Omega_i} (dd^c v)^M \leq \int_{\{u<v\}\cap\Omega_i} (dd^c u)^M.$$

It follows that

$$\int_{\{u<v\}} (dd^c v)^M = \sum_{i=1}^{d} \int_{\{u<v\}\cap\Omega_i} (dd^c v)^M \leq \sum_{i=1}^{d} \int_{\{u<v\}\cap\Omega_i} (dd^c u)^M = \int_{\{u<v\}} (dd^c u)^M.$$

□

With some care, this result would be all that we need for the remainder of this thesis. However, as already noted, we can obtain this result for $\Omega \subset \mathcal{V}$ which are not entirely generic and we will present that result for the sake of completeness.

The obstruction for when we cannot make a branch cut which avoids Ω is that π is only locally biholomorphic. This has two consequences; (i) if $u \in PSH(\mathcal{V})$ it is not necessarily true that $\pi^* u \in PSH(\mathbb{C}^M)$ (ii) the boundary data for Ω in the hypothesis may straddle any branch cut and may no longer being admissible upon being mapped by π as in the example from the introductory part of this section. Resolving these issues is more work than providing a direct proof.

We can proceed by direct proof because logically the only fact which the result depends on in the $\mathbb{C}^N$ case is integration by parts and Stokes theorem. Since both of these are valid on a smooth algebraic variety we can use the same deductions to deduce the result.

Recall the following results from [24].

Lemma 2.10 ([24], Lemma 1.9). *Let f_k be a sequence of usc functions converging to f on some separable locally compact space X and μ_k a sequences of positive measures converging weakly to μ on X. Then every weak limit ν of $f_k \mu_k$ satisfies $\nu \leq f\mu$.*

Theorem 2.11 ([24], Theorem 1.7). *Let $u_1, ..., u_q$ be locally bounded plurisubharmonic functions and let $u_1^k, ..., u_q^k$ be decreasing sequences of plurisubharmonic functions converging pointwise to $u_1, ..., u_q$. Then*

(a) $u_1^k dd^c u_2^k \wedge ... \wedge dd^c u_q^k \wedge T \longrightarrow u_1 dd^c u_2 \wedge ... \wedge dd^c u_q \wedge T$ *weakly.*

(b) $dd^c u_1^k \wedge ... \wedge dd^c u_q^k \wedge T \longrightarrow dd^c u_1 \wedge ... \wedge dd^c u_q \wedge T$ *weakly.*

Where T is an arbitrary positive current.

We need a slightly stronger version of this theorem, originally due to Bedford-Taylor.

Theorem 2.12. *Suppose that $\mathcal{V}$ is a smooth M-dimensional algebraic variety. Let $u_1, ..., u_q$ be locally bounded plurisubharmonic functions and let $u_1^k, ..., u_q^k$ be monotone (either increasing or decreasing) sequences of plurisubharmonic functions converging almost everywhere to $u_1, ..., u_q$. Then*

(a) $u_1^k dd^c u_2^k \wedge ... \wedge dd^c u_q^k \wedge T \longrightarrow u_1 dd^c u_2 \wedge ... \wedge dd^c u_q \wedge T$ *weakly.*

(b) $dd^c u_1^k \wedge ... \wedge dd^c u_q^k \wedge T \longrightarrow dd^c u_1 \wedge ... \wedge dd^c u_q \wedge T$ *weakly.*

Where T is an arbitrary positive current.

Proof. We only prove (a) as (b) follows along identical lines. The result is local so, given an atlas $\{(X_j, \phi_j)\}$ for $\mathcal{V}$, we prove the result on a chart (X_α, ϕ_α). Now each $\phi_\alpha^* u_l^k \in PSH \cap L_{loc}^\infty(\mathbb{C}^M)$ so we may use the $\mathbb{C}^M$ version of the result (Theorem 7.2, [3]). Using this we obtain

$$\phi_\alpha^* u_1^k dd^c \phi_\alpha^* u_2^k \wedge ... \wedge dd^c \phi_\alpha^* u_q^k \wedge \phi_\alpha^* T \longrightarrow \phi_\alpha^* u_1 dd^c \phi_\alpha^* u_2 \wedge ... \wedge dd^c \phi_\alpha^* u_q \wedge \phi_\alpha^* T \text{ weakly}$$

Pushing forward to X_α proves the claim.

□

With this theorem we have all the necessary tools to prove the comparison theorem for varieties along the lines of Klimek [37] (which in turn is a derivative of Cegrell [20]). We follow a version of this argument due to Błocki [9].

Theorem 2.13 (Comparison Theorem for Varieties)**.** *Let Ω be a bounded domain in a smooth algebraic variety $\mathcal{V}$ of dimension M. Let $u, v \in PSH \cap \mathcal{L}^\infty(\Omega)$ be such that for every $w \in \partial\Omega$,*

$$\liminf_{z \to w, z \in \Omega} (u(z) - v(z)) \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^M \leq \int_{\{u<v\}} (dd^c u)^M.$$

Lemma 2.14. *The above theorem holds for when u and v are continuous.*

Proof. Since u, v are continuous, $\Omega' = \{u < v\}$ is open, u, v are continuous on $\overline{\Omega}'$ and we may assume $u = v$ on $\partial\Omega'$. Let $u_\varepsilon := \max\{u + \varepsilon, v\}$. Then $u_\varepsilon \downarrow v$ on Ω' as $\varepsilon \downarrow 0$ and $u_\varepsilon = u + \varepsilon$ in a neighbourhood of Ω'. By Stokes theorem

$$\begin{aligned}\int_{\Omega'}(dd^c u_\varepsilon)^M &= \int_{\partial\Omega'} d^c u_\varepsilon \wedge (dd^c u_\varepsilon)^M \\ &= \int_{\partial\Omega'} d^c(u+\varepsilon)\wedge(dd^c(u+\varepsilon))^M \\ &= \int_{\partial\Omega'} d^c u \wedge (dd^c u^M) \\ &= \int_{\Omega'}(dd^c u)^M.\end{aligned}$$

Since u_ε is monotone decreasing in ε to v, by Theorem 2.12 we have

$$\int_{\{u<v\}}(dd^c v)^M \leq \lim_{\varepsilon\to 0}\int_{\{u<v\}}(dd^c u_\varepsilon)^M,$$

which proves the theorem if u and v are continuous. □

Proof of Theorem 2.13. We may assume that $\liminf(u-v) \geq \delta > 0$ for $\delta > 0$ on $\partial\Omega$ by considering $u+\delta$ in place of u since $\{u+\delta<v\}\uparrow\{u<v\}$ as $\delta\downarrow 0$.

Let W be a domain such that $\{u \leq v+\delta/2\}\subset \overline{W}\subset\Omega$ where inclusion is strict. We can find sequences u_j and v_k of smooth psh functions in a neighbourhood of $\overline{W}$ decreasing to u and v respectively and such that $u_j \geq v_k$ on ∂W for every j,k. We may assume that $-1\leq u_j, v_k \leq 0$ as per our usual regularisation process. We have

$$\int_{\{u<v\}}(dd^c v)^M = \lim_{j\to\infty}\int_{\{u_j<v\}}(dd^c v)^M. \tag{3}$$

Let $\varepsilon > 0$, by the Quasicontinuity Theorem we can find an open set G in Ω such that $cap_{\mathcal{V}}(G,\Omega) < \varepsilon$ and u, v continuous on $F = \Omega\backslash G$. There is a continuous function ϕ on Ω such that $v = \phi$ on F. Since $\{u_j < v\}\subset\{u_j<\phi\}\cup G$ and since $\{u_j<\phi\}$ is open,

$$\int_{\{u_j<v\}}(dd^c v)^M \leq \left(\int_{\{u_j<\phi\}} + \int_G\right)(dd^c v)^M \leq \liminf_{k\to\infty}\int_{\{u_j<\phi\}}(dd^c v_k)^M + \varepsilon. \tag{4}$$

Where we have used the quasicontinuity estimate in the second inequality. From the continuous case we have

$$\int_{\{u_j<v_k\}}(dd^c v_k)^M \leq \int_{\{u_j<v_k\}}(dd^c u_j)^M. \tag{5}$$

From equations (3), (4), the fact that $\{u_j < \phi\} \subset \{u_j < v_k\}$ and (5) we deduce

$$\begin{aligned}
\int_{\{u<v\}} (dd^c v)^M &= \liminf_{j\to\infty} \int_{\{u_j<v\}} (dd^c v)^M \\
&\leq \liminf_{j\to\infty} \liminf_{k\to\infty} \int_{\{u_j<\phi\}} (dd^c v_k)^M + \varepsilon \\
&\leq \liminf_{j\to\infty} \liminf_{k\to\infty} \int_{\{u_j<v_k\}\cap F} (dd^c v_k)^M + 2\varepsilon \\
&\leq \liminf_{j\to\infty} \liminf_{k\to\infty} \int_{\{u_j<v_k\}\cap F} (dd^c u_j)^M + 2\varepsilon \\
&\leq \liminf_{j\to\infty} \int_{\{u_j\leq v\}} (dd^c u_j)^M + 2\varepsilon. \qquad (6)
\end{aligned}$$

We use the quasicontinuity estimate again to deduce that

$$\int_{\{u_j\leq v\}} (dd^c u_j)^M \leq \int_{\{u_j\leq v\}\cap F} (dd^c u_j)^M + \varepsilon \qquad (7)$$

and since the set $\{u \leq v\} \cap F$ is compact, and $\{u_j \leq v\} \subset \{u \leq v\}$,

$$\limsup_{j\to\infty} \int_{\{u_j\leq v\}\cap F} (dd^c u_j)^M \leq \int_{\{u\leq v\}\cap F} (dd^c u)^M \leq \int_{\{u\leq v\}} (dd^c u)^M. \qquad (8)$$

Appending the deduction in (8) to the deduction from (7), and then appending that to the deduction in (6) we obtain,

$$\int_{\{u<v\}} (dd^c v)^M \leq \int_{\{u\leq v\}} (dd^c u)^M + 3\varepsilon.$$

Because $\varepsilon > 0$ was arbitrary we have

$$\int_{\{u<v\}} (dd^c v)^M \leq \int_{\{u\leq v\}} (dd^c u)^M.$$

This implies that for every $\eta > 0$,

$$\int_{\{u+\eta<v\}} (dd^c v)^M \leq \int_{\{u+\eta\leq v\}} (dd^c (u+\eta))^M = \int_{\{u+\eta\leq v\}} (dd^c u)^M.$$

The theorem follows since $\{u + \eta < v\} \uparrow \{u < v\}$ and $\{u + \eta \leq v\} \uparrow \{u < v\}$ as $\eta \downarrow 0$. □

2.2 Mass Comparison Theorem

The aim of this section is to prove an analogue of the following 'Mass Comparison Theorem' for a smooth algebraic variety.

Theorem 2.15 (Mass Comparison Theorem, Theorem 5.5.1, [37])**.** *Let* $u, v \in PSH \cap L^\infty_{loc}(\mathbb{C}^N)$.

If $v > 0$ outside a bounded subset of $\mathbb{C}^N$ and $u(z) = v(z) + o(z)$ as $\|z\| \to \infty$ then

$$\int_{\mathbb{C}^N} (dd^c u)^N \leq \int_{\mathbb{C}^N} (dd^c v)^N$$

Theorem 2.16 (Mass Comparison Theorem)**.** *Suppose that $\mathcal{V}$ is a smooth M-dimensional algebraic variety. Let $u, v \in PSH \cap L^\infty_{loc}(\mathcal{V})$. If $v > 0$ outside a bounded subset of $\mathcal{V}$ and $u(z) = v(z) + o(z)$ as $\|z\| \to \infty, (z \in \mathcal{V})$ then*

$$\int_{\mathcal{V}} (dd^c u)^M \leq \int_{\mathcal{V}} (dd^c v)^M$$

Proof. The proof follows the ideas of Klimek [37]. Let $\varepsilon, c > 0$ and define $w_{\varepsilon,c} := (1+\varepsilon)v - c$ and $W_{\varepsilon,c} = \{z \in \mathcal{V} : w_{\varepsilon,c}(z) < u(z)\}$. As $\varepsilon > 0$, the set $W_{\varepsilon,c}$ is bounded. From the Comparison Theorem we have

$$\int_{W_{\varepsilon,c}} (dd^c u)^M \leq \int_{W_{\varepsilon,c}} (dd^c w_{\varepsilon,c})^M = (1+\varepsilon)^M \int_{W_{\varepsilon,c}} (dd^c v)^M \leq (1+\varepsilon)^M \int_{\mathcal{V}} (dd^c v)^M.$$

Letting $c \to \infty$ we obtain

$$\int_{\{u > -\infty\}} (dd^c u)^M \leq (1+\varepsilon)^M \int_{\mathcal{V}} (dd^c v)^M.$$

Since $u \in L^\infty_{loc}(\mathcal{V})$ we know that $\int_{\{u=-\infty\}} (dd^c u)^M = 0$ so

$$\int_{\mathcal{V}} (dd^c u)^M \leq (1+\varepsilon)^M \int_{\mathcal{V}} (dd^c v)^M.$$

Letting $\varepsilon \to 0$ gives the result. □

Corollary 2.17. *If $u, v \in \mathcal{L}^+(\mathcal{V})$ then $\int_{\mathcal{V}} (dd^c u)^M = \int_{\mathcal{V}} (dd^c v)^M$.*

Proof. Apply previous theorem again but in reverse. □

We will encounter the following ideas frequently. While both are obvious, we record this result for clarity.

Lemma 2.18 (Conservation of Mass)**.** *Let $\mathcal{V}$ be a smooth M-dimensional algebraic variety. Suppose that C is a branch cut for $\mathcal{V}$ over $\mathbb{C}^M$ and $V_1, ..., V_d$ the corresponding branches of $\mathcal{V}$. Let T be a closed, positive, degree $2M$-current on $\mathcal{V}$.*

(i) $\displaystyle\int_{\mathcal{V}} T = \int_{\mathcal{V} \setminus B_\pi} T.$

(ii) If additionally $\operatorname{supp} T \subset \bigcup_{i=1}^d V_i$ *then* $\displaystyle\int_{\mathcal{V}} T = \sum_{i=1}^d \int_{V_i} T = \sum_{i=1}^d \int_{\pi_i(V_i)} \pi_i^* T.$

Proof. The first claim follows since B_π is a $M-1$ dimensional set by the Zariski-Nagata purity theorem. Since T is a $2M$-current it follows that the resulting integral is zero. For the second claim;

$$\int_{\mathcal{V}} T = \sum_{i=1}^{d} \int_{V_i} T + \int_{\pi^{-1}(C)} T.$$

The second integral is 0 because T is not supported on $\pi^{-1}(C)$. The final equality is just a change of variables. $\square$

2.3 Mass of Functions in the Class $\mathcal{L}^+(\mathcal{V})$ and Noether Presentations

We want to compute the (Monge-Ampère) mass of a particular function in $\mathcal{L}^+(\mathcal{V})$ and use Theorem 2.16 to deduce the mass of $u \in \mathcal{L}^+(\mathcal{V})$. The obvious candidate is $\log(1 + \|z\|)$ or $\log^+ \|z\|$ since either of these are easy to calculate in $\mathbb{C}^N$. However, for an arbitrary algebraic variety this computation isn't clear. We will find 'good' coordinates $z = (x, y)$ such that $\log^+ \|x\| \in \mathcal{L}^+(\mathcal{V})$. We open with an example to illustrate that this is not always the case.

Example 2.19. Take the curve $\{x^3 - xy + 1 = 0\} \subset \mathbb{C}^2$. Then $\pi : (x, y) \to x$ is not onto $\mathbb{C}^M$ ($x = 0$ is not on the curve). Now

$$\log^+ \|z\| = \frac{1}{2} \log^+(|x|^2 + |x^2 + x^{-1}|^2)$$

so $\log^+ \|z\| \to \infty$ when $x \to \infty$ or $x \to 0$. However $\log^+ |x| \to 0 \neq \infty$ as $x \to 0$ so $\log^+ |x|$ cannot be in $\mathcal{L}^+(\mathcal{V})$.

Example 2.20. $\mathbb{C}[\mathcal{V}]$ being finite over $\mathbb{C}[x]$ is not a sufficient condition either. Consider the curve $\mathcal{V} = \{x^3 - y = 0\}$. Then $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$. We calculate

$$\log^+ \|z\| = \frac{1}{2} \log^+(|x|^2 + |y|^2) = \frac{1}{2} \log^+(|x|^2 + |x|^6).$$

Since

$$\begin{aligned} & \log^+ |x|^6 + \alpha \leq \log^+ |x|^2 \\ \iff & \log^+ |x|^6 - \log^+ |x|^2 \leq -\alpha \\ \iff & \qquad \log^+ |x|^4 \leq -\alpha \qquad \lightning. \end{aligned}$$

It follows that there is no constant α such that $\log^+ \|z\| + \alpha \leq \log^+ |x|$ since $\log^+(|x|^6 + |x|^2) \geq \log^+ |x|^6$.

The proof of the following theorem uses a combination of ideas from Rudin [47], Sadullaev [50] and Demailly [24].

Theorem 2.21. *For a smooth irreducible algebraic variety $\mathcal{V}$ of (pure) dimension M there are $\mathbb{C}^N$ coordinates (x, y) satisfying the following properties*

(i) $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$.

(ii) For all $(x, y) \in \mathcal{V}$ we have the growth estimate $\|y\| \leq A(1 + \|x\|)$.

By a theorem due to Rudin [47] we know that a necessary and sufficient condition for $\mathcal{V}$ to be an algebraic variety is that we can find coordinates (x, y) that satisfy the estimate $\|y\| \leq A(1 + \|x\|)^B$ for some choice of positive constants A, B. Our theorem says that an algebraic variety has coordinates satisfying the estimate for $B = 1$ and that (x, y) satisfy the Noether normalisation theorems. Coordinates satisfying only the second condition were studied by Zeriahi in [57].

Proof. We use induction on the co-dimension of the variety.

Base Case: Co-dimension 1

$\mathcal{V}$ is defined by the zero set of a single polynomial P. If d is the total degree of P, then we can find a linear change of coordinates so that P is monic in y and that the highest power of y is d. That is, we can write

$$P(x, y) = y^d + \sum_{j=0}^{d-1} y^j Q_j(x),$$

where $\deg(Q_j) \leq d - j$. It is clear that $\mathbf{I}(\mathcal{V}) \cap \mathbb{C}[x] = \{0\}$ since $\mathbf{I}(\mathcal{V}) = \langle \mathrm{LT}(P) \rangle = \langle y^d \rangle$. We now fix a monomial ordering of 1-elimination type (where $y^\alpha \geq x^\beta$ for any $\alpha \in \mathbb{N}, \beta \in \mathbb{N}^{d-1}$). Since $P(x, y)$ is trivially a Gröbner basis for $\mathbf{I}(\mathcal{V})$ (Definition 1.103) with respect to this ordering, it follows that $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$ (Theorem 1.122). Fix some $x \in \mathbb{C}^{N-1}$ so that P is a one variable polynomial in y. Then any root of $P(x, y)$ satisfies the estimate

$$|y| \leq 2 \max_{1 \leq j \leq d-1} |Q_j(x)|^{1/d-j},$$

since otherwise

$$\begin{aligned}
|P(x, y)y^{-d}| &= |1 + Q_{d-1}(x)y^{-1} + ... + Q_0(x)y^{-d}| \\
&\geq 1 - (|Q_{d-1}(x)y^{-1}| + ... + |Q_0(x)y^{-d}|) \\
&\geq 1 - (2^{-1} + ... + 2^{-k}) \\
&= 2^{-d} > 0
\end{aligned}$$

which contradicts (x, y) being a root of P. The total degree of Q_j is at most $d - j$ so $|Q_j(x)|^{1/d-j} \leq O(\|x\|)$. This implies that $|y| \leq C(1 + \|x\|)$ for some $C \in \mathbb{R}$ which completes the proof of the base case.

Induction Step: Co-dimension $N - M$

Suppose that the proposition holds for co-dimension $1, ..., N - M - 1$. For this case $\mathcal{V} = \{P_1(x_y) = ... = P_{N-M}(x, y) = 0\}$ since $\mathcal{V}$ is irreducible. If $d = \deg(P_1)$, we can find a linear change of coordinates so that $\mathrm{LT}(P_1) = y_{N-M}^d$ as in the first step of the proof. Write

$y' = (y_1, ..., y_{N-M-1})$. In these coordinates P_1 is in a form which allows us to use Corollary 1.110. It follows from this corollary that there exists $R_1, ..., R_s \in \mathbb{C}[x, y']$ such that (x, y') is a common zero of all R_i if and only if there exists $\zeta \in \mathbb{C}$ such that (x, y', ζ) is a common zero of all P_i. If η is the orthogonal projection of $\mathbb{C}^N$ to $\mathbb{C}^{N-1}$ sending $(x, y', y_{N-M}) \mapsto (x, y')$, then

$$\eta\mathcal{V} = \{R_1(x, y') = ... = R_s(x,' y) = 0\}.$$

Hence $\eta\mathcal{V}$ is an algebraic variety. From the Noether Normalisation Theorem (Theorem 1.125) it follows we can find coordinates (x', y'') so that $\mathbb{C}[\eta\mathcal{V}]$ is finite over $\mathbb{C}[x']$ and k dimensional where k is the number of x' variables. Applying the same coordinate transformation to (x, y', y_{N-M}) to obtain (x', y'', y_{N-M}) and using the fact that $\mathbb{C}[z]/\mathbf{I}(P_1)$ is finite over $\mathbb{C}[x', y'']$ (since it is a Gröbner basis) we deduce that $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x']$ as well, hence the dimension of $\mathcal{V}$ must also be k. It follows that $\eta\mathcal{V}$ is M dimensional. Hence co-dimension $N - M - 1$ in $\mathbb{C}^{N-1}$, so we may apply the inductive hypothesis to the coordinates (x', y''). This produces new coordinates $(\tilde{x}, \tilde{y})$ such that for some $A > 0$

$$\|\tilde{y}\| \leq A(1 + \|\tilde{x}\|). \tag{9}$$

As in the first step, write

$$P_1(\tilde{x}, \tilde{y}, y_{N-M}) = y_{N-M}^d + \sum_{j=0}^{d-1} y_{N-M}^j Q_j(\tilde{x}, \tilde{y})$$

where using the same procedure from the base case we obtain the estimate

$$|y_{N-M}| \leq 2 \max_{1 \leq j \leq d-1} |Q_j(\tilde{x}, \tilde{y})|^{1/d-j},$$

which shows that $|y_{N-M}| \leq A(1 + \|(\tilde{x}, \tilde{y})\|)$. Applying equation (9), we obtain $|y_{N-M}| \leq A'(1 + \|\tilde{x}\|)$. Finally it follows that $\|(\tilde{y}, y_{N-M})\| \leq A''(1 + \|\tilde{x}\|)$.

We must still show that $\mathbb{C}[\tilde{x}, \tilde{y}, y_{N-M}]/\mathbf{I}(\mathcal{V})$ is finite over $\mathbb{C}[\tilde{x}]$. By the inductive hypothesis we know that $\mathbb{C}[\tilde{x}, \tilde{y}]/\mathbf{I}(\eta\mathcal{V})$ is finite over $\mathbb{C}[\tilde{x}]$. By construction of P_1, $\mathbb{C}[\tilde{x}, \tilde{y}, y_{N-M}]/\mathbf{I}(P_1)$ is finite over $\mathbb{C}[\tilde{x}, \tilde{y}]$. Since relative finiteness is transitive, it follows that $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[\tilde{x}]$ as required. □

Definition 2.22. *Coordinates (x, y) which satisfy the conclusion of Theorem 2.21 will be called a Noether presentation for $\mathcal{V}$.*

Example 2.23. The proof of Theorem 2.21 gives an algorithm to compute Noether presentations for a given smooth algebraic variety. We present an example to further illustrate how the proof works. We define the following variety:

$$\mathcal{V} = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1^3 - z_1 z_2 z_3 + z_2^3 = z_1 - z_4 z_2 = 0\}.$$

Our goal is to find a Noether presentation for $\mathcal{V}$. To this end we study the elimination ideal arising from eliminating the z_1 variable. We obtain this by substituting $z_1 = z_4 z_2$ into the other equation. This yields

$$(z_4 z_2)^3 - z_4 z_2^2 z_3 + z_2^3 = z_2^2(z_2 z_4^3 - z_4 z_3 + z_2) = z_2^2((1 + z_4^3)z_2 - z_4 z_3) = 0.$$

The variety generated by $(1 + z_4^3)z_2 - z_4 z_3 = 0$ consists of points which extend to solutions in $\mathcal{V}$ (i.e. is an R_i polynomial from the Noether presentation proof, moreover the R_i polynomials correspond to an elimination ideal with respect to a coordinate). This is a degree 4 polynomial with leading term which is not monic, so we must make a linear change of variables to make this so. Let $u_1 = \frac{1}{2}(z_4 + z_2)$ and $u_2 = \frac{1}{2}(z_4 - z_2)$. In these coordinates we have

$$(u_1 + u_2)^3(u_1 - u_2) - (u_1 + u_2)z_3 = u_1^4 + 2u_1^3 u_2 - u_1 u_2^3 - u_2^4 - u_1 z_3 - u_2 z_3 = 0.$$

We now write this in a polynomial in u_1 with polynomial coefficients in u_2, z_3

$$u_1^4 + 2u_1^3 u_2 - u_1(u_2^3 + z_3) - (u_2^4 + u_2 z_3) = 0.$$

We have the estimate from Theorem 2.21 which asserts that

$$|u_1| \leq 2 \max\{|2u_2|, |u_2^3 + z_3|^{1/3}, |u_2^4 + u_2 z_3|^{1/4}\}.$$

Which implies that

$$|u_1| \leq O(\|(u_2, z_3)\|).$$

We now applying this linear transformation to the original variety to obtain the following equations

$$\begin{aligned} 0 &= z_1^3 - z_1(u_1 - u_2)z_3 + (u_1 - u_2)^3 \\ z_1 &= (u_1 + u_2)(u_1 - u_2). \end{aligned}$$

Using the same estimate as before we obtain

$$|z_1| \leq 2 \max\{|(u_1 - u_2)z_3|^{1/2}, |u_1 - u_2|\}.$$

Which implies

$$|z_1| \leq O(\|(u_1, u_2, z_3)\|).$$

When combined with the first estimate of this type we have

$$\|(z_1, u_1)\| \leq O(\|(u_2, z_3)\|).$$

We declare coordinates $x_1 = u_2, x_2 = z_3, y_1 = u_1, y_2 = z_1$. Writing out the defining equations for $\mathcal{V}$ in these coordinates (with respect to grevlex order) we have

$$\begin{aligned} 0 &= y_2^3 - y_1^3 - y_1 y_2 x_2 + y_1^2 x_1 - y_2 x_1 x_2 - y_1 x_1^2 + x_1^3 \\ 0 &= y_1^2 - x_1^2 - y_1. \end{aligned}$$

Since the leading term in each defining equation is a monic in y_i this implies that the conditions of the Relative Finiteness Theorem (Theorem 1.122) are satisfied and hence $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$. Hence, the coordinates (x_1, x_2, y_1, y_2) are a Noether presentation for $\mathcal{V}$. That is,

$$\mathcal{V} = \{(x_1, x_2, y_1, y_2) \in \mathbb{C}^4 : y_2^3 - y_1^3 - y_1 y_2 x_2 + y_1^2 x_1 - y_2 x_1 x_2 - y_1 x_1^2 + x_1^3 = y_1^2 - x_1^2 - y_2 = 0\},$$

with the estimate

$$\|(y_1, y_2)\| \leq A(1 + \|(x_1, x_2)\|),$$

and $\mathbb{C}[V]$ is finite over $\mathbb{C}[x]$.

We can now prove the following useful theorem.

Theorem 2.24. *Suppose that $z = (x, y)$ is a Noether presentation for a smooth algebraic variety $\mathcal{V}$. Then $\log^+ \|x\| \in \mathcal{L}^+(\mathcal{V})$.*

Proof. Since the projection π is onto $\mathbb{C}^M$ it follows that $\log |x|$ is defined for all $(x, y) \in \mathcal{V}$. Moreover since the fiber $\pi^{-1}(x)$ is finite it follows that $\sup\{\log^+ \|(x, y)\| : \pi(x, y) = x\} < \infty$ for any $x \in \mathbb{C}^M$. Thus $\log^+ \|z\| \to \infty \iff \log^+ \|x\| \to \infty$.

We need to show that $(\log^+ \|x\| - \log^+ \|z\|) = O(1)$. Firstly, we have the obvious inequality $\frac{1}{2} \log \|x\|^2) \leq \frac{1}{2} \log(\|x\|^2 + \|y\|^2$ which shows that $\log^+ \|x\| \leq \log^+ \|z\|$. For the other side we need to show that the basis (x, y) satisfies the condition $\|y\| < A(1+\|x\|)$ for some $A > 0$. This is the defining condition of a Noether presentation.

We can use this estimate to deduce that for $\|x\| \geq 1$,

$$(\log^+ \|x\| - \log^+ \|(x, y)\|) = \frac{1}{2} \log \left(1 + \frac{\|y\|^2}{\|x\|^2}\right) \leq \frac{1}{2} \log \left(1 + \frac{A^2(1 + \|x\|)^2}{\|x\|^2}\right) \leq \frac{1}{2} \log(1 + C) \in \mathbb{R}$$

for an appropriately chosen constant C. This finishes the proof. $\square$

Corollary 2.25. *If $u \in \mathcal{L}^+(\mathbb{C}^M)$ then $u'(x, y) := u(x)$ is in $\mathcal{L}^+(\mathcal{V})$.*

Our aim in this section is to calculate the mass of functions in $\mathcal{L}^+$. The following example gives an explicit calculation of the mass, and provides a method to calculate the mass in general.

Example 2.26. Let $\mathcal{V} = \{y^3 - xy + 1 = 0\}$ so that $\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$. $\mathcal{V}$ is defined by an irreducible polynomial so the ideal $\mathbf{I}(\mathcal{V})$ is radical. When one fixes $x = b$ the polynomial $y^3 - by + 1 = 0$ is reducible by the fundamental theorem of algebra and the ideal $\mathbf{I}(\mathcal{V})$ is radical for almost all choices of b since its factors are all unique. Let $\zeta = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$. When $b_k = \zeta^k 3/4^{1/3}$ for $k = 1, 2, 3$ we have

$$y^3 - yb_k + 1 = (y - \zeta^{3-k}/2^{1/3})^2(y + \zeta^k 4^{1/3})$$

and so the ideal $\mathbf{I}(y^3 - b_k y + 1)$ is not radical. One can calculate that the cardinality of the fiber $\pi^{-1}(b_k)$ is 2 while at every other point it is 3. This means that $\{(b_k, \zeta^{3-k}/2^{1/3}) : k = 1, 2, 3\}$ is the branch locus of $\mathcal{V}$. We can choose a branch cut C to be the union of rays $\{rb_k : r \geq 1, k = 1, 2, 3\}$.

$\mathbb{C}[\mathcal{V}]$ is finite over $\mathbb{C}[x]$ and additionally since $x = y^2 - y^{-1}$ we can find $A \in \mathbb{R}$ such that $|y| \leq A(1 + |y^2 + y^{-1}|) = A(1 + |x|)$. It follows that (x, y) is a Noether presentation for $\mathcal{V}$. By the calculation in the previous paragraph, $\mathcal{V}$ decomposes into three branches. Suppose that V_1, V_2, V_3 be the branches of $\mathcal{V}$ and let $\pi_i : V_i \to \mathbb{C}$ be the projection onto x. From one variable theory we know that $dd^c \log^+ |x|$ is Lebesgue measure on the unit circle, hence $dd^c \log^+ |x|$ is supported in $\cup V_i$. Hence

$$\int_{V_i} (dd^c \log^+ |x|) = \int_{\mathbb{C}\setminus C} (dd^c \log^+ |x|) = 2\pi.$$

It follows from the Conservation of Mass Lemma (Lemma 2.18) that

$$\int_{\mathcal{V}} (dd^c \log^+ |x|) = \sum_{i=1}^{3} \int_{V_i} (dd^c \log^+ |z|) = 6\pi.$$

Hence by Corollary 2.17 the mass of any $u \in \mathcal{L}^+(\mathcal{V})$ is 6π.

This example is generalised by the following theorem.

Theorem 2.27. *Suppose that $\mathcal{V}$ is a smooth algebraic variety with a Noether presentation (x, y). Let $u \in \mathcal{L}^+(\mathcal{V})$. Then $\int_{\mathcal{V}} (dd^c u)^M = d(2\pi)^M$ where d is the number of branches of $\mathcal{V}$.*

Proof. By Corollary 2.17 it suffices to compute the mass of any function in $\mathcal{L}^+(\mathcal{V})$ since every function in this class has the same mass. We choose $\log^+ |x|$ which is in $\mathcal{L}^+(\mathcal{V})$ by Theorem 2.24. Without loss of generality (i.e. by translating the variety if necessary) we may assume that $\overline{B}_1(0) \cap \pi(B_\pi) = \varnothing$ and C is a branch cut for $\mathcal{V}$ over $\mathbb{C}^M$ which avoids the closed ball $\overline{B}_1(0)$. Let $V_1, ..., V_d$ be the branches resulting from this branch cut. Then from classical pluripotential theory we have

$$\int_{V_j} (dd^c \log^+ |x|)^M = \int_{\mathbb{C}^M\setminus C} (dd^c \log^+ |x|)^M = \int_{\mathbb{C}^M} (dd^c \log^+ |x|)^M = (2\pi)^M,$$

where the fact $dd^c(\log^+ |x|)^M$ is supported in $\overline{B}_1(0) \subset \mathbb{C}^M\setminus C$ allows the second equality. From this observation it follows that $(dd^c \log^+ |x|)^M$ is supported in $\cup_{i=1}^d V_i$ and hence by the Conservation of Mass Lemma (Lemma 2.18),

$$\int_{\mathcal{V}} (dd^c u)^M = \sum_{j=1}^{d} \int_{V_j} (dd^c u)^M = d(2\pi)^M$$

as claimed. □

We recall the following result from algebraic geometry which gives an estimate for the number of branches d for an arbitrary algebraic variety, and computes the number of branches for d under certain conditions.

Theorem 2.28 ([22], §4.3 Proposition 8). *Suppose $\mathbf{I}(\mathcal{V}) \subset \mathbb{C}[x,y]$ is an ideal and that $\mathbb{C}[\mathcal{V}] = \mathbb{C}[x,y]/\mathbf{I}(\mathcal{V})$ is finite over $\mathbb{C}[x]$. If $m_i \in \mathbb{N}$ is such that $y_i^{m_i} \in \mathbf{I}(\mathcal{V})$ then the number of branches of $\mathcal{V}$ is at most $m_1 \cdot \ldots \cdot m_{N-M}$. If $\mathbf{I}(\mathcal{V})$ is radical and m_i minimal, then equality holds.*

2.4 Definitional issues for the Robin function

2.4.1 Desingularisation

While we almost exclusively work with smooth (non-singular) algebraic varieties it is still possible that an algebraic variety may be singular at infinity. We wish to study this case. We won't need any explicit desingularisation for our general results, but we will do explicit examples which will need this machinery. The following deep result due to Hironaka.

Theorem 2.29 (Resolution of Singularities, [35]). *Any reduced singular scheme X of finite type over a field of characteristic zero admits a strong resolution of its singularities. This is, for every closed embedding X into a regular ambient scheme W, there is a proper birational morphism ε from a regular scheme W' onto W that satisfies explicitness, embeddedness, excision, equivariance and effectiveness (see [34] for the definition of each of these properties). The induced morphism $\eta : X' \to X$ is called a strong desingularisation of X.*

The following Theorem is simply a restatement of the result above that captures the important aspects for application in our work.

Theorem 2.30 ([34]). *Suppose that $\mathcal{V}$ is an algebraic variety in $\mathbb{C}^N$. Then there exists a resolution of its singularities given by η. η is a surjective differentiable map from a complex manifold $\tilde{\mathcal{V}}$ to $\mathcal{V}$ which is almost everywhere a diffeomorphism. The points at which η fails to be a diffeomorphism is the set $\eta^{-1}(\mathcal{V}^{sing})$. Moreover, η is a composition of blowups of $\mathcal{V}$ in regular closed centers Z transversal to the exceptional loci.*

Simply put, the diffeomorphism η can be constructed by repeatedly 'blowing up' up the variety $\mathcal{V}$ until it is no longer singular. Our examples will exclusively be concerning algebraic curves in $\mathbb{C}^2$ so we will explain how the blow up process works in this instance. There are many fantastic resources for general blow ups and modern treatments of Hironaka's work, of note Hauser's rendition [34] is particularly accessible.

Definition 2.31. *Let Z be the origin in $\mathbb{C}^N$. Let $\mathbb{P}^{N-1}$ be projective space with homogeneous coordinates $w_1, ..., w_N$. We call the space $\mathbb{C}^N_{\mathcal{B}} := \{((z_1, ..., z_N), [w_1 : ... : w_N]) : z_i w_j = z_j w_i\}$ paired with the projection $\eta : \mathbb{C}^N_{\mathcal{B}} \to \mathbb{C}^N$ (induced by the projection from $\mathbb{C}^N \times \mathbb{P}^{N-1} \to \mathbb{C}^N$) the blow up of $\mathbb{C}^N$ at the point Z.*

By changing the origin one can blow up any point $Z \in \mathbb{C}^N$. Suppose that $\mathcal{V} \subset \mathbb{C}^2$ is an algebraic curve with a singularity at 0. Then the blow up of $\mathcal{V}$ at 0 is given by

$$\mathcal{V}_{\mathcal{B}} = \{((z_1, z_2), [t_1 : t_2]) \in \mathcal{V} \times \mathbb{P} : z_1 t_2 = t_1 z_2\}.$$

The importance of desingularisation is captured in the following two simple and well known results.

Lemma 2.32 (Proposition 2.2, [28])**.** *Let $\eta : \hat{X} \to X$ be a desingularisation of X. If u is a weakly psh function on X then there is a psh function $\hat{u}$ on $\hat{X}$ such that $u(x) = \max_{\eta^{-1}(x)} \hat{u}$ for $x \in X$. Conversely, if $\hat{u}$ is psh on $\hat{X}$ then $x \mapsto \max_{\eta^{-1}(x)} \hat{u}$ defines a weakly psh function on X.*

Lemma 2.33 (Proposition 2.3, [28])**.** *Let $\eta : \hat{X} \to X$ be a desingularisation of X and u a weakly psh function on X. Suppose that T is a positive current on X and $\hat{T}$ is the positive current on $\hat{X}$ equal to T on $\hat{X} \backslash \left(\eta^{-1} X^{sing}\right)$ and 0 otherwise. Then $dd^c u \wedge T = \eta_*(dd^c \hat{u} \wedge \hat{T})$.*

2.4.2 Obstructions at Infinity

We open with an example due to Coman-Guedj-Zeriahi [21] which illustrates some of the difficulty in defining the Robin function.

Example 2.34 ([21], Example 3.3). We define an algebraic curve $\mathcal{V} \subset \mathbb{C}^2$ by

$$\mathcal{V} = \{(z_1, z_2) \in \mathbb{C}^2 : p(z_1, z_2) = z_2^3 - z_1 z_2 + 1 = 0\}.$$

$\mathcal{V}$ is a smooth algebraic curve; to see this we compute $\frac{\partial}{\partial z_1} p(z_1, z_2) = -z_2$ and $\frac{\partial}{\partial z_2} p(z_1, z_2) = 3z_2^2 - z_1$. Since $\frac{\partial}{\partial z_1} p(z_1, z_2) = \frac{\partial}{\partial z_2} p(z_1, z_2) = 0$ only at $(0, 0) \notin \mathcal{V}$, it follows that $\mathcal{V}$ is non-singular at every point. Note that the points on $\mathcal{V}$ are equivalent to those on the rational curve $\{z_2^2 + z_2^{-1} = z_1\}$. This observation shows us that there are two 'paths to infinity', one as $z_2 \to \infty$ and another as $z_2 \to 0$. The projectivisation of $\mathcal{V}$ is

$$\tilde{\mathcal{V}} = \{[z_0 : z_1 : z_2] : z_2^3 - z_0 z_1 z_2 + z_0^3 = 0\} \subset \mathbb{P}^2.$$

Setting $z_0 = 0$ we can see there is only one point at infinity given by $a = [0 : 1 : 0]$ in projective coordinates. Define

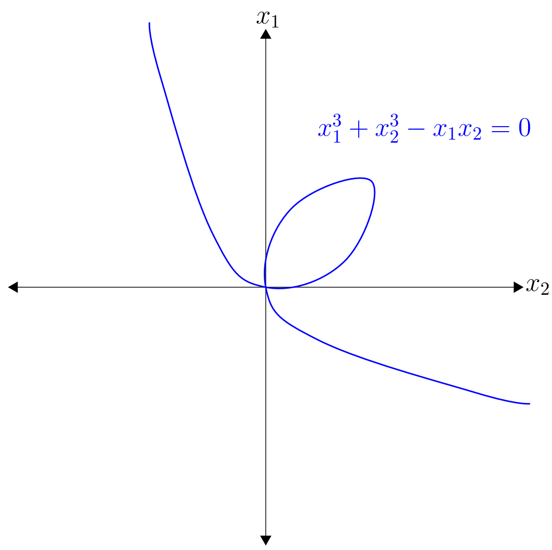
$$u(z_1, z_2) = \max\{-\log|z_2|, 2\log|z_2| + 1\},$$

then the naïve Robin function (i.e. $\tilde{\rho}_u(z) = \limsup_{\|z\| \to \infty} u(z) - \log\|z\|$) for u can be computed by

$$\limsup_{\substack{\|z\|\to\infty \\ |z_2|\to 0}} u(z_1, z_2) - \log\|z\| = 0$$
$$\limsup_{\substack{\|z\|\to\infty \\ |z_2|\to\infty}} u(z_1, z_2) - \log\|z\| = 1.$$

It follows that the value of $\tilde{\rho}_u(a)$ depends on the path taken to infinity. Moreover, $\tilde{\rho}_u(a)$ loses information about one of the 'branches' of the algebraic curve (since it takes the maximum of the two values) voiding the usefulness of the naïve Robin function. We remark that the results of Coman-Guedj-Zeriahi imply that when the Robin function for u depends on the path taken to infinity then $u \in \mathcal{L}(\mathcal{V})$ does not extend to a function in $\mathcal{L}(\mathbb{C}^N)$.

To understand what is going on we need to study the projective variety $\tilde{\mathcal{V}}$. We will examine the singularity at infinity by choosing a coordinate chart where the singularity is present. Let $X = \tilde{\mathcal{V}} \cap \{z_1 = 1\}$ to give the curve $\{(x_1, x_2) : x_1^3 + x_2^3 - x_1 x_2 = 0\}$ where $x_1 = z_0/z_1$ and $x_2 = z_2/z_1$.



Now the singularity is a local one (at $(0,0)$) and we can desingularise by blowing up at $(0,0)$.

$$X_{\mathcal{B}} = \{((x_1, x_2), [t_1 : t_2] \in X \times \mathbb{P} : t_2 x_1 = t_1 x_2\}.$$

In the coordinate chart $t_1 = t, t_2 = 1$ we have the condition

$$(tx_2)^3 + x_2^3 - tx_2^2 = 0$$
$$x_2^2(x_2(t^3+1) - t) = 0$$

and so for $x_2 \neq 0$ we must have $x_2 = \frac{t}{t^3+1}$. Substituting this into the original equation we get

$$x_1^3 - \frac{x_1 t}{t^3+1} + \frac{t^3}{(t^3+1)^3} = 0$$

which can be solved analytically in terms of t. One solution is $x_1 = t^2/(t^3+1)$. Then the parametrically defined curve

$$\hat{X} = \left\{ \left(\frac{t^2}{t^3+1}, \frac{t}{t^3+1}, t \right) \in \mathbb{C}^3 : t \in \mathbb{C} \right\}$$

is smooth and isomorphic to X except at $(0,0)$. It is easy to show that if η is the projection from $\hat{X}$ to X that $\eta^{-1}(0,0) = \{(0,0,0), (0,0,\infty)\}$. In $X_{\mathcal{B}}$ this corresponds to the points

$$\{((0,0)\,, [1:0]), ((0,0), [0:1])\}.$$

We can repeat this analysis at any point $y \in \mathbb{C}$ to deduce that the blowup $\tilde{\mathcal{V}}_{\mathcal{B}}$ consists of two points at infinity ($t = 0$)

$$\begin{aligned} a &= ([0:1:0], [1:0]) \in \tilde{\mathcal{V}}_{\mathcal{B}} \\ b &= ([0:1:0], [0:1]) \in \tilde{\mathcal{V}}_{\mathcal{B}}. \end{aligned} \tag{10}$$

The point a corresponds to taking the limit $\|z\| \to \infty, |z_2| \to 0$ while the point b corresponds to taking the limit $\|z\| \to \infty, |z_1| \to \infty$. In this space the Robin function has two values at infinity corresponding to the points a and b, moreover the lim sup along any path to a or b is independent of path. We draw attention to the fact that the point a has multiplicity 2 (i.e. branching order 2). This also poses a problem to defining the Robin function as we will see in Section 2.5.3.

Some Notation

At this point it is desirable to clarify the notation which we will use going forward.

Notation	Name	Ambient Space
$\mathcal{V}$	(Original) Variety	$\mathbb{C}^N$
$\tilde{\mathcal{V}}$	Projectivised Variety	$\mathbb{P}^N$
$\mathcal{V}_{\uparrow}$	Lifted Variety	$\mathbb{C}^{N+1}$
$\hat{\mathcal{V}}_{\uparrow}$	Desingularised (lifted) Variety	$\mathbb{C}^{N+1} \times X$ (X is unknown)
$\mathcal{V}^h$	Homogenised Variety	$\mathbb{C}^N$

The following relations describe the relationship between the objects in the table. We will always assume that $\mathcal{V} = \{P_j(z) = 0\}$ where $\{P_j\}$ form a Gröbner basis for $\mathcal{V}$.[†] Then,

- $\tilde{\mathcal{V}} := \{[t : z] \in \mathbb{P}^N : \zeta^{\deg P_j} P_j(z/\zeta) = 0\}$;
- $\mathcal{V}_\uparrow := \{(t, z) \in \mathbb{C}^{N+1} : \zeta^{\deg P_j} P_j(z/\zeta) = 0\}$;
- $\mathcal{V}^h = \mathcal{V}_\uparrow \cap H_\infty$;
- One way to obtain $\hat{V}_\uparrow$ is to repeatedly blow-up $\mathcal{V}_\uparrow$ until it is no longer singular.
- Given a branch cut C, we will use the notation $V_{1\uparrow}$ to be the ith branch of $\mathcal{V}_\uparrow$, as per Definition 1.46.

It will be notationally convenient to define $H_\infty = \{(t, z) \in \mathbb{C}^{N+1} : t = 0\}$, e.g. $\mathcal{V}_\uparrow \cap H_\infty = \mathcal{V}^h$. If $\pi : \mathcal{V} \to \mathbb{C}^M$, $(x, y) \mapsto x$ then we define $\pi_\uparrow : \mathcal{V}_\uparrow \to \mathbb{C}^{M+1}, (t, x, y) \mapsto (t, x)$. Note that $\pi_\uparrow$ induces a projection on both $\mathcal{V}$ and $\mathcal{V}^h$. Observe that $\tilde{\mathcal{V}} = \overline{\mathcal{V}}_{\mathbb{P}}$.

2.5 Construction of the Robin Function

As observed in the example, the Robin function is best understood from studying the lifted variety $\mathcal{V}_\uparrow$. To make this precise suppose that $u \in \mathcal{L}^+(\mathcal{V})$ and let

$$\mathcal{V}_\uparrow := \{(t, z) \in \mathbb{C}^{N+1} : t^{\deg P_1} P_1(z/t) = ... = t^{\deg P_{N-M}} P_{N-M}(z/t) = 0\}. \tag{11}$$

First we will assume that $\mathcal{V}_\uparrow$ is locally irreducible away from 0 and $\mathcal{V}_\uparrow \cap H_\infty \not\subset B_{\pi_\uparrow} \cap H_\infty$ and deal with other two cases shortly. This first case is the most important for this thesis since we will eventually show that 'distinct intersections with infinity' in the sense of Definition 1.133 satisfies this hypothesis. We include the other cases for completeness.

Definition 2.35. *Let π be the projection for an (affine) algebraic variety $\mathcal{V}$. We will say that $\mathcal{V}$ is branched at infinity if there is a component of $\mathcal{V}_\uparrow \cap H_\infty$ which is contained in $B_{\pi_\uparrow}$.*

2.5.1 $\mathcal{V}$ irreducible and not branched at infinity

The function $\tilde{u} : \mathcal{V}_\uparrow \backslash H_\infty \to [-\infty, \infty)$ defined by $(t, z) \mapsto u(z/t) + \log |t|$ is in $\mathcal{L}^+(\mathcal{V}_\uparrow \backslash H_\infty)$. But also for any $(t, z) \in \mathcal{V}_\uparrow \backslash H_\infty$ there are $\alpha, \beta \in \mathbb{R}$ such that

$$\log \|z/t\| + \alpha \le u(z/t) \le \log \|z/t\| + \beta$$

hence

$$\log \|z\| + \alpha \le \log \|z/t\| + \log |t| + \alpha \le u(z/t) + \log |t| \le \log \|z/t\| + \log |t| + \beta \le \log \|z\| + \beta.$$

[†]This assumption avoids some technical cases where $\tilde{\mathcal{V}} \neq \overline{\mathcal{V}}_{\mathbb{P}}$.

So $\tilde{u}(t,z)$ is locally bounded everywhere where is it defined. Since the hyperplane H_∞ is pluripolar it follow that $\tilde{u}(t,z)$ can be extended across $t=0$ by taking a lim sup. Precisely,

$$\tilde{u}(0,\zeta) = \limsup_{\substack{(t,z)\to(0,\zeta)\\(t,z)\in\mathcal{V}_\uparrow\setminus H_\infty}} u(z/t) + \log|t|. \tag{12}$$

Of course, excluding H_∞ from the lim sup is not necessary. Also note that $\tilde{u}(0,0) = -\infty$. This is important since $\mathcal{V}^h$ is always locally reducible at $z=0$, but because $\tilde{u}(0,0)$ is always $-\infty$ we can ignore the reducibility here (as we have done throughout).

Remark 2.36. At this point in the classical case one defines the formula

$$\rho_u(z) = \limsup_{t\to 0} u(z/t) + \log|t|, \tag{13}$$

to capture the behavior of $\tilde{u}(0,z)$. However this is not possible for us:

(i) The obvious problem is that z/t may not be on the variety yielding a nonsense expression.

(ii) To amend the problem in (i) we can restrict the limsup in the following way

$$\limsup_{\substack{t\to 0\\ z/t\in\mathcal{V}}} u(z/t) + \log|t|.$$

However this is still nonsense – the set $\{z/t : t\in\mathbb{C}\setminus\{0\}\}$ defines a hyperplane in $\mathbb{C}^{N+1}$ and hence, by Bezout's theorem, intersects $\mathcal{V}_\uparrow$ at only finitely many points.

As such simplifying the limsup in equation (12) is not a trivial matter.

Observe that $\mathcal{V}_\uparrow$ is a d-sheeted algebraic variety of codimension $N-M$ for some $d\in\mathbb{N}$, so we can find a Noether presentation (t,x,y) (where $\mathbb{C}[\mathcal{V}_\uparrow]$ is finite over $\mathbb{C}[t,x]$) and a branch cut C of $\mathbb{C}^{M+1}$ for $\mathcal{V}_\uparrow$ so that $\pi_{i\uparrow}^{-1} : \mathbb{C}^{M+1}\setminus C \to V_{i\uparrow}$ is biholomorphic for each i. Let $C^* = \pi_\uparrow^{-1}(C)$. We have $\pi_{i\uparrow}^*\tilde{u} \in \mathcal{L}^+(\mathbb{C}^M\setminus C)$ and moreover for any $(0,x,y)\in V_{i\uparrow}\setminus C^*$,

$$\limsup_{t\to 0} \pi_{i\uparrow}^*\tilde{u}(t,x) = \limsup_{(t,x,y(t,x))\to(0,x,y(0,x))} \tilde{u}(t,x,y(t,x)).$$

Given any point $(0,x,y)\in\mathcal{V}_\uparrow$ not contained in the branch locus $B_{\pi_\uparrow}$ we can find a branch cut C which avoids the point $(0,x,y)$. Hence we can emulate the $\mathbb{C}^N$ formula (13) on a variety in the following way:

$$\lim_{t\to 0} y(x/t) = \tilde{y}(x) \tag{14}$$

$$\rho_u(x,\tilde{y}(x)) = \limsup_{t\to 0} u(x/t, y(x/t)) + \log|t| \tag{15}$$

where we identify $\mathcal{V} \ni (x/t, y(x/t)/t)$ with $(t,x,y(t,x))\in\mathcal{V}_\uparrow$ and $(x,\tilde{y}(x))$ with $(0,x,y(0,x))\in$

$\mathcal{V}_\uparrow$. This identification is justified since by definition

$$\begin{aligned}(t, x, y(t, x)) \in \mathcal{V}_\uparrow \backslash H_\infty &\Longleftrightarrow (x/t, y(t, x)/t) \in \mathcal{V}. \\ &\Longleftrightarrow (x/t, y(x/t)) \in \mathcal{V}\end{aligned}$$

This makes sense because, subject to a suitable branch cut, y only depends on the value of the x coordinate on $\mathcal{V}$. As $(x/t, y(t,x)/t)$ is a point on $\mathcal{V}$, this logic forces $y(t,x)/t = y(x/t)$.

Example 2.37. Let $\mathcal{V} = \{x^2 + y^2 = 1\}$. Taking a suitable branch cut (e.g. $C = [-1, 1] \subset \mathbb{C}$) and projection we have $y(x) = \pm\sqrt{1 - x^2}$. Now $\mathcal{V}_\uparrow = \{x^2 + y^2 = t^2\}$ and so subject to a suitable branch cut and projection,

$$y(t, x) = \pm t\sqrt{1 - \frac{x^2}{t^2}}.$$

Clearly, $y(t,x)/t = y(x/t)$.

Consequently, we understand the limit in (14) to be taken within a branch of $\mathcal{V}$ with respect to a branch cut of $\mathcal{V}_\uparrow$ avoiding $(0, x, y(0, x)) \in \mathcal{V}_\uparrow$. This construction yields a well defined Robin function (since it captures the behavior of $\tilde{u}$ along H_∞) for all $(x, \tilde{y}(x))$ away from the branch locus of $\mathcal{V}^h$. But since the branch locus B_{π_h} is pluripolar (where π_h is the restriction of $\pi_\uparrow$ to $\mathcal{V}^h$), we can extend the Robin function across B_π by taking a lim sup.

Definition 2.38. *Suppose that $\mathcal{V}$ is an algebraic variety which is irreducible and not branched at infinity. Let $u \in \mathcal{L}^+(\mathcal{V})$. Suppose that*

$$\pi_i^{-1}(x) = (x, y_i(x)), \quad \textit{for a suitable branch cut } C \tag{16}$$

$$\lim_{\substack{t\to 0 \\ x/t \in \mathbb{C}^M \backslash C}} y_i(x/t) = \tilde{y}_i(x) \tag{17}$$

Then we define the Robin function $\rho_u : \mathcal{V}^h \backslash B_{\pi_h} \to [-\infty, \infty)$ pointwise by

$$\rho_u(x, \tilde{y}(x)) = \limsup_{\substack{t\to 0 \\ x/t \in \mathbb{C}^M \backslash C}} u(x/t), y(x/t)) + \log|t|. \tag{18}$$

We extend the Robin function to be defined everywhere on $\mathcal{V}^h$ by

$$\rho_u(x, y) = \limsup_{\mathcal{V}^h \backslash B_{\pi_h} \ni (\zeta, \eta) \to (x, y)} \rho_u(\zeta, \eta).$$

We will often omit the preamble (16, 17) and the extra hypothesis on the limsup in (18) to simplify notation. That is, the statement

$$\limsup_{t\to 0} u(x/t, y(x/t)) + \log|t| = \rho_u(x, y)$$

is to be understood as equation (18).

Remark 2.39. For clarity, by definition of $\pi_\uparrow$, the branch locus $B_\pi \subset \mathcal{V}_\uparrow$ induces the corresponding branch locus B_{π_h} for $\mathcal{V}^h$ and B_π for $\mathcal{V}$, and a branch cut for $\mathcal{V}_\uparrow$ over $\mathbb{C}^{M+1}$ induces a branch cut of $\mathcal{V}$ over $\mathbb{C}^M$ and $\mathcal{V}^h$ over $\mathbb{C}^M$.

2.5.2 $\mathcal{V}$ (locally) reducible and not branched at infinity

For the case when $\mathcal{V}$ is (locally) reducible at infinity (equivalently, $\mathcal{V}_\uparrow$ (locally) reducible along H_∞) we need to invoke desingularisation. Precisely, let $\eta : \hat{\mathcal{V}}_\uparrow \to \mathcal{V}_\uparrow$ be a desingularisation of $\mathcal{V}_\uparrow$. Then $\eta : \hat{\mathcal{V}}\backslash\eta^{-1}(\mathcal{V}_\uparrow^{sing}) \to \mathcal{V}_\uparrow^{reg}$ is a diffeomorphism. Let C be a branch cut for $\mathcal{V}_\uparrow$, then $\pi_{i\uparrow} : \mathcal{V}_i\backslash\pi^{-1}(C) \to \mathbb{C}^{M+1}\backslash C$ is a diffeomorphism. Recall that $\mathcal{V}_i \subset \mathcal{V}^{reg}$ by definition and that $\mathcal{V}_\uparrow^{sing} \subset H_\infty$. It follows then that the composition $\pi_{i\uparrow} \circ \eta : \hat{\mathcal{V}}\backslash\eta^{-1}(H_\infty) \to \mathbb{C}^{M+1}\backslash C$ is a diffeomorphism.

Our strategy now is to track points in $\hat{\mathcal{V}}\backslash\eta^{-1}(H_\infty)$ via points in $\mathbb{C}^{M+1}$ using these 'projections' and repeat the process laid out in the previous section.

In light of Example 2.34, for our strategy to be succesful we must ensure that our construction allows us to take limsups that capture all the required information. This is equivalent to the following.

Lemma 2.40. *Let $\mathcal{V}$ be an M-dimensional, d sheeted, smooth algebraic variety, and suppose that $V_\uparrow^{sing} \subset H_\infty$. Then there exists a branch cut C that satisfies the following: Let $Z_{\eta,C} = \{\eta^{-1}(0,z) : z \in \mathcal{V}_\uparrow\backslash(H_\infty\backslash\pi_\uparrow^{-1}(C))\}$, for any $\zeta \in Z_{\eta,C}$ there is a sufficiently small open neighbourhood $\zeta \in N_\zeta \subset \tilde{\mathcal{V}}$ such that $\eta(N_\zeta\backslash Z_{\eta,C}) \subset \mathcal{V}_{i_\uparrow}$ for exactly one $i \in \{1, ..., d\}$.*

Proof. Let $\pi_\uparrow$ be the projection to (t, x) and C' a branch cut for $\mathcal{V}_\uparrow$. Observe that $\eta^{-1}(H_\infty\backslash B_{\pi_\uparrow})$ has d fibers. Enumerate the fibers $F_1, ..., F_d$ (each F_i will be disconnected due to the exclusion of $B_{\pi_\uparrow}$). Let N_{F_i} be an open neighbourhood of F_i and declare that $\eta(N_{F_i}\backslash F_i) \subset \mathcal{V}'_i$ for each i. Now amend the branch cut C' to a branch cut C'' so that each $\mathcal{V}'_i$ belongs to one component of $\mathcal{V}_\uparrow\backslash C''$. Declare that the remaining components belong to one of the $\mathcal{V}'_i$ subject to the condition that $\pi : \mathcal{V}'_i \to \mathbb{C}^M$ is one to one. The branch cut C'' satisfies the conclusion of the Lemma. □

Remark 2.41. Observe that the branch cut constructed above, C'', does not give rise to connected branches $\mathcal{V}'_i$. This was a byproduct of ensuring that the limits are unique. This is not an issue for our work, but we note this curiosity nonetheless. Inspired by this Lemma we make a definition.

Definition 2.42. *Let $\pi : \mathcal{V} \to \mathbb{C}^M$ be a projection. We say a branch cut C is a distinguished branch cut for $\mathcal{V}$ over $\mathbb{C}^M$ if each sheet $\mathcal{V}_i$ has the following property. Let*

$$\mathcal{V}_{i\uparrow} := \{(t, z) \in \mathcal{V}_\uparrow : z/t \in \mathcal{V}_i\}.$$

Then each point of the set $\{(0, z) : \lim_{V_{i\uparrow}\ni(t,\zeta)\to(0,z)}(t, \zeta) = (0, z)\}$ is contained in the same local irreducible component of $\mathcal{V}_\uparrow$ near $(0, z)$ for all $1 \le i \le d$.

We define $\mathcal{V}_i^h := \left\{(0,z) \in \mathcal{V}^h : \lim_{\mathcal{V}_{i\uparrow} \ni (t,\zeta) \to (0,z)} (t,\zeta) = (0,z)\right\}$.

Remark 2.43. Any branch cut of an irreducible algebraic variety $\mathcal{V}$ which is irreducible at infinity is trivially a distinguished branch cut. The property formulated in the definition is the formalisation of saying that limits to infinity are independent of path in the sense of the preceding example.

Remark 2.44. The sets $\mathcal{V}_i^h$ form branches of $\mathcal{V}^h$.

Lemma 2.45. *If* $(x,y) \notin B_\pi$ *then there is a distinguished branch cut* C *such that* $x \notin C$ *and* $\pi_i^{-1}(x) = (x,y)$ *for some* i.

Proof. This follows since the induced branch cut from the desingularised variety can be chosen to avoid (x,y). □

Example 2.46. We construct a distinguished branch cut for $\mathcal{V} = \{y^3 - xy + 1 = 0\}$. Note that $\mathcal{V}$ is branched at infinity but the concept of distinguished branched cuts makes sense in this case as well. Recall that $B_\pi = \{\zeta^k 3/4^{1/3} : \zeta = (-1/2 + \sqrt{3}i/2), k = 1,2,3\}$. Let $C = \{rb : r \in [1,\infty), b \in B_\pi\}$ and $C' = \{rb : r \in [0,\infty), b \in B_\pi\}$ be branch cuts with projections $\pi : \mathcal{V} \to \mathbb{C}\backslash C$ and $\pi' : \mathcal{V} \to \mathbb{C}\backslash C'$.

Since $x = y^2 + y^{-1}$ as $|x| \to \infty$ either $y \to 0$ or $|y| \to \infty$. Since $\mathbb{C}\backslash C$ is connected $\pi_i^{-1} : \mathbb{C}\backslash C \to \mathcal{V}$ must contain regions where $y \to 0$ and $|y| \to \infty$. As noted in Section 2.4.2, the multiplicity of the $|y| \to \infty$ path with $\mathcal{V}^h$ is 2 and the $y \to 0$ path with $\mathcal{V}^h$ is 1. We can deduce that two of the three regions partitioned by C have π^{-1} mapping to components of $\mathcal{V}$ where $|y| \to \infty$ and the other region maps to a component of $\mathcal{V}$ where $y \to 0$. C is consequently not a distinguished branch cut.

Now consider the branch cut C'. Since $\mathbb{C}\backslash C'$ is partitioned into three regions we can choose a projection π' such that ${\pi'}_i^{-1}$ maps $\mathbb{C}\backslash C$ to a branch of $\mathcal{V}$ which preserves path to infinity. That is, we can choose the inverse projections ${\pi'}_1^{-1}, {\pi'}_2^{-1}, {\pi'}_3^{-1}$ to satisfy

$$\begin{aligned}
\lim_{|x|\to\infty} {\pi'}_1^{-1}(x) &= \lim_{|x|\to\infty} (x, y_1(x)) = (\infty, 0) \\
\lim_{|x|\to\infty} {\pi'}_2^{-1}(x) &= \lim_{|x|\to\infty} (x, y_2(x)) = (\infty, \infty) \\
\lim_{|x|\to\infty} {\pi'}_3^{-1}(x) &= \lim_{|x|\to\infty} (x, y_3(x)) = (\infty, \infty).
\end{aligned}$$

Of course, C' partitioning $\mathbb{C}$ into three regions is essential for ${\pi'}_i^{-1}$ to be biholomorphic since we cannot analytically continue ${\pi'}_i^{-1}$ across C' at any point. It follows that C' is a distinguished branch cut for $\mathcal{V}$ over $\mathbb{C}$.

Example 2.47. A distinguished branch cut need not partition $\mathbb{C}^M$ into disjoint regions. For instance, we can choose a branch cut of $x^2 + y^2 = 1$ to be the compact set $[-1, 1] \subset \mathbb{C}$. Since this curve is irreducible at infinity it follows that it is automatically distinguished.

We can now formulate a definition for the Robin function that includes the possibility that $\mathcal{V}_\uparrow$ is locally reducible along $t = 0$. The drawback is that it must be understood 'piecewise'.

Definition 2.48 (Robin functions for $\mathcal{V}$ (locally) reducible and not branched at infinity)**.** *Suppose that $\mathcal{V}$ is a smooth d-sheeted algebraic variety with Noether presentation (x, y). Let $u \in \mathcal{L}^+(\mathcal{V})$. Let C be a distinguished branch cut for $\mathcal{V}$ over $\mathbb{C}^M$. Suppose that*

$$\pi_i^{-1}(x) = (x, y_i(x)), \quad x \in \mathbb{C}^M \backslash C$$
$$\lim_{\substack{t \to 0 \\ x/t \in \mathbb{C}^M \backslash C}} y_i(x/t) = \tilde{y}_i(x).$$

Then we define the Robin function $\rho_u^i : \mathcal{V}_i^h \backslash B_\pi \to [-\infty, \infty)$ pointwise by

$$\rho_u(x, \tilde{y}_i(x)) = \limsup_{\substack{t \to 0 \\ x/t \in \mathbb{C}^M \backslash C}} u(x/t), y_i(x/t)) + \log |t|.$$

We extend the Robin function to be defined everywhere on V_i^h by

$$\rho_u^i(x, y) = \limsup_{\mathcal{V}_i^h \backslash B_{\pi_h} \ni (\zeta, \eta) \to (x, y)} \rho_u^i(\zeta, \eta).$$

Lemma 2.49. *If $X_1, ..., X_q$ are the local irreducible components of $\mathcal{V}_\uparrow \cap H_\infty$ (and hence are algebraic subvarieties) then for any $u \in \mathcal{L}^+(\mathcal{V})$ there is a well defined Robin function associated to each of $X_1, ..., X_q$.*

Proof. It suffices to show that we can define a Robin function on one of the X_j as the other cases are similar. Suppose that C is a distinguished branch cut, then each V_i^h belongs to exactly one of $X_1, ..., X_q$. Without loss of generality assume that $V_1^h, ..., V_n^h \subset X_1$ are all of the V_i^h's which are in X_1. Then $\bigcup_{j=1}^n V_j^h = X_1 \backslash S$ where S is a real $2M - 1$ dimensional set. Then setting $\rho_u = \rho_u^i(z)$ when $z \in V_i^h$ defines a psh function on $X_1 \backslash S$.

Next recall $\tilde{u}(t, z)$ can be extended to be psh in X_1 by taking the lim sup within X_1 (alternatively, use a desingularisation). Denote by $\tilde{u}_1(0, z)$ this extension of $\tilde{u}$. Then $\tilde{u}_1(0, z)$ coincides with ρ_u on $X_1 \backslash S$ and is psh. Since it agrees almost everywhere with ρ_u it follows that ρ_u has an extension to all of X_1 given by $\rho_u = \tilde{u}_1$. Repeating this for each X_j gives the result. $\square$

2.5.3 $\mathcal{V}$ branched at infinity

From the discussion in the previous section, if $\mathcal{V}$ is reducible at infinity we can instead study a desingularisation so it suffices to assume that $\mathcal{V}$ is smooth at infinity for the purpose of understanding this case. The basic problem of this section is that given $(0, x, y) \in \mathcal{V}_\uparrow \cap H_\infty$ then

$o_\pi(0, x, y) \geq 2$ (recall Definition 1.38). Equivalently, in the sense of Definition 2.38 (precisely equation (17)) there are i, j with $i \neq j$ such that

$$\lim_{t\to 0} y_i(x/t) = \tilde{y}_i(x) = \tilde{y}_j(x) = \lim_{t\to 0} y_j(x/t)$$

Example 2.46 exhibits this behavior along the $y \to \infty$ path. Situations like this it need not be true that the projection of the Robin function correctly captures the behavior of $\tilde{u}$ along $t = 0$ since the projection captures one of possibly many paths to that point. Taking the maximum over all possible paths is the way we will resolve this problem.

Definition 2.50 (Robin function for $\mathcal{V}$ irreducible and branched at infinity)**.** *Suppose that $\mathcal{V}$ is a smooth d-sheeted algebraic variety with Noether presentation (x, y) which is irreducible and branched at infinity. Let $u \in \mathcal{L}^+(\mathcal{V})$. Then we define the Robin function $\rho_u : \mathcal{V}^h \backslash B_\pi \to [-\infty, \infty)$ pointwise in the following way. Let C be a branch cut for $\mathcal{V}$ over $\mathbb{C}^M$.*

$$\begin{aligned}
&\pi_i^{-1}(x) = (x, y_i(x)), \quad x \in \mathbb{C}^M \backslash C \\
&\lim_{\substack{t\to 0 \\ x/t \in \mathbb{C}^M\backslash C}} \quad y_i(x/t) = \tilde{y}_i(x) \\
&J(x, \tilde{y}) = \{j : \tilde{y}_j(x) = \tilde{y}\} \\
&\rho_u(x, \tilde{y}(x)) = \max_{j\in J(x,\tilde{y})} \limsup_{\substack{t\to 0 \\ x/t \in \mathbb{C}^M\backslash C}} u(x/t), y_j(x/t)) + \log|t|.
\end{aligned}$$

We extend the Robin function to be defined everywhere on $\mathcal{V}^h$ by

$$\rho_u(x, y) = \limsup_{\mathcal{V}^h\backslash B_\pi \ni (\zeta,\eta)\to(x,y)} \rho_u(\zeta, \eta).$$

The obvious adaption can be made for when $\mathcal{V}$ is reducible and branched at infinity.

2.5.4 Geometry of Distinct Intersection with Infinity

Recall the following definition from Chapter 1.

Definition (Definition 1.133)**.** *We say a d-sheeted algebraic variety $\mathcal{V}$ has distinct intersections with infinity if it satisfies the following properties.*

(i) $\mathbb{C}[z_1, ..., z_M] \subset \mathbb{C}[\mathcal{V}]$ *is a Noether normalisation for $\mathcal{V}$.*

(ii) Let $P = \{\mathbf{V}(\{z_0, ..., z_{M-1}\}) \subset \mathbb{P}^M$. The set $\overline{\mathcal{V}}_{\mathbb{P}} \cap P = \tilde{\mathcal{V}} \cap P$ consists of d distinct points.

(iii) Let $\overline{\mathcal{V}}_{\mathbb{P}} \cap P = \{\lambda_1, ..., \lambda_d\}$ with $\lambda_i = [0 : ... : 0 : \lambda_{iM} : ... : \lambda_{iN}]$. Then for each i, $\lambda_{iM} \neq 0$.

In the context of the preceding discussion, we need to understand how 'bad' $\mathcal{V}$ is at infinity under this hypothesis in order to make a sensible Robin function definition. The following theorem does this.

Theorem 2.51. *Let $\mathcal{V} \subset \mathbb{C}^N$ be a d-sheeted, M-dimensional algebraic variety which has distinct intersections at infinity and write $(x, y) = z$ where $x \in \mathbb{C}^M, y \in \mathbb{C}^{N-M}$. Then $\dim(\mathcal{V}_\uparrow^{sing}) \leq M - 1$ and $\mathcal{V}^h$ is a d-sheeted finite branched holomorphic covering over $\mathbb{C}^M$.*

Proof. Suppose that $\mathcal{V} = \{f_1 = ... = f_{N-M} = 0\}$ so that $\mathcal{V}_\uparrow = \{\tilde{f}_1 = ... = \tilde{f}_{N-M} = 0\}$ where $\tilde{f}_i(t, z) = t^{\deg f_i} f(z/t)$. To study the singularities of $\mathcal{V}_\uparrow$ we study the Jacobian

$$Jac_{(t,z)}(\tilde{f}_1, ..., \tilde{f}_{N-M}) = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial t}(t, z) & \frac{\partial \tilde{f}_1}{\partial x_1}(t, z) & ... & \frac{\partial \tilde{f}_1}{\partial y_M}(t, z) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \tilde{f}_{N-M}}{\partial t}(t, z) & \frac{\partial \tilde{f}_{N-M}}{\partial x_1}(t, z) & ... & \frac{\partial \tilde{f}_{N-M}}{\partial y_M}(t, z) \end{pmatrix}.$$

First suppose that $t \neq 0$. Then observe that

$$\frac{\partial \tilde{f}_i}{\partial x_j} = t^{\deg f_i} \frac{\partial f_i(z/t)}{\partial z_j} = t \tilde{f}_{i,z_j}(z/t)$$

where $\tilde{f}_{i,z_j}$ is the homogenisation of $f_{i,z_j}(z) = \frac{\partial f_i(z)}{\partial z_j}$. We can hence write

$$\begin{aligned} Jac_{(t,z)}(\tilde{f}_1, ..., \tilde{f}_{N-M}) &= \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial t}(t, z) & t\tilde{f}_{1,x_1}(z/t) & ... & t\tilde{f}_{1,y_M}(z/t) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \tilde{f}_{N-M}}{\partial t}(t, z) & t\tilde{f}_{N-M,x_1}(z/t) & ... & t\tilde{f}_{N-M,y_M}(z/t) \end{pmatrix} \\ &= t^{N-M} \begin{pmatrix} \frac{1}{t}\frac{\partial \tilde{f}_1}{\partial t}(t, z) & | \\ \vdots & Jac_{z/t}(\tilde{f}_1, ..., \tilde{f}_{N-M}) \\ \frac{1}{t}\frac{\partial \tilde{f}_{N-M}}{\partial t}(t, z) & | \end{pmatrix} \end{aligned}$$

Where $Jac_{z/t}(\tilde{f}_1, ..., \tilde{f}_{N-M})$ is the Jacobian only over the z derivatives. By writing $\zeta = z/t$ we observe that $Jac_{z/t}(\tilde{f}_1, ..., \tilde{f}_{N-M}) = Jac_\zeta(f_1, ..., f_{N-M})$. We know that $Jac_\zeta(f_1, ..., f_{N-M})$ has rank $N - M$ except possibly on an $M - 1$ dimensional set, so the matrix above must have at least rank $N - M$ everywhere except possibly an $M - 1$ dimensional set. However since there are $N - M$ rows it follows that the rank cannot exceed $N - M$ which shows that, when $\{t \neq 0\}$, $\mathcal{V}_\uparrow$ is singular at most at an $M - 1$ dimensional set.

Now suppose that $t = 0$. If $\hat{f}$ denotes the top degree homogeneous part of f, then observe that

$$\left.\frac{\partial \tilde{f}_i}{\partial x_j}\right|_{t=0} = t^{\deg f_i} \left.\frac{\partial f_i(z/t)}{\partial z_j}\right|_{t=0} = \frac{\partial \hat{f}_i}{\partial z_j}(z) = \hat{f}_{i,j}(z).$$

$$\left.\frac{\partial \tilde{f}_i}{\partial t}\right|_{t=0} = \hat{f}_{i,0}(z), \qquad f_{i,0}(z) = f_i(z) - \hat{f}_i(z).$$

Suppose that the rank of $Jac_{(0,z)}(\tilde{f}_1, ..., \tilde{f}_{N-M})$ is strictly less than $N - M$ for all $(0, z) \in \mathcal{V}_\uparrow$. Then one of the rows must be a linear combination of the other rows, that is, there is some $k \in \{1, ..., N - M\}$ such that there exists c_i, $1 \leq i \leq N - M$, such that

$$\sum_{i \neq k} c_i \hat{f}_{i,j} = c_k \hat{f}_{k,j},$$

for all $0 \leq j \leq N$. Without loss of generality we may assume that $c_k = 1$. For $j \neq 0$ we have

$$\sum_{i \neq k} \int c_i \hat{f}_{i,j} \, dz_j = \int \hat{f}_{k,j} \, dz_j,$$

$$\sum_{i \neq k} c_i \hat{f}_i = \hat{f}_k + a(z_1, ..., z_{j-1}, z_{j+1}, ..., z_N).$$

Doing this for all j shows that a is does not depend on any $z_1, ..., z_N$ and hence is a constant. Each $\hat{f}_i$ is homogeneous so for any $\lambda \in \mathbb{C}$

$$\lambda^{\deg f_k} f_k(z) + a = \hat{f}_k(\lambda z) + a = \sum_{i \neq k} c_i \hat{f}_i(\lambda z) = \sum_{i \neq k} \lambda^{\deg f_i} c_i \hat{f}_i(z).$$

This shows that when $\lambda = 0$, $a = 0$ and since a is a constant it follows that

$$\hat{f}_k(z) = \sum_{i \neq k} c_i \hat{f}_i(z).$$

This implies that $\mathcal{V}^h (= \mathcal{V}_\uparrow \cap H_\infty \subset \mathbb{C}^N)$ is defined by $N - M - 1$ equations (since $\hat{f}_k = 0$ is a linear combination of the other equations). So $\mathcal{V}^h \cap P = \{z_1 = ... = z_{M-1} = \hat{f}_1 = ... = \hat{f}_{N-M} = 0\}$ has dimension at least 1. This contradicts property (ii) of Definition 1.133 i.e. $\mathcal{V}_\uparrow \cap P$ being distinct finite points $\{p_1, ..., p_d\}$. It follows that the rank of $Jac_{(0,z)}(\tilde{f}_1, ..., \tilde{f}_{N-M}) = N - M$ for $(0, z) \in \mathcal{V}_\uparrow$ satisfying $Rank\left(Jac_z\{\hat{f}_1, ..., \hat{f}_{N-M})\right) = N - M$. Hence the rank is $N - M - 1$ on a set of dimension at most $M - 1$, i.e. $\mathcal{V}_\uparrow$ is singular on a set of at most dimension $M - 1$. This shows that the M dimensional set $\mathcal{V}^h$ cannot be in the singular part of $\mathcal{V}_\uparrow$. Finally, condition (iii) of Definition 1.133 implies that $\mathcal{V}^h$ is d-sheeted. This completes the proof. □

In the context of defining the Robin function we have the following definition.

Definition 2.52 (Robin Function when $\mathcal{V}$ has distinct intersections with infinity). *Suppose that $\mathcal{V} \subset \mathbb{C}^N$ is an M-dimensional algebraic variety which has distinct intersections with infinity and Noether presentation (x, y). Further suppose that*

$$\pi_i^{-1}(x) = (x, y_i(x)), \quad \textit{for a suitable branch cut } C$$

$$\lim_{\substack{t \to 0 \\ x/t \in \mathbb{C}^M \setminus C}} y_i(x/t) = \tilde{y}_i(x).$$

Let $u \in \mathcal{L}^+(\mathcal{V})$. Then we define the Robin function $\rho_u : \mathcal{V}^{h,reg} \backslash B_\pi :\to [-\infty, \infty)$ pointwise by

$$\rho_u(x, \tilde{y}(x)) = \limsup_{\substack{t \to 0 \\ x/t \in \mathbb{C}^M \backslash C}} u(x/t), y(x/t)) + \log|t|.$$

We extend the Robin function to be defined everywhere on $\mathcal{V}^h$ by

$$\rho_u(x, y) = \limsup_{\mathcal{V}^{h,reg} \backslash B_{\pi_h} \ni (\zeta, \eta) \to (x,y)} \rho_u(\zeta, \eta).$$

Corollary 2.53. *If $u \in \mathcal{L}^+(\mathcal{V})$ then ρ_u is psh on $\mathcal{V}^h$.*

Lemma 2.54. *If $u \in \mathcal{L}^+(\mathcal{V})$ then $\rho_u \in \mathcal{L}^+(\mathcal{V}^h)$.*

Proof. Given a Noether presentation (x, y) it suffices to show that $\rho_u(x, \tilde{y}) = \log\|x\| + O(1)$. Since $u \in \mathcal{L}^+(\mathcal{V})$ it follows that there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\log\|x\| + \alpha \leq u(x, y) \leq \log\|x\| + \beta.$$

Making the substitution $(x, y) \mapsto (x/t, y(x/t))$ yields

$$\log\|x/t\| + \alpha \leq u(x/t, y(x/t)) \leq \log\|x/t\| + \beta.$$

Adding $\log|t|$ through the inequality yields

$$\log\|x\| + \alpha \leq u(x/t, y(x/t)) + \log|t| \leq \log\|x\| + \beta.$$

Taking the lim sup as $t \to 0$ and $y(x/t) \to \tilde{y}$ yields

$$\log\|x\| + \alpha \leq \rho_u(x, \tilde{y}) \leq \log\|x\| + \beta.$$

This proves the claim. □

The vast majority of the work we will do in the remainder of this thesis will be in the following setting.

Definition 2.55. *We say that an algebraic variety $\mathcal{V}$ satisfies the 'standard hypothesis' if it satisfies the following:*

(i) $\mathcal{V}$ is smooth, affine, irreducible, and M dimensional.

(ii) (x, y) is a Noether presentation for $\mathcal{V}$ with $x \in \mathbb{C}^M$ and $y \in \mathbb{C}^{N-M}$ and $\mathcal{V}$ is d-sheeted with respect to these coordinates.

(iii) $\mathcal{V}$ has distinct intersections with infinity.

2.5.5 Projective Definition

Observe the following log-homogeneity property of the Robin function.

Lemma 2.56. *Let $\mathcal{V}$ satisfy the standard hypothesis. Suppose that $u \in \mathcal{L}^+(\mathcal{V})$ then ρ_u is log homogeneous.*

Proof. Let $\lambda \in \mathbb{C}\backslash\{0\}$. Since $\mathcal{V}^h$ is homogeneous we have the following property:

$$(x, y) \in \mathcal{V}^h \iff \lambda(x, y) \in \mathcal{V}^h \iff (\lambda x, y(\lambda x)) \in \mathcal{V}^h.$$

That is,

$$\lambda y = y(\lambda x)$$

where y on the LHS is a point and the y on the RHS is an analytic function determined by the projection. With this observation we calculate

$$\begin{aligned}
\rho_u(\lambda(x, \tilde{y})) &= \limsup_{t\to 0} u(\lambda x, y(\lambda x/t)) + \log|t| \\
&= \limsup_{t\to 0} u\left(\frac{x}{t/\lambda}, y\left(\frac{x}{t/\lambda}\right)\right) + \log|t/\lambda| + \log|\lambda| \\
&= \limsup_{t/\lambda\to 0} u\left(\frac{x}{t/\lambda}, y\left(\frac{x}{t/\lambda}\right)\right) + \log|t/\lambda| + \log|\lambda| \\
&= \rho_u(x, \tilde{y}) + \log|\lambda|.
\end{aligned}$$

□

Lemma 2.56 says that the Robin function is determined on a complex line by a single point on that line. It follows that it is also natural to talk about the Robin function defined on the projective variety $\tilde{\mathcal{V}}^h$.

Definition 2.57. *Let $\mathcal{V}$ satisfy the standard hypothesis. Let $u \in \mathcal{L}^+(\mathcal{V})$. Recall that*

$$\tilde{\mathcal{V}}^h := \{[z] \in \mathbb{P}^{N-1} : \lambda z \in \mathcal{V}^h, \lambda \in \mathbb{C}\backslash\{0\}\}.$$

Then we define the projective Robin function to be

$$\tilde{\rho}_u([z]) = \rho_u(z) - \log\|z\| = \rho_u(z/\|z\|).$$

The motivation for this definition is that it is occasionally convenient to view $\tilde{\mathcal{V}}$ (the projectivisation of $\mathcal{V}$) as the disjoint union $\mathcal{V} \cup \tilde{\mathcal{V}}^h$. With this identification, if $[z_0 : \ldots : z_N]$ are projective coordinates for $\tilde{\mathcal{V}}$ then the chart given by $z_0 = 1$ corresponds to $\mathcal{V}$ whilst setting $z_0 = 0$ (which is the only set outside of the $z_0 = 1$ chart) gives $\tilde{\mathcal{V}}^h$. The projective Robin function often arises in this context; in particular in the work of Bedford-Taylor [4].

2.6 Bedford-Taylor Generalisations

We want to generalise results from [4] to the case of an algebraic variety. In particular the following theorem.

Theorem 2.58 (Theorem 5.5, [4]). *Let $u, v, w_2, ..., w_N \in \mathcal{L}^+(\mathbb{C}^N)$. Then*

$$\int_{\mathbb{C}^N} (udd^c v - vdd^c u) \wedge dd^c w_2 \wedge ... \wedge dd^c w_N = \int_{\mathbb{P}^{N-1}} (\tilde{\rho}_u - \tilde{\rho}_v) \wedge (dd^c \tilde{\rho}_{w_2} + \omega) \wedge ... \wedge (dd^c \tilde{\rho}_{w_N} + \omega)$$

where $\omega = \frac{1}{2} dd^c \log(1 + \|z\|^2)$.[‡]

While it would be interesting to recover this result in the most general case ($\mathcal{V}$ possibly reducible, branched at infinity) using our method, there are a number of technical obstructions to the proof we give here. As our primary concern is understanding the case when $\mathcal{V}$ has distinct intersection with infinity we elect to omit proofs of the more general case. A simpler approach for the general case would be to use the results of Berman-Boucksom. In particular, Proposition 4.7 [7] is in essence the complex manifold analogue of Theorem 2.58. With some care, one can use desingularisations and the Proposition to deduce the general case. While this approach could be used here, we elect to pursue the following method since it is more in the spirit of the work preceding it. Our strategy is to use projections to relate everything to the $\mathbb{C}^M$ case using techniques we have already encountered. We will need the following two results from [4].

Definition 2.59. *Let $\Omega \subset \mathbb{C}^M$ be open and $Z = \{z_1 = 0\}$. Assume that $\Omega \cap Z \neq \varnothing$. We define the class $\mathcal{L}^+(\Omega, Z) := \{u \in PSH \cap L^\infty_{loc}(\Omega \backslash Z) : u(z) = -\log|z_1| + O(1),\ z_1 \to 0\}$.*

For $u \in \mathcal{L}^+(\Omega, Z)$ we let $\tilde{u}(z) = u(z) + \log|z_1|$ so that $\tilde{u}$ is a bounded psh function on Ω.

Lemma 2.60 (Lemma 5.1, [4]). *Let $u, v, w_2, ..., w_M \in \mathcal{L}^+(\Omega, Z)$ and set $T = dd^c w_2 \wedge ... \wedge dd^c w_M$. Then there is a $(2M-1)$ current, S, defined on Ω by the formula*

$$\langle S, \alpha \rangle = \lim_{\varepsilon \to 0} \int_{|z_1| > \varepsilon} \alpha \wedge (u\, d^c v - v\, d^c u) \wedge T$$

for every test 1-form α.

Lemma 2.61 (Lemma 5.2, [4]). *Under the hypothesis of Lemma 2.60 we have*

$$dS = (u\, dd^c v - v\, dd^c u) \wedge T - 2\pi \chi_{\{z_1 = 0\}} [\tilde{u}(0, z') - \tilde{v}(0, z')] \wedge dd^c \tilde{w}_2(0, z') \wedge ... \wedge dd^c \tilde{w}_M(0, z'),$$

where $z' = (z_2, ..., z_M)$.

To prove a Bedford-Taylor formula as in Theorem 2.58 for varieties we first want to realise $u \in \mathcal{L}^+(\mathcal{V})$ (or $\mathcal{L}^+(\mathbb{C}^M)$) as the restriction of a function u' defined on $\tilde{\mathcal{V}}$ (or $\mathbb{P}^M$) to the chart $x_0 = 1$.

[‡] This is the usual Kähler form for $\mathbb{P}^N$.

Lemma 2.62. *Let $u \in \mathcal{L}^+(\mathcal{V})$. Then*

$$u'(x_0, x, y]) := \begin{cases} u(x,y), & x_0 = 1, \\ u\left(\frac{x_1}{x_0}, ..., \frac{x_{j-1}}{x_0}, \frac{1}{x_0}, \frac{x_{j+1}}{x_0}, ..., \frac{x_M}{x_0}, y\left(\frac{x}{x_0}\right)\right), & x_j = 1, \end{cases}$$

defines a psh function on $\mathcal{V}_\uparrow$ with logarithmic singularity along $x_0 = 0$.

Proof. The fact that u' is psh is immediate. To show that u' has a logarithmic singularity along $x_0 = 0$ it suffices to choose a chart containing $x_0 = 0$, we'll choose $x_1 = 1$. Since $u \in \mathcal{L}^+(\mathcal{V})$ and (x, y) is a Noether presentation for $\mathcal{V}$ it follows that

$$\begin{aligned} u\left(\frac{1}{x_0}, \frac{x_2}{x_0}, ..., \frac{x_M}{x_0}, y\left(\frac{x}{x_0}\right)\right) &= \log\left\|\left(\frac{1}{x_0}, \frac{x_2}{x_0}, ..., \frac{x_M}{x_0}\right)\right\| + O(1) \\ &= \log\|(1, x_2, ..., x_M)\| - \log|x_0| + O(1). \end{aligned}$$

It follows that $u'(x_0, x, y) = -\log|x_0| + O(1)$ near x_0 which finishes the proof. ☐

Corollary 2.63. *Let $u \in \mathcal{L}^+(\mathcal{V})$. Then $\tilde{u}(x_0, x, y) = u'(x_0, x, y) + \log|x_0|$ defines a locally bounded psh function near $x_0 = 0$. Moreover we can define*

$$\tilde{u}(0, x, y) = \limsup_{(x_0,\xi,\eta)\to(0,x,y)} u'(x_0, \xi, \eta) + \log|x_0|.$$

Lemma 2.64. *$dd^c\tilde{\rho}_u^*([x : y]) + \omega = dd^c\tilde{u}(0, x, y)$.*

Proof. Observe that by definition of ρ_u we have

$$\rho_u^*(x, y) = \tilde{u}(0, x, y).$$

Since $\rho_u^*(x, y) - \log\|(x, y)\| = \tilde{\rho}_u^*([x : y])$ and $dd^c \log\|(x, y)\| = \omega$ when restricted to $\mathbb{P}^{N-1}$ the result follows. ☐

Remark 2.65. The same constructions work for $\mathbb{C}^N$ functions. See Section 3 of [4] for details.

Lemma 2.66. *The current $(u'dd^cv' - v'dd^cu') \wedge dd^cw_2' \wedge ... \wedge dd^cw_M'$ has locally finite mass near $x_0 = 0$.*

Proof. Rewriting u' in terms of $\tilde{u}$ (and similar for other terms) we obtain

$$\log|x_0|dd^c(\tilde{u} - \tilde{v}) \wedge dd^c\tilde{w}_2 \wedge .. \wedge dd^c\tilde{w}_M + (\tilde{u}dd^c\tilde{v} - \tilde{v}dd^c\tilde{u}) \wedge dd^c\tilde{w}_2 \wedge ... \wedge dd^c\tilde{w}_M.$$

The second term is obviously has locally finite mass near $x_0 = 0$ since all functions are locally bounded there. For the first term let K be a compact set containing points of $\{x_0 = 0\}$ and

$L \subset int(K)$. Theorem 1.79 gives an inequality

$$\|\log|x_0|dd^c(\tilde{u}-\tilde{v})\wedge dd^c\tilde{w}_2\wedge ...\wedge dd^c w_M\|_L$$
$$\leq C_{K,L}\|\log|x_0|\|_{L^1(K)}\|\tilde{u}-\tilde{v}\|_{L^\infty(K)}\|\tilde{w}_2\|_{L^\infty(K)}...\|w_M\|_{L^\infty(K)}$$

which implies that the RHS has locally finite mass near $x_0 = 0$. □

Theorem 2.67. *Let $\mathcal{V}$ be a smooth irreducible algebraic variety with Noether presentation (x, y) which has distinct intersections at infinity. Let $u, v, w_2, ..., w_M \in \mathcal{L}^+(\mathcal{V})$ Then*

$$\int_{\mathcal{V}}(u\, dd^c v - v\, dd^c u)\wedge T = 2\pi\int_{\tilde{\mathcal{V}}^h}(\tilde{\rho}_u^* - \tilde{\rho}_v^*)\wedge\tilde{T}$$

where $T = dd^c w_2 \wedge ... \wedge dd^c w_M$.

Proof. Without loss of generality we may assume that $w_2 = ... = w_M$, and suppose that w is that function.[§] Let C be a branch cut for $\mathcal{V}$ over $\mathbb{C}^M$ with respect to the projection π and let C' be a second branch cut so that $C' \cap C \subset \pi(B_\pi)$ and let the projection for C' be denoted π'. Let the branches of $\mathcal{V}$ with respect to π be denoted $V_1, ..., V_d$ and with respect to π' be denoted $V_1', ..., V_d'$. Let $U_1, ..., U_{2d}$ such that $V_1 = U_1, ..., V_d = U_d$ and $V_1' = U_{d+1}, ..., V_d' = U_{2d}$ so that $\bigcup_{j=1}^q U_j = \mathcal{V}$. Then

$$\int_{\mathcal{V}}(udd^c v - vdd^c u)\wedge T = \sum_{j=1}^{2d}\int_{U_j}(udd^c v - vdd^c u)\wedge T - \sum_{j=1}^{2d}\sum_{i=j+1}^{2d}\int_{U_j\cap U_i}(udd^c v - vdd^c u)\wedge T. \tag{19}$$

Note that since B_π is pluripolar this decomposition is valid. Observe that $U_j \cap U_i \subset \mathcal{V}\backslash\pi^{-1}(C\cup C')$ so π_j and π_i' are biholomorphic there. For simplicity we will make the following conventions:

$$\theta = \theta(u, v, T) = (udd^c v - vdd^c u)\wedge T,$$
$$S(\alpha) = \langle S(u, v, T), \alpha\rangle = \lim_{R\to\infty}\int_{|x|<R}\alpha\wedge(u\, d^c v - v\, d^c u)\wedge T,$$

where $\omega = \frac{1}{2}\log(1 + \|z\|^2)$. Then we have that (19) is equal to

$$\sum_{j=1}^{d}\int_{\pi_j(U_j)}\pi_j^*\theta + \sum_{j=d+1}^{2d}\int_{\pi_j'(U_j)}\pi'^*_j\theta - \sum_{j=1}^{d}\sum_{i=j+1}^{2d}\int_{\pi_j(U_j\cap U_i)}\pi_j^*\theta. \tag{20}$$

We will now do the calculation for one of the U_j, the others are similar. Say we choose U_j (with $1 \leq j \leq d$). Define

$$\mathbb{P}^{M-1} \supset \tilde{U}_j = \overline{\{[1:x] : x \in \pi_j(U_j)\}}\backslash\overline{\{[1:x]\in C\}} = \overline{U}_{j,\mathbb{P}}\backslash\overline{C}_{\mathbb{P}}$$

[§]To see this, let $w = w_2 + ... + w_M$ and match coefficients after distributing dd^c over the terms in w.

and the homogeneous part of $\tilde{U}_j$ to be

$$\tilde{U}_j^h = \tilde{U}_j \cap \{x_0 = 0\}.$$

Since $\pi_j^* u, \pi_j^* v, \pi_j^* w \in \mathcal{L}^+(\pi_j(U_j))$ we can use the $\mathbb{C}^M$ version of Lemma 2.62 to realise each of these functions as the restriction of psh functions u', v' and w' on $\tilde{U}_j$ respectively. We claim that, for any test 1-form α, $\langle S(u', v', T'), \alpha \rangle$ defines a current on $\tilde{U}_j$. It is enough to prove this for test 1-forms α which have support in the coordinate patches of $\tilde{U}_j$. The coordinate patch $x_0 = 1$ has the form

$$\lim_{R\to\infty} -\int_{|z|<R} \alpha \wedge (u\, d^c v - v\, d^c u) \wedge T$$

which is clearly defines a current (since it is the limit of well defined currents). The $x_1, ..., x_M$ cases are similar so we only prove the claim for the $x_1 = 1$ chart. In this chart we have local coordinates

$$(s, t_2, ..., t_M) = (x_0/x_1, x_2/x_1, ..., x_M/x_0).$$

In these coordinates u' has the form

$$\pi_j^* u \left(\frac{1}{s}(1, t_2, ..., t_M) \right) = -\log |s| + O(1)$$

as in Lemma 2.62. So in these coordinates, $u' \in \mathcal{L}^+(\tilde{U}_j \cap \{x_1 = 1\}, Z)$ where $\tilde{U}_j \cap \{x_1 = 0\}$ is $\tilde{U}_j$ written in (s, t) coordinates and $Z = \{s = 0\}$ and

$$\langle S(u', v', T'), \alpha \rangle = \lim_{\varepsilon\to 0} -\int_{|x_0|>\varepsilon} \alpha \wedge (u'\, d^c v' - v'\, d^c u') \wedge T'.$$

So we are in the situation of Lemma 2.60, which implies that S defines a current on $\tilde{U}_j$. This proves that S defines a current on $\tilde{U}_j$, invoking Lemma 2.61 we obtain

$$dS(u', v', T') = \theta(\pi_j^* u, \pi_j^* v, \pi_j^* T) - 2\pi \chi_{\{x_0=0\}} (\tilde{u}(0, z) - \tilde{v}(0, z)) \wedge (dd^c \tilde{w}(0, z))^M. \tag{21}$$

By Lemma 2.66 the current θ can be trivially extended to a current on $\tilde{U}_j$ (by letting $\theta = 0$ on $\tilde{U}_j \backslash U_j$). Let ψ be a test form, then the equality above becomes

$$\int_{\tilde{U}_j} \psi dS(u', v', T') = \int_{\tilde{U}_j} \psi \theta(\pi_j^* u, \pi_j^* v, \pi_j^* T) - 2\pi \int_{\tilde{U}_j} \chi_{\{x_0=0\}} \psi (\tilde{u}(0, z) - \tilde{v}(0, z)) \wedge (dd^c \tilde{w}(0, z))^M.$$

Observe that $\pi_j^* \theta(u, v, T) = \theta(\pi_j^* u, \pi_j^* v, \pi_j^* T)$. Then using equation (21) we can deduce

$$\begin{aligned} \int_{U_j} \theta(u, v, T) &= \int_{\pi(U_j)} \pi_j^* \theta(u, v, T) = \int_{\pi_j(U_j)} \theta(\pi_j^* u, \pi_j^* v, \pi_j^* T) \\ &= \int_{\tilde{U}_j} dS(u', v', T') + 2\pi \int_{\tilde{U}_j^h} (\tilde{u}(0, z) - \tilde{v}(0, z)) \wedge (dd^c \tilde{w}(0, z))^M. \end{aligned} \tag{22}$$

Now

$$\begin{aligned}
\rho_{\pi_j^* u}(x) &= \limsup_{\substack{t\to 0\\ x/t\in\pi_j(U_j)}} \ \pi_j^* u(x/t) + \log|t| \\
&= \limsup_{\substack{t\to 0\\ x/t\in\pi_j(U_j)}} \ u(x/t, y_j(x/t)) + \log|t| \\
&= \rho_u(x,\tilde{y}_j),
\end{aligned}$$

where $(x,\tilde{y}_j) \in V_j^h$ (recalling that $U_j = V_j$). Observe that $\pi_j^{-1}(\tilde{U}_j)$ corresponds to $V_j \cup \tilde{V}_j^h = \tilde{V}_j$. Using Lemma 2.64 and the previous observation we can rewrite equation (22) as

$$\begin{aligned}
\int_{V_j} \theta(u,v,t) &= \int_{\tilde{U}_j} dS(u',v',T') + 2\pi \int_{\tilde{U}_j^h} (\tilde{\rho}^*_{\pi_j^* u} - \tilde{\rho}^*_{\pi_j^* v}) \wedge (dd^c \tilde{\rho}^*_{\pi_j^* w} + \pi_j^* \omega)^M \\
&= \int_{\tilde{V}_j} \pi_{j*} dS(u',v',T') + 2\pi \int_{\tilde{V}_j^h} (\tilde{\rho}^*_u - \tilde{\rho}^*_v) \wedge (dd^c \tilde{\rho}^*_w + \omega)^M \\
&= \int_{\tilde{V}_j} dS(\pi_{j*}u', \pi_{j*}v', \pi_{j*}T') + 2\pi \int_{\tilde{V}_j^h} (\tilde{\rho}^*_u - \tilde{\rho}^*_v) \wedge (dd^c \tilde{\rho}^*_w + \omega)^M.
\end{aligned}$$

Repeating this for each U_j means the sum in equation (19) becomes

$$\int_{\mathcal{V}} \theta(u,v,t) = \int_{\tilde{\mathcal{V}}} dS(\pi_{j*}u', \pi_{j*}v', \pi_{j*}T') + 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}^*_u - \tilde{\rho}^*_v) \wedge (dd^c \tilde{\rho}^*_w + \omega)^M$$

where we are justified in ignoring the pluripolar set B_π by the logic in the Conservation of Mass Lemma (Lemma 2.18). dS is an (N,N) current on $\tilde{\mathcal{V}}$ so can be understood by integrating against test forms. In particular 1 is a test form on $\tilde{\mathcal{V}}$ since $\tilde{\mathcal{V}}$ is compact. Since $\tilde{\mathcal{V}}$ has no boundary it follows from Stokes theorem for currents (Theorem 1.64) that

$$\int_{\tilde{\mathcal{V}}} dS = \int_{\partial\tilde{\mathcal{V}}} S = 0.$$

This completes the proof. □

2.7 Explicit Computation

While the proof of the Bedford-Taylor formula was conducted under imposing the standard hypothesis on $\mathcal{V}$ it's plausible that given suitable adaption it is valid in more general settings. Precisely, we must respect multiplicity and use distinguished branch cuts. We will verify the Bedford-Taylor formula for a particular example on an algebraic curve which is both singular and branched at infinity. Recall $dd^c = 2i\partial\bar{\partial}$. We wish to verify

$$\int_{\mathcal{V}} (u\, dd^c v - v\, dd^c u) \wedge T = 2\pi \sum_{i=1}^{d} \int_{\tilde{\mathcal{V}}_i^h} (\tilde{\rho}^*_{u,i} - \tilde{\rho}^*_{v,i}) \wedge \tilde{T}$$

where $u = \max\{-\log|y|, 2\log|y|+1\}$, $v = \max\{-\log|y|, 2\log|y|\}$ and $\mathcal{V} = \{y^3 - xy + 1 = 0\}$. Since $\mathcal{V}$ is 1 dimensional we have $T = 1$.

We now try to understand the integral on the RHS. We know from prior analysis that $\mathcal{V}^h = \{(x,0) : x \in \mathbb{C}\}$ which is one sheeted. From Example 2.46 we know that $\mathcal{V}$ is three sheeted and two sheets correspond to '$y \to \infty$' and one sheet corresponds to '$y \to 0$'. Each sheet corresponds to a copy of $\{(x,0) : x \in \mathbb{C}\}$ at infinity. Using a desingularisation of $\mathcal{V}_\uparrow$ (e.g. equation (10)) we can take $\mathcal{V}_1^h = \{(x,0;a) : x \in \mathbb{C}, a \in \mathbb{P}\}$, $\mathcal{V}_2^h = \{(x,0;a) : x \in \mathbb{C}, a \in \mathbb{P}\}$ and $\mathcal{V}_3 = \{(x,0;b) : x \in \mathbb{C}, b \in \mathbb{P}\}$ where a corresponds to the limit as $y \to \infty$ and b corresponds to the limit as $y \to 0$. We have included two copies of the point a because it has multiplicity 2. It follows that the integral on the RHS becomes the discrete evaluation of the Robin functions at $\zeta_1 = (1,0;a), \zeta_2 = (1,0;a)$ and $\zeta_3 = (1,0;b)$. Since this evaluation is discrete, we need not worry about regularising ρ_u or ρ_v. Observe that $x = y^2 + y^{-1}$ so $2\log|y| + 1 = \log|x - y^{-1}|$. Write $\rho_{u,i}$ to indicate the Robin function on $\mathcal{V}_i$. With this we calculate

$$\begin{aligned}
\rho_{u,1}(x) &= \limsup_{\substack{\lambda \to 0 \\ y \to \infty}} \log|(x - y^{-1})/\lambda| + \log|\lambda| = \log|x| + 1, \\
\rho_{u,1}(\zeta_1) &= 1.
\end{aligned}$$

Using a similar calculation one can deduce $\rho_{u,2}(\zeta_2) = 1$, $\rho_{u,3}(\zeta_3) = 0$ and $\rho_{v,1}(\zeta_1) = \rho_{v,2}(\zeta_2) = \rho_{v,3}(\zeta_3) = 0$. Hence the integral on the RHS can be evaluated as

$$\begin{aligned}
&2\pi\left((\rho_{u,1}(x_1) - \rho_{v,1}(x_1)) + (\rho_{u,2}(x_2) - \rho_{v,2}(x_2)) + (\rho_{u,3}(x_3) - \rho_{v,3}(x_3))\right) \\
&\quad = 2\pi((1-0) + (1-0) + (0-0)) = 4\pi.
\end{aligned}$$

Observe that $y^3 - xy + 1 = 0$ can be parameterised by $s = y$, $x = \frac{s^3+1}{s}$. In s coordinates we have $\mathcal{V} = \left\{\left(\frac{s^3+1}{s}, s\right) : s \in \mathbb{C}\backslash\{0\}\right\}$

We have the following dd^c of a maximum formula from [5].

Theorem 2.68 (Theorem 1, [5]). *Let $\Omega \subset \mathbb{C}$. If $u_1, u_2 \in C^3 \cap PSH(\Omega)$ let $u = \max\{u_1, u_2\}$. Then*

$$dd^c u = d^c(u_1 - u_2)|_{\{u_1 = u_2\}} + \int_{\{u_1 > u_2\}} dd^c u_1 + \int_{\{u_2 > u_1\}} dd^c u_2.$$

Using this formula we see that the only term which is nonzero for $dd^c u$ and $dd^c v$ is the d^c

term. In conjunction with the parameterisation in s we have;

$$\begin{aligned}
dd^c u(s) &= d^c(2\log|s| + 1 - (-\log|s|))|_{|s|=e^{-1/3}} = d^c(3\log|s|+1)|_{|s|=e^{-1/3}} = d^c(3\log|s|)|_{|s|=e^{-1/3}} \\
&= \frac{3i}{2}\left((\bar{\partial} - \partial)\log(s\bar{s})\right)|_{|s|=e^{-1/3}} = \frac{3i}{2}\left(\left(\frac{1}{\bar{s}}d\bar{s} - \frac{1}{s}ds\right)\right)\Big|_{|s|=e^{-1/3}} \\
&= \frac{3i}{2}\left(\frac{-ie^{-1/3}e^{-it}}{e^{-1/3}e^{-it}}dt - \frac{ie^{-1/3}e^{it}}{e^{-1/3}e^{it}}dt\right) \\
&= \frac{3i}{2}(-2i\ dt) = 3\ dt
\end{aligned}$$

i.e. for any test function ϕ, $\int \phi(s)\ dd^c u(s) = 3\int_0^{2\pi} \phi(e^{-1/3}t)\ dt$. That is, $dd^c u(s)$ is three times Lebesgue measure on $\{|s| = e^{1/3}\}$, the same calculation shows $dd^c v$ is three times Lebesgue measure on on the unit circle.

Let $d\lambda$ be Lebesgue measure. We calculate

$$\begin{aligned}
\int_{\mathcal{V}} u\, dd^c v - v\, dd^c u = &\int_{\mathbb{C}\setminus\{0\}} u\, dd^c v - v\, dd^c u \\
&3\int_{|s|=1} 2\log|s| + 1\, d\lambda - 3\int_{|s|=e^{-1/3}} -\log|s|\, d\lambda \\
&= 6\pi - 2\pi = 4\pi.
\end{aligned}$$

Which verifies the formula.

2.8 Justification of the Standard Hypothesis

While the motivations behind requiring $\mathcal{V}$ to have distinct intersections with infinity and a Noether presentation (x, y) has been thoroughly discussed, the assumption that $\mathcal{V}$ is irreducible and smooth has had no such discussion. This section is dedicated to illuminating how the singular and/or reducible case can be deduced from the smooth, irreducible case.

2.8.1 $\mathcal{V}$ singular

By quasicontinuity and the fact that the singular points of $\mathcal{V}$ lie in an $M-1$ dimensional algebraic subvariety, one might expect that the singular case can be deduced from the smooth case. It transpires that this is indeed the case. Dinh and Sibony [28] provide an efficient way to translate results from the smooth case to the singular case. The key results are the following lemmas that we encountered in Section 2.4.

Lemma 2.69 (Proposition 2.2, [28]). *Let $\eta : \hat{X} \to X$ be a desingularisation of X. If u is a wpsh function on X then there is a psh function $\hat{u}$ on $\hat{X}$ such that $u(x) = \max_{\eta^{-1}(x)} \hat{u}$ for $x \in X$. Conversely, if $\hat{u}$ is psh on $\hat{X}$ then $x \mapsto \max_{\eta^{-1}(x)} \hat{u}$ defines a wpsh function on X.*

Lemma 2.70 (Proposition 2.3, [28]). *Let $\eta : \hat{X} \to X$ be a desingularisation of X and u a wpsh function on X. Suppose that T is a positive current on X and $\hat{T}$ is the positive current on $\hat{X}$ equal to T on $\hat{X} \backslash \eta^{-1} X^{sing}$ and 0 otherwise. Then $dd^c u \wedge T = \eta_*(dd^c \hat{u} \wedge \hat{T})$.*

These two lemmas allow us to transfer almost all results from the smooth case to the singular case. We'll refer to the application of these two lemmas as 'transposition to the smooth case'. We illustrate this with an example.

Theorem 2.71 (c.f. Theorem 2.12). *Let $u_1, ..., u_q$ be locally bounded weakly plurisubharmonic functions and let $u_1^k, ..., u_q^k$ be monotone (either increasing or decreasing) sequences of weakly plurisubharmonic functions converging almost everywhere to $u_1, ..., u_q$. Then for any positive current T.*

(a) $u_1^k dd^c u_2^k \wedge ... \wedge dd^c u_q^k \wedge T \longrightarrow u_1 dd^c u_2 \wedge ... \wedge dd^c u_q \wedge T$ *weakly.*

(b) $dd^c u_1^k \wedge ... \wedge dd^c u_q^k \wedge T \longrightarrow dd^c u_1 \wedge ... \wedge dd^c u_q \wedge T$ *weakly.*

Proof. As in the smooth case, it suffices to prove (a) since (b) follows immediately from it. The result is true for the smooth case so we may apply the standard transposition to the smooth case. The details are as follows. The functions $\hat{u}_1, ..., \hat{u}_q$ are locally bounded psh functions on $\hat{X}$ so it follows from the smooth case that

$$\hat{u}_1^k dd^c \hat{u}_2^k \wedge ... \wedge \hat{u}_q^k \wedge \hat{T} \longrightarrow \hat{u}_1 dd^c \hat{u}_2 \wedge ... \wedge \hat{u}_q \wedge \hat{T}, \qquad \text{weakly.} \tag{23}$$

But now

$$\eta_*(\hat{u}_1^k dd^c \hat{u}_2^k \wedge ... \wedge \hat{u}_q^k \wedge T) = u_1^k dd^c u_2^k \wedge ... \wedge u_q^k \wedge T,$$
$$\eta_*(u_1 dd^c \hat{u}_2 \wedge ... \wedge \hat{u}_q \wedge \hat{T}) = u_1 dd^c u_2 \wedge ... \wedge u_q \wedge T.$$

So applying η_* to both sides of (23) gives the result. □

Define

$$w\mathcal{L}(\mathcal{V}) := \{u \in \mathcal{L}(\mathcal{V}^{reg}) : u \text{ is usc on } \mathcal{V}\}.$$

For results demanding a higher level of precision, such as the calculation of mass for $u \in w\mathcal{L}^+(\mathcal{V})$, we can exploit that $\mathcal{V}^{sing}$ is pluripolar. To highlight this we calculate the mass of functions in $w\mathcal{L}^+(\mathcal{V})$. Note that, provided that $\mathcal{V}$ is irreducible, the argument proving the existence of a Noether presentation goes through without modification.

Theorem 2.72. *Suppose that $\mathcal{V}$ is an M dimensional, irreducible, singular algebraic variety with Noether presentation (x, y). Let $u \in w\mathcal{L}^+(\mathcal{V})$, then $\int_{\mathcal{V}} (dd^c u)^M = d(2\pi)^M$ where d is the number of branches of $\mathcal{V}$.*

Proof. By transposition to the smooth case, the Mass Comparison Theorem (Theorem 2.16) is valid for a singular variety. As such it suffices to compute the mass of one function in the class $w\mathcal{L}^+(\mathcal{V})$. As in the smooth case, we will choose $u = \log^+|x|$. Without loss of generality (e.g. by a linear translation if necessary), we may assume that no singular points or branch points are contained in $\pi^{-1}(\overline{B}_1(0))$. So we may take a branch cut C which avoids $B_1(0)$. But now $\mathrm{supp}(dd^c u)^M$ is supported in $\mathcal{V}^{reg}\backslash\pi^{-1}(C)$ and so we may apply the Conservation of Mass Lemma (Lemma 2.18) to conclude

$$\int_{\mathcal{V}}(dd^c u)^M = \int_{\mathcal{V}^{reg}\backslash\pi^{-1}(C)}(dd^c u)^M = \sum\int_{V_i\cap\mathcal{V}^{reg}}(dd^c u)^M = \sum\int_{\overline{B}_1(0)}(dd^c u)^M = d(2\pi)^M.$$

$\square$

2.8.2 $\mathcal{V}$ Reducible

The only kind of reducible varieties we are willing to discuss are those of the form $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are smooth irreducible varieties. When $\dim(\mathcal{V}_1) > \dim(\mathcal{V}_2)$ the situation is uninteresting since $\mathcal{V}_2$ is pluripolar with respect to $PSH(\mathcal{V}_1)$. As such we will primarily be concerned with varieties of the same dimension M with $\mathcal{V}_1 \neq \mathcal{V}_2$. The logic in our discussion will extend to a finite union of smooth irreducible varieties so it is sufficient to use $\mathcal{V}_1$ and $\mathcal{V}_2$ to illuminate the concepts.

The general strategy is similar to the singular case. We use desingularisation as our main tool to obtain results from the smooth irreducible case. In section 4.7 we explicitly show how this can be done. The results that cannot be obtained this way are those which make geometric claims about $\mathcal{V}$. In light of this, the most important result necessary for us to establish is the existence of a Noether presentation. Suppose that

$$\mathcal{V}_1 = \{P_1(z) = ... = P_{N-M}(z) = 0\}$$
$$\mathcal{V}_2 = \{Q_1(z) = ... = Q_{N-M}(z) = 0\}$$

are smooth irreducible varieties. Then their union is defined to be

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 := \{P_i(z)Q_j(z) = 0 : 1 \leq i, j \leq N-M\}.$$

We will assume that $P_i \neq Q_j$ for all i, j for simplicity (if there is equality we can modify the polynomials defining $\mathcal{V}$ to accommodate this fact).

Lemma 2.73. *With the setup as above, there exists a Noether presentation for $\mathcal{V}$.*

Proof. The proof given in Theorem 2.21 works in this situation with no modification. To see this note the following:

- $\mathcal{V}$ is a variety defined by $(N-M)^2$ equations, but many of these equations contain redundant information. Since $\mathcal{V}$ is M-dimensional, it is straightforward to see that $\mathcal{V} = \{P_i(z)Q_i(z) : 1 \leq i \leq N-M\}$ and so $\mathcal{V}$ has co-dimension $N-M$.

- Reducibility has no impact on the existence of a Groöbner basis.
- The proof never appeals to any fact that might depend on irreducibility e.g. no structure of the ideal $\mathbf{I}(\mathcal{V})$ is assumed beyond the properties of any polynomial ideal.

$\square$

Remark 2.74. If $\mathcal{V}$ is singular or reducible and has distinct intersections with infinity, by the methods discussed in this Section we can make useful definition for the Robin function in the analogous way to the smooth case. This idea will be used in Section 4.6.

3 Properties of the Robin Function

3.1 Siciak's $\mathcal{H}$-principle

Recall the classical $\mathcal{H}$-principle due to J. Siciak.

Theorem 3.1 ($\mathcal{H}$-principle, [51])**.** *Let $P_n(\mathbb{C}^N)$ (resp. $H_n(\mathbb{C}^N)$) denote the space of all polynomials of degree at most n (resp. homogeneous polynomials degree at most n) in N complex variables. Let $\mathcal{L}(\mathbb{C}^N)$ (resp. $\mathcal{H}(\mathbb{C}^N)$) denote the class of logarithmic psh functions (resp. log homogeneous psh functions). The maps*

(i.) $H_n(\mathbb{C} \times \mathbb{C}^N) \ni Q_n(t, z) \to Q_n(1, z) \in P_n(\mathbb{C}^N)$,

(ii.) $\mathcal{H}(\mathbb{C} \times \mathbb{C}^N) \ni u(t, z) \to u(1, z) \in \mathcal{L}(\mathbb{C}^N)$

are one-to-one. If $P \in P_n(\mathbb{C}^N)$ then the unique element $\tilde{P} \in H_n(\mathbb{C} \times \mathbb{C}^N)$ such that $P(z) = \tilde{P}(1, z)$ is given by the formula $\tilde{P}(t, z) = t^n P(z/t)$. If $u \in \mathcal{L}(\mathbb{C}^N)$ then the unique element $\tilde{u} \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$ such that $\tilde{u}(1, z) = u(z)$ on $\mathbb{C}^N$ is given by

$$\tilde{u}(t, z) := \begin{cases} \log|t| + u(z/t), & t \neq 0 \\ \limsup\limits_{(t,\zeta)\to(0,z)} \log|t| + u(\zeta/t), & t = 0. \end{cases}$$

A corresponding version of the $\mathcal{H}$-principle for algebraic varieties essentially formalises the discussion from Section 2.5. We will also discuss a weighted version in Section 4.

Notation 3.2. We will employ the following notation for convenience; $P_n(\mathcal{V})$ (resp. $H_n(\mathcal{V})$) will denote the polynomials of degree n (resp. homogeneous polynomials of degree n) which are not in the ideal $I(\mathcal{V})$ (resp. homogeneous ideal $I_h(\mathcal{V})$).

The variety $\mathcal{V}_\uparrow$ always is singular at the origin (since it is a homogeneous variety). The singularity at the origin will never pose any problem for us since we will be exclusively consider log-homogeneous functions on $\mathcal{V}_\uparrow$ which all take the value $-\infty$ at the origin. It is possible that $\mathcal{V}_\uparrow$ has additional singularities, this necessitates the following definition.

Definition 3.3. *Suppose that $\mathcal{V}$ is an irreducible algebraic variety (possibly singular). We define the class of weakly log homogeneous psh functions to be*

$$w\mathcal{H}(\mathcal{V}) := \{u \in \mathcal{H}(\mathcal{V}^{reg}) : u \text{ is usc on } \mathcal{V}\}.$$

Theorem 3.4 ($\mathcal{H}$-principle for Algebraic Varieties)**.** *Suppose that $\mathcal{V}$ is an algebraic variety satisfying the standard hypothesis, with (possibly singular) lift $\mathcal{V}_\uparrow$.*

(i) The map $H_n(\mathbb{C} \times \mathbb{C}^N) \ni Q_n(t, z) \to Q_n(1, z) \in P_n(\mathbb{C}^N)$, is one to one. Moreover, $Q_n(t, z) \in I(\mathcal{V}_\uparrow)$ if and only if $Q_n(1, z) \in I(\mathcal{V})$.

(ii) If $P \in P_n(\mathcal{V})$ then the unique element $\tilde{P} \in H_n(\mathcal{V}_\uparrow)$ such that $P(z) = \tilde{P}(1,z)$ is given by the formula $\tilde{P}(t,z) = t^n P(z/t)$ and $P \in I(\mathcal{V})$ implies $\tilde{P} \in I(\mathcal{V}_\uparrow)$.

(iii) The map $w\mathcal{H}(\mathcal{V}_\uparrow) \ni u(t,z) \to u(1,z) \in \mathcal{L}(\mathcal{V})$ exists.

(iv) If $u \in \mathcal{L}(\mathcal{V})$ then the unique element $\tilde{u} \in w\mathcal{H}(\mathcal{V}_\uparrow)$ such that $\tilde{u}(1,z) = u(z)$ is given by

$$\tilde{u}(t,z) := \begin{cases} \log|t| + u(z/t), & t \neq 0 \\ \limsup_{(t,\zeta)\to(0,z)} \log|t| + u(\zeta/t), & t = 0. \end{cases}$$

In particular, the map in (iii) is one to one.

Proof. (i) The first claim is a direct consequence of the $\mathcal{H}$-principle (Theorem 3.1). The second claim follows since $Q_n \in I(\mathcal{V}_\uparrow)$ implies $Q_n \equiv 0$ at every point of $\mathcal{V}_\uparrow$. Since $\{1\} \times \mathcal{V} \subset \mathcal{V}_\uparrow$ it follows that $Q_n(1,z) \equiv 0$ on $\mathcal{V}$.

(ii) The first claim is a direct consequence of the $\mathcal{H}$-principle. The second claim follows since every point $(t,z) \in \mathcal{V}_\uparrow$ is a linear multiple of a corresponding point $(1, z/t) \in \mathcal{V}$. Hence for any $(t,z) \in \mathcal{V}_\uparrow$ we have $\tilde{P}(t,z) = t^n P(1, z/t) = 0$, hence $\tilde{P}(t,z) \in I(\mathcal{V}_\uparrow)$.

(iii) By Theorem 2.51, the only place where $\mathcal{V}_\uparrow$ can be singular is on the set $\{t = 0\} \cap \mathcal{V}_\uparrow$. It follows that the only place that $u(t,z) \in w\mathcal{H}(\mathcal{V}_\uparrow)$ may not be psh is on the set $\{t = 0\} \cap \mathcal{V}_\uparrow$ which does not contain $\{1\} \times \mathcal{V}$. It follows then that $u(1,z) \in PSH(\mathcal{V})$ so we need only check that u has logarithmic growth. This follows since by being a member of $w\mathcal{H}(\mathcal{V}_\uparrow)$ there exists an $\alpha \in \mathbb{R}$ such that

$$u(t,z) = u(1, z/t) + \log|t| \leq \log\|(t,z)\| + \alpha$$
$$u(1, z/t) \leq \log\|(1, z/t)\| + \alpha.$$

After letting $t = 1$ the result is immediate and $u(1,z) \in \mathcal{L}(\mathcal{V})$.

(iv) Clearly $\tilde{u}$ is psh away from $\{t = 0\}$ and since $\{t = 0\} \cap \mathcal{V}_\uparrow$ is pluripolar in $\mathcal{V}_\uparrow$ the upper semicontinuous regularisation yields a wpsh function on $\mathcal{V}_\uparrow$. By construction $\tilde{u}(\lambda t, \lambda z) = \tilde{u}(t,z) + \log|\lambda|$ for $t \neq 0$. We can use the argument from Lemma 2.56 to show that $\tilde{u}$ is log homogeneous on $\{t = 0\} \cap \mathcal{V}_\uparrow$ and hence log homogeneous everywhere.* Since $\tilde{u}$ is log homogeneous it is determined q.e. by the values taken on $\{1\} \times \mathcal{V}$ which implies uniqueness. □

Corollary 3.5. *We have the following inclusion of ideals in $\mathbb{C}[t,z]$; $I(\mathcal{V}_\uparrow) \subset I(\{1\} \times \mathcal{V}) \subset I_h(\mathcal{V}_\uparrow)$ where $I_h(\mathcal{V}_\uparrow) \subset H(\mathbb{C}^N)$ is a homogeneous polynomial ideal.*

*The hypothesis of Lemma 2.56 requires $u \in \mathcal{L}^+(\mathcal{V})$, however this merely ensures that ρ_u is not identically $-\infty$ on $\mathcal{V}^h$. $\tilde{u}$ is obviously still log homogeneous if it is identically $-\infty$ on $\{t = 0\} \cap \mathcal{V}_\uparrow$ so using the argument given there is valid.

Corollary 3.6. *Suppose that $\mathcal{V}$ is an algebraic variety (possibly singular). Then the $\mathcal{H}$-principle holds with $\mathcal{L}(\mathcal{V})$ replaced with*

$$w\mathcal{L}(\mathcal{V}) = \{u \in \mathcal{L}(\mathcal{V}^{reg}) : u \text{ is usc on } \mathcal{V}\}.$$

Proof. Follows from the standard desingularisation argument. □

Example 3.7. Suppose that $\mathcal{V} = \{y^3 - xy + 1 = 0\}$ and $u(x,y) = \max\{2\log|y| + 1, -\log|y|\}$. We saw in Section 2.4 that the function is either 0 or 1 at infinity depending on the path we take. The lifted variety is $\mathcal{V}_\uparrow := \{x^3 - xyt + t^3 = 0\}$. When $t = 0$, $\mathcal{V}_\uparrow$ consists of the set $\{0\} \times \mathbb{C} \times \{0\}$. The lift of u given by the $\mathcal{H}$-principle is

$$\tilde{u}(t,x,y) = \begin{cases} u(x/t, y(x/t)) + \log|t|, & t \neq 0 \\ \limsup\limits_{t\to 0} u(x/t, y(x/t)) + \log|t|, & t = 0. \end{cases}$$

We can find this explicitly by using the fact that $x = y^2/t - t^2/y$ and $t \to 0$ implies either $y \to 0$ or $y \to \infty$.

$$\limsup_{t\to 0} u(x/t, y(x/t)) + \log|t| = \limsup_{t\to 0} \left(\max\{2\log|x/t| + 1, -\log|x/t|\} + \log|t|\right).$$

When $y \to \infty$ then $2\log|y| + 1 > -\log|y|$ so in this case

$$\begin{aligned} \limsup_{t\to 0} 2\log|y(x/t)| + 1 + \log|t| &= \limsup_{t\to 0} \log(|t||y(x/t)|^2) + 1 \\ &= \limsup_{t\to 0} \log|t||x/t + t/y(x/t)| + 1 = \log|x| + 1. \end{aligned} \tag{24}$$

When $y \to 0$ then $-\log|y| > 2\log|y| + 1$ so

$$\begin{aligned} \limsup_{t\to 0} -\log|y(x/t)| + \log|t| &= \limsup_{t\to 0} \log(|t|/|y(x/t)|) \\ &= \limsup_{t\to 0} \log|t||x/t - y(x/t)^2/t| = \log|x|. \end{aligned} \tag{25}$$

Taking the max of equations (24) and (25) yields $\tilde{u}(0,x,y) = \log|x| + 1$. So

$$\tilde{u}(t,x,y) = \begin{cases} u(x/t, y(x/t)) + \log|t|, & t \neq 0 \\ \log|x| + 1, & t = 0. \end{cases}$$

Clearly $\tilde{u}$ is log homogeneous, but fails to be psh along $t = 0$. To see this, note that $\mathcal{V}_\uparrow$ is locally reducible near $\{t = 0\}$ and decomposes into two holomorphic components with one component corresponding to $y \to 0$ and the other corresponding to $y \to \infty$. The restriction map to either of these components is holomorphic. Restricting $\tilde{u}$ to the $y \to 0$ component fails to yield a psh function as it is not usc (it approaches the value $\log|x|$ at $t=0$, but takes the value $\log|x|+1$). Hence $\tilde{u}$ cannot be psh. It is readily seen that $\tilde{u}$ is in $w\mathcal{H}$ which is phenomena which does not occur in the classical case.

3.2 Bedford-Taylor Consequences

Our first consequence of Theorem 2.67 is a generalisation of Theorem 6.1 from [4]. For this we need an identity due to Bedford-Taylor.

Proposition 3.8. *Suppose that $u, v \in PSH(X)$. We have the following algebraic identity;*

$$\begin{aligned} u\,(dd^c v)^M - v\,(dd^c u)^M = (u\,dd^c v - v\,dd^c u) \wedge \sum_{j=0}^{n-1} (dd^c u)^j \wedge (dd^c v)^{n-j-1} \\ + (v-u)\sum_{j=1}^{n-1}(dd^c u)^j \wedge (dd^c v)^{n-j}. \end{aligned}$$

Proof. For notational convenience we write

$$T_{i,k}^{l,m}(u,v) = T_{i,k}^{l,m} = \sum_{j=i}^{l} (dd^c u)^{j+k} \wedge (dd^c v)^{m-j}$$

so that the RHS of the formula becomes

$$(u\,dd^c v - v\,dd^c u) \wedge T_{0,0}^{M-1,M-1} + (v-u)\,T_{1,0}^{M-1,M}.$$

Note the symbolic formula $T_{i,k}^{l,m} = T_{i+1,k-1}^{l+1,m+1}$. Observe that

$$\begin{aligned} v(dd^c v)^M - u(dd^c u)^M &= v(dd^c v)^M - u(dd^c u)^M + u\,T_{0,0}^{M-1,M} - u\,T_{0,0}^{M-1,M} + v\,T_{0,1}^{M-1,M-1} \\ &\quad - v\,T_{0,1}^{M-1,M-1} \\ &= u\,T_{0,0}^{M-1,M} - u\,T_{0,0}^{M,M} + v\,T_{-1,1}^{M-1,M-1} - v\,T_{0,1}^{M-1,M-1} \\ &= u\,T_{0,0}^{M-1,M} - v\,T_{0,1}^{M-1,M-1} + v\,T_{0,0}^{M,M} - u\,T_{0,0}^{M,M} \\ &= u\,T_{0,0}^{M-1,M} - v\,T_{1,0}^{M,M} + (v-u)\,T_{0,0}^{M,M} \\ &= (u\,dd^c v) \wedge T_{0,0}^{M-1,M-1} - (v\,dd^c u) \wedge T_{1,-1}^{M,M} + (v-u)\,T_{0,0}^{M,M} \\ &= (u\,dd^c v) \wedge T_{0,0}^{M-1,M-1} - (v\,dd^c u) \wedge T_{0,0}^{M-1,M-1} + (v-u)\,T_{0,0}^{M,M} \\ &= (u\,dd^c v - v\,dd^c u) \wedge T_{0,0}^{M-1,M-1} + (v-u)\,T_{0,0}^{M,M}. \end{aligned}$$

Note that

$$\begin{aligned} (v-u)\,T_{0,0}^{M,M} &= (v-u)\sum_{j=0}^{M}(dd^c u)^j \wedge (dd^c v)^{M-j} \\ &= (v-u)\sum_{j=1}^{M-1}(dd^c u)^j \wedge (dd^c v)^{M-j} + (v-u)\left((dd^c u)^M + dd^c v)^M\right). \end{aligned}$$

By adding $-(v-u)\left((dd^cu)^M + dd^cv)^M\right)$ to both sides of the first deduction and cancelling terms, we have

$$u(dd^cv)^M - v(dd^cu)^M = (u\,dd^cv - v\,dd^cu) \wedge T_{0,0}^{M-1,M-1} + (v-u)\,T_{1,0}^{M-1,M}$$

which proves the formula. □

Theorem 3.9. *Suppose that $\mathcal{V}$ is an algebraic variety which satisfies the standard hypothesis. If $u, v \in \mathcal{L}^+(\mathcal{V})$ and $u \geq v$ then*

$$\int_{\mathcal{V}} u\,(dd^cv)^M \leq \int_{\mathcal{V}} v\,(dd^cu)^M + 2\pi \sum_{j=0}^{M-1} \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_u - \tilde{\rho}_v)\,(dd^c\tilde{\rho}_v + \omega)^{M-1-j} \wedge (\tilde{\rho}_u + \omega)^j$$

where $\omega = \frac{1}{2}dd^c \log(1 + \|z\|^2)$.

Proof. We retain the notation from Proposition 3.8 for convenience. By integrating the algebraic formula Proposition 3.8 we have

$$\int_{\mathcal{V}} u(dd^cv)^M - v(dd^cu)^M = \int_{\mathcal{V}} (u\,dd^cv - v\,dd^cu) \wedge T_{0,0}^{M-1,M-1} + \int_{\mathcal{V}} (v-u)\,T_{1,0}^{M-1,M}.$$

Since $u \leq v$ it follows that the last term must be negative from which we deduce

$$\int_{\mathcal{V}} u(dd^cv)^M - v(dd^cu)^M \leq \int_{\mathcal{V}} (u\,dd^cv - v\,dd^cu) \wedge T_{0,0}^{M-1,M-1}. \tag{26}$$

Observe that by definition

$$T_{0,0}^{M-1,M-1} = \sum_{j=0}^{M-1} (dd^cu)^j \wedge (dd^cv)^{M-1-j}.$$

Invoking the Bedford-Taylor formula (Theorem 2.67) to each term of the sum above we obtain

$$\begin{aligned} &\int_{\mathcal{V}} (u\,dd^cv - v\,dd^cu) \wedge \sum_{j=0}^{M-1} (dd^cu)^j \wedge (dd^cv)^{M-1-j} \\ &\qquad = 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_u - \tilde{\rho}_v) \wedge \sum_{j=0}^{M-1} (dd^c\tilde{\rho}_u + \omega)^j \wedge (dd^c\tilde{\rho}_v + \omega)^{M-1-j}. \end{aligned} \tag{27}$$

Applying the deduction from (27) to the deduction in (26) yields the result. □

We want to prove a consequence of Theorem 3.9 due to Bloom (Lemma 2.1, [11]). To do so we need the following lemma originally due to Bedford-Taylor ([4], Lemma 6.5).

Lemma 3.10. *Suppose that $u \in \mathcal{L}(\mathcal{V}), v \in \mathcal{L}^+(\mathcal{V})$ and $u \leq v$ for $(dd^cv)^M$-almost all of the points in the support of $(dd^cv)^M$. Then $u \leq v$ in $\mathcal{V}$.*

Proof. First note that $\text{supp}(dd^cv)^M$ is contained in $\{u \leq v\}$. Suppose that there exists $z_0 \in \mathcal{V}$ such that $u(z_0) > v(z_0)$. Let E be an open neighbourhood of z_0 where $E \subset \{u(z) > v(z)\}$ and let $w \in \mathcal{L}(\mathcal{V})$ with $\operatorname{supp}(dd^c w)^M \subset E$. Since $w \in \mathcal{L}(\mathcal{V})$, by addition of a constant if necessary, we may assume that $w(z) \leq v(z)$.

Let $0 < \delta < \varepsilon$ be such that $S = \{u(z) + \delta w(z) \geq (1+\varepsilon)v(z)\}$ is compact. Supposing that S is compact is valid since $v(z) \geq \log \|z\| + \beta$ for some $\beta \in \mathbb{R}$. Hence, for an appropriately chosen ε and δ, $(1+\varepsilon)v(z)$ grows faster than $u(z) + \delta w(z)$. Now for sufficiently small ε we have $E \subset S$, precisely when ε is chosen so that $0 \leq \varepsilon v(z) - \delta w(z) \leq \min\{u(z) - v(z) : z \in E\}$. Observe that if $S \cap \operatorname{supp}(dd^c v)^M \neq \varnothing$ then any point in the intersection fails to satisfy the lower bound for $v(z)$; i.e.

$$(1+\varepsilon)v \leq u(z) + \delta w(z) \leq v(z) + \delta w(z) \iff \varepsilon v(z) \leq \delta w(z) \lightning .$$

This implies that $S \cap \operatorname{supp}(dd^c v)^M = \varnothing$. Hence $\int_S (dd^c v)^M = 0$. We may use the comparison theorem (Theorem 2.13) since ∂S has admissible boundary data.

$$\begin{aligned}
0 &< \int_E (dd^c \delta w)^M + \int_E (dd^c u)^M \\
&\leq \int_E (dd^c(u + \delta w))^M \\
&\leq \int_S (dd^c(u + \delta w))^M \\
&\leq (1+\varepsilon)^M \int_S (dd^c v)^M = 0 \lightning .
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.11. *Let K be a nonpluripolar compact set in $\mathcal{V}$ and suppose that $V_K(z) \leq 0$ iff $z \in K$. Let $v \in \mathcal{L}^+(\mathcal{V})$ and suppose that $v \leq 0$ on K. If $\tilde{\rho}_v^* = \tilde{\rho}_K^*$ then $V_K = v$.*

Proof. Without loss of generality, we may assume $v \geq 0$ by consider $\max\{v, 0\}$ if necessary. The formula from Theorem 3.9 yields

$$\begin{aligned}
\int_{\mathcal{V}} V_K \, (dd^c v)^M &\leq \int_{\mathcal{V}} v \, (dd^c V_K)^M + 2\pi \sum_{j=0}^{M-1} \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_K^* - \tilde{\rho}_v^*) \, (dd^c \tilde{\rho}_v^* + \omega)^{M-1-j} \wedge (dd^c \tilde{\rho}_u^* + \omega)^j \\
&= \int_{\mathcal{V}} v \, (dd^c V_K)^M = 0.
\end{aligned}$$

We have used the fact that $\operatorname{supp}(dd^c V_K)^M \subset K$ and Stokes theorem to conclude that the final integral is 0. This implies that

$$0 = \int_{\mathcal{V}} V_K (dd^c v)^M = \int_{\{V_K > 0\}} V_K (dd^c v)^M.$$

Hence $\{V_K > 0\}$ is a set of $(dd^c v)^M$-measure zero. But since $V_K, v \geq 0$ it follows that $\{V_K > v\} \subset \{V_K > 0\}$ and so $\{V_K > v\}$ must also have $(dd^c v)^M$-measure zero. By Lemma 3.10 it follows that $V_K \leq v$ in $\mathcal{V}$. Hence there must be equality since V_K is maximal on $\mathcal{V} \backslash K$. This completes the proof. □

Remark 3.12. Bloom [11] proves the analogous proposition for $v \in \mathcal{L}(\mathbb{C}^N)$. We can weaken the hypothesis of Theorem 3.11 from $\mathcal{L}^+$ to $\mathcal{L}$ by considering $\max\{v, 0, \log|z|+c\}$ instead of v, with c chosen so that $\log|z| + c \leq 0$ on K. Bloom also imposes polynomial convexity and regularity on K. Polynomial convexity is equivalent to the condition $V_K(z) \leq 0$ iff $z \in K$ while we make no regularity assumption.

We are now interested in studying the continuity properties of the Robin function. Classical results along these lines are due to Levenberg [40], Bloom [11], and Nguyen and Zeriahi [45]. Since our standard hypothesis allows the possibility of singularities on $\mathcal{V}^h$ we can only hope for weak continuity. Precisely,

Definition 3.13. *Suppose that $\mathcal{V}$ is an algebraic variety. We say a function u is weakly continuous if u is continuous on $\mathcal{V}^{reg}$ and upper semicontinuous on $\mathcal{V}^{sing}$.*

Definition 3.14. *Suppose that $\mathcal{V}$ is irreducible but possibly singular. We say that a compact set $K \subset \mathcal{V}$ is regular if the extremal function V_K is weakly continuous on $\mathcal{V}$. Equivalently, $V_K \equiv V_K^*$ on K.*

Corollary 3.15. *If $V_K^* \equiv 0$ on K then K is regular.*

Proof. Note that $V_K^* \in w\mathcal{L}(\mathcal{V})$. Then $V_K^* = 0$ on K implies $V_K^* \leq V_K$. □

Corollary 3.16. *If $K \subset \mathcal{V}$ is regular then V_K is weakly continuous on $\mathcal{V}$. If additionally $\mathcal{V}$ is smooth then V_K is continuous.*

Remark 3.17. This is somewhat surprising since it is not guaranteed that V_K will be weakly continuous at singular points. V_K being an upper envelope of continuous functions guarantees that V_K is lower semicontinuous so one might expect that at singular points V_K is lower semicontinuous while V_K^* is upper semicontinuous. The corollary says that this is never the case when K is regular.

We note the following (often overlooked) property of regularity for future use.

Lemma 3.18 (Negligibility Lemma for Regular Sets)**.** *Let K be a non-pluripolar compact set in $\mathcal{V}$. Let $K^* := \{V_K^*(z) \leq 0\}$. Then $K \backslash K^*$ is pluripolar.*

Proof. The result is local so it suffices to prove the result on charts (E_α, ϕ_α) which cover K. Let

$$\mathcal{U}_\alpha := \{u \in PSH(\phi_\alpha(E_\alpha)) : (u \circ \phi_\alpha)(z) = v(z)|_{E_\alpha} \text{ where } v \in \mathcal{L}(\mathcal{V}), v|_K \leq 0\}.$$

Then $\mathcal{U}_\alpha$ is a family of psh functions which is locally bounded above (by $\phi_\alpha^* V_K^*$). If $u_\alpha = \sup_{u \in \mathcal{U}_\alpha} u(z)$ then the set $\{u_\alpha(z)^* > u_\alpha(z)\}$ is pluripolar by Theorem 4.7.6 [37]. By construction $u_\alpha = \phi_\alpha^* V_K$ and $u_\alpha^* = \phi_\alpha^* V_K^*$. It follows that $(K \backslash K^*) \cap E_\alpha$ is pluripolar which completes the proof. □

Corollary 3.19. *The set $\{V_K^* > V_K\}$ is pluripolar.*

Proof. $\mathcal{V}^{sing}$ is pluripolar so it suffices to check $\{V_K^* > V_K\} \cap \mathcal{V}^{reg}$ is pluripolar. The argument in Lemma 3.18 does this. $\square$

Theorem 3.20. *Suppose that $\mathcal{V}$ satisfies the standard hypothesis and let $K \subset \mathcal{V}$ be regular. Let $\tilde{V}_K(t, z)$ be the lift of V_K to $\mathcal{V}_\uparrow$ given by the $\mathcal{H}$-principle. Let $E = \{(t, z) \in \mathcal{V}_\uparrow : \tilde{V}_K(t, z) \leq 0\}$. Then*

$$\tilde{V}_K^+(t, z) = \max\{0, \tilde{V}_K(t, z)\} = V_E(t, z).$$

Moreover, both functions are weakly continuous on $\mathcal{V}_\uparrow$.

Proof. Firstly, $\tilde{V}_K^+(t, z) = 0$ on E by construction of E. Since $V_K = V_K^*$ on $\mathcal{V}$ it follows that $V_K \in \mathcal{L}(\mathcal{V})$ so by the $\mathcal{H}$-principle $\tilde{V}_K(t, z) \in w\mathcal{H}(\mathcal{V}_\uparrow)$ and so $\tilde{V}_K^+(t, z) \in w\mathcal{L}(\mathcal{V}_\uparrow)$. It follows that $\tilde{V}_K^+ \leq V_E$, so we seek the reverse inequality. Observe that E is a circled set since $\tilde{V}_K(t, z)$ is log homogeneous. It follows that V_E is homogeneous and hence coincides with the homogeneous extremal function. Consequently we need only consider log homogeneous functions to define V_E. If $h \in w\mathcal{H}(\mathcal{V}_\uparrow)$ then by the $\mathcal{H}$-principle $h(1, z) \in \mathcal{L}(\mathcal{V})$, and in particular for $t \neq 0$

$$\begin{aligned} & h(t, z) \leq V_E(t, z) \\ \Rightarrow\ & h(1, z/t) + \log|t| \leq V_E(1, z/t) + \log|t|,\ t \neq 0 \\ \Rightarrow\ & h(1, \zeta) \leq V_E(1, \zeta) \leq V_K(\zeta),\ \zeta = z/t. \end{aligned}$$

It follows that $h(t, z) \leq \tilde{V}_K(t, z)$ for all $(t, z) \in \mathcal{V}_\uparrow \backslash (\{t = 0\} \cup E)$. It follows then that $V_E = 0$ on $E \backslash \{t = 0\}$, and hence $V_E^* \equiv 0$ on E since $\{t = 0\} \cap E$ is pluripolar (in $\mathcal{V}_\uparrow$). Since $\tilde{V}_K^+ = V_E$ quasi-everywhere it follows that they must agree everywhere since they are both psh functions, hence $\tilde{V}_K^+ \equiv V_E$. Weak continuity is an immediate consequence of $V_E = V_E^* = 0$ on E. $\square$

Corollary 3.21. *Suppose that $K \subset \mathcal{V}$ is regular. Then ρ_K is weakly continuous on $\mathcal{V}^h$.*

Proof. By the previous theorem the function $\tilde{V}_K^+$ is weakly continuous on $\mathcal{V}_\uparrow$. Note that the restriction of $\tilde{V}_K^+$ to $\{t = 0\} \cap \{\tilde{V}_K \geq 0\}$ defines the values for the Robin function on that set. It follows that the Robin function is weakly continuous there. Since Robin functions are log homogeneous by Lemma 2.56 they are continuous along complex lines. It follows that weak continuity extends to all points of $\mathcal{V}^h$ (taking the value $-\infty$ at 0) which completes the proof. $\square$

3.3 Polynomial Formulae

Theorem 3.22. *Let $K \subset \mathcal{V}$ be compact and V_K be the corresponding extremal function. Define*

$$\Phi_K(z) = \sup\{|p(z)|^{1/\deg(p)} : p \in \mathbb{C}[\mathcal{V}], \|p\|_K \leq 1\}.$$

Then $V_K = \log \Phi_K$.

Proof. By Klimek [37] Theorem 5.1.7 the corresponding result holds in $\mathbb{C}^N$. In this light, treating K as a subset of $\mathbb{C}^N$ gives the desired result provided that

$$\Phi_K(z) = \sup\{|p(z)|^{1/\deg(p)} : p \in \mathbb{C}[z], \|p\|_K \leq 1\}. \tag{28}$$

To this end observe that $\{p + \mathbf{I}(\mathcal{V}) : p \in \mathbb{C}[\mathcal{V}]\} \supset \mathbb{C}[z]$ and all polynomials q in this class have the same values on $\mathcal{V}$. It follows that for any fixed $p \in \mathbb{C}[\mathcal{V}]$ the supremum

$$\sup\left\{|q(z)|^{1/\deg q} : q \in \{p + \mathbf{I}(\mathcal{V})\}\right\}$$

is attained by $q \in \{p+\mathbf{I}(\mathcal{V})\}$ of minimal degree in this class. Observe that p has minimal degree in this class since $\deg(p+q) = \max\{\deg p, \deg q\}$ so $\deg(p+q) \geq \deg(p)$ for all q. As such it suffices for the supremum in (28) to be taken only over $p \in \mathbb{C}[\mathcal{V}]$. $\square$

Remark 3.23. Despite K being pluripolar in $\mathbb{C}^N$ the above result still holds without contradicting that V_K is finite on $\mathcal{V}$. However, $V_K = \infty$ on $\mathbb{C}^N \backslash \mathcal{V}$ if K is viewed as a subset of $\mathbb{C}^N$ (which implies that K is pluripolar in $\mathbb{C}^N$). Our approach simplifies the approach of Zeriahi [55] who proves the formula intrinsically on $\mathcal{V}$ rather than by restriction from $\mathbb{C}^N$.

Theorem 3.24. *Suppose that $\mathcal{V}$ satisfies the standard hypothesis and let $K \subset \mathcal{V}$ be compact and regular. Define*

$$\Psi_K(z) = \sup\{|\hat{p}(z)|^{1/\deg(p)} : p \in \mathbb{C}[\mathcal{V}], \|p\|_K \leq 1\}.$$

where $\hat{p}(z)$ is the top degree homogeneous part of p (recall Definition 1.150). Then

$$\rho_K(z) = \log \Psi_K(z).$$

Proof. We use the notation from Theorem 3.20. Let E be as in the hypothesis of Theorem 3.20. Then E is a circled set so, by Lemma 1.148, $\Phi_E = \Psi_E$. Then $V_E = \Psi_E(z)$ by Theorem 3.24. Note $V_E(0,z) = \rho_K(z)$ hence $\Phi_E(0,z) = \rho_K(z)$. We make two observations. Firstly, if $p(t,z)$ is a homogeneous polynomial in $N+1$ variables, then $p(0,z)$ is a homogeneous polynomial in N variables and is equal to $\hat{p}(1,z)$. Secondly, if $\frac{1}{\deg(p)} \log|p(t,z)| \leq V_E(t,z)$ then $\log|p(1,z)| \leq 0$ for $z \in K$. The conclusion of the theorem follows from these two observations. $\square$

Corollary 3.25. *Let $\hat{K} = \{z \in \mathcal{V} : |p(z)| \leq \|p\|_K$ for all $p \in \mathbb{C}[\mathcal{V}]\}$ be the polynomial convex hull of K. Then $V_K = V_{\hat{K}}$. Moreover if $K = \{z \in \mathcal{V} : V_K(z) \leq 0\}$ then K is polynomially convex.*

Proposition 3.26. *Suppose that $\mathcal{V}$ satisfies the standard hypothesis and let $K \subset \mathcal{V}$ be compact and not regular. Then*

$$\rho_K(z)^* = \log \Psi_K(z)^*.$$

Proof. Let V be the function

$$V := \sup\{\tilde{u}(t,z) : u|_K \leq 0, u \in \mathcal{L}^+(\mathcal{V})\}$$

where $\tilde{u}(t,z)$ is the lift guaranteed by the $\mathcal{H}$-principal. Then V is the extremal function for its zero set. That is, if $E = \{V \leq 0\}$ then $V_E = V$. Moreover E is circled since each $\tilde{u}$ is homogeneous. It follows then that $V_E = \log \Psi_E(t,z)$. If $K^h := \{z \in \mathcal{V}^h : \Psi_K(z) \leq 0\}$ then since any p which is a competitor for $\Psi_{K^h}(z)$ is a competitor for $\Psi_E(0,z)$ and visa versa,

$$\Psi_K(z) = \Psi_{K^h}(z) = \Psi_E(0,z).$$

By Corollary 3.19 $V_K(z)^* = V_E(1,z)$ q.e. since $V_E(1,z) = V_K(z)$. It follows that $\tilde{V}_K(t,z)^* = V_E(t,z)$ q.e. Taking the usc regularisation of both sides we obtain the equality $V_K(t,z)^* = V_E(t,z)^*$ and in particular for $t = 0$. That is,

$$\rho_K(z)^* = \tilde{V}_K(0,z)^* = V_E(0,z)^* = \Psi_E(0,z)^*$$

We are done if $\Psi_E(0,z)^* = \Psi_K(z)^*$ (note there may not be equality since the usc regularisation on the LHS is over (t,z) while the usc regularisation on the RHS is only over z). Let $Z = \{z \in \mathcal{V} : V_K(z)^* > V_K(z)\}$ and $\tilde{Z} = \{(t,z) \in \mathcal{V}_\uparrow : (1, z/t) \in Z, t \neq 0\}$. Then the set $\tilde{Z}$ coincides with $\{V_E(t,z)^* > V_E(t,z), t \neq 0\}$ by construction of V_E. Then for any point $(0,z)$ outside of an open neighbourhood of $\tilde{Z}$ we have

$$V_E(0,z) = \limsup_{\substack{(t,\zeta)\to(0,z) \\ (t,z)\in\mathcal{V}_\uparrow}} V_E(t,z) = \limsup_{\substack{(t,\zeta)\to(0,z) \\ (t,z)\in\mathcal{V}_\uparrow}} V_E(t,z)^* = V_E(0,z)^*.$$

Since $\tilde{Z}$ is pluripolar in $\mathcal{V}_\uparrow$ it follows that $V_E(0,z)$ is determined q.e. by values on $\mathcal{V}_\uparrow \backslash \tilde{Z}$. This shows that $\Psi_K(z) = \Psi_E(0,z) = V_E(0,z) = V_E(0,z)^*$ q.e. on $\mathcal{V}_\uparrow \cap \{t = 0\}$. Taking the usc regularisation (as functions of z and not (t,z)) yields $\Psi_K(z)^* = \Psi_E(0,z)^*$ which completes the proof. □

Lastly we show that the Robin function induces its own extremal function.

Lemma 3.27. *Suppose that X is an M-dimensional complex manifold with the property that $x \in X$ implies $\lambda x \in X$ for all $\lambda \in \mathbb{C}$. Suppose that $u : X \to \mathbb{R} \cup \{-\infty\}$ is a log homogeneous psh function. Then u is maximal on the set $X \backslash \{u \leq 0\}$.*

Proof. Suppose that $v \in \mathcal{L}(X)$ such that $v \leq u$ on $\{u \leq 0\}$. If $\zeta \in \{u = 0\}$ then $v(\lambda\zeta) \leq u(\lambda\zeta)$ for $|\lambda| \leq 1$. For $\lambda > 1$ we have $u(\lambda\zeta) = u(\zeta) + \log|\lambda|$. Since $\log|\lambda|$ is harmonic in λ for $|\lambda| > 1$ it follows that u is maximal along the line $L_\lambda = \{\lambda\zeta : \lambda \in \mathbb{C}\}$. Hence by Lemma 1.19, $u \geq v$ on L_λ. Repeating this argument for all $\zeta \in \{u = 0\}$ shows that $u \geq v$ everywhere, and hence u is maximal. □

Corollary 3.28. $\rho_u^+ = \max\{0, \rho_u\}$ *is the extremal function for the set* $\{\rho_u \leq 0\}$. *In particular,* $V_{K_\rho} = \rho_K^+$ *where* $K_\rho := \{\rho_K \leq 0\}$.

We also note the following homogeneity property for future use.

Lemma 3.29. *Suppose that* $\mathcal{V}$ *is an algebraic variety which satisfies the standard hypothesis. Suppose that* u *is a log-homogeneous function in the* x *variables, i.e.*

$$u(\lambda x, y(\lambda x)) = u(x, y(x)) + \log|\lambda|.$$

Then u *is maximal on the set* $\mathcal{V}\backslash\{u \leq 0\}$.

Proof. Let $z \in \mathcal{V}\backslash\{u \leq 0\}$. Then there is $\lambda \in \mathbb{C}, |\lambda| > 1$ and $(x, y(x)) \in \{u = 0\}$ such that $\lambda z = (\lambda x, y(\lambda x))$ since the projection $z \in \mathcal{V} \to \mathbb{C}^M \ni x$ is onto. The argument is now the same as Lemma 3.27. $\square$

3.4 Projective Capacity

Definition 3.30. *Suppose that* $K \subset \mathcal{V}$ *is a compact set. Let* $\mathbb{C}[\mathcal{V}]$ *be the usual reduced monomial basis for* $\mathcal{V}$ *with Noether presentation* (x, y). *Let* $\zeta \in \mathcal{V}^h$ *with* $\|\zeta\| = 1$. *Then we define the projective Chebyshev constant in the direction* ζ *to be*

$$\kappa_n(K, \zeta) = \inf\{\|p\|_K : p \in P_n(\mathcal{V}), |\hat{p}(\zeta)| = 1\}$$
$$\kappa(K, \zeta) = \limsup_{n\to\infty} \kappa_n(K, \zeta)^{1/n}.$$

A polynomial $p \in P_n(\mathcal{V})$, $|\hat{p}(\zeta)| = 1$ *such that* $\|p\|_K = \kappa_n(K, \zeta)$ *will be called a* ζ*-Chebyshev polynomial of degree* n.

Remark 3.31. The terminology 'Projective Chebyshev constant' comes from the fact that $\zeta \in \mathcal{V}^h$ and $\|\zeta\| = 1$ means that ζ naturally corresponds to a point in $\tilde{\mathcal{V}}^h$. The lim sup in the definition can be replaced by a limit, we will see this in Corollary 3.34.

Lemma 3.32. ζ*-Chebyshev polynomials exist.*

Proof. Let $K \subset \mathcal{V}$ be compact, we show the existence of the ζ-Chebyshev polynomial of degree n for this set. By definition of $\kappa_n(K, \zeta)$ there exists a sequence of polynomials $\{p_j\}$ of at most degree n such that $\lim_{j\to\infty} \|p_j\|_K = \kappa_n(K, \zeta)$. Choose $j_0 \in \mathbb{N}$ sufficiently large so that $\|p_j\|_K \leq 2\kappa_n(K, \zeta)$ and any set of $m(n)$ points in K enumerated as $z_1, ..., z_{m(n)}$, where $m(n)$ is the number of monomials of degree at most n.

Let $L : P_n(\mathcal{V}) \to \mathbb{C}^{m(n)}$ be the linear map that maps $p \mapsto (p(z_1), ..., p(z_{m(n)}))$. As L is a linear map between vector spaces of the same dimension and the kernel of L is 0, it follows that L is invertible. Each coordinate of Lp_j is bounded by $2\kappa_n(K, \zeta)$, and hence forms a bounded sequence of complex numbers and so there is a convergent subsequence Lp_{j_k}. Inverting under

L yields a sequence p_{j_k} converging to some $p \in P_n(K)$. As $\|p_{j_k}\|_K \to \kappa_n(K,\zeta)$ it follows that $\|p\|_K = \kappa_n(K,\zeta)$. Finally, since $\hat{p}_{j_k}(\zeta) = 1$ for all k, it follows that $\hat{p}(\zeta) = 1$ and hence p is a ζ-Chebyshev polynomial. $\square$

Corresponding classical notions have been well studied by a number of authors, in particular Alexander [1], Levenberg and Bloom [43] [11], Kołodziej and Cegrell [38] and Nivoche [46]. In particular it is known that for any $\zeta \in \mathbb{P}^{N-1}$ that $\kappa(K,\zeta) = \tilde{\rho}_K(\zeta)$ (e.g. Proposition 4.2, [46]). In [33] we showed that this equality holds in the case of an algebraic curve (essentially the contents of Theorem 4.2). We will show that the results here supersede the results found there. The main result in this section is the following equality.

Theorem 3.33. *Suppose that K is compact and nonpluripolar. Then for all $\zeta \in \mathcal{V}^h$ with $\|\zeta\| = 1$ we have $\kappa(K,\zeta) = \Psi_K(\zeta)$.*

Proof. First we establish $\kappa(K,\zeta) \geq \Psi_K(\zeta)$. Let p be any polynomial with $|\hat{p}(\zeta)| = 1$ of degree n and let $q(z) = p(z)/\|p\|_K$. Then $\|q\|_K \leq 1$ so we have by definition

$$|\hat{q}(z)| \leq \Psi(z)^n.$$

Letting $z = \zeta$ we obtain

$$\Psi(\zeta)^n \geq |\hat{q}(\zeta)| = \frac{|\hat{p}(\zeta)|}{\|p\|_K} = \frac{1}{\|p\|_K}.$$

This implies that

$$\|p\|_K \geq \Psi(\zeta)^n.$$

But this inequality is valid for any p satisfying the hypothesis, so in particular for a ζ-Chebyshev polynomial hence

$$\kappa_n(K,\zeta) \geq \Psi(\zeta)^n.$$

Taking nth roots and the the lim inf as $n \to \infty$ yields

$$\liminf_{n\to\infty} \kappa_n(K,\zeta) \geq \Psi_K(\zeta).$$

For the converse, let p_j be a polynomial of degree j such that $\|p_j(\zeta)\|_K \leq 1$ and $\limsup_{j\to\infty} |\hat{p}(z)|^{1/j} = \Psi_K(\zeta)$ and write $q_j(z) = p_j(z)/|\hat{p}_j(\zeta)|$. Then $\|q_j\|_K \geq \kappa_j(K,\zeta)$ so for all $j \in \mathbb{N}$

$$\kappa_j(K,\zeta) \leq \|q_j\|_K = \frac{\|p_j\|_K}{|\hat{p}_j(z)|} \leq \frac{1}{|\hat{p}_j(\zeta)|}.$$

Taking the jth root of both sides and lim sup we obtain

$$\kappa(K,\zeta) = \limsup_{j\to\infty} \kappa_j(K,\zeta)^{1/j} \leq \limsup_{j\to\infty} \frac{1}{|\hat{p}_j(\zeta)|^{1/j}} = \Psi_K(\zeta).$$

Putting these two results together we obtain

$$\Psi_K(\zeta) \le \liminf_{n\to\infty} \kappa_n(K,\zeta) \le \kappa(K,\zeta) \le \Psi_K(\zeta).$$

This completes the proof. $\square$

Corollary 3.34. *The* lim sup *in Definition 3.30 can be replaced by a limit.*

Corollary 3.35. *Suppose that K is compact, regular and nonpluripolar. Then for all $\zeta \in \mathcal{V}^h$ with $\|\zeta\| = 1$ we have $\kappa(K,\zeta) = e^{-\rho_K(\zeta)}$.*

Proof. Follows from Theorems 3.24 and 3.33. $\square$

Remark 3.36. Observe that $\rho_K(\zeta) = \tilde{\rho}_K([\zeta])$ when $\|\zeta\| = 1$ which allows for the classical identification observed in Remark 3.31.

Corollary 3.35 can be seen as a generalisation (and minor simplification) of Theorem 5.7 in [33]. To see this we need some further terminology.

Definition 3.37. *Suppose that $\mathcal{V}$ satisfies the standard hypothesis. Let $d\sigma$ be normalised 'spherical' measure on $\tilde{\mathcal{V}}^h$ so that $\sigma(\tilde{\mathcal{V}}^h) = 1$. We define the projective capacity to be*

$$\kappa(K) := \exp\left(\int_{\tilde{\mathcal{V}}^h} \ln \kappa(K,\zeta)\, d\sigma\right).$$

Remark 3.38. The normalisation constant for $d\sigma$ can be checked to be $d(2\pi)^M$.

In the situation of Hart-Ma'u [33], $\mathcal{V}$ is an all algebraic curve are of degree d with d intersections with the hyperplane at infinity denoted $\lambda_1, ..., \lambda_d$ and also called directions. Their projective coordinates can be normalised so that $\|\lambda_j\| = 1$ for each j, but in [33] the normalisation is chosen so that the first non-zero coordinate of λ_j is 1. The choice of normalisation has no impact on the result which we will see shortly. To each direction there is a unique polynomial $\mathbf{v}_{\lambda_j}(z) \in \mathbb{C}[\mathcal{V}]$ satisfying

(a) $\mathbf{v}_{\lambda_j}$ is of minimal degree,

(b) $\mathbf{v}_{\lambda_j}(\lambda_j) = 1$ and $\mathbf{v}_{\lambda_j}(\lambda_i) = 0$ for any direction $i \neq j$,

(c) For any polynomial in $\mathbb{C}[z]$, $p(z)\mathbf{v}_{\lambda_j}(z) = \hat{p}(\lambda_j) z_1^{\deg p} \mathbf{v}_{\lambda_j}(z) + q(z)$ where q has degree at most $\deg p + \deg \mathbf{v}_{\lambda_j} - 1$.

From these conditions it is clear the only difference that results from a different choice of normalisation for λ_j is multiplication of $\mathbf{v}_{\lambda_j}$ by a constant. In particular there is a constant c_λ so that $c_{\lambda_j}\mathbf{v}_{\lambda_j} = v_j$ where v_j is one of the polynomials discussed in Lemma 1.134 which are used in the construction of the set $\mathcal{C}$ (Definition 1.138). Moreover it is readily seen that this is a specific instance of our more general notion of distinct intersections at infinity (check Section 1.3.1).

Definition (Definition 1.139)**.** *For $K \subset \mathcal{V}$ compact we define*

$$\begin{aligned} T_n(K, \lambda_j) &:= \inf\{\|p\| : \deg(p) = n, p = \mathbf{v}_{\lambda_j} z_1^{n-\deg \mathbf{v}_\lambda} + l.o.t.\} \\ \tau(K, \lambda_j) &:= \lim_{n\to\infty} T_n(K, \lambda_j)^{1/n} \\ \tau(K) &= \left(\prod_{j=1}^{d} \tau(K, \lambda_j) \right)^{1/d}. \end{aligned}$$

Where l.o.t. stands for terms of lower order than $\mathbf{v}_{\lambda_j} z_1^{n-\deg \mathbf{v}_\lambda}$ with respect to grevlex ordering for the basis $\mathcal{C}$.

Theorem 3.39 (Theorem 5.7, [33])**.** *Let $K \subset \mathcal{V}$ be compact and regular and let λ_j be a direction of $\mathcal{V}$. Then*

$$e^{-\rho_K(\lambda_j)} = \tau(K, \lambda_j).$$

Proof. It suffices to show that $\kappa(K, \lambda_j) = \tau(K, \lambda_j)$. Let $(x, y_2, ..., y_N)$ be a Noether presentation for $\mathcal{V}$. Using polynomial long division we can decompose any $p \in \mathbb{C}[\mathcal{V}]$ so that

$$p(z) = \left(\sum_{i=1}^{d} a_i x^{n-\deg \mathbf{v}_{\lambda_i}} \mathbf{v}_{\lambda_i}(x, y) \right) + l.o.t. \tag{29}$$

Suppose that q is a polynomial of degree n satisfying $\|q\|_K = \kappa_n(K, \lambda_j)$. Then $|\hat{q}(\lambda_j)| = 1$ so using the decomposition in (29) we deduce that $a_j = 1$. It follows by using properties (b) and (c) of $\mathbf{v}_{\lambda_j}$ and the decomposition above that

$$\|\mathbf{v}_{\lambda_j} q\|_K = \|\hat{q}(\lambda_j) x^n \mathbf{v}_{\lambda_j} + l.o.t.\|_K = \|a_j x^n \mathbf{v}_{\lambda_j} + l.o.t.\|_K = \|x^n \mathbf{v}_{\lambda_j} + l.o.t.\|_K \geq T_{n+\deg \mathbf{v}_{\lambda_j}}(K, \lambda)$$

After taking nth roots and letting $n \to \infty$ it follows that $\kappa(K, \lambda_j) \geq \tau(K, \lambda_j)$. The opposite inequality follows since there exists a polynomial such that $T_n(K, \lambda) = \|\mathbf{v}_{\lambda_j} x^{n-\deg \mathbf{v}_\lambda} + l.o.t.\|_K \geq \kappa_n(K, \lambda_j)$ for each n. The conclusion of the theorem now follows from Corollary 3.35. □

Corollary 3.40. *Let $K \subset \mathcal{V}$ be compact and regular, and let $\{\lambda_j\}_{j=1,...,d}$ be the directions of $\mathcal{V}$.*

$$\tau(K) = \exp\left(\frac{1}{d} \sum_{j=1}^{d} -\rho_K(\lambda_j) \right) = \kappa(K).$$

To generalise this result to an algebraic variety we need a generalisation of Rumely's formula [49].

3.5 Rumely's Formula

Theorem 3.41 (Rumely's Formula,[49]). *Suppose that $K \subset \mathbb{C}^N$ is compact, polynomially convex and regular. Then*

$$
\begin{aligned}
-\log\delta(K) =& \frac{1}{N}\Bigg[\frac{1}{(2\pi)^{N-1}}\int_{\mathbb{C}^{N-1}} \rho_K(1,z_2,...,z_N)(dd^c\rho_K(1,z_2,...,z_N))^{N-1} \\
&+ \frac{1}{(2\pi)^{N-2}}\int_{\mathbb{C}^{N-1}} \rho_K(0,1,z_3,...,z_N)(dd^c\rho_K(0,1,z_3,...,z_N))^{N-2} \\
&+ ... + \frac{1}{2\pi}\int_{\mathbb{C}} \rho_K(0,...,0,1,z_N)(dd^c\rho_K(0,...,0,1,z_N)) + \rho_K(0,...,0,1)\Bigg]. \qquad (30)
\end{aligned}
$$

Equivalently, we have an "energy version" of this formula

$$
-\log\delta(K) = \frac{1}{N(2\pi)^{N-1}}\int_{\mathbb{P}^{N-1}} [\tilde{\rho}_K - \tilde{\rho}_T] \sum_{j=0}^{N-1} (dd^c\tilde{\rho}_K + \omega)^j \wedge (dd^c\tilde{\rho}_T + \omega)^{M-j-1}, \qquad (31)
$$

where T is the unit torus in $\mathbb{C}^N$.

We will check multiple things to obtain a generalisation of Rumely's formula. We firstly state the result then realise our situation in the context of Berman-Boucksom ([7]) which allows us to rely on their machinery to do the heavy lifting to derive the result.

Theorem 3.42 (Rumely-Type Formula for Algebraic Varieties). *Suppose that $K \subset \mathcal{V}$ is compact, polynomially convex and regular. Then*

$$
\begin{aligned}
-\log\delta(K) =& \frac{1}{dM}\Bigg[\frac{1}{(2\pi)^{M-1}}\int_{\mathcal{V}_i^h} \rho_K(1,z_2,...,z_N)(dd^c\rho_K(1,x_2,...,x_M,y))^{M-1} \\
&+ \frac{1}{(2\pi)^{M-2}}\int_{\mathcal{V}_i^h} \rho_K(0,1,x_3,...,x_M,y)(dd^c\rho_K(0,1,x_3,...,x_M,y))^{M-2} \\
&+ ... + \frac{1}{2\pi}\int_{\mathcal{V}_i^h} \rho_K(0,...,0,1,x_N,y)(dd^c\rho_K(0,...,0,1,x_N,y)) \\
&+ \sum_{i=1}^{d} \rho_K(0,...,0,1,y_i)\Bigg].
\end{aligned}
$$

Equivalently, we have an "energy version" of this formula

$$
-\log\delta(K) = \frac{1}{dM(2\pi)^{M-1}}\int_{\mathcal{V}^h} [\tilde{\rho}_K - \tilde{\rho}_{T_\mathcal{V}}] \sum_{j=0}^{N-1} (dd^c\tilde{\rho}_K + \omega)^j \wedge (dd^c\tilde{\rho}_{T_\mathcal{V}} + \omega)^{M-j-1},
$$

where $T_\mathcal{V} = \{|x_1| \leq 1, ..., |x_M| \leq 1\} \cap \mathcal{V}$.

3.5.1 Projectivised Energy Formula

The following result is a consequence of Theorem 2.67. Using the notation of Levenberg (Section 3, [41]), we define the Monge-Ampère energy bracket to be

$$\mathcal{E}(u,v) = \int_{\mathcal{V}} (u-v) \sum_{j=0}^{M} (dd^c u)^j \wedge (dd^c v)^{M-j}.$$

Proposition 3.43. *Let $u, v \in \mathcal{L}^+(\mathcal{V})$ with* $\operatorname{supp}(dd^c u)^M$ *and* $\operatorname{supp}(dd^c v)^M$ *compact. Then*

$$\mathcal{E}(u,v) = \int_{\mathcal{V}} u(dd^c u)^M - \int_{\mathcal{V}} v(dd^c v)^M + 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_u^* - \tilde{\rho}_v^*) \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1}.$$

Proof. Using the algebraic identity for the difference of powers we obtain

$$(dd^c u)^M - (dd^c v)^M = (dd^c u - dd^c v) \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1} = dd^c(u-v) \wedge \sum_{j=0}^{M-1} (dd^c u)^j. \tag{32}$$

This allows us to write the energy bracket as

$$\mathcal{E}(u,v) \int_{\mathcal{V}} (u-v) \left[(dd^c u)^M + dd^c v \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1} \right].$$

Since the supports of $(dd^c u)^M$ and $(dd^c v)^M$ are compact we can split the integral up to obtain

$$\begin{aligned}
\mathcal{E}(u,v) &= \int_{\mathcal{V}} u(dd^c u)^M - \int_{\mathcal{V}} v(dd^c u)^M + \int_{\mathcal{V}} (u-v) dd^c v \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1} \\
&= \int_{\mathcal{V}} u(dd^c u)^M - \int_{\mathcal{V}} v(dd^c v)^M + \int_{\mathcal{V}} v[(dd^c v)^M - (dd^c u)^M] \\
&\qquad + \int_{\mathcal{V}} (u-v) dd^c v \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1}.
\end{aligned}$$

Using identity (32) again in the third term we obtain

$$\begin{aligned}
\mathcal{E}(u,v) &= \int_{\mathcal{V}} u(dd^c u)^M - \int_{\mathcal{V}} v(dd^c v)^M + \int_{\mathcal{V}} v\, dd^c(v-u) \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1} \\
&\quad + (u-v) dd^c v \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1} \\
&= \int_{\mathcal{V}} u(dd^c u)^M - \int_{\mathcal{V}} v(dd^c v)^M + \int_{\mathcal{V}} (u\, dd^c v - v\, dd^c u) \wedge \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1}.
\end{aligned}$$

Now we apply Theorem 3.9 to the last term to obtain.

$$\mathcal{E}(u,v) = \int_{\mathcal{V}} u(dd^c u)^M - \int_{\mathcal{V}} v(dd^c v)^M + 2\pi \int_{\tilde{\mathcal{V}}_i^h} (\tilde{\rho}_u^* - \tilde{\rho}_v^*) \sum_{j=0}^{M-1} (dd^c u)^j \wedge (dd^c v)^{M-j-1}.$$

□

3.5.2 Berman-Boucksom on Algebraic Varieties

Define $T_{\mathcal{V}} = \{|x_j| \leq 1\} \cap \mathcal{V}$.

Proposition 3.44. $V_{T_{\mathcal{V}}} = \max\{\log^+ |x_1|, ..., \log^+ |x_M|\}$

Proof. Clearly, $V_{T_{\mathcal{V}}}$ is log homogeneous. By Lemma 3.29 it must be maximal outside of its zero set. It is readily seen that $\{V_{T_{\mathcal{V}}} = 0\} = \pi^{-1}(T_{\mathcal{V}})$ which proves the claim. □

Proposition 3.45. *Let D_k be the monomials of degree k in $\mathbb{C}[\mathcal{V}]$, O_k be an $L^2(\mu)$-orthogonal basis for the monomials of at most degree k in $\mathbb{C}[\mathcal{V}]$ with respect to the probability measure $\mu = \frac{1}{d(2\pi)^M}(dd^c T_{\mathcal{V}})^M$ and S_k the corresponding orthonormal basis. Let $c_k = \max\{\|p_k\|_\mu : p_k \in O_k\}$. Then assuming $c_k^{1/k} \to 1$ as $k \to \infty$ we have*

$$\lim_{k\to\infty} \frac{1}{kN_k} \log \|\det[D_k]\|_{L^\infty(K)} = \lim_{k\to\infty} \frac{1}{kN_k} \log \|\det[O_k]\|_{L^\infty(K)} = \lim_{k\to\infty} \frac{1}{kN_k} \log \|\det[S_k]\|_{L^\infty(K)},$$

where we are using the notation that

$$\det[O_k] = VDM_{O_k}(z_1, ..., z_s)$$

where s is the number of monomials of at most degree k in O_k and $[O_k]$ is the corresponding Vandermonde matrix.

Proof. Observe that $\log \|\det[D_k]\|_{L^\infty(K)}$ is unchanged if rows are interchanged due to taking the absolute value and adding a multiple of a row to another leaves the determinant unchanged. It follows then that we can apply a Gram-Schmidt procedure to the monomials in D_k to produce an orthogonal (but not necessarily orthonormal) basis O_k. Such a procedure would take the form

$$\tilde{e}_j(z) = e_j(z) - \sum_{i=1}^{j-1} \langle e_j(z), \tilde{e}_i(z)\rangle_\mu \, \tilde{e}_i(z)$$

where $e_j \in D_k, \tilde{e}_j \in O_k$. It follows that adding $\langle e_j, \tilde{e}_i\rangle_\mu$ times the ith row of $[D_k]$ to the jth row (inductively) for every $0 \leq j \leq N_k$ yields $[O_k]$. Hence we have equality between $\log \|\det[D_k]\|_{L^\infty(K)} = \log \|\det[O_k]\|_{L^\infty(K)}$.

To obtain $[S_k]$ from $[O_k]$ we normalise each row by multiplying the ith row by a factor of

$\langle \tilde{e}_i(z), \tilde{e}_i(z)\rangle_\mu^{-1/2} = a_{i,k}$. Set $c_k = \max_{0\le i\le N_k}\{a_{i,k}\}$ and $c'_k = \min_{0\le i\le N_k}\{a_{i,k}\}$. Observe that

$$\begin{aligned}\log \|\det[O_k]\|_{L^\infty(K)} =& \log \|a_1 ... a_{N_k} \det[S_k]\|_{L^\infty(K)} \le \log \|c_k^{N_k} \det[S_k]\|_{L^\infty(K)},\\ & \log \|a_1 ... a_{N_k} \det[S_k]\|_{L^\infty(K)} \ge \log \|c_k'^{N_k} \det[S_k]\|_{L^\infty(K)}.\end{aligned}$$

From which it follows that

$$\begin{aligned}&\lim_{k\to\infty} \frac{1}{kN_k} \log \|\det[O_k]\| \le \log \|c_k^{N_k} \det[S_k]\|\\ &\qquad = \lim_{k\to\infty} \frac{N_k \log |c_k|}{kN_k} + \frac{1}{kN_k} \log \|\det[S_k]\|_{L^\infty(K)} = \frac{1}{kN_k} \log \|\det[S_k]\|_{L^\infty(K)},\end{aligned}$$

and

$$\begin{aligned}&\lim_{k\to\infty} \frac{1}{kN_k} \log \|\det[O_k]\|_{L^\infty(K)} \ge \log \|c_k'^{N_k} \det[S_k]\|_{L^\infty(K)}\\ &\qquad = \lim_{k\to\infty} \frac{N_k \log |c'_k|}{kN_k} + \frac{1}{kN_k} \log \|\det[S_k]\|_{L^\infty(K)} = \lim_{k\to\infty} \frac{1}{kN_k} \log \|\det[S_k]\|_{L^\infty(K)}.\end{aligned}$$

□

We now aim to show that normalising with respect to $\mu = (dd^c V_{T_\mathcal{V}})^M / d(2\pi^M)$ satisfies the growth condition on c_k in the hypothesis of Proposition 3.45.

Proposition 3.46. *Suppose that $\mathcal{V}$ is an algebraic variety and $\mu = \frac{(dd^c V_{T_\mathcal{V}})^M}{d(2\pi)^M}$. The the constants c_k required to normalise O_k from Proposition 3.45 satisfy the growth condition $c_k^{1/k} \to 0$ as $k \to \infty$.*

Proof. Let D_k and S_k be as in Proposition 3.45. Let $\tilde{D}_k$ be the basis generated by normalising each monomial in D_k with respect to μ. Precisely,

$$D_k \ni e_i(z) \mapsto \frac{e_i(z)}{\langle e_i, e_i\rangle_\mu^{1/2}} \in \tilde{D}_k.$$

Since (x, y) is a Noether presentation for $\mathcal{V}$ it follows that $\mathcal{V}$ is finite over x, which implies that there exists $R \ge 1$ such that $|y_i| \le R$ for all $1 \le i \le N - M$ and $(x, y) \in T_\mathcal{V}$. We now estimate

$$\begin{aligned}\langle x^\alpha y^\beta, x^\alpha y^\beta\rangle &= \int_{T_\mathcal{V}} x^\alpha y^\beta \overline{x^\alpha y^\beta}\, d\mu\\ &= \int_{T_\mathcal{V}} \prod_{i=1}^{M} |x_i|^{\alpha_i} \prod_{j=1}^{N-M} |y_i|^{\beta_i}\, d\mu\\ &\le \int_{T_\mathcal{V}} R^{\beta_1+\ldots+\beta_{N-M}}\, d\mu\\ &= R^{\beta_1+\ldots+\beta_{N-M}}.\end{aligned} \tag{33}$$

It follows that for any $e_i(z) \in D_k$ that $\langle e_i, e_i\rangle_\mu \le R^{m_Y}$ where $y_i^{m_i} \in \mathbb{C}[\mathcal{V}], y_i^{m_i+1} \notin \mathbb{C}[\mathcal{V}]$ and

$m_Y = \prod_{i=1}^{N-M} m_i$. Hence

$$\log \| \det[D_k] \|_{L^2(\mu)} \leq \log \| R^{m_Y N_k} \det[\tilde{D}_k] \|_{L^2(\mu)}$$

$$\lim_{k \to \infty} \frac{1}{kN_k} \log \| \det[D_k] \|_{L^2(\mu)} \leq \lim_{k \to \infty} \frac{1}{kN_k} \log \| R^{m_Y N_k} \det[\tilde{D}_k] \|_{L^2(\mu)} = \lim_{k \to \infty} \frac{1}{kN_k} \log \| \det[\tilde{D}_k] \|_{L^2(\mu)}.$$

We can obtain an orthogonal basis $\tilde{O}_k$ from $\tilde{D}_k$ by applying Gram-Schmidt without normalisation. To obtain S_k from $\tilde{O}_k$ we need only normalise the polynomials in $\tilde{O}_k$. Let f_j be an element of $\tilde{O}_k$ and $\tilde{f}_j$ the corresponding element of S_k and e_j the corresponding element of $\tilde{D}_k$. Then

$$\begin{aligned}
f_j(z) &= e_j(z) - \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu \tilde{f}_i(z) \\
\langle f_j, f_j \rangle_\mu &= \langle e_j, e_j \rangle_\mu - \left\langle e_j, \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu \tilde{f}_i \right\rangle_\mu - \left\langle \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu \tilde{f}_i, e_j \right\rangle_\mu \\
&\quad + \left\langle \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu \tilde{f}_i, \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu \tilde{f}_i \right\rangle_\mu \\
&= 1 - \sum_{i=1}^{j-1} \langle e_j, \tilde{f}_i \rangle_\mu^2 - \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu^2 + \sum_{i=1}^{j-1} \langle \tilde{f}_i, e_j \rangle_\mu^2 \langle \tilde{f}_i, \tilde{f}_i \rangle_\mu \\
&= 1 - \sum_{i=1}^{j-1} \langle e_j, \tilde{f}_i \rangle_\mu^2 \\
\langle f_j, f_j \rangle_\mu &\leq 1 + \sum_{i=1}^{j-1} |\langle \tilde{f}_i, e_j \rangle_\mu|^2 \leq 1 + \sum_{i=1}^{j-1} |\langle \tilde{f}_i, \tilde{f}_i \rangle_\mu| |\langle e_j, e_j \rangle_\mu| = j,
\end{aligned}$$

where we have used the fact that $\tilde{f}_i$ and e_i are normalised in lines 2 and 3, and used the Cauchy-Schwartz inequality in the final step. Taking the largest possible j yields the inequality

$$|\langle f_j, f_j \rangle_\mu| \leq N_k.$$

Hence

$$\lim_{k \to \infty} \frac{1}{kN_k} \log \| \det[\tilde{O}_k] \|_{L^2(\mu)} \leq \lim_{k \to \infty} \frac{1}{kN_k} \log \| N_k \det[S_k] \|_{L^2(\mu)} = \lim_{k \to \infty} \frac{1}{kN_k} \log \| \det[S_k] \|_{L^2(\mu)}.$$

This shows that the constants c_k from Proposition 3.45 satisfy the required growth condition. □

We can now relate the results of Berman-Boucksom to our situation.

Definition 3.47. *Let K be a compact set and μ a probability measure. We say that pair (K, μ) has the Bernstein-Markov property if for any polynomial p_j of degree at most j satisfies*

$$\|p_j\|_K \leq M_j \|p_j\|_{L^2(\mu)} \text{ with } \limsup_{j \to \infty} M_j^{1/j} = 1.$$

Equivalently, given $\varepsilon > 0$ there exists $C = C(\varepsilon, K)$ such that for all $k \in \mathbb{N}$

$$\|p_k\|_K \leq C(1+\varepsilon)^k \|p_k\|_{L^2(\mu)}.$$

Lemma 3.48. *The measure $\mu = (dd^c V_{T_\mathcal{V}})^M / d(2\pi)^M$ has the Bernstein-Markov property.*

Proof. We first consider a fixed monomial $x^\alpha y^\beta \in \mathbb{C}[\mathcal{V}]$. Observe that by the maximum principle $\|x^\alpha y^\beta\|_{T_\mathcal{V}} = \|x^\alpha y^\beta\|_{\partial T_\mathcal{V}}$. Since $\|x^\alpha\|_{\partial T_\mathcal{V}} = 1$ for any α it follows that

$$\|x^\alpha y^\beta\|_{\partial T_\mathcal{V}} = \max\{\|y^\beta\|_{\partial T_\mathcal{V}} : (x,y) \in \partial T_\mathcal{V}\} = M_\beta.$$

Next by equation (33) we know $\|x^\alpha y^\beta\|_{L^2(\mu)} \leq R^{m_Y}$. It follows that

$$\|x^\alpha y^\beta\|_K \leq \frac{M_\beta}{R^{m_Y}} \|x^\alpha y^\beta\|_{L^2(\mu)}.$$

Now consider an arbitrary monomial $x^\alpha y^\beta$. Set

$$M_0 = \max_{\beta : y^\beta \in \mathbb{C}[\mathcal{V}]} \frac{M_\beta}{R^{m_Y}}$$

which is a finite maximum because there are only finitely many β since (x,y) is a Noether presentation. It follows then that

$$\|x^\alpha y^\beta\|_K \leq M_0 \|x^\alpha y^\beta\|_{L^2(\mu)}.$$

Since M_0 is constant, it is trivial that $M_0^{1/k} \to 0$ as $k = |\alpha| + |\beta| \to \infty$. Now let p be an arbitrary polynomial of the form $p(x,y) = \sum c_{\alpha,\beta} x^\alpha y^\beta$. We have

$$\begin{aligned}
\|p\|_K &\leq \sum \|c_{\alpha,\beta} x^\alpha y^\beta\|_K \\
&\leq M_0 \|c_{\alpha,\beta} x^\alpha y^\beta\|_{L^2(\mu)} \\
&= M_0 \int_{T_\mathcal{V}} \sum c_{\alpha,\beta} x^\alpha y^\beta \overline{c_{\alpha,\beta} x^\alpha y^\beta}\, d\mu \\
&= M_0 \int_{T_\mathcal{V}} p\overline{p} - f(x,y)\, d\mu \\
&\leq M_0 \int_{T_\mathcal{V}} p\overline{p}\, d\mu = M_0 \|p\|_{L^2(\mu)}
\end{aligned}$$

where $f(x,y)$ consists of all of the cross terms of $p\overline{p}$ so that $f(x,y) + \sum c_{\alpha,\beta} x^\alpha y^\beta \overline{c_{\alpha,\beta} x^\alpha y^\beta} = p\overline{p}$. In particular, with this construction f is positive (and hence the inequality). This completes the proof. $\square$

Theorem 3.49 (Corollary A, [7]). *Let $E \subset \mathcal{V}$ and ν a probability measure on E with the Bernstein-Markov property. Let S_k be an $L^2(\mu)-$orthonormal basis for $\mathbb{C}[\mathcal{V}]$ and N_k the dimension of S_k.*

- *For every subset K we have*

$$\lim_{k\to\infty}\frac{1}{kN_k}\log\|\det[S_k]\|_{L^\infty(K)}=\frac{\mathcal{E}(V_E^*,V_K^*)}{M+1}.$$

- *If μ is a probability measure with the Bernstein-Markov property for K then*

$$\lim_{k\to\infty}\frac{1}{kN_k}\log\|\det[S_k]\|_{L^2(\mu)}=\frac{\mathcal{E}(V_E^*,V_K^*)}{M+1}.$$

Where we using the notation that

$$\|\det[S_k]\|_{L^\infty(K)}=\sup_{(z_1,...,z_{N_k})\in K}|VDM_{S_k}(z_1,...,z_{N_k})|$$
$$\|\det[S_k]\|_{L^2(\mu)}=\int_{(z_1,...,z_{N_k})\in\mathcal{V}}|VDM_{S_k}(z_1,...,z_{N_k})|^2\,\mu(dx_1)...\mu(dx_N).$$

We have made simplifications to the statement of the theorem so that they are directly relevant to our context. The more general statement is available in [7]. The set E above effectively serves as a ground energy level or an energy reference state. In classical pluripotential theory one chooses $E=T$, the unit torus, because the associated normalised Monge-Ampère operator of V_T induces an inner product for which the monomials are orthonormal. This allows the relationship between an energy bracket and monomials which is essential for the Rumely formula to be valid.

Proposition 3.50. *Let $T_\mathcal{V},\mu$ be as in Proposition 3.46. Then the conclusion of Theorem 3.49 is satisfied.*

Proof. We must find (i) a probability measure μ with the Bernstein-Markov property and (ii) an orthonormal basis S_k respect to $\langle.,.\rangle_\mu$. Claim (i) follows since the mass of $(dd^cV_{T_\mathcal{V}})^M$ is $d(2\pi)^M$ by Theorem 2.27 and that mass is concentrated on $T_\mathcal{V}$ by maximality. That μ has the Bernstein-Markov property follows from Lemma 3.48. For Claim (ii), Propositions 3.45 and 3.46 shows that we can find an orthonormal basis S_k with respect to μ such that the Fekete-Leja transfinite diameter with respect to D_k is the same as S_k. □

3.5.3 Cox-Ma'u Transfinite Diameter is a Berman-Boucksom Transfinite Diameter

Important to our final result is the precise relationship between the Cox-Ma'u transfinite diameter $\delta(K)$ (Definition 1.143) and the Berman-Boucksom transfinite diameter $\lim_{k\to\infty}\frac{1}{kN_k}\log\|\det[S_k]\|_{L^\infty(K)}$.

Theorem 3.51. *Let $K\subset\mathcal{V}$ be a compact set, $\mu=(dd^cT_\mathcal{V})^M$ and S_k be an $L^2(\mu)$-orthonormal basis. Then*

$$\log(K)=\frac{M+1}{M}\lim_{k\to\infty}\frac{1}{kN_k}\log\|\det[S_k]\|_{L^\infty(K)}.$$

Proof. We make the following notation conventions:

1. N_k is the number of monomials in $\mathbb{C}[\mathcal{V}]$ of degree at most k.

2. $l_k = \sum_{i=1}^{k} i(N_i - N_{i-1})$ is the sum of the degrees of the monomials in $\mathbb{C}[\mathcal{V}]$ of degree at most k.

3. $N_k^{(x,M)} = N_k^x$ is the number of monomials in $\mathbb{C}[\mathcal{V}]$ in x of degree at most k.

4. $l_k^x = \sum_{i=1}^{k} i(N_i^x - N_{i-1}^x)$ is the sum of the degrees of the monomials in $\mathbb{C}[\mathcal{V}]$ in x of degree at most k.

5. Let m_i be the smallest integer such that $y_i^{m_i+1} \notin \mathbb{C}[\mathcal{V}]$ and $m_Y = \sum m_i$.

6. Let $\mathbb{C}[\mathcal{V}]_{\leq k}$ be the monomials in $\mathbb{C}[\mathcal{V}]$ of degree at most k and let $\mathbb{C}[\mathcal{V}]_{\leq k}^x$ be the monomials in x in $\mathbb{C}[\mathcal{V}]_{\leq k}$

It is a standard result that $N_k^x = \begin{pmatrix} M+k \\ M \end{pmatrix}$ and $l_k^x = \dfrac{MkN_k^x}{M+1}$. For $k \geq m_Y$ we can decompose $\mathbb{C}[\mathcal{V}]$ in terms of $\mathbb{C}[\mathcal{V}]_{\leq m_Y}$ for various j in the following way

$$\mathbb{C}[\mathcal{V}]_{\leq k} = \mathbb{C}[\mathcal{V}]_{\leq m_Y - 1} \oplus \bigoplus_{j=0}^{k-m_Y} \bigoplus_{|\alpha|=j} x^\alpha \mathbb{C}[\mathcal{V}]_{=m_Y}.$$

For notational convenience we will write $\sigma(k) = k - m_Y$. Using this decomposition and writing B to be the number of monomials in $\mathbb{C}[\mathcal{V}]_{\leq m_Y - 1}$ and C to be the number of monomials in $\mathbb{C}[\mathcal{V}]_{=m_Y}$ we can compute N_k as

$$N_k = B + CN_{\sigma(k)}^x.$$

So for $\sigma(k) \geq 0$ we can compute $l_k - l_{m_Y}$ as

$$\begin{aligned}
l_k - l_{m_Y} &= \sum_{i=0}^{k} i(N_i - N_{i-1}) - l_{m_Y} = \sum_{i=m_Y+1}^{k} i(N_i - N_{i-1}) \\
&= \sum_{j=1}^{\sigma(k)} (j + m_Y) C(N_j^x - N_{j-1}^x) \\
&= \sum_{j=1}^{\sigma(k)} jC(N_j^x - N_{j-1}^x) + \sum_{j=1}^{\sigma(k)} m_Y C(N_j^x - N_{j-1}^x) \\
&= Cl_{\sigma(k)}^x + m_Y C(N_{\sigma(k)}^x - 1) \\
&= \frac{CM\sigma(k)N_{\sigma(k)}^x}{M+1} + m_Y C(N_{\sigma(k)}^x - 1).
\end{aligned}$$

Where we have used the fact that the second sum on the third line is telescoping. Using this calculation we can compare the growth of l_k and $\sigma(k)N^x_{\sigma(k)}$;

$$\begin{aligned}\lim_{k\to\infty}\frac{l_k}{\sigma(k)N_{\sigma(k)}} &= \lim_{k\to\infty}\frac{CM\sigma(k)N^x_{\sigma(k)}}{\sigma(k)N_{\sigma(k)}(M+1)} + \frac{m_Y C(N^x_{\sigma(k)}-1)}{\sigma(k)N_{\sigma(k)}} \\ &= \lim_{k\to\infty}\frac{CMN^x_{\sigma(k)}}{N_{\sigma(k)}(M+1)}. \end{aligned} \tag{34}$$

Where we have used the fact that

$$\begin{aligned}\lim_{k\to\infty}\frac{N^x_{\sigma(k)}}{N^x_{\sigma(k-m_Y)}} &= \begin{pmatrix} M+\sigma(k) \\ M\end{pmatrix}\begin{pmatrix} M+\sigma(k-m_Y) \\ M\end{pmatrix} \\ &= \lim_{k\to\infty}\frac{(M+\sigma(k))!}{M!\sigma(k)!}\cdot\frac{M!\sigma(k-m_Y)!}{(M+\sigma(k-m_Y))!} \\ &= \lim_{k\to\infty}\frac{(M+\sigma(k))\cdot\ldots\cdot(M+\sigma(k)-m_Y+1)}{\sigma(k)\cdot\ldots\cdot(\sigma(k)-m_Y+1)} \\ &= \lim_{k\to\infty}\left(\frac{M}{\sigma(k)}+1\right)\cdot\ldots\cdot\left(\frac{M}{\sigma(k)-m_Y+1}+1\right) \\ &= 1 \end{aligned} \tag{35}$$

so

$$\lim_{k\to\infty}\frac{m_Y C(N^x_{\sigma(k)}-1)}{\sigma(k)N_{\sigma(k)}} = \lim_{k\to\infty}\frac{m_Y C}{\sigma(k)}\cdot\frac{N^x_{\sigma(k)}}{B+CN^x_{\sigma(k-m_Y)}} = \lim_{k\to\infty}\frac{m_Y C}{\sigma(k)} = 0.$$

To calculate the limit in equation (34) we use equation (35) again:

$$\begin{aligned}\lim_{k\to\infty}\frac{CMN^x_{\sigma(k)}}{(M+1)N_{\sigma(k)}} &= \lim_{k\to\infty}\frac{CMN^x_{\sigma(k)}}{(M+1)(B+CN^x_{\sigma(k-m_Y)})} \\ &= \frac{M}{M+1}\lim_{k\to\infty}\frac{C\left(N^x_{\sigma(k)}/N^x_{\sigma(k-m_Y)}\right)}{B/N^x_{\sigma(k-m_Y)}+C} = \frac{M}{M+1}.\end{aligned}$$

Using Proposition 3.45, Proposition 3.50 and Theorem 1.145 we conclude that†

$$\begin{aligned}\log\delta(K) &= \lim_{k\to\infty}\frac{1}{l_k}\log\|\det D_k\|_{L^\infty(K)} = \lim_{k\to\infty}\frac{M+1}{MkN_k}\log\|\det D_k\|_{L^\infty(K)} \\ &= \lim_{k\to\infty}\frac{M+1}{MkN_k}\log\|\det S_k\|_{L^\infty(K)}.\end{aligned}$$

This completes the proof. □

†Theorem 1.145 is used in the first equality here, i.e. that the transfinite diameter in the basis $\mathcal{C}$ is the same as the usual monomial basis for $\mathbb{C}[\mathcal{V}]$.

3.5.4 Integral Formula

Proposition 3.52. *Let $\mathcal{V}$ satisfy the standard hypothesis with corresponding homogeneous variety $\mathcal{V}^h$. Let (x,y) be a Noether presentation for $\mathcal{V}^h$ and $X_j = \mathcal{V}^h \cap \{x_1 = ... = x_{j-1} = 0\} \cap \{x_j = 1\}$. Suppose that $u, v \in \mathcal{L}^+(\mathcal{V})$ with continuous Robin functions. Then*

$$\frac{1}{d(2\pi)^{M-1}} \int_{\tilde{\mathcal{V}}^h} [\tilde{\rho}_u - \tilde{\rho}_v] \sum_{j=0}^{M-1} (dd^c\tilde{\rho}_u + \omega)^j \wedge (dd^c\tilde{\rho}_v + \omega)^{M-j-1}$$
$$= \sum_{j=1}^{M} \frac{1}{d(2\pi)^{M-j}} \left(\int_{X_j} \rho_u(x,y)\,(dd^c\rho_u)^{M-j} - \int_{X_j} \rho_v(x,y)\,(dd^c\rho_v)^{M-j} \right)$$

where we interpret $\int_{X_M} \rho_u(x,y)$ as the discrete sum $\sum_{(x,y)\in X_M} \rho_u(x,y)$.

Note that the normalisation factors $1/(2\pi)^{M-j}$ are necessary to normalise the currents in the integrals. These terms are absent in Levenberg's formulation of this result because he makes the convention that $dd^c = 2i\partial\bar{\partial}/2\pi$ so that the normalisation factors of $1/(2\pi)^{M-j}$ are built into the formula. We could do the same for our situation, leaving only the normalisation factor of $1/d$ but elect not to do this for clarity. The requirement that u and v have continuous Robin functions is made for notational convenience and because this is the setting of the Rumely formula.

Proof. Upon examining the proof given by Levenberg (Proposition 8.1, [41]) we see that the proof of the corresponding result in $\mathbb{C}^N$ is largely algebraic up to respecting the normalisation for $(dd^c\rho_u)^{M-j}$ and respecting our definition of the Robin function. As such our proof is essentially the one given by Levenberg, up to region of integration. Let $u, v \in \mathcal{L}^+(\mathcal{V})$ with continuous Robin functions. We make the following definitions:

$$T_u = dd^c\tilde{\rho}_u + \omega \tag{36}$$

$$T_j = dd^c \log|x_j| = 2\pi[x_j = 0] \tag{37}$$

Where $[x_j = 0]$ is the current of integration on $\{x_j = 0\}$ and $\omega = \frac{1}{2}dd^c\log(1 + \|z\|^2)$ is the usual

Kähler form. We observe the following algebraic identity for any $0 \le j \le M-1$

$$\begin{aligned}
&T_u^{M-j-1} \wedge \omega^j \\
&\quad = T_u^{M-j-1} \wedge \omega^{j-1} \wedge (T_v - dd^c\tilde{\rho}_v) \\
&\quad\quad = T_u^{M-j-1} \wedge T_v \wedge \omega^{j-1} - dd^c\tilde{\rho}_v \wedge T_u^{M-j-1} \wedge \omega^{j-1} \\
&\quad = T_u^{M-j-1} \wedge T_v \wedge \omega^{j-2} \wedge (T_v - dd^c\tilde{\rho}_v) - dd^c\tilde{\rho}_v \wedge T_u^{M-j-1} \wedge \omega^{j-1} \\
&\quad\quad = T_u^{M-j-1} \wedge T_v^2 \wedge \omega^{j-2} - \sum_{k=1}^{2} (dd^c\tilde{\rho}_v) \wedge T_u^{M-j-1} \wedge \omega^{j-k} \wedge T_v^{k-1} \\
&\quad \vdots \\
&\quad = T_u^{M-j-1} \wedge T_v^{j-1} \wedge (T_v - dd^c\tilde{\rho}_v) - \sum_{k=1}^{j-1} (dd^c\tilde{\rho}_v) \wedge T_u^{M-j-1} \wedge \omega^{j-k} \wedge T_v^{k-1} \\
&\quad\quad = T_u^{M-j-1} \wedge T_v^{j} - \sum_{k=1}^{j} (dd^c\tilde{\rho}_v) \wedge T_u^{M-j-1} \wedge \omega^{j-k} \wedge T_v^{k-1}
\end{aligned}$$

From this we deduce (after an integration by parts to pass from terms involving $dd^c\rho_v$ to terms involving $dd^c\rho_u$)

$$\begin{aligned}
\tilde{\rho}_u T_u^{M-j-1} \wedge \omega^j &= \tilde{\rho}_u T_u^{M-j-1} \wedge T_v^j - \tilde{\rho}_u \sum_{k=1}^{j} (dd^c\tilde{\rho}_v) \wedge T_u^{M-j-1} \wedge \omega^{j-k} \wedge T_v^{k-1} \\
&= \tilde{\rho}_u T_u^{M-j-1} \wedge T_v^j - \tilde{\rho}_v \sum_{k=1}^{j} (dd^c\tilde{\rho}_u) \wedge T_u^{M-j-1} \wedge \omega^{j-k} \wedge T_v^{k-1}.
\end{aligned}$$

Taking the sum over $0 \le j \le M-1$ we obtain (interpreting $\sum_{k=1}^{j=0}$ as 0)

$$\sum_{j=0}^{M-1} \tilde{\rho}_u T_u^{M-j-1} \wedge \omega^j = \sum_{j=0}^{M-1} \left(\tilde{\rho}_u T_u^{M-j-1} \wedge T_v^j - \sum_{k=1}^{j} \tilde{\rho}_v (dd^c\tilde{\rho}_u) \wedge T_u^{M-j-1} \wedge \omega^{j-k} \wedge T_v^{k-1} \right). \tag{38}$$

Applying the same argument to $\sum_{j=0}^{M-1} T_v^{M-j-1} \wedge \omega^j$ but without using the integration by parts step (so that we retain terms involving $dd^c\rho_u$) we deduce

$$\sum_{j=0}^{M-1} \tilde{\rho}_v T_v^{M-j-1} \wedge \omega^j = \sum_{j=0}^{M-1} \left(\tilde{\rho}_v T_v^{M-j-1} \wedge T_u^j - \sum_{k=1}^{j} \tilde{\rho}_v (dd^c\tilde{\rho}_u) \wedge T_v^{M-j-1} \wedge \omega^{j-k} \wedge T_u^{k-1} \right). \tag{39}$$

Subtracting equation (39) from equation (38) yields

$$\sum_{j=0}^{M-1} \tilde{\rho}_u T_u^{M-j-1} \wedge \omega^j - \sum_{j=0}^{M-1} \tilde{\rho}_v T_v^{M-j-1} \wedge \omega^j = (\tilde{\rho}_u - \tilde{\rho}_v) \sum_{j=0}^{M-1} T_u^{M-j-1} \wedge T_v^j. \tag{40}$$

We can integrate both sides of equation (40) to conclude that

$$\int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_u - \tilde{\rho}_v) \sum_{j=0}^{M-1} T_u^{M-j-1} \wedge T_v^j = \int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} \tilde{\rho}_u T_u^{M-j-1} \wedge \omega^j - \int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} \tilde{\rho}_v T_v^{M-j-1} \wedge \omega^j. \quad (41)$$

The result now follows from writing the above integrals on the right hand side of (41) over X_j. Firstly, let $\tilde{X}_j := \tilde{\mathcal{V}}^h \cap \{x_1 = ... = x_{j-1} = x_j = 0\}$. Firstly note that $\tilde{\mathcal{V}}^h = X_1 \cup \tilde{X}_2$ and $\tilde{X}_j = X_j \cup X_{j-1}$. We can apply this deduction inductively to get

$$\tilde{\mathcal{V}}^h = X_1 \cup \tilde{X}_2 = X_1 \cup X_2 \cup \tilde{X}_3 = ... = X_1 \cup ... \cup X_M. \quad (42)$$

We need local representations of the integrands in order to use (41). For this we need x-coordinate-wise Robin functions

$$g_{j,u}(x, y(x)) = g_j(x, y(x)) := \limsup_{\lambda \to \infty} u(\lambda(x, y(\lambda x))) - \log |\lambda x_j|.$$

These Robin functions are well defined by the logic in Section 2.5 after noting that $t \to 0 \iff 1/t \to \infty$. Observe for future use that we have locally the identity

$$dd^c g_j = T_u - T_j.$$

It will be convenient to use the notation

$$S_j = T_1 \wedge ... \wedge T_j, \quad S_0 = 1,$$

which makes a well defined current since each T_j is the current of integration on $\{x_j = 0\}$ multiplied by a factor of 2π. It follows that S_j is the current of integration on $\{x_1 = ... = x_j = 0\}$ multiplied by $(2\pi)^j$ by Definition 1.71. For $0 \leq j \leq M-1$ we have the identity

$$\begin{aligned}
T_u^{M-j-1} &= T_1 \wedge T_u^{M-j-2} + dd^c g_1 \wedge T_u^{M-j-2} \\
&= T_1 \wedge T_2 \wedge T_u^{M-j-3} + T_1 \wedge dd^c g_2 \wedge T_u^{M-j-3} + dd^c g_1 \wedge T_u^{M-j-2} \\
&\vdots \\
&= S_{M-j-1} + S_{M-j-2} \wedge dd^c g_{M-j-1} + ... + dd^c g_1 \wedge T_u^{M-j-2} \\
&= S_{M-j-1} + \sum_{k=0}^{M-j-2} dd^c g_{k+1} \wedge S_k \wedge T_u^{M-j-k-2}. \quad (43)
\end{aligned}$$

Integrating (43) against $\tilde{\rho}_u$ we observe using integration by parts that

$$\begin{aligned}\int_{\tilde{\mathcal{V}}^h} \tilde{\rho}_u T_u^{M-j-1} \wedge \omega^j &= \int_{\tilde{\mathcal{V}}^h} \tilde{\rho}_u \left(S_{M-j-1} + \sum_{k=0}^{M-j-2} dd^c g_{k+1} \wedge S_k \wedge T_u^{M-j-k-2} \right) \wedge \omega^j \\ &= \int_{\tilde{\mathcal{V}}^h} \tilde{\rho}_u S_{M-j-1} \wedge \omega^j + \left(\sum_{k=0}^{M-j-2} g_{k+1} dd^c \tilde{\rho}_u \wedge S_k \wedge T_u^{M-j-k-2} \wedge \omega^j \right).\end{aligned} \tag{44}$$

If we denote the sum in (44) by E_j then using the identity $dd^c\tilde{\rho}_u = T_u - \omega$ we deduce that for $1 \leq j \leq M-2$

$$\begin{aligned}&E_j + E_{j-1} \\ &= \sum_{k=0}^{M-j-2} g_{k+1} dd^c \tilde{\rho}_u \wedge S_k \wedge T_u^{M-j-k-2} \wedge \omega^j + \sum_{k=0}^{M-j-1} g_{k+1} dd^c \tilde{\rho}_u \wedge S_k \wedge T_u^{M-j-k-1} \wedge \omega^{j-1} \\ &= \sum_{k=0}^{M-j-2} g_{k+1} (T_u - \omega) S_k \wedge T_u^{M-j-k-2} \wedge \omega^j + \sum_{k=0}^{M-j-1} g_{k+1} (T_u - \omega) \wedge S_k \wedge T_u^{M-j-k-1} \wedge \omega^{j-1} \\ &= \sum_{k=0}^{M-j-2} g_{k+1} S_k \wedge T_u^{M-j-k-1} \wedge \omega^j - \sum_{k=0}^{M-j-2} g_{k+1} S_k \wedge T_u^{M-j-k-2} \wedge \omega^{j+1} \\ &\quad + \sum_{k=0}^{M-j-1} g_{k+1} \wedge S_k \wedge T_u^{M-j-k} \wedge \omega^{j-1} - \sum_{k=0}^{M-j-1} g_{k+1} S_k \wedge T_u^{M-j-k-1} \wedge \omega^j \\ &= -g_{M-j} S_{M-j-1} \wedge \omega^j - \sum_{k=0}^{M-j-2} g_{k+1} S_k \wedge T_u^{M-j-k-2} \wedge \omega^{j+1} \\ &\qquad + \sum_{k=0}^{M-j-1} g_{k+1} \wedge S_k \wedge T_u^{M-j-k} \wedge \omega^{j-1}.\end{aligned} \tag{45}$$

It follows then that

$$\begin{aligned}E_{M-2} + ... + E_0 &= -\sum_{j=0}^{M-2} g_{M-j} S_{M-j-1} \wedge \omega^j - g_1 \omega^{M-1} + \sum_{k=0}^{M-1} g_{k+1} S_k \wedge T_u^{M-k-1} \\ &= -\sum_{j=0}^{M-1} g_{M-j} T_u^{M-j-1} \wedge \omega^j + \sum_{k=0}^{M-1} g_{k+1} S_k \wedge T_u^{M-k-1}.\end{aligned}$$

Summing over $0 \le j \le M-1$ in (44) we get

$$\int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} \tilde{\rho}_u S_{M-j-1} \wedge \omega^j + \left(\sum_{k=0}^{M-j-2} g_{k+1} dd^c \tilde{\rho}_u \wedge S_k \wedge T_u^{M-j-k-2} \wedge \omega^j \right)$$
$$= \int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} \tilde{\rho}_u S_{M-j-1} \wedge \omega^j - \sum_{j=0}^{M-1} g_{M-j} S_{M-j-1} \wedge \omega^j + \sum_{k=0}^{M-1} g_{k+1} S_k \wedge T_u^{M-k-1}$$
$$= \int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} (\tilde{\rho}_u - g_{M-j}) S_{M-j-1} \wedge \omega^j + \sum_{k=0}^{M-1} g_{k+1} S_k \wedge T_u^{M-k-1}$$
$$= \int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} (\log \|(x,y)\| - \log |x_{M-j}|) S_{M-j-1} \wedge \omega^j + \sum_{k=0}^{M-1} g_{k+1} S_k \wedge T_u^{M-k-1}. \tag{46}$$

Where in the last line we have used the fact that

$$\rho_u(x, \tilde{y}(x)) - g_{M-j}(x, \tilde{y}(x)) = \limsup u(\lambda x, y(\lambda x)) - \log \|(\lambda x, y(\lambda x)\| - \limsup u(\lambda x, y(\lambda x)) - \log |\lambda x_j|$$
$$= \log \|(x, \tilde{y})\| - \log \|x_j\|.$$

Where the lim sups are taken over appropriate values as in Section 2.5. Note that the first term in the integrand is independent of u so when repeating this calculation for v we obtain the same term. Hence by considering the u and v versions of equation (46) substituted into equation (41) we obtain

$$\int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} \tilde{\rho}_u T_u^{M-j-1} \wedge \omega^j - \int_{\tilde{\mathcal{V}}^h} \sum_{j=0}^{M-1} \tilde{\rho}_v T_v^{M-j-1} \wedge \omega^j$$
$$= \int_{\tilde{\mathcal{V}}^h} \sum_{k=0}^{M-1} g_{u,k+1} S_k \wedge T_u^{M-k-1} - \int_{\tilde{\mathcal{V}}^h} \sum_{k=0}^{M-1} g_{v,k+1} S_k \wedge T_v^{M-k-1}. \tag{47}$$

Note that $S_k = dd^c \log |x_1| \wedge ... \wedge dd^c \log |x_k|$ is $(2\pi)^k$ times the current of integration on $\{x_1 = ... = x_k = 0\} = \tilde{X}_{k+1}$ (noting that S_0 gives the integration on $\tilde{X}_1 = \tilde{\mathcal{V}}^h$). Applying this to the first term on the RHS of (47) we obtain

$$\int_{\tilde{\mathcal{V}}^h} \sum_{k=0}^{M-1} g_{k+1} S_k \wedge T_u^{M-k-1} = \sum_{j=0}^{M-1} (2\pi)^j \int_{\tilde{X}_{j+1}} g_{j+1} T_u^{M-j-1}. \tag{48}$$

Note that the calculation for the v term is identical. We can now write $g_j T_u^{M-j-1}$ in local coordinates on X_j using the following identifications.

$$\begin{aligned}
&g_j([0:...:0:x_j:...:x_M:y]) \\
&\qquad = g_j([0:...:0:1:x_{j+1}/x_j:...:x_M/x_j:y(x/x_j)]) + \log|x_j| \\
&\qquad = \limsup_{\lambda\to 0} u(0,...,0,1/\lambda, x_{j+1}/\lambda x_j, ..., x_M/\lambda x_j, y(x/\lambda x_j)) + \log|\lambda x_j| + \log|x_j| \\
&\qquad = \limsup_{\lambda\to 0} u(0,...,0,1/\lambda, t_{j+1}/\lambda, ..., t_M/\lambda, u(t/\lambda)) + \log|\lambda| \\
&\qquad = \rho_u(0,...,0,1,t_{j+1},...,t_M,u(t)), \qquad (49) \\
&T_u^{M-j-1} = dd^c\tilde{\rho}_u + \omega \\
&\qquad = dd^c\Big(\rho_u(0,...,0,1,t_{j+1},...,t_M,u(t)) - \frac{1}{2}\log|1+t_{j+1}^2+...+u_{N-M}^2|\Big) + \omega \\
&\qquad = dd^c\rho_u(0,...,0,1,t_{j+1},...,t_M,u(t)) - \omega + \omega \\
&\qquad = dd^c\rho_u(0,...,0,1,t_{j+1},...,t_M,u(t)), \qquad (50)
\end{aligned}$$

where $(0,...,1,t_{j+1},...,t_M,u(t))$ are local coordinates on X_j and where we have made use of the lim sup convention for Robin functions to obtain equation (49). Using (49) and (50) in (48) we obtain

$$\sum_{j=0}^{M-1}(2\pi)^j\int_{\tilde{X}_{j+1}} g_{j+1}T_u^{M-j-1} = \sum_{j=0}^{M-1}(2\pi)^j\int_{X_{j+1}}\rho_u(t,u)\,(dd^c\rho_u)^{M-j-1}. \qquad (51)$$

Using (51) in (47) for both the u and v terms we obtain (after combining with (41))

$$\begin{aligned}
\int_{\tilde{\mathcal{V}}^h}(\tilde{\rho}_u - \tilde{\rho}_v)\sum_{j=0}^{M-1}T_u^{M-j-1}\wedge T_v^j = \sum_{j=1}^{M}&(2\pi)^{j-1}\int_{X_j}\rho_u(t,u)\,(dd^c\rho_u)^{M-j} \\
&-(2\pi)^{j-1}\int_{X_j}\rho_v(t,u)\,(dd^c\rho_v)^{M-j} \qquad (52)
\end{aligned}$$

Replacing the dummy variables (t,u) with (x,y) and dividing through by the normalisation factor of $d(2\pi)^{M-1}$ yields the proposition. □

3.5.5 Derivation of Rumely's Formula

The final step is to derive a relationship between $\log\|\det[S_k]\|_{L^2(\mu)}$ and $\delta(K)$ from Proposition 3.50. By Propositions 3.45 and 3.46 it suffices to consider $E = T_{\mathcal{V}}$ in Proposition 3.50.

Theorem 3.53 (Rumely-Type Formula for Algebraic Varieties)**.** *Suppose that $K \subset \mathcal{V}$ is com-*

pact, regular and $K = \{V_K(z) \leq 0\}$.[‡] *Then*

$$-\log\delta(K) = \frac{1}{dM}\left[\frac{1}{(2\pi)^{M-1}}\int_{\mathcal{V}^h}\rho_K(1,z_2,...,z_N)(dd^c\rho_K(1,x_2,...,x_M,y))^{M-1}\right.$$
$$+\frac{1}{(2\pi)^{M-2}}\int_{\mathcal{V}^h}\rho_K(0,1,x_3,...,x_M,y)(dd^c\rho_K(0,1,x_3,...,x_M,y))^{M-2}$$
$$+...+\frac{1}{2\pi}\int_{\mathcal{V}^h}\rho_K(0,...,0,1,x_N,y)(dd^c\rho_K(0,...,0,1,x_N,y))$$
$$\left.+\sum_{j=1}^{d}\rho_K(0,...,0,1,y_i)\right]. \tag{53}$$

Equivalently, we have an "energy version" of this formula

$$-\log\delta(K) = \frac{1}{dM(2\pi)^{M-1}}\int_{\mathcal{V}^h}[\tilde{\rho}_K - \tilde{\rho}_{T_{\mathcal{V}}}]\sum_{j=0}^{N-1}(dd^c\tilde{\rho}_K+\omega)^j\wedge(dd^c\tilde{\rho}_{T_{\mathcal{V}}}+\omega)^{M-j-1}. \tag{54}$$

Proof. From Theorem 3.51 we have

$$\log\delta(K) = \frac{M+1}{M}\frac{\mathcal{E}(V_{T_{\mathcal{V}}},V_K^*)}{(M+1)d(2\pi)^M} = \frac{\mathcal{E}(V_{T_{\mathcal{V}}},V_K^*)}{Md(2\pi)^M}. \tag{55}$$

Where the $d(2\pi)^M$ term comes from normalising $(dd^cV_{T_{\mathcal{V}}})^M$ and $(dd^cV_K^*)^M$ so that they are probability measures. Note that $V_K = V_K^*$ since K is regular. Hence $\int V_K(dd^cV_K)^M = \int V_{T_{\mathcal{V}}}(dd^cV_{T_{\mathcal{V}}})^M = 0$ since $(dd^cV_K)^M$ and $(dd^cV_{T_{\mathcal{V}}})^M$ are supported on K and $T_{\mathcal{V}}$ respectively.

It follows by Proposition 3.43 and Corollary 3.21 that

$$\mathcal{E}(V_{T_{\mathcal{V}}},V_K) = \int_{\mathcal{V}}V_{T_{\mathcal{V}}}(dd^cV_{T_{\mathcal{V}}})^M + \int_{\mathcal{V}}V_K(dd^cV_K)^M + 2\pi\int_{\tilde{\mathcal{V}}^h}(\tilde{\rho}_{T_{\mathcal{V}}}-\tilde{\rho}_K)\sum_{j=0}^{M-1}T^j_{V_{T_{\mathcal{V}}}}\wedge T^{M-j-1}_{V_K}$$
$$= 2\pi\int_{\tilde{\mathcal{V}}^h}(\tilde{\rho}_{T_{\mathcal{V}}}-\tilde{\rho}_K)\sum_{j=0}^{M-1}T^j_{V_{T_{\mathcal{V}}}}\wedge T^{M-j-1}_{V_K}.$$

Dividing through by $Md(2\pi)^M$ (in the spirit of Theorem 3.50) we obtain

$$\frac{\mathcal{E}(V_{T_{\mathcal{V}}},V_K)}{Md(2\pi)^M} = \frac{1}{Md(2\pi)^{M-1}}\int_{\tilde{\mathcal{V}}^h}(\tilde{\rho}_{T_{\mathcal{V}}}-\tilde{\rho}_K)\sum_{j=0}^{M-1}T^j_{V_{T_{\mathcal{V}}}}\wedge T^{M-j-1}_{V_K}.$$

Applying Proposition 3.52 to the above equation we obtain

$$\frac{\mathcal{E}(V_{T_{\mathcal{V}}},V_K)}{Md(2\pi)^M} = \sum_{j=1}^{M}\frac{1}{Md(2\pi)^{M-j}}\left(\int_{X_j}\rho_{T_{\mathcal{V}}}(x,y)\,(dd^c\rho_{T_{\mathcal{V}}})^{M-j} - \int_{X_j}\rho_K(x,y)\,(dd^c\rho_K)^{M-j}\right).$$

[‡]Recall that this is equivalent to K being polynomially convex. We write the hypothesis in this form to emphasise its application in the proof.

Observe that

$$\rho_{T_{\mathcal{V}}} = \lim_{\lambda\to\infty} \log^{+} \|\lambda x\| - \log|\lambda| = \log\|x\|.$$
$$\rho_{T_{\mathcal{V}}}|_{X_j} = \log\|(1, x_{j+1}, ..., x_M)\|.$$

Which means that the support of $(dd^c\rho_{T_{\mathcal{V}}})^{M-j}$ on X_j is $\{(1, 0, ..., 0)\}$ and on this set $\rho_{T_{\mathcal{V}}} = 0$ so the first integral is zero. Consequently,

$$\frac{\mathcal{E}(V_{T_{\mathcal{V}}}, V_K)}{Md(2\pi)^M} = -\sum_{j=1}^{M} \frac{1}{dM(2\pi)^{M-j}} \int_{X_j} \rho_K(x, y)\,(dd^c\rho_K)^{M-j}. \tag{56}$$

Finally, combining equations (55) and (56) we obtain

$$-\log\delta(K) = \frac{1}{Md} \sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_K(x, y)\,(dd^c\rho_K)^{M-j},$$

which is equation (53) and the proof is complete. □

4 Weighted Pluripotential Theory on Varieties

Weighted theory has been well studied on complex manifolds by many authors including Berman-Boucksom [7] and Branka-Stawiska [18]. Our aim here is not so much as to re-develop the basic theory but to obtain analogues of Levenberg-Bloom's result [15] which related Chebyshev constants in $\mathbb{C}^N$ to weighted Chebyshev constants in $\mathbb{C}^{N-1}$. We include a preliminary section using our own methods for completeness.

4.1 Weighted Pluripotential Theory Preliminaries

The following results are standard in $\mathbb{C}^N$, e.g. see [10]. Our goal in this section is to establish relationships between weighted polynomials and weighted extremal functions. In particular, a polynomial formula for the extremal function.

Definition 4.1. *Let $\mathcal{V} \subset \mathbb{C}^N$ be a (possibly singular) algebraic variety. Let $E \subset \mathcal{V}$ be a closed set. We say that $w \geq 0$ is an admissible weight function if the following properties are satisfied.*

(i) w is upper semi-continuous,

(ii) the set $\{z \in E : w(z) > 0\}$ is not pluripolar,

(iii) if E is unbounded then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in E$.

If w satisfies properties (i) and (ii) on a compact set E then we will say that w is a reduced weight on E.

Unlike creating sensible analogues of psh functions on a variety, we need not impose any further conditions on w from those that are usually placed on w in the $\mathbb{C}^N$ case.

Corollary 4.2. *Let $Q = Q_w = -\log w$. Then Q is lower semicontinuous on E, the set $\{z \in E : Q(z) < \infty\}$ is not pluripolar and if E is unbounded $Q(z) - \log|z| \to \infty$ as $|z| \to \infty, z \in E$.*

Given an admissible weight w we will call Q the corresponding admissible log-weight.

Definition 4.3. *Suppose that Q is an admissible log-weight on $E \subset \mathcal{V}$ where $\mathcal{V}$ is possibly singular. Then the weighted extremal function is given by*

$$V_{E,Q}(z) := \sup\{u(z) : u \in w\mathcal{L}, u \leq Q \text{ on } E\}.$$

(Recall that $w\mathcal{L}(\mathcal{V}) = \{u \in \mathcal{L}(\mathcal{V}^{reg}) : u \text{ is usc on } \mathcal{V}\}$.)

We will use the convention that when $Q = 0$, $V_{E,Q} = V_E$. We first check to see if this definition makes sense when E is compact.

Lemma 4.4. *Suppose that $E \subset \mathcal{V}$ is compact and Q is an admissible log-weight on E. Then $V^*_{E,Q} \in \mathcal{L}(\mathcal{V})$.*

Proof. Let $F = \{Q(z) < \infty\}$. First suppose that F is closed. Set $c = \max\{Q(z) : z \in F\}$ so that if $u \in \mathcal{L}(\mathcal{V})$ satisfies $u \leq Q$ on F then $u - c \leq 0$ on F. It follows then that $u - c \leq V_F$. Taking the supremum over all candidates for u yields the conclusion $V_{F,Q} - c \leq V_F$. It follows that $V_{F,Q} \in \mathcal{L}(\mathcal{V})$.

Now suppose that F is not closed, meaning that there is a subset U of ∂F where $Q(z) = \infty$. We want to show that the set of $u \in \mathcal{L}(\mathcal{V})$ such that $u \leq Q$ on F is bounded above on F and then use the argument in the previous paragraph to deduce the result. To this end, suppose that there were such a sequence $\{u_j\}$ with $u_j \to \infty$ on U. Since F is non-pluripolar we can find a closed susbset G of F which is non-pluripolar. G is non-pluripolar so $V_G^* \neq +\infty$. Set $c = \max\{Q(z) : z \in G\}$, then $u_j - c \leq 0$ for each j on G and hence $u_j - c \leq V_G$. But by construction of u_j, as $j \to \infty$, $u_j - c \to \infty$ on U which forces $V_G \to \infty$ on U which contradicts $V_G^* \neq \infty$. It follows that there exists some $C > 0$ such that any $u \in \mathcal{L}(\mathcal{V})$ such that $u \leq Q$ is bounded above by C on F and hence $V_{F,Q} \in \mathcal{L}(\mathcal{V})$. $\square$

One of the reasons why we need not impose additional conditions on our weight functions is that in the case that E is unbounded there exists a compact set E_R which captures information pertaining to the extremal function. This also shows that our definition when E is unbounded makes sense.

Lemma 4.5. *Suppose that $E \subset \mathcal{V}$ is an unbounded closed set with admissible log-weight Q. Define $E_R = \{z \in E : \|z\| \leq R\}$. Then for R sufficiently large $V_{E,Q} = V_{E_R,Q}$.*

Proof. Note that by definition $V_{E_r} \geq V_E$. We seek the reverse inequality.

Since E is non-pluripolar it follows that at least one of E_r or $E \backslash E_r$ is non-pluripolar for all r. Then there is a point $a \in E$ such that for all neighbourhoods V of a and psh functions $u \in PSH(V)$ the set $\{u(z) = -\infty\} \neq E \cap V$. This means we can find a bounded non-pluripolar set in E hence for sufficiently large r this set is contained in E_r and hence E_r is non-pluripolar.

It follows that since $V_{E_r,Q}^* \in \mathcal{L}(\mathcal{V})$ we can find $c \in \mathbb{R}$ such that $V_{E_r,Q}(z) \leq \log^+ |z| + c$. Then for $R > r$ sufficiently large we have $Q(z) - \log |z| > c + 1$ for $|z| > R$ since $Q(z) - \log |z| \to \infty$. Then for $u \in \mathcal{L}(\mathcal{V})$ and $u(z) \leq Q(z)$ on E_r we have $u \leq V_{E_r,Q}^*$. It follows that on $E \backslash E_R$ that $u(z) \leq \log |z| + c \leq Q(z)$ hence $u \leq Q$ on E_r implies $u \leq Q$ on E. From this is follows that $V_{E_r} \leq V_E$ and the result follows. $\square$

Corollary 4.6. *The support of $(dd^c V_{E,Q})^M$ is compact.*

Proof. Choose R sufficiently large so that $V_{E_R,Q} = V_{E,Q}$. Set $c = \max\{V_{E_R,Q}(z) : z \in E_R\}$ and let $Z = \{z \in \mathcal{V} : V_{E_R,Q} - c \leq 0\}$. Then $V_Z = V_{E_R,Q}$ on $\mathcal{V} \backslash Z$ and $(dd^c V_Z)^M$ has compact support. It follows that $(dd^c V_{E_R,Q})^M$ is supported in Z and hence has compact support. $\square$

Corollary 4.7. *For any admissible weight w on a set E there exists a reduced weight w' on a set K such that $V_{E,Q} = V_{K,Q'}$.*

The main obstruction at this point to obtaining a polynomial formula for the extremal function is the lack of an analogous result to the following one due to Siciak.

Theorem 4.8 (Siciak)**.** *Suppose that $u \in \mathcal{L}(\mathbb{C}^N)$. Then there exists a sequence of functions $\{u_j\}$, $j \in \mathbb{N}$ satisfying for all $z \in \mathbb{C}^N$*

(i) $u_{j+1}(z) \leq u_j(z)$ *for all* $j \in \mathbb{N}$*;*

(ii) $\lim_{j\to\infty} u_j = u(z)$*;*

(iii) for each j there exists finitely many polynomials $\{p_{k,j}\}_{1\leq k\leq n_{k,j}}$ each of degree $\leq n_{k,j}$ such that

$$u_j(z) = \sup_{1\leq k\leq n_{k,j}} \frac{1}{\deg p_{k,j}} \log |p_{k,j}(z)|.$$

The main barrier to generalisation is that membership in $\mathcal{L}$ is a global property so the local techniques of pullback are generally not useful. We choose not to prove an analogous theorem to Theorem 4.8 for $u \in \mathcal{L}(\mathcal{V})$ and instead will deduce the result from the $\mathbb{C}^N$ polynomial formula using an approximation technique.

Recall that in the unweighted case (Theorem 3.22) our strategy was to view E as a subset of $\mathbb{C}^N$ and invoke $\mathbb{C}^N$ theory to deduce the result. The problem with the weighted case is that the weight must be positive on a non-pluripolar set, but necessarily $\mathcal{V}$ is pluripolar in $\mathbb{C}^N$. The unweighted case was equivalent to imposing the log-weight $Q = 0$ on $E_\varepsilon = \{z \in \mathbb{C}^N : \mathrm{dist}(z, E) \leq \varepsilon\}$ which is a non-pluripolar set and then letting $\varepsilon \to 0$. While there is no natural extension of a general weight to a 'fat' set E_ε, the following lemma gives a possible extension which has the desired properites which will allow us to emulate this process for the weighted case.

Lemma 4.9. *Suppose that (x, y) is a Noether presentation for $\mathcal{V}$ with branches $V_1, ..., V_d$ and Q is an admissible weight on $E \subset \mathcal{V}$. Define $E_\varepsilon = \{z \in \mathbb{C}^N : \mathrm{dist}(z, E) \leq \varepsilon\}$ and $E^i_\varepsilon = \{z \in \mathbb{C}^N : \mathrm{dist}(z, E \cap V_i) \leq \varepsilon\}$. Let $L_x := \{(x, y) : y \in \mathbb{C}^{N-M}\}$ be the $N - M$ dimensional hyperplane through x and write (x, y^i_x) for the only point in $L_x \cap E^i_0$. The weight $Q_\varepsilon(z)$ given by*

$$Q_{i,\varepsilon}(x, y) = \begin{cases} V_{E,Q}(x, y^i_x), & \text{when } (x, y) \in L_x \cap E^i_\varepsilon, \\ +\infty, & \text{otherwise.} \end{cases}$$
$$Q_\varepsilon(z) = \min_{1\leq i\leq d} \{Q_{i,\varepsilon}(z)\},$$

is admissible on E_ε.

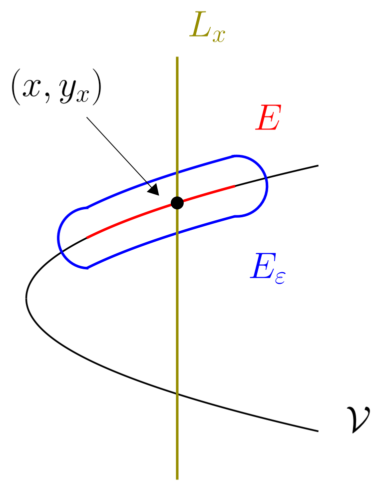


Figure 1: Illustration of the construction of (x, y_x).

Proof. First observe that since (x, y) is a Noether presentation for $\mathcal{V}$ the branches $V_1, ..., V_d$ each have exactly one intersection with L_x. Hence setting (x, y_x^i) to be the only point in $L_x \cap E_0^i$ is valid. Next observe that $Q_{i,\varepsilon}(x, y)$ is a lower semicontinuous function since

$$\begin{aligned}\liminf_{(\xi,\zeta)\to(x,y)} Q_{i,\varepsilon}(\xi, \zeta) = \liminf_{(\xi,\zeta_\xi)\to(x,y_x)} Q_{i,\varepsilon}(\xi, \zeta_\xi) &= \liminf_{(\xi,y_\xi)\to(x,y_x)} V_{E,Q}(\xi, y_\xi) \\ &\geq V_{E,Q}(x, y_x) = Q_{i,\varepsilon}(x, y_x) = Q_{i,\varepsilon}(x, y).\end{aligned}$$

Where we have used the fact that $V_{E,Q}$ is lower semicontinuous. As $Q'_\varepsilon(z)$ is the minimum of finitely many lower semicontinuous functions it too is lower semicontinuous. Finally we need to show that $Q_\varepsilon(z) < \infty$ on a non-pluripolar set. This follows since E_ε is a non-pluripolar set and $Q_\varepsilon(z)$ is finite at each point of E_ε and at any point $(x, y) \in E_\varepsilon$,

$$Q_\varepsilon(x, y) \leq \max_{(x,y_x)\in L_x\cap\mathcal{V}} V_{E,Q}(x, y_x) < \infty.$$

This shows that Q_ε is admissible on E_ε. $\square$

Remark 4.10. For an unbounded set E we use use Lemma 4.5 to construct a bounded set F which shares the same extremal function as E and then use Lemma 4.9 to construct the extension of the weight on the extension of F.

Lemma 4.11. *Suppose that $E \subset \mathcal{V}$ and Q is admissible on E. Then* $\operatorname{supp}(dd^c V^*_{E,Q})^M \subset \{V^*_{E,Q}(z) \geq Q(z)\}$.

Proof. Suppose that $z \in \{V_{E,Q}(z) < Q(z)\}$. Suppose we have an atlas (U_α, ϕ_α) for which $z \in U_1$. Since $V^*_{E,Q}$ is usc and Q there is an open neighbourhood N containing z such that

$$\sup_{z\in N} V^*_{E,Q} < \inf_{z\in N\cap E} Q(z). \tag{57}$$

Taking a smaller neighbourhood if necessary so that $N \subset U_1$, $\phi_1^* Q$ is an admissible weight on $\{w \in \phi^{-1}(E \cap U_1)\} \subset \mathbb{C}^M$ and the pull back of the inequality (57) remains valid. Therefore we may find $r > 0$ such that $B_r(w) \subset \phi_1(N)$ and we can find $u \in \mathcal{L}(\mathbb{C}^M)$ satisfying: (i) $u = \phi_1^* V_{E,Q}^*$ on $\partial B_r(w)$, (ii) $(dd^c u)^M \equiv 0$ on $B_r(w)$ (this follows from solving the Dirichlet problem in $\mathbb{C}^M$). By the maximum principle for psh functions it follows that on $B_r(w)$

$$u(\zeta) \leq \sup_{\zeta \in B_r(w)} \phi_1^* V_{E,Q}^*(\zeta) < \int_{\zeta \in B_r(w) \cap \phi_1(E \cap U_1)} Q(z).$$

It follows from maximality of u that $u(\zeta) = \phi_1^* V_{E,Q}^*(\zeta)$ on $B_r(w)$. The maximality of u is preserved upon push-forward hence $(\phi_1)_* u(z) = V_{E,Q}^*$ on $\phi_1^{-1}(B_r(w))$ and hence $(dd^c V_{E,Q})^M = 0$ in a neighbourhood of z. This proves the lemma. $\square$

Lemma 4.12. *Suppose that $E \subset \mathcal{V}$ and Q is admissible on E. Then $V_{E,Q}^* = Q$ q.e. on E.*

Proof. We first show that $\{V_{E,Q} \leq V_{E,Q}^*\}$ is pluripolar. Since pluripolarity is a local property it suffices to check $\{V_{E,Q} \leq V_{E,Q}^*\}$ is pluripolar at an arbitrary z. With the atlas setup as in the proof of Lemma 4.12 we can find a neighbourhood N of z such that $N \subset U_1$. By a result of Klimek ([37], Proposition 2.9.16) $(V_{E,Q}^* \circ \phi_1^{-1}) = (V_{E,Q} \circ \phi_1^{-1})^*$. By another result ([37], Theorem 4.7.6) the set $\{(V_{E,Q} \circ \phi_1^{-1}) < (V_{E,Q} \circ \phi_1^{-1})^*\}$ is pluripolar in $\phi_1(N)$ (and in particular at $\phi_1(z)$). Since pluripolarity is preserved under holomorphic maps, it follows that $\phi^{-1}\{(V_{E,Q} \circ \phi_1^{-1}) < (V_{E,Q} \circ \phi_1^{-1})^*\} = \{V_{E,Q} < V_{E,Q}^*\}$ is pluripolar.

Thus we have shown that $V_{E,Q}^* \leq Q$ q.e. on E. Combining this result with Lemma 4.11 we deduce that $V_{E,Q}^* = Q$ q.e. on E which finishes the proof. $\square$

Corollary 4.13. *If Q and Q' are both admissible on E and $Q = Q'$ q.e. then $V_{E,Q} = V_{E,Q'}$ q.e.*

Proof. Let $F = \{Q \neq Q'\} \cup \{V_{E,Q} \neq Q\} \cup \{V_{E,Q'} \neq Q'\}$. Then F is pluripolar and $V_{E,Q} = Q = Q' = V_{E,Q'}$ on $E \backslash F$. Maximality of $V_{E,Q}$ implies equality outside of E. Hence $V_{E,Q} = V_{E,Q'}$ q.e. $\square$

Lemma 4.14. *Set $Q_0 := \lim_{\varepsilon \to 0} Q_\varepsilon$. Q_0 is admissible on E and $V_{E,Q_0} = V_{E,Q}$ quasi-everywhere on $\mathcal{V}$.*

Proof. Note that for $\delta_1 > \delta_2$ we have $Q_{\delta_1} \leq Q_{\delta_2}$ since the minimum is taken over terms satisfying the same property, i.e. for $\varepsilon_1 > \varepsilon_2$ we have $Q_{\varepsilon_1,i} \leq Q_{\varepsilon_2,i}$ for any i. It follows that $\lim_{\varepsilon \to 0} Q_\varepsilon = \sup_{\varepsilon > 0} Q_\varepsilon$ and a supremum of lower semicontinuous functions is again lower semicontinuous. It follows that Q_0 is lower semicontinuous.

If for some i and x we have $L_x \cap E_\varepsilon^i \cap E_\varepsilon^j$ for all $i \neq j$ then $Q_0 = Q$. Hence we need only worry about points $(x,y) \in E$ for which $(x,y) \in L_x \cap E_\varepsilon^i \cap E_\varepsilon^j$ for every $\varepsilon > 0$ and for some $i \neq j$. But the only place where this can occur is the branch locus B_π. We know that the branch locus is pluripolar in $\mathcal{V}$ hence $Q_0 = Q$ q.e. on $\mathcal{V}$. It follows that Q_0 is admissible on E.

Moreover since Q_0 and Q only differ on a pluripolar set it follows that $V_{E,Q} = V_{E,Q_0}$ by Corollary 4.13 as claimed. $\square$

Definition 4.15. *Suppose that w is an admissible weight on $E \subset \mathcal{V}$. We define*

$$\Phi_{E,Q}(z) = \sup \left\{ \frac{1}{\deg P} \log |w^{\deg P} P| : \|w^{\deg P} P\|_E \leq 1 \right\}.$$

Theorem 4.16. *Let Q be an admissible weight on E. Then $V_{E,Q}(z) = \Phi_{E,Q}(z)$ q.e.*

Proof. We instead show that $V_{E,Q_0} = \Phi_{E,Q_0}(z)$ q.e. which implies immediately the desired conclusion. To this end, note that $V_{E_\varepsilon,Q_\varepsilon}$ is a weighted extremal function in $\mathbb{C}^N$ hence by classical theory $V_{E_\varepsilon,Q_\varepsilon} = \Phi_{E_\varepsilon,Q_\varepsilon}(z)$ q.e. for all $\varepsilon > 0$. Observe that $V_{E_\varepsilon,Q_\varepsilon}$ increases to V_{E,Q_0} as $\varepsilon \to 0$ since Q_ε increases to Q_0. In particular note that $V_{E_\varepsilon,Q_\varepsilon}(z) \leq V_{E,Q_0}(z) < \infty$ when $z \in \mathcal{V}$ and so the restrictions of the families $\{V_{E_\varepsilon,Q_\varepsilon}\}$ and $\{\Phi_{E_\varepsilon,Q_\varepsilon}(z)\}$ to $\mathcal{V}$ form monotone increasing sequences which are bounded above and hence converge to V_{E,Q_0} (q.e. in the case of Φ).

To complete the proof we want to show that $V_{E,Q_0} \leq \Phi_{E,Q_0}(z)$ q.e. on $\mathcal{V}$ since the converse follows immediately by definition. Observe that for any $j \in \mathbb{N}$ and a polynomial p satisfying we have $\frac{1}{\deg p} \log |p(z)| \leq Q_j(z)$ we have $\frac{1}{\deg p} \log |p(z)| \leq Q_0(z)$ since $Q_j \leq Q_0$. It follows then that $\Phi_{E_j,Q_j}(z) \leq \Phi_{E,Q_0}(z)$ for every $j \in \mathbb{N}$ hence in the limit we have

$$\lim_{j\to\infty} \Phi_{E_j,Q_j}(z) = \lim_{j\to\infty} V_{E_j,Q_j}(z) = V_{E,Q_0} \leq \Phi_{E,Q_0}(z) \quad \text{q.e.},$$

as desired. $\square$

4.2 Weighted $\mathcal{H}$-principle

An important technique which is exploited in the $\mathbb{C}^N$ case is a weighted analogue of Siciak's $\mathcal{H}$-principle. First let us recall the original result from Section 3 for comparison.

Theorem 4.17 ($\mathcal{H}$-Principle, [51])**.** *Let $P_n(\mathbb{C}^N)$ (resp. $H_n(\mathbb{C}^N)$) denote the space of all polynomials of degree at most n (resp. homogeneous polynomials degree at most n) in N complex variables. Let $\mathcal{L}(\mathbb{C}^N)$ (resp. $\mathcal{H}(\mathbb{C}^N)$) denote the class of logarithmic psh functions (resp. log homogeneous psh functions). The maps*

(i.) $H_n(\mathbb{C} \times \mathbb{C}^N) \ni Q_n(t,z) \to Q_n(1,z) \in P_n(\mathbb{C}^N)$,

(ii.) $\mathcal{H}(\mathbb{C} \times \mathbb{C}^N) \ni u(t,z) \to u(1,z) \in \mathcal{L}(\mathbb{C}^N)$

are one-to-one. If $P \in P_n(\mathbb{C}^N)$ then the unique element $\tilde{P} \in H_n(\mathbb{C} \times \mathbb{C}^N)$ such that $P(z) = \tilde{P}(1,z)$ is given by the formula $\tilde{P}(t,z) = t^n P(z/t)$. If $u \in \mathcal{L}(\mathbb{C}^N)$ then the unique element

$\tilde{u} \in \mathcal{H}(\mathbb{C} \times \mathbb{C}^N)$ *such that* $\tilde{u}(1,z) = u(z)$ *on* $\mathbb{C}^N$ *is given by*

$$\tilde{u}(t,z) := \begin{cases} \log|t| + u(z/t), & t \neq 0 \\ \limsup_{(t,\zeta)\to(0,z)} \log|t| + u(\zeta/t), & t = 0. \end{cases}$$

A weighted $\mathcal{H}$-principle has been used (without being called as such) by a number of authors, in particular in the papers [15], [13] and [16].

Definition 4.18. *Let w be a reduced weight on $K \subset \mathbb{C}^N$. We define the following classes of functions.*

(i) $P_n(\mathbb{C}^N, K, w) := \{P \in P_n(\mathbb{C}^N) : \|w^n P\|_K \leq 1\}$.

(ii) $H_n(\mathbb{C}^N, K, w) := \{Q \in H_n(\mathbb{C}^N) : \|w^n Q\|_K \leq 1\}$.

(iii) $\mathcal{L}(\mathbb{C}^N, K, Q) := \{u \in \mathcal{L}(\mathbb{C}^N) : u|_K \leq Q\}$.

(iv) $\mathcal{H}(\mathbb{C}^N, K, Q) := \{u \in \mathcal{H}(\mathbb{C}^N) : u|_K \leq Q\}$.

We make the convention that if $Q = 0$ (resp. $w = 1$) then we omit the weight so that $P_n(K, 1) = P_n(K)$ and $\mathcal{L}(\mathbb{C}^N, K, 0) = \mathcal{L}(\mathbb{C}^N, K)$ and similar.

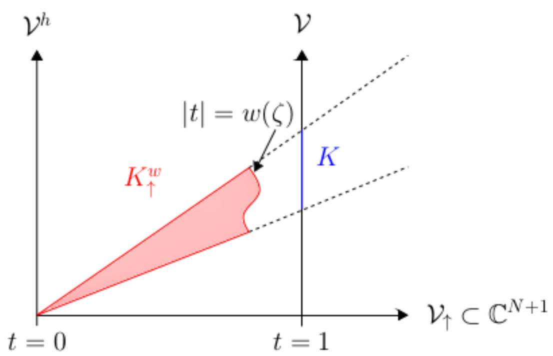
The definitions above make sense when w is admissible, but we will only study cases where w is reduced. Of course, by Lemma 4.5 there is no loss in generality by restricting to reduced weights.

Theorem 4.19 (Weighted $\mathcal{H}$-Principle)**.** *Suppose w is a reduced weight on $K \subset \mathbb{C}^N$. Define $K_\uparrow^w := \{(t,z) = (t,t\zeta) : |t| = w(\zeta), \zeta \in K\}$. Then the maps*

(i.) $H_n(\mathbb{C}^{N+1}, K_\uparrow^w) \ni Q_n(t,z) \to Q_n(1,z) \in P_n(\mathbb{C}^N, K, w)$,

(ii.) $\mathcal{H}(\mathbb{C}^{N+1}, K_\uparrow^w) \ni u(t,z) \to u(1,z) \in \mathcal{L}(\mathbb{C}^N, K, Q)$,

are one to one. Moreover, the functions guaranteed by the maps above are the same as Theorem 4.17.



Proof. By Theorem 4.17 it suffices to check that the functions determined by the maps in that theorem satisfy the bounding conditions imposed. To this end, suppose that $Q \in H_{n+1}(\mathbb{C}^N, K, w)$ and let P be the corresponding polynomial determined by Theorem 4.17. We check that

$$\|w(\zeta)^n P(\zeta)\|_K = \|t^n P(z/t)\|_{K_\uparrow^w} = \|Q(t,z)\|_{K_\uparrow^w}$$

which implies the result for (i). Similarly, for (ii) let $\tilde{u} \in \mathcal{H}(\mathbb{C}^N, K_\uparrow^w)$ and u be the associated function in $\mathcal{L}(\mathbb{C}^N)$. Then we check that for $(t,z) \in K_\uparrow^w$,

$$\begin{aligned} 0 \geq \tilde{u}(t,z) &= \tilde{u}(1,\zeta) + \log|t| = u(z) + \log|t| \\ -\log|t| &\geq u(\zeta) \\ -\log w(z) &\geq u(\zeta). \end{aligned}$$

By construction of $K_\uparrow^w$ it follows that $Q(\zeta) = -\log w(\zeta) \geq u(\zeta)$ holds for all $\zeta \in K$. The argument is clearly reversible and so the result for (ii) holds.* □

The main utility of the weighted $\mathcal{H}$-principle is the ability to relate weighted theory in $\mathbb{C}^N$ to unweighted theory in $\mathbb{C}^{N+1}$ in a precise way. Thereby being able to prove weighted results from using unweighted theory in one higher dimension. This idea is the basis of the paper of Bloom-Levenberg [15] which culminates in relating Chebyshev constants between dimensions. Since we are headed in the same direction we need an analogue of this result.

Theorem 4.20 (Weighted $\mathcal{H}$-principle for Varieties)**.** *Suppose that w is admissible on $K \subset \mathcal{V}$. Define $K_\uparrow^w := \{(t,tz) \in \mathcal{V}_\uparrow : |t| = w(z)\} \subset \mathcal{V}_\uparrow$. Then the following maps are one to one with associated functions given by Theorem 4.19.*

(i.) $H_n(\mathcal{V}_\uparrow, K_\uparrow^w) \ni Q_n(t,z) \to Q_n(1,z) \in P_n(\mathcal{V}, K, w)$.

(ii.) $w\mathcal{H}(\mathcal{V}_\uparrow, K_\uparrow^w) \ni u(t,z) \to u(1,z) \in \mathcal{L}(\mathcal{V}, K, Q)$.

Proof. First we verify that $K_\uparrow^w$ is a subset of $\mathcal{V}_\uparrow$. Suppose that $\mathcal{V} = \{\zeta \in \mathbb{C}^N : P_i(\zeta) = 0, 1 \leq i \leq N-M\}$. Then $\mathcal{V}_\uparrow = \{(t,z) \in \mathbb{C}^{N+1} : t^{\deg P_i} P_i(z/t) = 0, 1 \leq i \leq N-M\}$. Write $z = t\zeta$ and $Q_i(t,z) = Q_i(t,t\zeta) = t^{\deg P_i} P_i(\zeta)$. It follows that $V_\uparrow = \{(t,t\zeta) : Q_i(t,t\zeta) = 0, 1 \leq i \leq N-M\}$ and hence $K_\uparrow^w$ is clearly a subset of $\mathcal{V}_\uparrow$. The maps above are well defined by the same logic as in Theorem 4.19.

Part (i) follows from the same argument given in Part (i) of Theorem 4.19. Part (ii) follows in the smooth case from the same argument as in Part (ii) of Theorem 4.19 since in this case $w\mathcal{H}(\mathcal{V}_\uparrow, K_\uparrow^w) = \mathcal{H}(\mathcal{V}_\uparrow, K_\uparrow^w)$. We can thus apply the standard desingularisation argument to $\mathcal{V}_\uparrow$ in the singular case and obtain the result. □

*The result makes sense because ζ is a dummy variable so in the last steps in each of (i) and (ii) it can be replaced with a z.

Remark 4.21. As a point of clarification, a point $\zeta \in \mathcal{V}$ maps to a *circle* $(t, t\zeta) \in \mathcal{V}_\uparrow$ and we let $t\zeta = z$.

Remark 4.22. We can simplify the notation that we have been using thus far. Suppose that w is an admissible weight on $K \subset X$ where $X = \mathbb{C}^N$ or $X = \mathcal{V}$. Then we can extend w to be defined on all of X by setting $w = 0$ on $X \backslash K$. Such a globally defined weight w on X carries all the necessary information to define $P_n(X, K, w)$ so instead we can write $P_n(w)$ with no ambiguity. Similarly we can define $H_n(w)$, $\mathcal{L}(Q)$ and $\mathcal{H}(Q)$. With the special notation convention when $w = 1$ on K as set out in Definition 4.18 we employ that same convention; i.e. $P_n(w) = P_n(K)$ and so on.

As a first application we immediately get a polynomial formula for the weighted Robin function.

Corollary 4.23. *If $\mathcal{V}$ satisfies the standard hypothesis and $K \subset \mathcal{V}$ is compact and regular then*

$$\rho_{K,Q} = \rho_{V^*_{E,Q}} = \Phi_{K,Q}(z) = \sup \left\{ \frac{1}{\deg P} \log |\hat{P}| : \|w^{\deg P} P\|_E \leq 1 \right\}.$$

Proof. Follow the same argument as in Theorem 3.24 with the $\mathcal{H}$-principle replaced with the weighted $\mathcal{H}$-principle. □

We also have convergence of weighted Chebyshev constants.

Definition 4.24. *Let $K \subset \mathcal{V}$ be a compact set and w an admissible weight on K. We define the weighted α-Chebyshev constant in the direction λ_j (recall Definition 1.133) to be*

$$T^w(K, \alpha, \lambda) = \inf\{\|w^{|\alpha|} p\|_K : \textsc{lt}(p) = z^\alpha v_{\lambda_j}\}.$$

We define other weighted quantities $\tau^w(K, \theta, \lambda_j)$ and $\tau^w(K)$ in the analogous way to Definition 1.139.

Remark 4.25. We will also use the notation $d^w(K) = \tau^w(K)$ for convenience (observing that in the unweighted case $\tau^1(K) = d^1(K) = d(K) = \delta(K)$ by Zakharjuta [54]).

Corollary 4.26. $\lim\limits_{\substack{|\alpha| \to \infty \\ \alpha/|\alpha| \to \theta}} T^w(K, \alpha, \lambda_j)^{1/|\alpha|} = \tau^w(K, \theta, \lambda_j)$.

Proof. The weighted $\mathcal{H}$-principle gives a correspondence to a homogeneous α'-Chebyshev constants (where α' is a multiindex in $\mathbb{Z}^{M+1}_{\geq 0}$ corresponding to the multiindex of $\textsc{lt}(\tilde{p})$). Since $K_\uparrow$ is circled it follows that homogeneous α'-Chebyshev constants converge by Corollary 1.149, which implies the convergence of the limit in the hypothesis. □

4.3 Homogeneous Transfinite Diameter

The homogeneous transfinite diameter has many practical applications in relating the weighted transfinite diameter and the unweighted transfinite diameter. While the concepts discussed here should be valid in complete generality, it is sufficient for our purposes to work exclusively with circled sets. In this setting we are assured convergence of the homogeneous Chebyshev constants (see below) as they agree with the usual Chebyshev constants from Chapter 1 (Definition 1.139). This avoids invoking the theory of weakly submultiplicative functions to prove convergence. The framework developed in [23] combined with the arguments here is enough to prove the convergence of the homogeneous transfinite diameter for arbitrary K.

Definition 4.27. *Let (x, y) be a Noether presentation for $\mathcal{V}$. Suppose that $\mathbf{e}_i$ is the ith (reduced) basis monomial for $\mathbb{C}[\mathcal{V}]$ ordered by grevlex. We make the following definitions.*

- $m^{(\mathcal{V})}(i) =$ *the number of monomials of degree at most i for $\mathbb{C}[\mathcal{V}]$,*
- $h^{(\mathcal{V})}(i) = m^{(\mathcal{V})}(i) - m^{(\mathcal{V})}(i-1) =$ *the number of monomials degree exactly i,*
- $l^{(\mathcal{V})}(i) = \sum_{j=1}^{i} j h^{(\mathcal{V})}(j) =$ *the sum of degrees of the monomials of at most degree i.*

Lemma 4.28. *Let $\mathbf{e}_i$ be the ith basis monomial for $\mathbb{C}[\mathcal{V}]$ in grevlex. Define a graded ordering for the monomials of $\mathbb{C}[\mathcal{V}_\uparrow]$ by letting the new variable $t = x_0$ and then impose grevlex on $(x_0, ..., x_M, y_1, ..., y_{N-M})$. Then there is a bijective correspondence between the monomials of degree at most n in $\mathbb{C}[\mathcal{V}]$ and the monomials of exactly degree n in $\mathbb{C}[\mathcal{V}_\uparrow]$. In particular, $m^{(\mathcal{V})}(n) = h^{(\mathcal{V}_\uparrow)}(n)$.*

Proof. The $\mathcal{H}$-principle for varieties provides the desired one to one map. It suffices then to restrict the domain of the map so that it is a bijection. Then monomials of the form $t^n \mathbf{e}_i(z/t)$ where $0 \leq i \leq m^{(\mathcal{V})}(n)$ are the only candidates we can form to make degree n monomials in $\mathbb{C}[\mathcal{V}_\uparrow]$. But by the $\mathcal{H}$-principle each of these elements is unique. It follows that this is onto the degree n monomials in $\mathbb{C}[\mathcal{V}_\uparrow]$ which completes the proof. □

Lemma 4.29. *Suppose that (x, y) is a Noether presentation for $\mathcal{V}$ and $m_Y = \max\{|\beta| : y^\beta \in \mathbb{C}[\mathcal{V}]\}$. We have the inequality*

$$m^{(\mathcal{V})}(n)\frac{Mn}{M+1} \leq l^{(\mathcal{V})}(n) \leq m^{(\mathcal{V})}(n)\frac{m_Y + Mn}{M+1}.$$

Proof. Recall that for a Noether presentation (x, y) for $\mathcal{V}$ that there are finitely many multi-indices β such that $y^\beta \in \mathbb{C}[\mathcal{V}]$. Let $m_Y = \max\{|\beta| : y^\beta \in \mathbb{C}[\mathcal{V}]\}$. Observe that we have the decomposition for $n \geq m_Y$

$$\mathbb{C}_{\leq n}[\mathcal{V}] = \bigoplus_{\beta : y^\beta \in \mathbb{C}[\mathcal{V}]} y^\beta \mathbb{C}_{\leq n-|\beta|}[x].$$

Given a multiindex β such that $y^\beta \in \mathbb{C}[\mathcal{V}]$ and $n \geq m_Y$ we can count the monomials of degree n with a y^β term using $\mathbb{C}^M$ theory;

$$\begin{aligned} &\text{cardinarlity}(\{y^\beta x^\alpha \in \mathbb{C}[\mathcal{V}] : |\beta + \alpha| \leq n\}) = m^{(M)}(n - |\beta|), \\ &m^{(\mathcal{V})}(n) = \sum_\beta m^{(M)}(n - |\beta|), \end{aligned} \tag{58}$$

and

$$\begin{aligned}
l^{(\mathcal{V})}(n) &= \sum_{\beta} |\beta| m^{(M)}(n-|\beta|) + l^{(M)}(n-|\beta|) \\
&= \sum_{\beta} |\beta| m^{(M)}(n-|\beta|) + \frac{M}{M+1}(n-|\beta|) m^{(M)}(n-|\beta|) \\
&= \sum_{\beta} m^{(M)}(n-|\beta|) \left(\frac{|\beta| + Mn}{M+1} \right). \qquad (59)
\end{aligned}$$

Using the obvious min/max estimates for $|\beta|$ in equation (59) and the identification in equation (58) yields the inequality

$$m^{(\mathcal{V})}(n) \frac{Mn}{M+1} \leq l^{(\mathcal{V})}(n) \leq m^{(\mathcal{V})}(n) \frac{m_Y + Mn}{M+1}.$$

□

Definition 4.30. *Let $\mathcal{B}$ be a polynomial basis for $\mathbb{C}[\mathcal{V}]$ (either the usual basis for $\mathbb{C}[\mathcal{V}]$ or the distinguished basis $\mathcal{C}$). Let $\mathcal{B}_{=k}$ be the monomials in $\mathcal{B}$ of exactly degree k. For $s \leq h^{(\mathcal{V})}(k)$ we define the kth homogeneous Vandermonde determinant of the points $(\zeta_1, ..., \zeta_s)$ to be*

$$VDMH_{\mathcal{B}_{=k}}(\zeta_1, ..., \zeta_s) = \det \begin{pmatrix} \mathbf{e}_{1,k}(\zeta_1) & \mathbf{e}_{1,k}(\zeta_2) & ... & \mathbf{e}_{1,k}(\zeta_s) \\ \mathbf{e}_{2,k}(\zeta_1) & \mathbf{e}_{2,k}(\zeta_2) & ... & \mathbf{e}_{2,k}(\zeta_s) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{e}_{s,k}(\zeta_1) & \mathbf{e}_{s,k}(\zeta_2) & ... & \mathbf{e}_{s,k}(\zeta_s) \end{pmatrix}$$

where $\mathbf{e}_{j,k}$ is an enumeration of the elements in $\mathcal{B}_{=k}$ with respect to grevlex.

Definition 4.31. *Suppose that $\mathcal{V}$ satisfies the standard hypothesis and is homogeneous. Suppose that $K \subset \mathcal{V}$ is a circled set. We define the homogeneous transfinite diameter of K to be*

$$d_{H,h^{(\mathcal{V})}(k)}(K) := \max_{\zeta_1,...,\zeta_{h^{(\mathcal{V})}(k)} \in K} \left| VDMH_{\mathcal{B}_{=k}}(\zeta_1, ..., \zeta_{h^{(\mathcal{V})}(k)}) \right|.$$

$$d_H(K) := \limsup_{k \to \infty} d_{H,h^{(\mathcal{V})}(k)}(K)^{1/kh^{(\mathcal{V})}(k)}.$$

Definition 4.32. *Let $\mathcal{V}$, K be as above. Let*

$$\mathcal{C}_H(\alpha, i) = \{p \in \mathbb{C}[\mathcal{V}] : p(z) = z^\alpha + g(z), g(z) \prec z^\alpha v_i, p \text{ homogeneous}\}.$$

We define the homogeneous α-Chebyshev constant in the direction λ_i to be

$$T_H(K, \alpha, \lambda_i) = \inf\{\|p\|_K : p \in \mathcal{C}_H(\alpha, i)\}.$$

We define the θ-partial homogeneous Chebyshev constant in the direction λ_i to be

$$\tau_H(K,\theta,\lambda_i) = \limsup_{\substack{|\alpha|\to\infty \\ \alpha/|\alpha|\to\theta}} T(K,\alpha,\lambda_i)^{1/|\alpha|}.$$

We define the homogeneous Chebyshev constant in the direction λ_i to be

$$\tau_H(K,\lambda_i) = \exp\left(\frac{1}{vol(\Sigma)}\int_{\Sigma_0} \log\tau(K,\theta,\lambda_i)\,d\theta\right).$$

We define the principal Chebyshev constant for K to be

$$\tau_H(K) = \left(\prod_{i=1}^{d} \tau_H(K,\lambda_i)\right)^{1/d}.$$

Lemma 4.33. *For K circled, $\tau_H(K,\lambda_i) = \tau(K,\lambda_i)$ for all i.*

Proof. Since Chebyshev polynomials can be chosen to be homogeneous (since K is circled) it follows that $T_H(K,\alpha,\lambda_i) = T(K,\alpha,\lambda_i)$ for all α, i. From this the result follows. □

Corollary 4.34. *When K circled the lim sup in the definition above can be replaced with a limit.*

Corollary 4.35. *When K is circled we have $d(K) = \tau(K) = \tau_H(K)$.*

Notation 4.36. To prove the convergence of $d_H(K)$ it will be convenient to define

$$T_H(K,\alpha,\lambda_0) = \inf\{\|\hat{p}\|_K : \deg(p) \leq |\alpha| + \deg(v_i), \text{LT}(p) = z^\gamma z_M^l z^\beta, z_M z^\beta \in \mathcal{B}\}$$

and analogous quantities to be Chebyshev constants without direction. Labeling this the 'λ_0' direction is convenient in what is to follow, although we stress that there is no λ_0 point at infinity. We will also let $h^{(\mathcal{V})}(k) = h(k)$ and similar for this result since there is no ambiguity as to what this means in the proof. Finally we set

$$D_H(s,k) = \max_{\zeta_1,\ldots,\zeta_s\in K} |VDMH_{\mathcal{B}_{=k}}(\zeta_1,\ldots,\zeta_s)|.$$

Lemma 4.37. *Suppose $s \leq h(k)$ and $\mathbf{e}_{s,k} = z^\alpha v_i$ where $|\alpha| = k$. Then we have the following inequality*

$$T_H(K,\alpha,\lambda_i) \leq \frac{D_H(s,k)}{D_H(s-1,k)} \leq kT_H(K,\alpha,\lambda_i).$$

If instead $\mathbf{e}_s = z^\alpha z_M^l z^\beta$ and $s > 1$ then

$$T_H(K,\alpha,\lambda_0) \leq \frac{D_H(s,k)}{D_H(s-1,k)} \leq sT_H(K,\alpha,\lambda_0).$$

If $s = 1$ then

$$T_H(K, \alpha, \lambda_0) \leq D_H(1, k) \leq sT_H(K, \alpha, \lambda_0).$$

Proof. For the last inequality observe that matrix defining $D_H(1, k)$ consists only of the monomial $\mathbf{e}_{1,k}$, as such the inequality is trivial. The proof of either of the remaining inequalities is the same, so we only prove the first. To this end suppose that $\mathbf{e}_{s,k} = z^\alpha v_i$ with $|\alpha| = k$. Suppose that $(\zeta_1, ..., \zeta_{s-1})$ are points for which $VDMH_{B_{=k}}(\zeta_1, ..., \zeta_{s-1})$ obtains its maximum. Observe that

$$q_s(z) = \frac{VDMH_{B_{=k}}(\zeta_1, ..., \zeta_{s-1}, z)}{VDMH_{\mathcal{B}_{=k}}(\zeta_1, ..., \zeta_{s-1})}$$

is a homogeneous polynomial with leading term $\mathbf{e}_{s,k}(z)$. From the definition of the Chebyshev constants we have immediately

$$T(K, \alpha, \lambda_i) \leq \|q_s\|_K = \frac{D_H(s, k)}{D_H(s-1, k)}.$$

For the other side, suppose that $t(z)$ is a homogeneous polynomial such that $\|t\|_K = T_H(K, \alpha, \lambda_i)$. By adding multiple of one row to another we observe that

$$\begin{pmatrix} \mathbf{e}_{1,k}(\zeta_1) & ... & \mathbf{e}_{1,k}(\zeta_{s-1}) & \mathbf{e}_{1,k}(z) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_{s-1,k}(\zeta_1) & ... & \mathbf{e}_{s-1,k}(\zeta_{s-1}) & \mathbf{e}_{s-1,k}(z) \\ \mathbf{e}_{s,k}(\zeta_1) & ... & \mathbf{e}_{s,k}(\zeta_{s-1}) & \mathbf{e}_{s,k}(z) \end{pmatrix} \sim \begin{pmatrix} \mathbf{e}_{1,k}(\zeta_1) & ... & \mathbf{e}_{1,k}(\zeta_{s-1}) & \mathbf{e}_{1,k}(z) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_{s-1,k}(\zeta_1) & ... & \mathbf{e}_{s-1,k}(\zeta_{s-1}) & \mathbf{e}_{s-1,k}(z) \\ t(\zeta_1) & ... & t(\zeta_{s-1}) & t(z) \end{pmatrix}$$

Since the determinant is unchanged by these row operations it follows that maximising the determinant of the RHS is equal to $D_H(s, k)$. Expanding the determinant along the bottom row yield terms of the form

$$t(\zeta_j)VDMH_{\mathcal{B}_{=k}}(\zeta_1, ..., \zeta_{j-1}, \zeta_{j+1}, ..., \zeta_{s-1}, z)$$

or

$$t(z)VDMH_{\mathcal{B}_{=k}}(\zeta_1, ..., \zeta_s).$$

Maximising each term separately yields an inequality of the form

$$D_H(s, k) \leq |t(z)D_H(s-1, k)| + \sum_{j=1}^{s-1} |t(\zeta_j)D_H(s-1, k)|.$$

But since $\|t\|_K = T_H(K, \alpha, \lambda_i)$ we can maximise the t terms to obtain

$$D_H(s, k) \leq sT_H(K, \alpha, \lambda_i)D_H(s-1, k)$$

This proves the other side of the inequality and we are done. $\square$

Corollary 4.38. $\prod_{i=0}^{d} \prod_{|\alpha|=k} T_H(K, \alpha, \lambda_i) \leq D_H(h(k), k) \leq h(k)! \prod_{i=0}^{d} \prod_{|\alpha|=k} T_H(K, \alpha, \lambda_i)$.

Proof. Observe that

$$D_H(h(k), k) = \frac{D_H(h(k), k)}{D_H(h(k)-1, k)} \cdot \frac{D_H(h(k)-1, k)}{D_H(h(k)-2, k)} \cdot \ldots \cdot \frac{D_H(2, k)}{D_H(1, k)} \cdot D_H(1, k).$$

Applying the estimate from Lemma 4.37 to each of the terms in the above expansion yields:

$$\prod_{i=0}^{d} \prod_{|\alpha|=k} T_H(K, \alpha, \lambda_i) \leq D_H(h(k), k) \leq h(k)! \prod_{i=0}^{d} \prod_{|\alpha|=k} T_H(K, \alpha, \lambda_i).$$

$\square$

Lemma 4.39. $\lim_{k\to\infty} h(k)!^{1/kh(k)} = 1$.

Proof. Firstly,

$$1 \leq h(k)!^{1/kh(k)} \leq h(k)^{h(k)/kh(k)} = h(k)^{1/k}.$$

So it suffices to show that $h(k)^{1/k} \to 1$. If $\mathcal{V}_\downarrow = \{x_1 = 0\} \cap \mathcal{V}$, then

$$h(k) = m^{(\mathcal{V}_\downarrow)}(k) \leq m^{(M-1)}(k) = \begin{pmatrix} M-1+k \\ k \end{pmatrix} = \frac{(M-1+k)!}{k!(M-1)!}.$$

We calculate that

$$\lim_{k\to\infty} \left(\prod_{j=1}^{M-1} (k+j) \right)^{1/k} \leq \lim_{k\to\infty} (M-1+k)^{(M-1)/k} = 1.$$

Hence

$$\lim_{k\to\infty} \left(\frac{1}{(M-1)!} \right)^{1/k} \left(\frac{(M-1+k)!}{k!} \right)^{k} = \lim_{k\to\infty} \left(\frac{1}{(M-1)!} \right)^{1/k} \left(\prod_{j=1}^{M-1} (k+j) \right)^{1/k} \leq 1.$$

From which it follows that $\lim_{k\to\infty} h(k)^{1/k} = 1$ $\square$

Lemma 4.40. $\lim_{k\to\infty} \left(\prod_{i=0}^{d} \prod_{|\alpha|=k} T_H(K, \alpha, \lambda_i) \right)^{1/kh(k)} = \tau_H(K)$.

Proof. Using log to rewrite the product in the LHS as a sum we obtain

$$\exp\left(\sum_{j=0}^{d} \frac{1}{kh(k)} \sum_{|\alpha|=k} \log T_H(K, \alpha, \lambda_i) \right)$$

Firstly we deal with the λ_0 contribution. Recall that the construction of the distinguished basis $\mathcal{C}$ for $\mathcal{V}$ is such that any monomial without a v_i term precedes any term with a v_i term of the same degree (Definition 1.138). By the definition of $T_H(K, \alpha, \lambda_0)$ (Notation 4.36) it follows that the sets of polynomials that the infimum is taken over cosnsist only of linear combinations of type-1 monomials, and in particular has only finitely many powers of y_i. Suppose that p is a homogeneous polynomial such that $\|p\|_K = T_H(K, \alpha, \lambda_0)$ and $\mathrm{LT}(p) = x^{\gamma_i} y^{\beta_i}$ for some multiindices γ_i, β_i. Let π be the projection onto $x \in \mathbb{C}^M$ and let $q_i(x)$ be a homogeneous polynomial with the property that

$$\|q_i\|_{\pi(K)} = \inf\{\|\hat{p}\|_{\pi(K)} : p \in \mathbb{C}[x], \mathrm{LT}(p) = x^{\gamma_i}\}.$$

Let $R > 1$ be sufficient large so that $K \subset B_R(0)$. Then we have the following chain of estimates

$$T_H(K, \alpha, \lambda_0) = \|p\|_K \leq \|q_i y^{\beta_i}\|_K \leq \|q_i\|_{\pi(K)} \|y^{\beta_i}\|_K \leq \|q_i\|_{\pi(K)} R^{\beta_i}$$

By construction q_i is a homogeneous Chebyshev polynomial for $\pi(K)$ in the direction γ_i, so by $\mathbb{C}^M$ theory (i.e. [36] Section 6) we know that

$$\|q_i\|_{\pi(K)} \leq T_H(B'_R(0), \alpha) = R^{\deg q_i} \leq R^{|\alpha| - |\beta_i| + \deg v_i}$$

Hence we have the upper esimate

$$T_H(K, \alpha, \lambda_0) \leq \|q_i\|_{\pi(K)} R^{\beta_i} \leq R^{|\alpha| + \deg v_i}$$

If $T_H(K, \alpha, \lambda_0) = 0$ for some α then $\tau_H(K) = 0$ and equality above is trivial. Otherwise we know that $T_H(K, \alpha)$ is uniformly bounded below by some constant C independent of α as $|\alpha| \to \infty$ (i.e. [36] Section 4). It follows then that

$$\begin{aligned}
&\exp\left(\frac{k \log C}{kh(k)} + \sum_{j=1}^{d} \frac{1}{kh(k)} \sum_{|\alpha|=k} \log T_H(K, \alpha, \lambda_i)\right) \\
&\qquad \leq \exp\left(\sum_{j=0}^{d} \frac{1}{kh(k)} \sum_{|\alpha|=k} \log T_H(K, \alpha, \lambda_i)\right) \\
&\qquad\qquad \leq \exp\left(\frac{2(k + \deg(v_i)) \log R}{kh(k)} + \sum_{j=1}^{d} \frac{1}{kh(k)} \sum_{|\alpha|=k} \log T_H(K, \alpha, \lambda_i)\right) \qquad (60)
\end{aligned}$$

We now turn our attention to any one of the λ_j terms. Let $t = \deg(v_i)$. First note that $h(k) = a(k) + dh^{(M)}(k)$ where $a(k)$ is the number of type-1 monomials of degree k. Then $a(k) \leq Ah^{(M-1)}(k + t)$ where $A =$ the number of $z_M, ..., z_N$ monomials of degree strictly less than t. Then we calculate

$$\frac{a(k)}{h(k)} \leq \frac{Ah^{(M-1)}(k+t)}{dh^{(M)}(k)} \longrightarrow 0, \quad k \longrightarrow \infty. \tag{61}$$

This implies that

$$\frac{dh^{(M)}(k)}{h(k)} \longrightarrow 1, \quad k \longrightarrow \infty. \tag{62}$$

Using a standard trick originally due to Zakharjuta [54], observe that as $k \to \infty$

$$\frac{1}{h(k)} \sum_{|\alpha|=k} \log T_H(K,\alpha,\lambda_j)^{1/k} \longrightarrow \frac{1}{d \cdot vol(\Sigma)} \int_{\Sigma_0} \log \tau_H(K,\theta,\lambda_j)\, d\theta. \tag{63}$$

To see this note that $k = |\alpha|$ so $\log T_H(K,\alpha,\lambda_j)^{1/|\alpha|} \to \log T_H(K,\theta,\lambda_j)$ where $\alpha/|\alpha| \to \theta \in \Sigma_0$. Next, $\alpha/|\alpha|$ is a point in the simplex Σ_0. Treating $T_H(K,\alpha,\lambda)^{1/|\alpha|}$ as a function in α, the sum can be seen as evaluating T_H at $h^{(M)}(k)$ uniformly distributed points over the simplex Σ_0. But $h(k) \to dh^{(M)}(k)$ as $k \to \infty$ by equation (61) and (62). Then the sum converges to the definition of the Riemann integral of T_H (with respect to α) on Σ_0, up to a scale factor of $\frac{1}{d}$, hence the convergence in equation (63). Now we can take the limit as $k \to \infty$ in equation (60) to obtain

$$\begin{aligned}
&\exp\left(\frac{1}{d \cdot vol(\Sigma)} \int_{\Sigma_0} \log \tau_H(K,\theta,\lambda_j)\, d\theta\right) \\
&\qquad \leq \lim_{k\to\infty} \exp\left(\sum_{j=0}^{d} \frac{1}{kh(k)} \sum_{|\alpha|=k} \log T_H(K,\alpha,\lambda_i)\right) \\
&\qquad\qquad \leq \exp\left(\frac{1}{d \cdot vol(\Sigma)} \int_{\Sigma_0} \log \tau_H(K,\theta,\lambda_j)\, d\theta\right).
\end{aligned}$$

Hence

$$\lim_{k\to\infty} \exp\left(\sum_{j=0}^{d} \frac{1}{kh(k)} \sum_{|\alpha|=k} \log T_H(K,\alpha,\lambda_i)\right) = \log \tau_H(K)$$

which completes the proof. $\square$

Theorem 4.41. *If $\mathcal{V}$ satisfies the standard hypothesis and K is circled then $d_H(K) = \tau_H(K)$.*

Proof. Take $1/kh(k)$ powers of everything in Corollary 4.38 and then take the limit as $k \to \infty$. Using Lemmas 4.39, 4.40 and observing (by definition) that $D_H(h(k),k)^{1/k(hk)} \to d_H(K)$ implies the result. $\square$

Corollary 4.42. *With hypotheses as in Theorem 4.41 we have $\delta(K) = \tau(K) = \tau_H(K) = d_H(K)$.*

Proof. $\delta(K) = \tau(K)$ and $\tau_H(K) = d_H(K)$ are known. $\tau(K) = \tau_H(K)$ follows from the fact that K is circled, Corollary 1.149 implies that Chebyshev polynomials can be chosen to be homogeneous which completes the proof. $\square$

4.4 Weighted Transfinite Diameter

We are now ready to discuss the weighted transfinite diameter. We assume throughout that w is a reduced weight on a compact set $K \subset \mathcal{V}$. Recall the following definition from Chapter 1.

Definition 4.43. *Suppose that $\mathcal{V}$ satisfies the standard hypothesis and let $\mathcal{B}$ be either the standard basis for $\mathbb{C}[\mathcal{V}]$ or the distinguished basis. Let $\mathbf{e}_i$ be the ith basis monomial for $\mathcal{B}$ ordered by grevlex. Let w be an admissible weight on K. Then we define the weighted transfinite diameter for a compact set $K \subset \mathcal{V}$ to be*

$$\delta_n^w(K) := \max_{z_1,...,z_n \in K} VDM_{\mathcal{B}}(z_1,...,z_n) w(z_1)^{\alpha(n)} ... w(z_n)^{\alpha(n)}$$

$$= \max_{z_1,...,z_n \in K} \det \begin{pmatrix} \mathbf{e}_1(z_1) & \mathbf{e}_1(z_2) & ... & \mathbf{e}_1(z_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{e}_n(z_1) & \mathbf{e}_n(z_2) & ... & \mathbf{e}_n(z_n) \end{pmatrix} w(z_1)^{\alpha(n)} ... w(z_n)^{\alpha(n)}.$$

$$\delta^w(K) = \lim_{n\to\infty} \delta^w_{m(n)}(K)^{1/l(n)}.$$

We will also use the terminology that a Fekete n-set is an n-tuple of points $z_1, ..., z_n$ such that the maximum in $\delta_n^w(K)$ is obtained at that n-tuple.

Theorem 4.44. *The limit $\lim_{n\to\infty} \delta^w_{m(n)}(K)^{1/l(n)}$ exists.*

Proof. Write $m(n) = m^{(\mathcal{V})}(n), h(n) = h^{(\mathcal{V})}(n)$ and $l(n) = l^{(\mathcal{V})}(n)$. Our strategy is to show that $\delta_n^w(K) = d_{H,n}(K_\uparrow^w)$ and use the fact that $d_{H,m(n)}(K_\uparrow^w)^{1/nh^{(\mathcal{V}_\uparrow)}(n)}$ converges (noting that $h^{(\mathcal{V}_\uparrow)}(n) = m(n)$ by Lemma 4.28). Let $\xi_1, ..., \xi_n$ be an Fekete n-set for K. Let $\alpha(n) =$ multideg$(\mathbf{e}_n)$. Then the polynomial given by

$$p(\zeta) = VDM_{\mathbb{C}[\mathcal{V}]}(\xi_1, ..., \xi_{n-1}, \zeta) w(\xi_1)^{|\alpha(n)|} ... w(\xi_{n-1})^{|\alpha(n)|}$$

satisfies $\|w^{|\alpha(n)|}p\|_K = \delta_n^w(K)$. From the weighted $\mathcal{H}$-principle for varieties it follows that the associated polynomial satisfies $\frac{1}{\delta_n^w(K)} q(t,z) \in H_n(K_\uparrow^w)$. Using the identification from Remark

4.21 we observe that

$$\begin{aligned}
q(t,t\zeta) &= \det\begin{pmatrix} \mathbf{e}_1(\xi_1) & ... & \mathbf{e}_1(\xi_{n-1}) & \mathbf{e}_1(\zeta) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_n(\xi_1) & ... & \mathbf{e}_n(\xi_{n-1}) & \mathbf{e}_n(\zeta) \end{pmatrix} w(\xi_1)^{|\alpha(n)|} ... w(\xi_{n-1})^{|\alpha(n)|} t^{|\alpha(n)|} \\
&= \det\begin{pmatrix} w(\xi_1)^{|\alpha(n)|}\mathbf{e}_1(\xi_1) & ... & w(\xi_{n-1})^{|\alpha(n)|}\mathbf{e}_1(\xi_{n-1}) & t^{|\alpha(n)|}\mathbf{e}_1(\zeta) \\ \vdots & \ddots & \vdots & \vdots \\ w(\xi_1)^{|\alpha(n)|}\mathbf{e}_n(\xi_1) & ... & w(\xi_{n-1})^{|\alpha(n)|}\mathbf{e}_n(\xi_{n-1}) & t^{|\alpha(n)|}\mathbf{e}_n(\zeta) \end{pmatrix} \\
&= \det\begin{pmatrix} \tilde{e}_1(t_1', z_1') & ... & \tilde{e}_1(t_{n-1}', z_{n-1}') & \tilde{e}_1(t,z) \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{e}_n(t_1', z_1') & ... & \tilde{e}_n(t_{n-1}', z_{n-1}') & \tilde{e}_n(t,z) \end{pmatrix},
\end{aligned}$$

where $\tilde{e}_i(t,z) = \tilde{e}_i(t,t\zeta) = t^{|\alpha(n)|}\mathbf{e}_i(\zeta)$ and (t_i', z_i') are points in $K_\uparrow^w$. By the $\mathcal{H}$-principle $\tilde{e}_i(t,z)$ is a monomial of degree $|\alpha(n)|$ for all i. It follows that $q(t,z)$ is a homogeneous Vandermonde of degree $|\alpha(n)|$ monomials over the points $(t_1', z_1'), ..., (t_{n-1}', z_{n-1}'), (t,z)$ and hence $\delta_n^w(K) = \|q\|_{K_\uparrow^w} \leq d_{H,n}(K_\uparrow^w)$.

For the converse, suppose $(t_1', z_1'), ..., (t_{n-1}', z_{n-1}'), (t_n', z_n')$ is a homogeneous Fekete n-set and let $q(t,z) = VDMH((t_1', z_1'), ..., (t_{n-1}', z_{n-1}'), (t_n', z_n'))$. By the weighted $\mathcal{H}$-principle the argument given above is reversible and produces a polynomial p such that $d_{H,n}(K_\uparrow^w) = \|q\|_{K_\uparrow^w} = \|w^{|\alpha(n)|}p\|_K \leq \delta_n^w(K)$ which shows $\delta_n^w(K) = d_{H,n}(K_\uparrow^w)$. It follows that (presuming the limit exists)

$$\lim_{n\to\infty} \delta_{m(n)}^w(K)^{1/l(n)} = \lim_{n\to\infty} d_{H,m(n)}(K_\uparrow^w)^{1/l(n)}.$$

Using Lemmas 4.28 and 4.29 and we obtain

$$h^{(\mathcal{V}_\uparrow)}(n)\frac{Mn}{M+1} = m(n)\frac{Mn}{M+1} \leq l(n) \leq m(n)\frac{m_Y + Mn}{M+1} = h^{(\mathcal{V}_\uparrow)}(n)\frac{m_Y + Mn}{M+1}.$$

Hence

$$\frac{M}{M+1} \leq \frac{l(n)}{nh^{(\mathcal{V}_\uparrow)}(n)} \leq \frac{m_Y/n + M}{M+1}.$$

Hence we have $\lim_{n\to\infty} l(n)/nh^{(\mathcal{V}_\uparrow)}(n) = M/M+1$. So we deduce that:

$$d_H(K_\uparrow^w)^{(M+1)/M} = \lim_{n\to\infty} d_{H,m(n)}(K_\uparrow^w)^{(M+1)/(Mnh^{(\mathcal{V}_\uparrow)}(n))} = \lim_{n\to\infty} d_{H,m(n)}(K_\uparrow^w)^{1/l(n)}$$

which finishes the proof. $\square$

Corollary 4.45. $\delta^w(K) = d_H(K^w_\uparrow)^{(M+1)/M}$.

4.5 The K^w_ρ Lemmas.

The following are important technical lemmas that will be used in subsequent sections. These will be proven using Rumely's formula for varieties. Some of these results are known in the $\mathbb{C}^N$ case without needing to invoke the formula.

Lemma 4.46. *Suppose that $K \subset \mathcal{V}$ and w an admissible weight on K. Define*

$$K^w_\rho := \{z \in \mathcal{V}^h : \rho_{K,Q} \leq 0\}.$$

Then $\log \delta^w(K) = \log \delta(K^w_\rho) - \frac{1}{dM(2\pi)^M} \int_K Q(dd^c V_{K,Q})^M$.

Proof. First suppose that K is regular. By Corollaries 4.42 and 4.45,

$$\log(\delta^w(K)) = \frac{M+1}{M} \log(d_H(K^w_\uparrow)) = \frac{M+1}{M} \log(\delta(K^w_\uparrow)).$$

If $X_j = \{x_0 = ... = x_{j-1} = 0, x_j = 1\} \cap V_\uparrow$ then Rumely's formula for varieties says that

$$-\log(\delta(K^w_\uparrow)) = \frac{1}{(M+1)d} \sum_{j=0}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_{K^w_\uparrow}(x,y)\,(dd^c \rho_{K^w_\uparrow})^{M-j}.$$

Observe that since $K^w_\uparrow$ is circled, so $K^w_\rho = K^w_\uparrow \cap \{x_0 = 0\}$ is a circled set. For a circled set, the homogeneous extremal function equals the extremal function i.e. $\max\{0, \rho_{K^w_\rho}\} = V_{K^w_\rho}$ on $\mathcal{V}^h$. But $\rho_{K^w_\uparrow}(0,z)$ is a log homogeneous function with the same zero set as $\rho_{K^w_\rho}(z)$ hence by maximality (Lemma 3.27) they must be equal. It follows that

$$\begin{aligned}
&\frac{1}{d(M+1)} \sum_{j=0}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_{K^w_\uparrow}(x,y)\,(dd^c \rho_{K^w_\uparrow})^{M-j} \\
&= \frac{1}{d(M+1)} \frac{1}{(2\pi)^M} \int_{X_0} \rho_{K^w_\uparrow}(x,y)\,(dd^c \rho_{K^w_\uparrow})^{M} + \frac{1}{dM} \sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_{K^w_\uparrow}(x,y)\,(dd^c \rho_{K^w_\uparrow})^{M-j} \\
&= \frac{1}{d(M+1)} \frac{1}{(2\pi)^M} \int_{X_0} \rho_{K^w_\uparrow}(x,y)\,(dd^c \rho_{K^w_\uparrow})^{M} \\
&\qquad\qquad + \frac{1}{d(M+1)} \sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \int_{\{x_1 = ... = x_{j-1} = 0, x_j = 1\}} \rho_{K^w_\rho}(x,y)\,(dd^c \rho_{K^w_\rho})^{M-j} \\
&= \frac{1}{d(M+1)} \frac{1}{(2\pi)^M} \int_{X_0} \rho_{K^w_\uparrow}(x,y)\,(dd^c \rho_{K^w_\uparrow})^{M} - \frac{M}{M+1} \log d(K^w_\rho).
\end{aligned}$$

Hence

$$\log \delta^w(K) = \log \delta(K^w_\rho) - \frac{1}{dM(2\pi)^M} \int_{X_0} \rho_{K^w_\uparrow}(x,y)(dd^c \rho_{K^w_\uparrow})^M. \tag{64}$$

Since $K^w_\uparrow$ is a circled set, it follows by the same logic as before that $V_{K^w_\uparrow} = \rho_{K^w_\uparrow}$ and by the weighted $\mathcal{H}$-principle, $V_{K_\uparrow^w}(1,z) = V_{K,Q}(z)$. By Lemma 4.11 $\operatorname{supp} dd^c(V_{K,Q})^M \subset K$ and by Lemma 4.12 $V_{K,Q} = Q$ q.e. on K. It follows that

$$\int_{X_0} \rho_{K_\uparrow^w}(x,y)(dd^c \rho_{K_\uparrow^w})^M = \int_K Q(dd^c V_{K,Q})^M.$$

Putting this into equation (64) we obtain

$$\log \delta^w(K) = \log d(K_\rho^w) - \frac{1}{dM(2\pi)^M} \int_K Q(dd^c V_{K,Q})^M$$

as claimed.

Now suppose that K is not regular. Take a sequence of regular sets K_j decreasing to K and w_j continuous and admissible on K_j decreasing to K. Let

$$K_{j,\uparrow}^w(\Delta) = \{(t,z) \in \mathcal{V}_\uparrow : |t| \leq w_j(z)\}.$$

Then $K_{j,\uparrow}^w(\Delta)$ is the homogeneous polynomial hull of $K_{j,\uparrow}^w$ so $d_H(K_{j,\uparrow}^w) = d_H(K_{j,\uparrow}^w(\Delta))$. By construction, $K_{j,\uparrow}^w(\Delta) \supset K_{j+1,\uparrow}^w(\Delta)$. Hence

$$d_H(K_{j,\uparrow}^w) = d_H(K_{j,\uparrow}^w(\Delta) \geq d_H(K_{j+1,\uparrow}^w(\Delta) = d_H(K_{j+1,\uparrow}^w).$$

Since $K_{j+1,\uparrow}^w(\Delta)$ decreases to $K_\uparrow^w(\Delta)$ it follows that the transfinite diameters above converge to $d_H(K_\uparrow^w)$. Hence $\delta^{w_j}(K_j) \to \delta^w(K_j)$.

The convergence $\delta(K_{j,\rho}^{w_j}) \to \delta(K_\rho^w)$ follows since $K_{j,\rho}^{w_j}$ decreases to K_ρ^w. Finally the convergence of

$$\int_{K_j} Q_j(dd^c V_{K_j,Q_j})^M \to \int_K Q(dd^c V_{K,Q})^M$$

follows from Theorem 2.12. Putting all these convergences together yields the result. □

Lemma 4.47. *Suppose that $K \subset \mathcal{V}$ and w is an admissible weight on K. Let $C = \max\{V_{K,Q}^*(z) : z \in K\}$ and define*

$$Z(K) = \{z \in \mathcal{V} : V_{K,Q}^*(z) \leq C\}.$$

Then $d^w(K) = \delta(Z(K))e^{-C}$.

Recall Definition 4.24 for the definition of $d^w(K)$.

Proof. We show that for any λ, α that $e^{-|\alpha|C}T(Z(K),\alpha,\lambda) = T^w(K,\alpha,\lambda)$ which is enough to imply the result. To this end, suppose that q is an α-Chebyshev polynomial in the direction λ for $Z(K)$ (that is, $\|q\|_{Z(K)} = T(Z(K),\alpha,\lambda)$). Observe then that $\|w^{|\alpha|}q\|_K$ is a competitor for the weighted α-Chebyshev polynomial in the direction λ for K. This implies that

$$T^w(K,\alpha,\lambda) \le \|w^{|\alpha|}q\|_K \le \|w^{|\alpha|}\|_K\|q\|_K \le \|w^{|\alpha|}\|_K\|q\|_{Z(K)} \le e^{-|\alpha|C}T(Z(K),\alpha,\lambda).$$

Note that we have used the fact that $\|w^{|\alpha|}\|_K = e^{-|\alpha|\|Q\|_K} = e^{-|\alpha|C}$ by Lemma 4.12. For the converse, suppose that p is a weighted α-Chebyshev polynomial in the direction λ for K (that is, $\|w^{|\alpha|}p\|_K = T^w(K,\alpha,\lambda)$). Then for $z \in K$

$$\frac{1}{|\alpha|}\log\frac{|p(z)|}{\|w^{|\alpha|}p\|_K} = \frac{1}{|\alpha|}\log|w^{|\alpha|}p(z)| - \log|w(z)| - \frac{1}{|\alpha|}\log\|w^{|\alpha|}p\|_K \le -\log|w(z)| = Q(z).$$

This implies

$$\begin{aligned}\frac{1}{|\alpha|}\log\frac{|p(z)|}{\|w^{|\alpha|}p\|_K} &\le V_{K,Q}(z)\\ \log|p(z)| &\le |\alpha|V_{K,Q}(z) + \log\|w^{|\alpha|}p\|_K\\ |p(z)| &\le T^w(K,\alpha,\lambda)\exp(|\alpha|V_{K,Q}(z))\\ T(Z(K),\alpha,\lambda) \le \|p\|_{Z(K)} &\le T^w(K,\alpha,\lambda)\exp(|\alpha|C)\end{aligned}$$

Combining these inequalities we deduce $e^{-|\alpha|C}T(Z(K),\alpha,\lambda) = T^w(K,\alpha,\lambda)$ which completes the proof. ☐

Corollary 4.48. $\tau^w(K,\theta,\lambda) = e^{-C}\tau(Z(K),\theta,\lambda)$.

Corollary 4.49. *If $Z(K,C) = \{V_{K,Q} \le C\} \ne \varnothing$ then $T^w(K,\alpha,\lambda) = e^{-|\alpha|C}T(Z(K,C),\alpha,\lambda)$.*

Proof. Follows in exactly the same way as the proof of Lemma 4.47. ☐

Corollary 4.50. *If $Z(K,C) = \{z \in \mathcal{V} : V^*_{K,Q}(z) \le C\}$ then $d^w(K) = d(Z(K,C))e^{-C}$.*

Proof. Use the exact same argument as in Lemma 4.47. ☐

Lemma 4.51. *Let $K \subset \mathcal{V}$ and suppose that w is an admissible weight on K. Then $d^w(K) = d(K^w_\rho)$.*

Proof. By Lemma 4.47 $d^w(K) = d(Z(K))e^{-C}$. Observe that by construction of $Z(K)$ that $V_{Z(K)} = V_{K,Q} - C$ outside of $Z(K)$. It follows by taking Robin functions of both sides that $\rho_{Z(K)} = \rho_{K,Q} - C$. Since $d(Z(K)) = \delta(Z(K))$ we can use Lemma 4.46 to show that $d(Z(K)) = d(Z(K)_\rho)$ (recall that $Q = 0$ since this is the unweighted situation so there is no integral term). Now

$$Z(K)_\rho := \{z \in \mathcal{V}^h : \rho_{Z(K)} \le 0\} = \{z \in \mathcal{V}^h : \rho_{K,Q} \le C\}.$$

Noting that $\rho_{K,Q}$ is the weighted extremal function for K^w_ρ and using Corollary 4.50 in conjunction with the observation above it follows that $d(K^w_\rho) = e^{-C}d(Z(K))$. Hence, $d^w(K) = d(K^w_\rho)$ as claimed. ☐

Remark 4.52. Alternatively, we could invoke Rumely's formula to obtain the final conclusion.

$$\begin{aligned}
-\log d(Z(K)) &= \frac{1}{dM}\sum_{j=0}^{M-1}\frac{1}{(2\pi)^{M-j-1}}\int_{X_j}\rho_{Z(K)}(dd^c\rho_{Z(K)})^{M-j-1} \\
&= \frac{1}{dM}\sum_{j=0}^{M-1}\frac{1}{(2\pi)^{M-j-1}}\int_{X_j}\rho_{K,Q}-C(dd^c\rho_{K,Q})^{M-j-1} \\
&= -\log d(K_\rho^w) - \frac{C}{dM}\sum_{j=0}^{M-1}\frac{d(2\pi)^{M-j-1}}{(2\pi)^{M-j-1}} \\
&= -\log d(K_\rho^w) - C
\end{aligned}$$

which implies $d(K_\rho^w) = e^{-C}d(Z(K))$.

Corollary 4.53.

$$d^w(K) = \delta^w(K)\exp\left(\frac{1}{dM(2\pi)^M}\int_K Q(dd^cV_{K,Q})^M\right).$$

Proof. Apply Lemma 4.51 to Lemma 4.46. □

Corollary 4.54.

$$d^w(K) = d_H(K_\uparrow^w)^{(M+1)/M}\exp\left(\frac{1}{dM(2\pi)^M}\int_K Q(dd^cV_{K,Q})^M\right).$$

Proof. Apply Corollary 4.45 to Corollary 4.53. □

Lemma 4.55. *With notation as Lemma 4.51, $\tau^w(K,\theta,\lambda) = \tau(K_\rho^w,\theta,\lambda)$.*

Proof. Observe that $\rho_{K,Q} = \rho_{Z(K)} + C$ where $C = \max\{V_K(z) : z \in K\} \geq 0$. Hence we have

$$K_\rho^w = \{\rho_{K,Q} \leq 0\} = \{\rho_{Z(K)} \leq -C\}.$$

Observe that since K_ρ^w is circled, $\max\{\rho_{K_\rho^w}, 0\} = V_{K_\rho^w}$. Then

$$Z(K)_\rho = \{\rho_{Z(K)} \leq 0\} = \{\rho_{K_\rho^w} \leq C\} = \{V_{K_\rho^w} \leq C\} = Z(K_\rho^w, C).$$

It follows that

$$T(Z(K)_\rho,\alpha,\lambda) = T(Z(K_\rho^w,C),\alpha,\lambda) = e^{-|\alpha|C}T(K_\rho^w,\alpha,\lambda) \tag{65}$$

by Corollary 4.49. By Corollary 4.48 we have

$$T^w(K,\alpha,\lambda) = e^{-|\alpha|C}T(Z(K),\alpha,\lambda). \tag{66}$$

We must now relate $T(Z(K),\alpha,\lambda)$ and $T(Z(K)_\rho,\alpha,\lambda)$. To do this note that for an α-Chebyshev polynomial in the direction λ, say p, we have

$$\frac{1}{\deg p}\log\frac{|p(z)|}{\|p\|_{Z(K)}} \leq V_{Z(K)}(z).$$

Taking Robin functions of both sides;

$$\frac{1}{\deg p}\log\frac{|\hat{p}(z)|}{\|p\|_{Z(K)}} \leq \rho_{Z(K)}(z).$$

For any $z \in Z(K)_\rho$ we observe that

$$|\hat{p}(z)|^{1/\deg p} \leq \|p\|_{Z(K)}^{1/\deg p} = T(Z(K), \alpha, \lambda).$$

Since $\textsc{lt}(\hat{p}) = \textsc{lt}(p)$ it follows that $\hat{p}$ is a competitor for the α-Chebyshev polynomial in the direciton λ of $Z(K)_\rho$. Hence we conclude $T(Z(K)_\rho, \alpha, \lambda) \leq T(Z(K), \alpha, \lambda)$. This implies for any $\theta \in \Sigma^0$ and direction λ that

$$\tau(Z(K)_\rho, \theta, \lambda) \leq \tau(Z(K), \theta, \lambda). \tag{67}$$

Suppose that the inequality were strict for some θ and λ. Since τ is log-convex on θ (by Lemma 1.140) there exists some open subset $S \subset \Sigma^0$ such that the inequality is strict for all $\theta \in S$. By Lemma 4.51 we know that $d(Z(K)) = d(Z(K)_\rho)$. Unwinding definitions we obtain

$$\begin{aligned}
\log d(Z(K)) &= \frac{1}{d \cdot vol(\Sigma)} \sum_{i=1} d \int_{\Sigma^0} \log \tau(Z(K), \theta, \lambda_i)\, d\sigma \\
&= \frac{1}{d \cdot vol(\Sigma)} \sum_{i=1} d \int_{\Sigma^0} \log \tau(Z(K)_\rho, \theta, \lambda_i)\, d\sigma \\
&= d(Z(K)_\rho).
\end{aligned}$$

Since $\int_S \log \tau(Z(K)_\rho, \theta, \lambda)\, d\sigma < \int_S \tau(Z(K), \theta, \lambda)\, d\sigma$ it must be true that

$$\int_{\Sigma^0 \setminus S} \log \tau(Z(K)_\rho, \theta, \lambda)\, d\sigma > \int_{\Sigma^0 \setminus S} \tau(Z(K), \theta, \lambda)\, d\sigma,$$

which contradicts equation (67). It follows that the inequality in equation (67) is never strict, so we have equality.

The result now follows by comparing equations (65) and (66) by observing

$$\tau^w(K, \theta, \lambda) = e^{-C}\tau(Z(K), \theta, \lambda) = e^{-C}\tau(Z(K)_\rho, \theta, \lambda) = e^{-C}e^{C}\tau(K^w_\rho, \theta, \lambda).$$

$\square$

4.6 Convergence of Fekete Polynomials to the Extremal Function

Theorem 4.56. *Suppose that $F \subset E$ and $d(E) = d(F)$. Then $\rho_E^* = \rho_F^*$.*

Proof. We start with the integral formula Proposition 3.52.

$$\frac{1}{d(2\pi)^{M-1}} \int_{\tilde{\mathcal{V}}^h} [\tilde{\rho}_E^* - \tilde{\rho}_F^*] \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_E^* + \omega)^j \wedge (dd^c \tilde{\rho}_F^* + \omega)^{M-j-1}$$
$$= \sum_{j=1}^{M} \frac{1}{d(2\pi)^{M-j}} \left(\int_{X_j} \rho_E^*(x,y)\,(dd^c \rho_E^*)^{M-j} - \int_{X_j} \rho_F^*(x,y)\,(dd^c \rho_F^*)^{M-j} \right).$$

By the Rumely formula this is equal to

$$\sum_{j=1}^{M} \frac{1}{d(2\pi)^{M-j}} \left(\int_{X_j} \rho_E^*(x,y)\,(dd^c \rho_E^*)^{M-j} - \int_{X_j} \rho_F^*(x,y)\,(dd^c \rho_F^*)^{M-j} \right)$$
$$= \frac{1}{d(2\pi)^{M-1}} \int_{\tilde{\mathcal{V}}^h} [\tilde{\rho}_E^* - \tilde{\rho}_{T_\mathcal{V}}^*] \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_E^* + \omega)^j \wedge (dd^c \tilde{\rho}_{T_\mathcal{V}}^* + \omega)^{M-j-1}$$
$$- \frac{1}{d(2\pi)^{M-1}} \int_{\tilde{\mathcal{V}}^h} [\tilde{\rho}_F^* - \tilde{\rho}_{T_\mathcal{V}}^*] \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_F^* + \omega)^j \wedge (dd^c \tilde{\rho}_{T_\mathcal{V}}^* + \omega)^{M-j-1}$$
$$= -\log d(E) + \log d(F) = 0.$$

Hence

$$\frac{1}{d(2\pi)^{M-1}} \int_{\tilde{\mathcal{V}}^h} [\tilde{\rho}_E^* - \tilde{\rho}_F^*] \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_E^* + \omega)^j \wedge (dd^c \tilde{\rho}_F^* + \omega)^{M-j-1} = 0.$$

Recognising that

$$\sum_{j=0}^{M-1} (dd^c \tilde{\rho}_E^* + \omega)^j \wedge (dd^c \tilde{\rho}_F^* + \omega)^{M-j-1} = \sum_{j=0}^{M-1} (dd^c \tilde{V}_E^*(0,z))^j \wedge (dd^c \tilde{V}_F^*(0,z))^{M-j-1}$$

we see that the integral as a multiple of the integral in Theorem 3.9. Since $V_F^* \geq V_E^*$ the hypothesis of the theorem is satisfied. Using that result we observe

$$\int_\mathcal{V} V_F^* (dd^c V_E^*)^M \leq \int_\mathcal{V} V_E^* (dd^c V_F^*)^M + 2\pi \int_{\tilde{\mathcal{V}}^h} [\tilde{\rho}_E^* - \tilde{\rho}_F^*] \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_E^* + \omega)^j \wedge (dd^c \tilde{\rho}_F^* + \omega)^{M-j-1}$$
$$= \int_\mathcal{V} V_E^* (dd^c V_F^*)^M.$$

We now argue as in Theorem 3.11. Since $(dd^c V_F^*)^M$ is supported in F and $V_E^* \equiv 0$ on $E \supset F$ we conclude that the RHS of the above inequality is 0. Hence as $V_F^* \geq 0$ it follows that $\int_\mathcal{V} V_F^* (dd^c V_E^*)^M = 0$. This in turn implies that $\{V_F^* > 0\}$ is a set of $(dd^c V_E^*)^M$-measure 0. Since $\{V_F^* > V_E^*\} \subset \{V_F^* > 0\}$ it follows that $\{V_F^* > V_E^*\}$ is a $(dd^c V_E^*)^M$-set of measure 0. By Lemma

3.10 it follows that $V_F^* \leq V_E^*$ which can only be true if there is equality. Since these functions are the same, it follows that their Robin functions are the same as claimed. $\square$

Corollary 4.57. *If $F \subset E$ satisfies $d(E) = d(F)$ then $E \backslash F$ is pluripolar.*

Theorem 4.58. *Let $K \subset \mathcal{V}$ be compact, regular and polynomially convex. Let w be a continuous admissible weight function on K. For each i, let $\{p_{j,i}\}_{j\in\mathbb{N}}$ be a sequence of polynomials such that for all θ there exists a subsequence $Y_{\theta,i} \in \mathbb{Z}_{\geq 0}$ with $p_{j,i} \in \{p : \mathrm{LT}(p) = \boldsymbol{v}_{\lambda_i} x^{\alpha_j}\}, j \in Y_{\theta,i}$ and*

$$\lim_{j\in Y_{\theta,i}} \|w^{\deg p_{j,i}} p_{j,i}\|_K^{1/\deg p_{j,i}} = \tau^w(K,\theta,\lambda_i).$$

Then

$$\max_{1\leq i\leq d}\left[\limsup_{j\to\infty} \frac{1}{\deg p_{j,i}} \log \frac{|p_{j,i}(z)|}{\|w^{\deg p_{j,i}} p_{j,i}\|_K}\right]^* = V_{K,Q}(z), z \notin K.$$

Proof. Let v denote the LHS of the above equality. By Theorem 3.11 it suffices to show that $\rho_v = \rho_K$. Observe that $\{\rho_v \leq 0\} \supset \{\rho_{K,Q}\}$, we seek the converse. We employ methods due to Bloom ([12], Theorem 4.1) to obtain this. It suffices to show that $Z = \{z \in \mathcal{V}^h : \rho_v(z) \leq 0\}$ is the interior of K_ρ^w. To this end, suppose that $z_0 \in \partial K_\rho^w \cap Z$ and let $B = \{\|z_0 - z\| \leq r : z \in \mathcal{V}^h\}$ with r chosen so that $B \subset Z$. By hypothesis, given $\theta \in \Sigma^0$ and $1 \leq i \leq d$ there exists $Y_{\theta,i} \subset \mathbb{Z}_{\geq 0}$ with desirable convergence properties. For $z \in K_\rho^w \cup B$ we have $\rho_v \leq 0$ so for such z we have for any i (as v is a maximum)

$$\limsup_{j\in Y_{\theta,i}} \frac{1}{\deg p_{j,i}} \log|\hat{p}_{j,i}(z)| \leq \limsup_{j\in Y_{\theta,i}} \frac{1}{\deg p_{j,i}} \log\|w^{\deg p_{j,i}} p_{j,i}\|_K = \log \tau^w(K,\theta,\lambda_i).$$

By Hartogs lemma (Theorem 1.7) we conclude that

$$\limsup_{j\in Y_{\theta,i}} \frac{1}{\deg p_{j,i}} \log\|\hat{p}_{j,i}\|_{K_\rho^w \cup B} \leq \log \tau^w(K,\theta,\lambda_i).$$

Hence $\tau(K_\rho^w \cup B, \theta, \lambda_i) \leq \tau^w(K,\theta,\lambda_i) = \tau(K_\rho^w, \theta, \lambda_i)$ where we have used Lemma 4.55 in the equality on the RHS. But by monotonicity of τ it follows that $\tau(K_\rho^w \cup B, \theta, \lambda_i) \geq \tau(K_\rho^w, \theta, \lambda_i)$. Hence $\tau(K_\rho^w, \theta, \lambda_i) = \tau(K_\rho^w \cup B, \theta, \lambda_i)$. This is true for all θ and $1 \leq i \leq d$ so $d(K_\rho^w) = d(K_\rho^w \cup B)$. By Theorem 4.56 it follows that $\rho_{K_\rho^w} = \rho_{K_\rho^w \cup B}$. But then $\rho_{K_\rho^w} \leq 0$ on B which implies $B \subset K_\rho^w$. But $B \backslash K_\rho^w \neq \varnothing$ by construction (since B is a ball centered at boundary point of K_ρ^w) so we have a contradiction. It follows that $\{\rho_v \leq 0\} \subset \{\rho_{K,Q}\}$ which provides the desired equality. $\square$

Corollary 4.59. *Let $K \subset \mathcal{V}$ be compact, regular and polynomially convex. Let w be a continuous admissible weight function on K. Let $\{p_j\}_{j\in\mathbb{N}}$ be a sequence of polynomials such that for all θ and $1 \leq i \leq d$ there exists a subsequence $Y_{\theta,i} \in \mathbb{Z}_{\geq 0}$ with $p_j \in \{p : \mathrm{LT}(p) = \boldsymbol{v}_{\lambda_i} x^{\alpha_j}\}, j \in Y_{\theta,i}$ for* *all $j \in Y_{\theta,i}$ and*

$$\lim_{j\in Y_{\theta,i}} \|w^{\deg p_j} p_j\|_K^{1/\deg p_j} = \tau^w(K,\theta,\lambda_i).$$

Then

$$\left[\limsup_{j\to\infty} \frac{1}{\deg p_j} \log \frac{|p_j(z)|}{\|w^{\deg p_j} p_j\|_K}\right]^* = V_{K,Q}(z), z \notin K.$$

Proof. If v denotes the function on the LHS then it is clear that $v \leq V_{K,Q}$; we seek the converse. Suppose we have a sequence which satisfies the hypothesis and denote this sequence S. Define $S_i := \{p \in S : \text{LT}(p) = \mathbf{v}_{\lambda_i} x^\alpha\}$. Then each S_i satisfies the hypothesis of Theorem 4.58 so

$$\max_{1\leq i\leq d} \left[\limsup_{\substack{j\to\infty \\ p_j\in S_i}} \frac{1}{\deg p_j} \log \frac{|p(z)|}{\|w^{\deg p_j} p\|_K}\right]^* = V_{K,Q}(z).$$

Let u denote the function on the LHS. Since S contains each S_i it follows that $u \leq v$ and since $u = V_{K,Q}$ we have shown the converse and the proof is complete. □

Remark 4.60. The proof of Theorem 4.58 works for Corollary 4.59, so could alternatively be proven directly via that method.

With the formula in Corollary 4.59 we can solve the problem of convergence of Fekete polynomials to the extremal function. The proof given of the $\mathbb{C}^N$ case by Bloom in [12] invokes Lagrange interpolation theory. However this is needlessly complex and standard arguments based on those given by Zakharjuta [54] suffice.

Theorem 4.61. *Let $K \subset \mathcal{V}$ be compact, regular and polynomially convex. Let w be a continuous admissible weight function on K. Recalling the notation from Definition 4.43, we define the jth Fekete polynomial to be*

$$F_j(z) = \frac{VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j, z)}{VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j)}$$

where $\{\zeta_1, ..., \zeta_j\}$ are a Fekete j-set for K (where the elements may change as j changes). Then for $z \in \mathcal{V}\backslash K$ we have

$$\left[\limsup_{j\to\infty} \frac{1}{\deg F_j} \log |F_j(z)|\right]^* = V_{K,Q}(z). \tag{68}$$

Proof. Write $\hat{\zeta}_i = (\zeta_1, ..., \zeta_{i-1}, \zeta_{i+1}, ..., \zeta_j)$. Observe that

$$\begin{aligned}
|VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j, z)| &= \left| \det \begin{pmatrix} \mathbf{e}_1(\zeta_1) & ... & \mathbf{e}_1(\zeta_j) & \mathbf{e}_1(z) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_{j+1}(\zeta_1) & ... & \mathbf{e}_{j+1}(\zeta_j) & \mathbf{e}_{j+1}(z) \end{pmatrix} w(\zeta_1)^{\alpha(j+1)} ... w(z)^{\alpha(j+1)} \right| \\
&= \left| \mathbf{e}_{j+1}(z) VDM(\zeta_1, ..., \zeta_j) - \sum_{i=1}^{j} c_i \mathbf{e}_i(z) \right|.
\end{aligned}$$

It follows then that if $\mathbf{e}_{j+1}(z) = x^\alpha \mathbf{v}_\lambda(z)$ then F_j is a competitor for the weighted $\alpha(j+1)$-Chebyshev constant in the direction λ. That is,

$$T(K, \alpha(j+1), \lambda)^{|\alpha(j+1)|} \leq \|w^{|\alpha(j+1)|} F_j\|_K. \tag{69}$$

Suppose that $t = e_{j+1}(z) + \sum_{i=1}^{j} c_i e_i(z)$ is a weighted α Chebyshev polynomial in the direction λ. Let $\alpha(j+1) = \text{multideg}(e_j)$. Then observe that

$$\begin{aligned}
&|VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j, z)| \\
&= \left| \det \begin{pmatrix} \mathbf{e}_1(\zeta_1) & ... & \mathbf{e}_1(\zeta_j) & \mathbf{e}_1(z) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{e}_{j+1}(\zeta_1) & ... & \mathbf{e}_{j+1}(\zeta_j) & \mathbf{e}_{j+1}(z) \end{pmatrix} w(\zeta_1)^{|\alpha(j+1)|} ... w(z)^{|\alpha(j+1)|} \right| \\
&= \left| \det \begin{pmatrix} \mathbf{e}_1(\zeta_1) & ... & \mathbf{e}_1(\zeta_j) & \mathbf{e}_1(z) \\ \vdots & \ddots & \vdots & \vdots \\ t(\zeta_1) & ... & t(\zeta_j) & t(z) \end{pmatrix} w(\zeta_1)^{|\alpha(j+1)|} ... w(z)^{|\alpha(j+1)|} \right| \\
&= \left| \det \begin{pmatrix} w(\zeta_1)^{|\alpha(j+1)|} \mathbf{e}_1(\zeta_1) & ... & w(\zeta_j)^{|\alpha(j+1)|} \mathbf{e}_1(\zeta_j) & w(z)^{|\alpha(j+1)|} \mathbf{e}_1(z) \\ \vdots & \ddots & \vdots & \vdots \\ w(\zeta_1)^{|\alpha(j+1)|} t(\zeta_1) & ... & w(\zeta_j)^{|\alpha(j+1)|} t(\zeta_j) & w(z)^{|\alpha(j+1)|} t(z) \end{pmatrix} \right|
\end{aligned}$$

where we have added c_1 times the first row to the $j+1$st row, ..., and c_j times the jth row to the $j+1$st row and used the fact that determinants are unchanged under such operations. Doing a cofactor expansion along the bottom row and using the triangle inequality we have

$$\begin{aligned}
|VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j, z)| \leq & |w(z)^{|\alpha(j+1)|} t(z)| |VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j) w(\zeta_1) ... w(\zeta_j) \\
& + \sum_{i=1}^{j} |w(\zeta_i)^{|\alpha(j+1)|} t(\zeta_i)| |VDM_{\mathbb{C}[\mathcal{V}]}(\hat{\zeta}_i, z) w(\hat{\zeta}_i)|.
\end{aligned} \tag{70}$$

where $w(\hat{\zeta}_i) = w(\zeta_1) ... w(\zeta_{i-1}) w(\zeta_{i+1}) ... w(\zeta_j) w(z)$. Observe the following for $z \in K$;

(i) $|w^{|\alpha(j+1)|} t(\zeta_j)| \leq \|w^{|\alpha(j+1)|} t\|_K = T^w(K, \alpha, \lambda)^{|\alpha(j+1)|}$

(ii) $|VDM_{\mathbb{C}[\mathcal{V}]}(\hat{\zeta}_i, z)| \leq |VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j)$ since $\zeta_1, ..., \zeta_j$ is a j-Fekete set for K.

(iii) There is a constant C such that $\|w(\hat{\zeta}_i)\|_K \leq C$ for all i and $|w(\zeta_1)...w(\zeta_j)| \leq C$.

Using all these estimates in equation (70) we obtain

$$|VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j, z)| \leq C(j+1)T^w(K, \alpha(j+1), \lambda)^{|\alpha(j+1)|}|VDM_{\mathbb{C}[\mathcal{V}]}(\zeta_1, ..., \zeta_j)|. \tag{71}$$

From equations (69) and (71) we deduce that

$$T^w(K, \alpha(j+1), \lambda)^{|\alpha(j+1)|} \leq \|w^{|\alpha(j+1)|}F_j\|_K \leq C(j+1)T^w(K, \alpha(j+1), \lambda)^{|\alpha(j+1)|}.$$

This estimate shows that given θ, λ and a subsequence Y_θ of $\{F_j\}$ such that $\alpha(j+1)/|\alpha(j+1)| \to \theta$ and $\textsc{lt}(F_j) = x^\beta \mathbf{v}_\lambda$ we have

$$\lim_{j \in Y_\theta} T^w(K, \alpha(j+1), \lambda) \leq \lim_{j \in Y_\theta} \|w^{\alpha(j+1)}F_j\|_K^{1/|\alpha(j+1)|} \leq \lim_{j \in Y_\theta} (C(j+1))^{1/|\alpha(j+1)|}T^w(K, \alpha(j+1), \lambda)$$
$$\tau^w(K, \theta, \lambda) = \lim_{j \in Y_\theta} \|w^{\alpha(j+1)}F_j\|_K^{1/|\alpha(j+1)|} = \tau^w(K, \theta, \lambda).$$

Hence the sequence $\{F_j\}$ satisfies the hypothesis of Corollary 4.59 and the result follows. □

Remark 4.62. One might expect a factor of $\frac{1}{d}$ or d to appear in the formula (68) since we saw this factor turn up in other places where the transfinite diameter and the extremal function are related. However, because Fekete polynomials are 'normalised' by dividing through by a Vandermonde determinant any factor that would be incorporated is canceled out.

4.7 Unions of Sets on Different Varieties

For this section we assume that $\mathcal{V}_1 = \{p_i(z) = 0 : 1 \leq i \leq m_1\} \subset \mathbb{C}^N$ and $\mathcal{V}_2 = \{q_j(z) = 0 : 1 \leq j \leq m_2\} \subset \mathbb{C}^N$ are smooth algebraic varieties satisfying the standard hypothesis with d_1 and d_2 directions respectively. We assume that there are no overlap in directions, that is if λ_i are the directions for $\mathcal{V}_1$ and μ_j are the directions for $\mathcal{V}_2$ then $\lambda_i \neq \mu_j$ for all i, j. We want to study the variety

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 = \{p_i(z)q_j(z) = 0 : 1 \leq i \leq m_1, 1 \leq j \leq m_2\}.$$

This variety is singular at all points $z \in \mathcal{V}_1 \cap \mathcal{V}_2$, hence $\mathcal{V}^{sing}$ is an algebraic subvariety of both $\mathcal{V}_1$ and $\mathcal{V}_2$. By Demailly [24] Monge-Ampère is locally bounded near $\mathcal{V}^{sing}$ (as $\mathcal{V}^{sing}$ is pluripolar) and we may use the techniques discussed in Section 2.8 to study this case.

Theorem 4.63. *Suppose that $E \subset \mathcal{V}_1, F \subset \mathcal{V}_2$ are compact sets and w_E, w_F are admissible* *weight functions on E and F respectively. If* $w = \begin{cases} w_E(z), & z \in E, \\ w_F(z), & z \in F, \end{cases}$ *then*

$$V_{E\cup F,Q}(z) = \begin{cases} V_{E,Q_E}(z), & z \in \mathcal{V}_1 \backslash \mathcal{V}^{sing}, \\ V_{F,Q_F}(z), & z \in \mathcal{V}_2 \backslash \mathcal{V}^{sing}, \\ \max\{V_{E,Q_E}(z), V_{F,Q_F}(z)\}, & z \in \mathcal{V}^{sing}. \end{cases}$$

Proof. Suppose that $\eta : \hat{\mathcal{V}} \to \mathcal{V}$ is a desingularisation of $\mathcal{V}$. Let $\eta^{-1}(\mathcal{V}_1) = \hat{\mathcal{V}}_1$ and $\eta^{-1}(\mathcal{V}_2) = \hat{\mathcal{V}}_2$ and observe that $\hat{\mathcal{V}}_1 \cup \hat{\mathcal{V}}_2 = \hat{\mathcal{V}}, \hat{\mathcal{V}}_1 \cap \hat{\mathcal{V}}_2 = \varnothing$. Since being plurisubharmonic is a local property and $\hat{\mathcal{V}}_1$ and $\hat{\mathcal{V}}_2$ are disjoint it follows that $u \in PSH(\hat{\mathcal{V}})$ induces a psh function u_1 on $\mathcal{V}_1$ (by restriction and usc regularisation) and u_2 on $\mathcal{V}_2$.
By Lemma 2.32 there is a function $\widehat{\log} \in PSH(\hat{\mathcal{V}})$ such that $\log^+ \|z\| = \max_{x=\eta^{-1}(z)} \widehat{\log}^+(x)$. It follows then that the associated function to $u \in w\mathcal{L}(\mathcal{V})$ guaranteed by Lemma 2.32, $\hat{u}$, satisfies $\hat{u}(x) \leq \widehat{\log}^+(x) + \alpha$ for some $\alpha \in \mathbb{R}$. Write

$$\hat{\mathcal{L}}(\hat{\mathcal{V}}) := \{u \in PSH(\hat{\mathcal{V}}) : u(x) \leq \widehat{\log}^+(x) + \alpha, \text{ for some } \alpha \in \mathbb{R}\}.$$

Write $\hat{V}_{\hat{K},\hat{Q}} := \sup\{u \in \widehat{\mathcal{L}}(\hat{\mathcal{V}}) : u(x) \leq \hat{Q}(x), x \in K\}$. Then it follows that for any $K \subset \mathcal{V}$ with admissible weight w_K and associated set $\hat{K} \subset \hat{\mathcal{V}}$ with associated admissible weight $\hat{w}_K$ that

$$V_{K,Q_K}(z) = \max_{x=\eta^{-1}(z)} \hat{V}_{\hat{K},\hat{Q}_K}(x).$$

From the fact that $\hat{\mathcal{V}}_1$ and $\hat{\mathcal{V}}_2$ are disjoint, we can find $\hat{V}_{\widehat{E\cup F},\hat{Q}}$ by finding the supremum on $\hat{\mathcal{V}}_1$ and $\hat{\mathcal{V}}_2$ separately, that is

$$\hat{V}_{\widehat{E\cup F},\hat{Q}}(x) = \begin{cases} \hat{V}_{\hat{E},\hat{Q}_E}(x), & x \in \hat{\mathcal{V}}_1, \\ \hat{V}_{\hat{F},\hat{Q}_F}(x), & x \in \hat{\mathcal{V}}_2. \end{cases}$$

Pushing this forward to $\mathcal{V}$ we deduce

$$
\begin{aligned}
V_{E\cup F,Q}(z) &= \max_{x=\eta^{-1}(z)} \hat{V}_{\widehat{E\cup F},\hat{Q}}(x) \\
&= \max_{x=\eta^{-1}(z)} \begin{cases} \hat{V}_{\hat{E},\hat{Q}_E}(x), & x \in \hat{\mathcal{V}}_1, \\ \hat{V}_{\hat{F},\hat{Q}_F}(x), & x \in \hat{\mathcal{V}}_2. \end{cases} \\
&= \begin{cases} V_{E,Q_E}(z), & z \in \mathcal{V}_1 \backslash \mathcal{V}^{sing}, \\ V_{F,Q_F}(z), & z \in \mathcal{V}_2 \backslash \mathcal{V}^{sing}, \\ \max\{V_{E,Q_E}(z), V_{F,Q_F}(z)\}, & z \in \mathcal{V}^{sing}. \end{cases}
\end{aligned}
$$

□

Remark 4.64. Writing V to be the claimed weighted extremal function for $E \cup F$, we have the following observation;

$$
\begin{aligned}
\int_{\mathcal{V}} (dd^c V(z)^*)^M &= \int_{\mathcal{V}_1 \backslash \mathcal{V}^{sing}} (dd^c V(z)^*)^M + \int_{\mathcal{V}_2 \backslash \mathcal{V}^{sing}} (dd^c V(z)^*)^M + \int_{\mathcal{V}^{sing}} (dd^c V(z)^*)^M \\
&= \int_{\mathcal{V}_1 \backslash \mathcal{V}^{sing}} (dd^c V_E(z)^*)^M + \int_{\mathcal{V}_2 \backslash \mathcal{V}^{sing}} (dd^c V_F(z)^*)^M + 0 \\
&= \int_E (dd^c V_E(z)^*)^M + \int_F (dd^c V_F(z)^*)^M.
\end{aligned}
$$

That is, $(dd^c V^*)^M$ is (compactly) supported on $E \cup F$. A similar argument shows that $V = Q$ q.e. on $E \cup F$. From these observations one can deduce the equality of the regularisation of V and $V_{E\cup F,Q}$ which offers an alternate method to proving the result.

Corollary 4.65. *With setup as in Theorem 4.63,*

$$
\rho_{E\cup F} = \begin{cases} \rho_E(z), & z \in \mathcal{V}_1^h \backslash \mathcal{V}^{h,sing} \\ \rho_F(z), & z \in \mathcal{V}_2^h \backslash \mathcal{V}^{h,sing} \\ \max\{\rho_E(z), \rho_F(z)\}, & z \in \mathcal{V}^{h,sing}. \end{cases}
$$

We need a version of Rumely's formula for varieties of the form $\mathcal{V}_1 \cup \mathcal{V}_2$ which has distinct intersections at infinity. Careful observation of the arguments shows that the only claims that need to be verified are the two relations involving the Monge-Ampère energy bracket. The other arguments carry through without modification. This is captured in the following result.

Proposition 4.66. *Suppose that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ satisfies the standard hypothesis except for being smooth. Without loss of generality (e.g. by a linear transformation if necessary), assume that $T_{\mathcal{V}} \cap \mathcal{V} \subset \mathcal{V}^{reg}$. Let $E \subset \mathcal{V}_1$ and $F \subset \mathcal{V}_2$. Let $\nu = \frac{1}{d(2\pi)^M}(dd^c V_{T_{\mathcal{V}}})^M$ and S_k an $L^2(\nu)$-orthonormal basis for $\mathbb{C}[\mathcal{V}]$. Then* *(i)* $\lim_{k\to\infty} \frac{1}{kN_k} \log \| \det[S_k] \|_{L^\infty(E\cup F)} = \frac{\mathcal{E}(V_{T_\mathcal{V}}, V^*_{E\cup F})}{M+1}$

(ii) $\mathcal{E}(V_{T_\mathcal{V}}, V^*_{E\cup F}) = 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_{T_\mathcal{V}} - \tilde{\rho}_{E\cup F}) \wedge \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_{T_\mathcal{V}})^j \wedge (dd^c \tilde{\rho}_{E\cup F})^{M-j-1}.$

Proof. For (i) observe that

$$\begin{aligned}
\mathcal{E}(V_{T_\mathcal{V}}, V^*_{E\cup F}) &= \int_\mathcal{V} (V_{T_\mathcal{V}} - V^*_{E\cup F}) \sum_{j=0}^{M} (dd^c V_{T_\mathcal{V}})^j \wedge (dd^c V^*_{E\cup F})^{M-j} \\
&= \int_{\mathcal{V}\setminus\mathcal{V}^{sing}} (V_{T_\mathcal{V}} - V^*_{E\cup F}) \sum_{j=0}^{M} (dd^c V_{T_\mathcal{V}})^j \wedge (dd^c V^*_{E\cup F})^{M-j},
\end{aligned}$$

since the currents in the integral place no mass on pluripolar sets. Now the result follows from applying Corollary A [7] (i.e. Theorem 3.49 in full generality) to the complex manifold $\mathcal{V}^{reg} = \mathcal{V}\setminus\mathcal{V}^{sing}$. For (ii) observe that

$$\begin{aligned}
\mathcal{E}(V_{T_\mathcal{V}}, V^*_{E\cup F}) &= \int_\mathcal{V} (V_{T_\mathcal{V}} - V^*_{E\cup F}) \sum_{j=0}^{M} (dd^c V_{T_\mathcal{V}})^j \wedge (dd^c V^*_{E\cup F})^{M-j} && (72) \\
&= \left(\int_{\mathcal{V}_1} + \int_{\mathcal{V}_2} \right) (V_{T_\mathcal{V}} - V^*_{E\cup F}) \sum_{j=0}^{M} (dd^c V_{T_\mathcal{V}})^j \wedge (dd^c V^*_{E\cup F})^{M-j} && (73) \\
&= \int_{\mathcal{V}_1} (V_{T_\mathcal{V}\cap E} - V^*_E) \sum_{j=0}^{M} (dd^c V_{T_\mathcal{V}\cap E})^j \wedge (dd^c V^*_E)^{M-j} \\
&\quad + \int_{\mathcal{V}_2} (V_{T_\mathcal{V}\cap F} - V^*_F) \sum_{j=0}^{M} (dd^c V_{T_\mathcal{V}\cap F})^j \wedge (dd^c V^*_F)^{M-j} && (74) \\
&= 2\pi \int_{\tilde{\mathcal{V}}_1^h} (\tilde{\rho}_{T_\mathcal{V}\cap E} - \tilde{\rho}_E) \wedge \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_{T_\mathcal{V}\cap E})^j \wedge (dd^c \tilde{\rho}_E)^{M-j-1} \\
&\quad + 2\pi \int_{\tilde{\mathcal{V}}_2^h} (\tilde{\rho}_{T_\mathcal{V}\cap F} - \tilde{\rho}_F) \wedge \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_{T_\mathcal{V}\cap F})^j \wedge (dd^c \tilde{\rho}_F)^{M-j-1} && (75) \\
&= 2\pi \int_{\tilde{\mathcal{V}}^h} (\tilde{\rho}_{T_\mathcal{V}} - \tilde{\rho}_{E\cup F}) \wedge \sum_{j=0}^{M-1} (dd^c \tilde{\rho}_{T_\mathcal{V}})^j \wedge (dd^c \tilde{\rho}_{E\cup F})^{M-j-1}, && (76)
\end{aligned}$$

where we have used the following logic to pass from line to line; from (72) to (73) we have used the pluripolarity of $\mathcal{V}^{sing}$, from (73) to (74) we have used Theorem 4.63, from (74) to (75) we have used the smooth result (Theorem 3.43), from (75) to (76) we have used Corollary 4.65. This completes the proof. □

Corollary 4.67. *Rumely's formula for varieties (Theorem 3.53) is valid under the weaker hypothesis that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are smooth varieties and moreover that $\mathcal{V}$ satisfies the standard hypothesis except for the smooth condition.*

Theorem 4.68. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be smooth M-dimensional varieties with d_1 and d_2 branches respectively. Moreover suppose that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is M-dimensional and has distinct intersections at infinity. Suppose that $E \subset \mathcal{V}_1, F \subset \mathcal{V}_2$ are compact, polynomially convex and non-pluripolar (with respect to $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively). Then $d(E \cup F)^{d_1+d_2} = d(E)^{d_1} d(F)^{d_2}$.*

Proof. From the Rumely formula with $X_j = \mathcal{V}^h \cap \{z_1 = ... = z_{j-1} = 0, z_j = 1\}$, $X_{j,1} = \mathcal{V}_1^h \cap X_j$ and $X_{j,2} = \mathcal{V}_2^h \cap X_j$;

$$\begin{aligned}
-\log d(E \cup F) &= \frac{1}{M(d_1+d_2)} \sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_{E\cup F}(z)(dd^c \rho_{E\cup F})^{M-j} \\
&= \frac{1}{M(d_1+d_2)} \sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \left[\int_{X_{j,1}} \rho_{E\cup F}(z)(dd^c \rho_{E\cup F})^{M-j} + \int_{X_{j,2}} \rho_{E\cup F}(z)(dd^c \rho_{E\cup F})^{M-j} \right].
\end{aligned}$$

From Corollary 4.65 we have

$$\rho_{E\cup F} = \begin{cases} \rho_E(z), & z \in \mathcal{V}_1^h \backslash \mathcal{V}^{h,sing}, \\ \rho_F(z), & z \in \mathcal{V}_2^h \backslash \mathcal{V}^{h,sing}, \\ \max\{\rho_E(z), \rho_F(z)\}, & z \in \mathcal{V}^{h,sing}. \end{cases}$$

Since no mass is placed on the singular part of $\mathcal{V}^h$ (since ρ_E and ρ_F are locally bounded away from 0) we have

$$\begin{aligned}
-M(d_1+d_2)\log d(E \cup F) &= \sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \left[\int_{X_{j,1}} \rho_E(z)(dd^c \rho_E)^{M-j} + \int_{X_{j,2}} \rho_F(z)(dd^c \rho_F)^{M-j} \right] \\
&= -Md_1 \log d(E) - Md_2 \log d(F).
\end{aligned}$$

Hence $d(E \cup F)^{d_1+d_2} = d(E)^{d_1} d(F)^{d_2}$ as claimed. □

Proposition 4.69. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be smooth M_1-dimensional and M_2-dimensional varieties respectively with d_1 and d_2 branches respectively. Suppose further that $M_1 > M_2$ so that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ has dimension M_1. Suppose that $E \subset \mathcal{V}_1$ and $F \subset \mathcal{V}_2$ are compact, polynomially convex and non-pluripolar (with respect to $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively). Then*

$$d(E \cup F) = d(E)$$

where the transfinite diameter on the LHS is understood in the context of Berman-Boucksom (Theorem 3.49).

Proof. In this case $\mathcal{V}_2$ is a pluripolar subset of $\mathcal{V}$ so it suffices to study the non-pluripolar part of $\mathcal{V}$, i.e. $\mathcal{V}_1$. The result follows from this observation. □

Remark 4.70. This Theorem 4.68 captures the spirit of Proposition 3.5 from [2] due to Baleikorocau and Ma'u which showed that for algebraic curves $\mathcal{V}_1, \mathcal{V}_2$, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ and $E \subset V_1, F \subset \mathcal{V}_2$ we have

$$\tau(E \subset \mathcal{V}_1, \lambda) = \tau(E \subset \mathcal{V}, \lambda) = \tau(E \cup F \subset \mathcal{V}, \lambda).$$

where the $E \subset \mathcal{V}$ argument denotes the monomial basis from which the Chebyshev polynomials are taken from. Taking the geometric average over λ recovers the transfinite diameter and the result from Theorem 4.68. The following is a direct generalisation of that result. It also shows that we can drop the polynomial convexity hypothesis.

Theorem 4.71. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be smooth M-dimensional algebraic varieties such that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ has distinct intersections with infinity. Suppose that $E \subset \mathcal{V}_1, F \subset \mathcal{V}_2$ are non-pluripolar compact sets, λ is a direction of $\mathcal{V}_1$ and $E \cup F \subset \mathcal{V}$. Then $\tau(E \subset \mathcal{V}_1, \lambda) = \tau(E \subset \mathcal{V}, \lambda) = \tau(E \cup F \subset \mathcal{V}, \lambda)$ for all directions λ.*

Proof. Let $P(\mathcal{V}_1, E, \alpha, \lambda) \subset \mathbb{C}[\mathcal{V}_1]$ denote the set of competitor Chebyshev polynomials of degree α for the set $E \subset \mathcal{V}_1$ in the direction λ and similar for $P(\mathcal{V}, E \cup F, \alpha, \lambda) \subset \mathbb{C}[\mathcal{V}]$. If $I(\mathcal{V})$ is the ideal for the variety $\mathcal{V}$ observe that

$$\mathbb{C}[\mathcal{V}] = \mathbb{C}[z] \backslash I(\mathcal{V}) = \mathbb{C}[z] \backslash I(\mathcal{V}_1 \cup \mathcal{V}_2) = \mathbb{C}[z] \backslash (I(\mathcal{V}_1) I(\mathcal{V}_2)).$$

Note that $I(\mathcal{V}_1)I(\mathcal{V}_2) \subset I(\mathcal{V}_1)$ so $\mathbb{C}[z]\backslash(I(\mathcal{V}_1)I(\mathcal{V}_2)) \supset \mathbb{C}[z]\backslash I(\mathcal{V}_1)$. Hence $P(\mathcal{V}_1, E, \alpha, \lambda) \subset P(\mathcal{V}, E, \alpha, \lambda)$ so $\tau(E \subset \mathcal{V}_1, \theta, \lambda) \geq \tau(E \subset \mathcal{V}, \theta, \lambda)$. Integrating over $\theta \in \Sigma^0$ yields $\tau(E \subset \mathcal{V}_1, \lambda) \geq \tau(E \subset \mathcal{V}, \lambda)$.

Suppose that v_i is the polynomial from Lemma 1.134 which is associated to the direction λ. Choose a polynomial $g \in I(\mathcal{V}_2) \backslash I(\mathcal{V}_1) I(\mathcal{V}_2)$ which satisfies $\|g\|_E > 0$ and $\textsc{lt}(g) = v_i x^\beta$ and suppose $t \in P(\mathcal{V}, E, \alpha, \lambda)$ is a Chebyshev polynomial. Then (using the notation of Theorem 1.113) $\textsc{lt}([g \cdot t]) = C v_i x^{\alpha+\gamma}$ for some multi-index γ and constant C. It follows then that

$$CT(E \cup F \subset \mathcal{V}, \alpha + \gamma, \lambda)^{|\alpha+\gamma|} \leq \|[g \cdot t]\|_{E \cup F} = \|[g \cdot t]\|_E \leq \|g\|_E \|t\|_E = \|g\|_E T(E \subset \mathcal{V}, \alpha, \lambda)^{|\alpha|}.$$

Since g (and hence $\|g\|_E$, C and γ) are fixed, taking $|\alpha|$th roots and letting $\alpha/|\alpha| \to \theta$ as $|\alpha| \to \infty$ shows that $\tau(E \cup F \subset \mathcal{V}, \theta, \lambda) \leq \tau(E \subset \mathcal{V}, \theta, \lambda)$. Hence the inequality remains after integrating over $\theta \in \Sigma^0$ i.e. $\tau(E \cup F \subset \mathcal{V}, \lambda) \leq \tau(E \subset \mathcal{V}, \lambda)$.

Finally, if $t \in P(\mathcal{V}, E \cup F, \alpha, \lambda)$ is a Chebyshev polynomial then

$$T(E \cup F \subset \mathcal{V}, \alpha, \lambda)^{|\alpha|} = \|t\|_{E \cup F} \geq \|t\|_E \geq T(E \subset \mathcal{V}_1, \alpha, \lambda)^{|\alpha|}.$$

The same argument used in the previous cases yields $\tau(E \cup F \subset \mathcal{V}, \lambda) \geq \tau(E \subset \mathcal{V}_1, \lambda)$. Hence we have

$$\tau(E \subset \mathcal{V}_1, \lambda) \leq \tau(E \subset \mathcal{V}, \lambda) \leq \tau(E \cup F \subset \mathcal{V}, \lambda) \leq \tau(E \subset \mathcal{V}_1, \lambda)$$

which proves the result. □

Remark 4.72. Weighted versions of the previous results can be easily obtained using Corollaries 4.48 and 4.50.

5 Further Research

5.1 Rumely Formula for Principal Chebyshev Constants

Suppose that $\mathcal{V}$ is a smooth algebraic variety of dimension M with d branches with unique intersection at infinity, let $K \subset \mathcal{V}$ be a compact set. We know that $d(K)$ is the geometric average of the principal Chebyshev constants $T(K, \lambda_i)$ so we may rewrite the Rumely formula to be

$$-\frac{1}{d}\sum_{i=1}^{d} \log \tau(K, \lambda_i) = \frac{1}{dM}\sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_K(z)\,(dd^c \rho_K)^M.$$

Given how nicely the directional Chebyshev constants behave in Section 4 one might expect a formula like the following to hold.

Conjecture 5.1. *Suppose that X_j^i is the ith branch of X_j and that $\lambda_i \in X_j^i$ for all j, i.*

$$-\log \tau(K, \lambda_i) = \frac{1}{M}\sum_{j=1}^{M} \frac{1}{(2\pi)^{M-j}} \int_{X_j} \rho_K(z)\,(dd^c \rho_K)^M.$$

Proving this conjecture is somewhat different to proving the transfinite diameter relation. The machinery developed by Berman-Boucksom ([7, 41]) obtains the classical result through an auxiliary concept of taking ratios of polynomial ball volumes. Their work showed that the transfinite diameter and the Monge-Ampère bracket were both asymptotically equal to these ball volumes (subject to a suitable 'ground level' energy i.e. the unit torus).

The important point here is that everything is set up to compare to the transfinite diameter; or rather the Vandermonde matrix formulation of the transfinite diameter. A principle Chebyshev constant does not naturally lend itself to a 'Vandermonde' type formulation so it becomes difficult to understand these constants within the Berman-Boucksom framework.

One approach which may be fruitful is to change the ground level energy from $V_{T_{\mathcal{V}}}$ to another set which in some way incorporates the directionality of the Chebyshev constant.

The other approach to proving Rumely's formula given by Rumely in [49] involves deriving the Rumely formula from the 'sectional capacity' which under appropriate conditions equals the transfinite diameter. Again, this is an approach which chooses to relate the Robin function directly to the transfinite diameter rather than the Chebyshev constant. The theoretical underpinnings of Rumely work is not something that we understand fully so cannot offer a starting point to attempting to prove the conjecture in this manner.

5.2 Robin Function Formulae when $\mathcal{V}^h$ Singular & Branched at Infinity

Recall that we restricted our focus to $\mathcal{V}$ having 'nice' behaviour at infinity in order to deduce properties of the Robin function; e.g. the Bedford-Taylor formula which was a critical component of the Rumely formula. It is reasonable to ask whether similar results hold for when $\mathcal{V}$ is badly behaved at infinity. It is sufficient to consider how the proof of Theorem 2.67 (the Bedford-Taylor formula) would proceed in this circumstance.

The strategy to prove this result was to take two branch cuts, study the respective projection, recover the result there then project back to $\mathcal{V}$. Suppose that $\mathcal{V}$ is singular at infinity. Recalling that the branches of $\mathcal{V}$ 'feed into' an irreducible component of $\mathcal{V}_{\uparrow}$ along $\{t = 0\}$ given a distinguished branch cut, it appears as if the arguments should go straight through on each irreducible component. The end formula for this case should look like the following

Conjecture 5.2. *Let X_j be the jth irreducible component of $\mathcal{V}_{\uparrow}$ along $\{t = 0\}$ and $\rho_{u,j}$ the Robin function defined on X_j. Suppose that there are k irreducible components. Then*

$$\int_{\mathcal{V}} (udd^c v - vdd^c u) \wedge T = 2\pi \sum_{j=1}^{k} \int_{\tilde{X}_j} (\tilde{\rho}^*_{u,j} - \tilde{\rho}^*_{v,j}) \wedge \tilde{T}_j.$$

When $\mathcal{V}$ is branched at infinity we observed that the Robin function could not be determined solely by considering projections, ultimately we had to max over Robin functions defined on branch points to get a sensible definition. The arguments used in Theorem 2.67 should still work, up to being aware of when to max Robin functions. However this looks like an involved process and we wonder if there is an easier way to go about it. Nevertheless, we expect a formula such as the following.

Conjecture 5.3. *Let X_j be the jth irreducible component of $\mathcal{V}_{\uparrow}$ along $\{t = 0\}$ and $\rho_{u,j}$ the Robin function defined on X_j. Suppose that there are k irreducible components. Then*

$$\int_{\mathcal{V}} (udd^c v - vdd^c u) \wedge T = 2\pi \sum_{j=1}^{k} \int_{\tilde{X}_j} o_\pi([z])(\tilde{\rho}^*_{u,j} - \tilde{\rho}^*_{v,j}) \wedge \tilde{T}_j,$$

where $o_\pi([z])$ is the branching order at $[z] \in \tilde{X}_j$.

The $o_\pi([z])$ term accounts for the possible branching at $[z] \in \tilde{X}_j \subset \tilde{\mathcal{V}}^h$. It effectively plays the role of accounting for the possible multiplicity arising from the branching. This conjecture agrees with the example computed in Section 2.7.

With a sensible Bedford-Taylor formula we imagine that the results of Section 3 can be approached in a similar manner, in particular recovering a Rumely type formula for this case.

5.3 Polynomial Convexity

This section is a half-answer to an extension problem for V_K on an algebraic variety. Interesting machinery was developed along the way but we eventually ran into a roadblock which we could not resolve. See the end of this section for details on this roadblock.

We open with a motivating example. For this we assume $\mathcal{A} \subset \mathbb{C}^N$ is a hypersurface (an algebraic variety of codimension 1) with (x, y) a Noether presentation for $\mathcal{A}$. If $\mathcal{A}$ is quadratic $\mathbb{C}[\mathcal{A}] = \text{span}\{x^j, yx^j : j \in \mathbb{N}\}$. Motivated by polynomial convexity in $\mathbb{C}^N$ we define the following;

Definition 5.4. *Let $K \subset \mathcal{A}$. We define the reduced polynomial hull of K (in $\mathbb{C}^N$) to be the set* $\hat{K}_{\mathcal{A}} := \{z \in \mathbb{C}^2 : |p(z)| \leq \|p\|_K, \forall p \in \mathbb{C}[\mathcal{A}]\}$.

Lemma 5.5. $V_K(z) = V_{\hat{K}_{\mathcal{A}}}$.

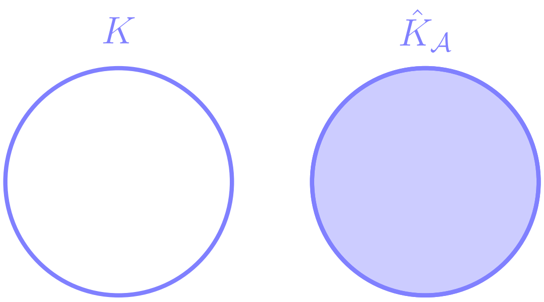
Proof. Recall that

$$V_K(z) = \sup \left\{ \frac{1}{\deg p} \log |p(z)| : \|p\|_K \leq 1, p \in \mathbb{C}[\mathcal{A}] \right\}.$$

But by definition if $p \in \mathbb{C}[\mathcal{A}]$ then $\|p\|_K \leq 1$ implies $\|p\|_{\hat{K}_{\mathcal{A}}} \leq 1$. It follows that $V_K(z) \leq 0$ on $\hat{K}_{\mathcal{A}}$ and hence by maximality $V_K(z) = V_{\hat{K}_{\mathcal{A}}}(z)$. □

If $\hat{K}_{\mathcal{A}}$ extends off of $\mathcal{A}$ then we have an explicit and natural extension of V_K to a function in $\mathcal{L}(\mathbb{C}^N)$. Hence we are interested in the question of which sets satisfy $\hat{K}_{\mathcal{A}} = K$, $\hat{K} = \hat{K}_{\mathcal{A}}$ or $\hat{K}_{\mathcal{A}} \cap \mathcal{A} \neq \hat{K}_{\mathcal{A}}$. This last case being of particular interest.

Example 5.6. Let $\mathcal{A} = \{z_1^2 + z_2^2 = 1\}$ and $K = \mathcal{A} \cap \mathbb{R} \times \mathbb{R} = \{(z_1, z_2) \in \mathcal{A} : z_1, z_2 \in \mathbb{R}\}$. This is just the circle in $\mathbb{R}^2$. We claim that $\hat{K}_{\mathcal{A}} = \hat{K}_{\text{convex}}$. First of all, note that $\mathbb{C}[\mathcal{A}]$ contains all linear functions in (z_1, z_2) and so necessarily $\hat{K}_{\mathcal{A}} \subset \hat{K}_{\text{convex}}$.



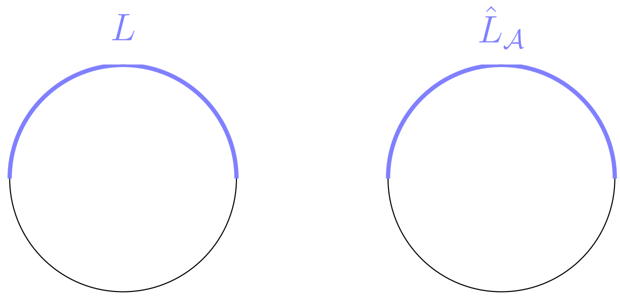
Suppose that $t \in (-1, 1)$ and consider the intersection $K_t = K \cap \{z_1 = t\}$. This will always give two points; $(t, \pm\sqrt{1 - t^2})$. Let $p \in \mathbb{C}[\mathcal{A}]$ and write $p = q_1(z_1) + z_2 q_2(z_1)$ where $q_1, q_2 \in \mathbb{C}[z_1]$. Note $p|_{K_t} = p(t, z_2)$ defines a one variable polynomial in z_2 (for fixed t) and so define $\tilde{p}(z_2) = p(t, z_2)$. Note that $\tilde{p}(z_2)$ is a linear function (since $q_1(t)$ and $q_2(t)$ are constant for fixed t). Under the $\sim$ map, $\tilde{K}_t$ is the z_2-projection of the set K_t, or $\tilde{K}_t = \{\pm\sqrt{1 - t^2}\}$. It follows that

$$\{z_2 \in \mathbb{C} : |\tilde{p}(z_2)| \leq \|p\|_K\} \supset \{z_2 \in \mathbb{C} : |\tilde{p}(z_2)| \leq \|\tilde{p}\|_{\tilde{K}_t}\} \supset (\hat{\tilde{K}}_t)_{\text{convex}}.$$

Since this is true for any polynomial in $\mathbb{C}[\mathcal{A}]$, it follows that the set $(\hat{K}_t)_{\text{convex}} \subset \hat{K}_{\mathcal{A}}$ for all $t \in (-1, 1)$. Since $\cup_{t\in[-1,1]}(K_t)_{\text{convex}} = \hat{K}_{\text{convex}}$, it follows that $\hat{K}_{\text{convex}} \subset \hat{K}_{\mathcal{A}} \subset \hat{K}_{\text{convex}}$.

Thus $\hat{K}_{\mathcal{A}} = \hat{K}_{\text{convex}}$ and so $\hat{K}_{\mathcal{A}}$ extends off of $\mathcal{A}$.

Example 5.7. Let $L = \mathcal{A} \cap \mathbb{R} \times \mathbb{R}_+ = \{(z_1, z_2) : z_1, z_2 \in \mathbb{R}, z_2 \geq 0\}$. We claim that $\hat{L}_{\mathcal{A}} = L$. Firstly note that the previous argument cannot be applied to this instance since the corresponding L_t sets would only intersect L once, and the convex hull of a single point is the same point. However, it is still true that $\hat{L}_{\mathcal{A}} \subset \hat{L}_{\text{convex}}$ so we need only show that any point in $\hat{L}_{\text{convex}} \backslash L$ is not in $\hat{L}_{\mathcal{A}}$.



Suppose that $w = (w_1, w_2) \in \hat{L}_{\text{convex}} \backslash L$. Consider the function $f(z) = e^{-az_1^2}(1 - z_2)$. Restricted to $\hat{L}_{\text{convex}}$, this function takes its maximum at $(0, 0)$ and minimum at $(0, 1)$ and restricted to L it has maximum at $(\pm 1, 0)$ and minimum at $(0, 0)$. The maximum at $(\pm 1, 0)$ has value e^{-a}. The function increases in the z_2 direction, so the points below the graph of $e^{-a} = e^{-az_1^2}(1 - z_2)$ satisfy $|f(z)| > \|f\|_L$. We solve

$$\begin{aligned} e^{-a} &= e^{-az_1^2}(1 - z_2) \\ \iff \quad -a &= -az_1^2 + \log(1 - z_2) \\ \iff \quad a &= \frac{\log(1 - z_2)}{z_1^2 - 1}. \end{aligned}$$

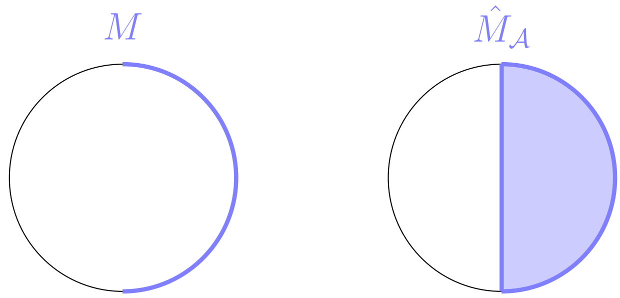
We note that on $\hat{L}_{\text{convex}} \backslash L$ that this equation for a is well defined. In particular, we can put in the point (w_1, w_2), find a using the formula above, and then choose an a slightly larger than this so ensure (w_1, w_2) falls below the graph. Of course, $f(z)$ is not a polynomial, but the Taylor polynomial approximation given by

$$P_n(z) = (z_2 - 1) \sum_{j=0}^{n} \frac{(-z_1)^{2j}}{j!}$$

converges uniformly to f on $\hat{L}_{\text{convex}}$ and $P_n(z) \in \mathbb{C}[\mathcal{A}]$ for all n. Thus given (w_1, w_2), we can find n such that $|P_n(w)| > \|P\|_L$. Thus $w \notin \hat{L}_{\mathcal{A}}$ and $L = \hat{L}_{\mathcal{A}}$.

This contrasts starkly with the next example.

Example 5.8. Let $M = \mathcal{A} \cap \mathbb{R}_+ \times \mathbb{R} = \{(z_1, z_2) : z_1, z_2 \in \mathbb{R}, z_1 \geq 0\}$. Then $\hat{M}_{\mathcal{A}} = \hat{M}_{\text{convex}}$ since the argument applied to the circle case (Example 5.6) can be used in this case.



Example 5.9. For this example take $\mathcal{A} := \{z_1^2 + z_2^2 = 0\}$, which is the curve consisting of the two lines $z_1 = z_2$ and $z_1 = -z_2$. Suppose that $K = \overline{B}_R(0) \cap \mathcal{A}$ for some $R > 0$. Using the argument from example 1 it follows that $\hat{K}_{\mathcal{A}} = \{z_2 = tz_1 : |t| \leq 1\} \cap \{|z_1| \leq R\}$.

Suppose now that $L \subset \{z_1 = z_2\}$. Note that the polynomial $z_1 - z_2 \in \mathbb{C}[\mathcal{A}]$ and $z_1 - z_2 = 0$ on L. It follows that $\hat{L}_{\mathcal{A}} \subset \mathcal{A}$ i.e. the reduced polynomial hull does not extend off of $\mathcal{A}$.

Lastly, consider M defined by $\{z \in \mathcal{A} : z_1, z_2 \in \mathbb{R}, z_2 \geq 0\}$. We can use the function $f(z) = z_2 e^{-z_1^2}$ and a similar argument to Example 5.7 to conclude that $\hat{M}_{\mathcal{A}} = M$.

These examples indicate that there is some interesting phenomena occurring here. In particular, the choice of Noether presentation for $\mathcal{A}$ can change the resulting reduced polynomial hull as seen in Examples 5.7 and 5.8. It follows that V_K (with K taken from Example 5.6) extends off of $\mathcal{A}$ and is equal to the extremal function of the real circle (which is known by Lundin's formula i.e. Theorem 5.4.6 [37] or see Burns-Levenberg-Ma'u [19]).

The primitive arguments used in the previous examples can be used to understand quadratics on a case by case basis. Higher degree curves are more difficult to understand because there is essentially no theory on quadratic, cubic, ... hulls of sets (only the convex and polynomial hulls are useful and hence studied). Hence we don't know if we have extensions for higher degree hypersurfaces.

Polynomial hulls in $\mathbb{C}^N$ are closely related to the study of uniform algebras (see Stout [52] or Gamelin [30] for this theory). Motivated by this, we lay the foundation for studying this problem in the context of uniform algebras and in doing so provide a complete description of the quadratic case. This essentially means building a uniform algebra theory on subspaces of $\mathbb{C}[z]$.

5.3.1 Restriction of $\mathbb{C}[z]$ to a Subspace

Our study is motivated by the algebraic curve case so we will primarily consider hypersurfaces of the form $\mathcal{V} = \{p(z) = 0\}$ with Noether presentation $z = (x_1, ..., x_{N-1}, y)$ so that $\mathbb{C}[z] = \mathbb{C}[x] + y\mathbb{C}[x] + ... + y^d\mathbb{C}[x]$ for some d. We remark that the construction given here is general enough to cover more complex cases, but the results are clearer in the hypersurface case.

Let A be the (commutative) algebra $\mathbb{C}[z]$ with identity 1. We are interested in studying the restriction of A to the subspace $S = \mathbb{C}[x] + y\mathbb{C}[x] + ... + y^d\mathbb{C}[x]$. A can be endowed with a norm and completed such that A becomes a Banach algebra. Precisely, a norm $\|.\|$ which satisfies $\|fg\| \leq \|f\|\|g\|$ for $f, g \in A$ and $\|1\| = 1$. For instance, such a norm could be the sup norm of polynomials over a compact set $K \subset \mathbb{C}^N$.

With this setup, S inherits a lot of structure from A. In particular, if $\|.\|$ is a norm on A then $\|.\|$ defines a norm on S. In our study S will always contain the multiplicative identity 1 and $\|1\| = 1$. Moreover, S can be completed with respect to this norm and become a complete subspace of A, again we denote this completion as S. This completion will be of great interest.

5.3.2 Ideals of a Subspace

Throughout this section we assume that $A = \mathbb{C}[z]$, not with any norm nor completion.

Definition 5.10. *We say J is an ideal of S if it is the restriction of an ideal I of A to S. We denote restriction of an ideal J to S by $J|_S := \{j \in J : j \in S\} = J \cap S$.*

Corollary 5.11. *Ideals of S are closed under multiplication by $\mathbb{C}[x]$.*

Corollary 5.12. *Ideals of S are subgroups of S (with respect to addition).*

Definition 5.13. *We say an ideal J of S is a maximal ideal if the only proper ideal containing J is S.*

Lemma 5.14. *For every maximal ideal J of S there is a maximal ideal $\tilde{J}$ of A which satisfies $\tilde{J}|_S = J$.*

Proof. Suppose that J is a maximal ideal in S. Then there is some $f \in S$ such that $f \notin J$. By definition of ideals in S, it follows there is an ideal J' of A such that $J'|_S = J$. Now J' is contained in some proper ideals, and in particular $J' = \cap_{J' \subset I_\alpha} I_\alpha$. Since $f \notin J'$ it follows that there exists I_α such that $f \notin I_\alpha$. Let $I_\alpha = \tilde{J}$ and claim that this is our desired maximal ideal i.e. that $\tilde{J}|_S = J$.

By construction of $\tilde{J}$ it follows that $J \subset \tilde{J}|_S$ so we only need to show the converse. Suppose that $g \in \tilde{J}|_S$ with $g \notin J$. Then $\tilde{J}|_S$ is an ideal of S which properly contains J. However since J is maximal in S we must have that $\tilde{J}|_S = S$. In particular, $f \in \tilde{J}|_S$ and hence $f \in \tilde{J}$ which is absurd. So by contradiction, the result is proven. □

Definition 5.15. *By $\langle f \rangle$ we mean the ideal generated by f using elements from A. By $\langle f \rangle_S$ we mean the ideal generated by f using elements from A and then restricted to S (the order of generating and then restricting is important!). The notation $\langle I \rangle$ where I is a set is used to denote the ideal generated by elements in that set (using elements from A). We will us the* *notation $\langle I, f\rangle$ to be the ideal generated by I and f (using elements from A) and $\langle I, f\rangle_S$ to be the restriction of $\langle I, f\rangle$ to S.*

Corollary 5.16. *If J is an ideal of S then $\langle J\rangle_S = J$.*

Proof. By definition of an ideal of S, there is some ideal J' such that $J = J'|_S$. Since $J \subset J'|_S$ it follows that $\langle J\rangle \subset \langle J'\rangle = J'$. In particular, $J \subset \langle J\rangle_S \subset J'|_S = J$. □

Lemma 5.17. *The only ideal of A that contains S is A. Equivalently, the only ideal of S containing 1 is S.*

Proof. Firstly, S is an ideal of S since $A|_S = S$ and A is an ideal of A. Now since S contains 1, it follows that any ideal of A which restricts to S must contain the element 1. But the only ideal which contains 1 is A itself. □

Lemma 5.18. *If I is a maximal ideal in A then $I|_S$ is a maximal ideal of S.*

Proof. The previous lemma shows that $I|_S$ is not S. Thus there exists $f \in S$ such that $f \notin I|_S$ and hence $f \notin I$. It follows that $\langle f, I\rangle = A$ from the maximality of I. From Corollary 5.16, $\langle f, I|_S\rangle_S = \langle f, I\rangle_S = S$. Since this is true for any choice of $f \notin I|_S$ it follows that $I|_S$ cannot be properly contained within any proper ideal of S. Hence $I|_S$ is a maximal ideal of S. □

Corollary 5.19. *An ideal J is a maximal ideal of S if and only if there exists a maximal ideal $\tilde{J}$ such that $\tilde{J}|_S = J$.*

We will use the notation M_S to mean the maximal ideal space of S, or precisely, $M_S := \{J : J \text{ is a maximal ideal of } S\}$.

Definition 5.20. *Let J be an ideal of S. We define the quotient S/J to be the equivalence classes of the relation $f \sim g \iff \exists j \in J, f = g + j$.*

Lemma 5.21. *$\sim$ is an equivalence. We denote equivalence classes under $\sim$ on S by $[f]_S$.*

Proof. Reflexivity is clear. Since J is an ideal, if $j \in J$ it follows that $-j \in J$ and so the relation is symmetric. If $j_1, j_2 \in J$, then $j_1 + j_2 \in J$ and from this it follows that the relation is transitive. □

Notation 5.22. By $[f] \in A/J$ we denote the usual equivalence $f \sim g \iff \exists j \in J, f = g + j$.

Lemma 5.23. *Suppose that J is an ideal of S and $\tilde{J}$ an ideal of A such that $\tilde{J}|_S = J$. Then the set $(A/\tilde{J})|_S = \{[f] \in A/\tilde{J} : \exists g \in [f], g \in S\}$ (the set of equivalence classes which have an element in S) are isomorphic to S/J.*

Proof. Observe that $[0]_S + J = [0]_S + \tilde{J}|_S$, so the restriction map $|_S : \tilde{J} \mapsto \tilde{J}|_S$ preserves the additive identity. Let $f, g \in (A/\tilde{J})|_S$ such that $[f] = [g]$. Then $f = g + j$ for some $j \in J$. It follows that $f - g = j \in J$ and hence $[f] - [g] = [0] \in A/\tilde{J}$. Applying the restriction map to both sides yields $([f] - [g])_S = [f]_S - [g]_S = [0]_S \in S/J$ (where we have used the fact that S is closed under addition to distribute the restriction map and the hypothesis to ensure the restriction is non-empty). Hence $[f]_S = [g]_S \in S/J$. Observe that this logic is reversible. So if $f, g \in S$ and $[f] = [g] \in A/\tilde{J}$ then $[f]_S = [g]_S \in S/J$ and conversely. From this the result follows. $\square$

Recall the following standard fact from uniform algebra theory.

Theorem 5.24 (Theorem 2.2, [30]). *Every maximal ideal of A is closed. If J is a maximal ideal of A, then A/J is isometrically isomorphic to the field of complex numbers.*

Theorem 5.25 (Theorem 2.3, [30]). *Let ϕ be a non-zero complex-valued homomorphism on A and A_ϕ the kernel of ϕ. The correspondence $\phi \to A_\phi$ is a bijective correspondence of non-zero complex valued homomorphisms of A and maximal ideals in A. In particular A_ϕ is a maximal ideal and if $f \in A$ then $\phi(f)$ is the unique complex number λ such that $f + A_\phi = \lambda + A_\phi$.*

Using these we can deduce the same facts concerning maximal ideals of S.

Lemma 5.26. *An ideal J of S is maximal if and only if S/J is a field isomorphic to the complex numbers.*

Proof. We know that J induces a maximal ideal $\tilde{J}$ of A and that $A/\tilde{J}$ is isomorphic to the complex numbers by Theorem 5.24. It follows that S/J is isomorphic to a subset of the complex numbers. From Theorem 5.25 maximal ideals are 'point evaluation', that is a maximal ideal $\tilde{J}$ takes the form $\tilde{J} = \langle x_1 - a_1, ..., x_{N-1} - a_{N-1}, y - a_N \rangle$ for some $(a_1, ..., a_N) \in \mathbb{C}^N$. But each of $x_1 - a_1, ..., y - a_N \in S$ since S contains all linear polynomials, so the corresponding maximal ideal in S guaranteed by Lemma 5.18 is given by $\langle x_1 - a_1, ..., y - a_N \rangle_S$. Let $\lambda \in \mathbb{C}$, then the polynomial $\lambda + (x_1 - a_1) + ... + (y - a_N) \equiv \lambda \mod J$. Since this is valid for any λ, it follows that J maximal implies S/J is isomorphic to all of $\mathbb{C}$.

Now suppose that S/J is isomorphic to $\mathbb{C}$. Suppose that I is an ideal properly containing J. Then there exists $f \in I$ such that $f + J$ is a nonzero element of S/J. Then there exists $f^{-1} + J \in S/J$ such that $(f + J)(f^{-1} + J) = ff^{-1} + J = 1 + J$ since S/J is a field. Now $ff^{-1} \in I$ as $f^{-1} \in S$ and from the calculation we just did $ff^{-1} - 1 \in J \subset I$. It thus follows that $1 = (1 - ff^{-1}) + ff^{-1} \in I$. The only ideal of S which contains 1 is S and so $I = S$ and it follows that J must be maximal. $\square$

Corollary 5.27. *Maximal ideal of S are 'point evaluation' i.e. generated by linear factors $\langle (x_1 - a_1), ..., (x_{N-1} - a_{N-1}), y - a_N \rangle_S$.*

Definition 5.28. *A (complex valued) homomorphism of S is a map $\phi : S \to \mathbb{C}$ which satisfies $\phi(f + g) = \phi(f) + \phi(g)$ for all $f, g, \in S$ and $\phi(fg) = \phi(f)\phi(g)$ when $fg \in S$.*

Note that we require the homomorphism to preserve the multiplicative structure only when the multiplication is defined. From Corollary 5.11 it follows that when ϕ is a homomorphism, $p(x) \in \mathbb{C}[x]$ and $f \in S$ then $\phi(p(x)f) = \phi(p(x))\phi(f)$ so our definition above is essentially a $\mathbb{C}[x]$-module homomorphism with the added restriction that if fg is ever defined that ϕ preserves this multiplication.

Lemma 5.29. *$\phi : S \to \mathbb{C}$ is a homomorphism if and only if there exists a homomorphism $\tilde{\phi} : A \to \mathbb{C}$ such that $\tilde{\phi}|_S = \phi$.*

Proof. The 'only if' direction is clear, since any complex valued homomorphism of A is a complex valued homomorphism of S. To prove the 'if' direction, note that $\phi(x_i)$ and $\phi(i)$ are defined for all $1 \leq i \leq N-1$. Given an element $f(x, y)$ of A define $\tilde{\phi}$ to be the function $\tilde{\phi}(f(x_1, ..., x_{N-1}, y)) := f(\phi(x_1), ..., \phi(x_{N-1}), \phi(y))$. Checking that $\tilde{\phi}$ is a homomorphism and that $\tilde{\phi}|_S = \phi$ is immediate. □

Lemma 5.30. *There is a bijection between the kernel of homomorphisms $\phi : S \to \mathbb{C}$ and the maximal ideals of S.*

Proof. Homomorphisms associated to the maximal ideals of A restricted so that their domain is S gives the desired homomorphisms on S. The fact that they are a bijection to the maximal ideals follows since every maximal ideal of A induces a maximal ideal in S and conversely. □

We will need the flexibility of working with canonically associated functions defined on the maximal ideal space to work up to a absolute fluidity between $\mathbb{C}^N$ and M_S in subsequent work. We formally identify each maximal ideal with the homomorphism whose kernel induces that maximal ideal. Since these are all point evaluation, we may take M_S to be a subset of $\mathbb{C}^N$ by identifying each homomorphism with the point to be evaluated at. Of course, with A and S set up as in this section, $M_S = M_A = \mathbb{C}^N$. This will change shortly.

5.3.3 The uniform algebra $A(K)$ and the 'uniform module' $S(K)$

Suppose that $\|.\|$ is a norm such that A completed with respect to this norm is a Banach algebra. In particular we will be interested in $\|.\|_K$, the supremum norm over a compact set K. In this section we will take $A(K) = A$ to be the completion of $\mathbb{C}[x, y]$ with respect to $\|.\|_K = \|.\|$. We will take $S(K) = S$ to be the completion of $\mathbb{C}[x] + ... + y^d\mathbb{C}[x]$ with respect to $\|.\|$. Note that $S \subset A$ but not conversely. As a consequence, it follows that $M_S \supset M_A$.

Definition 5.31. *The Gelfand transform of $f \in S$ is the complex-valued function $\check{f}$ on M_S defined by $\check{f}(\phi) = \phi(f)$.*

Note it is customary to denote the Gelfand transform by $\hat{f}$, however due to our frequent use of $\hat{}$ as a symbol we will use the notation $\check{f}$ when needed. That said, this distinction will not normally be necessary.

Lemma 5.32. *$\check{f} : M_S \to \mathbb{C}$ equals the usual Gelfand transform on A restricted to elements from S.*

Proof. Let us temporarily denote the Gelfand transform on A of a function f as $\tilde{f}$. We must check that when $f \in S$, $\tilde{f} = \check{f}$ for $\phi \in M_S$. Let $\phi \in M_S$ then $f(\phi) = \check{f}(\phi) = \tilde{f}(\phi)$ which verifies this claim. □

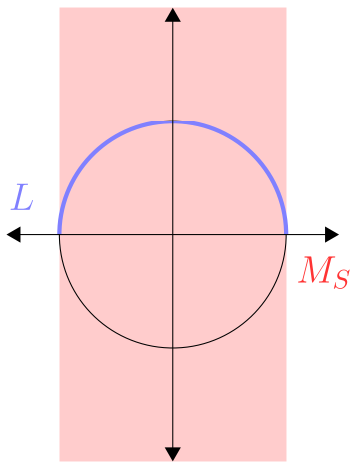
Corollary 5.33. *The Gelfand transform is a homomorphism (in the sense of Definition 5.28) of S onto the set $\check{S}$ of continuous functions of M_S. The subset $\check{S}$ separates points of M_S and $\check{S}$ contains the constants.*

Proof. The first claim follows as $(f \check{+} g)(\phi) = \phi(f+g) = \phi(f) + \phi(g) = \check{f} + \check{g}$ and $(\check{fg})(\phi) = \phi(fg) = \phi(f)\phi(g) = \check{f}\check{g}$ when $fg \in S$. The fact that the Gelfand transform separates points is checked by letting $\phi, \psi \in M_S$ be such that $\check{f}(\phi) = \check{f}(\psi)$ for all $f \in S$ which forces $\phi(f) = \psi(f)$ for all $f \in S$ and hence $\phi = \psi$. Finally, the Gelfand transform of the identity is the function which is constant everywhere on M_S and so $\check{S}$ contains the constants. □

M_S may be unbounded unlike the uniform algebra case. To see this first observe that without the uniform completion, $M_S = M_A = \mathbb{C}^N$ by Corollary 5.27. Suppose that $\mathcal{V} = \{x^2 + y^2 = 1\}$ and take L as in Example 5.7 (i.e. the upper half of the real circle) then M_S is unbounded in the y direction. To see this observe that the uniform completion of $S = S(L)$ is

$$\{f(x) + g(x)y : f, g \text{ holomorphic for } x \in [-1, 1]\}.$$

It follows by Corollary 5.27 that $\langle x - a_1, y - a_2 \rangle$ is a maximal ideal when $a_1 \in [-1, 1]$ and $a_2 \in \mathbb{C}$. In particular $M_S \cong \{(x, y) : x \in [-1, 1], y \in \mathbb{C}\}$.



This is in contrast to M_A which is always a compact subset of $\mathbb{C}^N$ (provided K is compact). For instance, $M_A \cong L$ in the previous example. The difference is due to the fact that M_A contains all holomorphic functions in y as well as x, so situations like what happened above cannot occur. We need to make the following definition to make the maximal ideal space a useful concept for the uniform module.

Definition 5.34. *We define $M_{S(K)} := \{\phi \in M_S : |\phi(f)| \leq \|f\|_K, \forall f \in S\}$ to be the reduced maximal ideal space for S.*

Lemma 5.35. *$M_{S(K)}$ is the largest subset of M_S such that the Gelfand transform is norm-decreasing.*

Proof. Suppose $\phi \in M_S \backslash M_{S(K)}$, then there exists $f \in S$ such that $|\check{f}(\phi)| = |f(\phi)| > \|f\|_K$ and so the Gelfand transform cannot be norm decreasing on $M_{S(K)} \cup \{\phi\}$ and so $M_{S(K)}$ cannot be enlarged without voiding this property. □

Lemma 5.36. *Suppose $\phi \in M_{S(K)}$, then $\|\phi\| = 1 = \phi(1)$ and ϕ is continuous.*

Proof. $\phi(1) = 1$ follows since $\phi(1)^2 = \phi(1)$ and so either $\phi(1) = 1$ or $\phi(1) = 0$. $\phi \in M_{S(K)}$ excludes the latter case since $\phi(1) = 0$ implies ϕ is identically zero. By construction of the set $M_{S(K)}$ we have $|\phi(f)| \leq \|f\|_K$ for all $f \in S$ with equality taken when $f = 1$, it follows that $\|\phi\| = 1$. Since ϕ is a bounded linear functional on S it follows that ϕ is continuous. □

Lemma 5.37. *$M_{S(K)}$ is a compact Hausdorff.*

Proof. The weak-star limit of homomorphisms satisfying $\phi(1) = 1$ is again a non-zero homomorphism. Hence $M_{S(K)}$ is a closed subset of the unit ball of S^*. By Alaoglu's theorem (Theorem 3.15, [48]), the unit ball of S^* is weak-star compact. Hence $M_{S(K)}$ is compact. □

We are interested in two things; firstly the relationship between the reduced maximal ideal space $M_{S(K)}$ and the reduced polynomial hull of K and secondly the extension of the module S by a function f which we will denote $[S, f]$.

Definition 5.38. *The reduced polynomial hull or S-polynomial hull of a compact set K (denoted $\hat{K}_S$) is the set*

$$\hat{K}_S = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K, \forall p \in S(K)\}.$$

Lemma 5.39. *Suppose that $K \subset \mathbb{C}^n$ is compact. Then $\hat{K}_S \subset M_{S(K)}$.*

Proof. Suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials converging uniformly on K to $f \in S$. From the definition of the reduced polynomial hull it follows that

$$\|p_n - p_m\|_{\hat{K}_S} \leq \|p_n - p_m\|_K$$

for all n and m. Consequently, $\{p_n\}_{n=1}^{\infty}$ converges uniformly on $\hat{K}_S$ to a function $\hat{f}$ on $\hat{K}_S$ which is an extension of f (i.e. $\hat{f}|_K = f$). It follows that the point evaluation homomorphisms of $\hat{f}$ are well defined on $\hat{K}_S$ and hence we can view $\hat{K}_S$ as a subset of $M_{S(K)}$. □

Theorem 5.40. *Suppose that $K \subset \mathbb{C}^n$ is compact. Then $M_{S(K)} = \hat{K}_S$ i.e. the reduced maximal ideal space of $S(K)$ is equal to the reduced polynomial hull of K.*

Proof. The homomorphisms in $M_{S(K)}$ are point evaluation for a particular subset of $\mathbb{C}^n$. Note that $\hat{K}_S \subset M_{S(K)}$ by Lemma 5.39, so we must check that the opposite inclusion holds. Let $x \in M_{S(K)}$, we want to show that for any $f \in S$ that $|f(x)| \leq \|f\|_K$. To see this write

$$|f(x)| = |\phi_x(f)| \leq \|\check{f}\|_{M_{S(K)}} \leq \|f\|_K,$$

which follows from the definition of $M_{S(K)}$. This completes the proof. $\square$

We can now characterise the elements of $S(K)$.

Lemma 5.41. *Elements in $S = S(K)$ are of the form $f_0(x) + yf_1(x) + ... + y^d f_d(x)$ where $f_0, ..., f_d$ are analytic functions on $M_{S(K)} \cap \{x \in \mathbb{C}^{N-1}\} = M_{S(K),x}$.*

Proof. Suppose $f(x, y) = f_0(x) + yf_1(x) + ... + y^d f_d(x)$ where $f_0, ..., f_d$ are analytic functions on $M_{S(K)} \subset \mathbb{C}^{N-1}$. $f_0, ..., f_d$ being analytic on $M_{S(K)}$ implies that there exists sequences $f_{0,j} ..., f_{d,j}$ of polynomials which unformly approximate f on $M_{S(K)}$. That is, given $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for all $j_0 \geq j$ we have $\|f_{i,j} - f_i\|_K \leq \|f_{i,j} - f_i\|_{M_{S(K),x}} < \varepsilon$. If $f_j(x, y) = f_{0,j}(x, y) + ... + y^d f_d(x, y)$ Then

$$\|f_j - f\|_K = \left\| \sum_{i=0}^{d} y^i (f_{i,j} - f_j) \right\|_K \leq \sum_{i=0}^{d} \|y^i\|_K \|(f_{i,j} - f_i)\|_K = \sum_{i=0}^{d} \|y^i\|_K \|(f_{i,j} - f_i)\|_{M_{S(K),x}} \leq M\varepsilon$$

where M depends on d and $\max_{0 \leq i \leq d} \|y^i\|_K$. It follows that $f_j \to f$ uniformly so f is an element of the uniform closure of S.

We must now show any element of the uniform closure has the form of the hypothesis. Let $\{f_j = f_{0,j}(x) + yf_{1,j} + ... + y^d f_{d,j}(x)\}_{j \in \mathbb{N}} \in \mathbb{C}[x, y]$ be a sequence uniformly convergent to $g(x, y)$ with respect to $\|.\|_K$. Fix a value of $x_0 \in K$ and let y vary. Then $f_j(x_0, y)$ is a polynomial of degree d in the variable y which is uniformly convergence to $g(x_0, y)$. Since finite dimensional spaces are complete, it follows that $g(x_0, y)$ is a polynomial of degree d in y. Of note the sequence of each coefficient converges for any x_0 and hence uniformly. That is,

$$\lim_{j \to \infty} \|f_{i,j} - g_i\|_K = 0 \quad \text{where} \quad g(x, y) = \sum_{i=0}^{d} y^i g_i(x).$$

Finally, write $r_{i,j}(x) = f_{i,j}(x) - g_i(x)$ for each j. Then

$$\|g_i(x)\|_{M_{S(K),x}} = \|f_{i,j} - r_j\|_{M_{S(K),x}} \leq \|f_{i,j}\|_{M_{S(K),x}} + \|r_{i,j}\|_{M_{S(K),x}} = \|f_{i,j}\|_K + \|r_{i,j}\|_{M_{S(K),x}}$$

where $\|r_{i,j}\|_{M_{S(K),x}} \to 0$ as $j \to \infty$. It follows that $\|g_i\|_{M_{S(K),x}} \leq \lim_{j \to \infty} \|f_{i,j}\|_K + \|r_{i,j}\|_{M_{S(K),x}} = \|g_i\|_K$ for each i. Since the coefficients of f_j are absolutely convergence in $M_{S(K),x}$ to g it follows that g is analytic on $M_{S(K),x}$ as claimed. $\square$

Definition 5.42. *We say a function f is S-holomorphic at a point $x \in M_{S(K)}$ if there is a neighbourhood $U \subset M_{S(K)}$ of x such that f can be approximated uniformly by elements in S. A function f is S-holomorphic on $E \subset M_{S(K)}$ if it is S-holomorphic at every point in E.*

Note that we must only consider points in the reduced maximal ideal space to ensure uniform convergence. Of course, one could take any compact set of M_S and this definition would make sense, however $M_{S(K)}$ has interesting properties when studied with this definition in mind. Also note that on M_S one could define an analogous concept by declaring S-holomophic require that f can be approximated locally uniformly by elements in S. Again, we will not pursue this direction.

Definition 5.43. *The (module) extension of S by a continuous function f (denoted $[S,f]$) is the uniform closure of the set $\mathbb{C}[x] + y\mathbb{C}[z_1] + ... + y^d\mathbb{C}[x] + f\mathbb{C}[x]$. The set $M_{[S,f](K)}$ is the reduced maximal ideal space of $[S,f]$ which is defined in the same manner as Definition 5.34.*

Theorem 5.44. *Suppose $K \subset E \subset M_{S(K)}$. Let f be a continuous function on E such that f is S-holomorphic on $E \backslash \{f^{-1}(0)\}$. Then $M_{[S,f](K)} = E$.*

Proof. Without loss of generality we may assume that $E \cap \{f^{-1}(0)\} = \varnothing$. We first must verify that $\phi(f)$ is defined for $\phi \in E$. Since f is S-holomorphic on E, for each $x \in E$ there exists an open neighbourhood U of x such that f can be uniformly approximated by elements of S. Suppose $\{f_k\}$ is such a sequence. Since $f_k \in S$ it follows that $\phi_x(f_k)$ is defined, and since ϕ_x is weak-star continuous on S, it follows that $\lim_{k\to\infty} \phi_x(f_k) = \phi_x(f)$ and so $\phi_x(f)$ is defined. Repeating this for all $x \in E$ verifies that $\phi(f)$ is defined for all $\phi \in E$.

Now suppose $v \in [S,f]$, i.e. v is locally the uniform limit of $v_j(z) = p_0^j(x) + ... + y^d p_d^j(z_1) + f(z)p_{d+1}^j(x)$ as $j \to \infty$ where $p_0^j, ..., p_d^j, p_{d+1}^j \in S$. Then v is a continuous function on E and so $\phi(v)$ is defined for $\phi \in E$. We must check that the Gelfand transform is norm decreasing for v. First we check that v_j satisfies this property. To see this note that for $\phi \in E$

$$\begin{aligned}
|\phi(v_j)| &= \left|\phi\left(\lim_{k\to\infty} \sum_{j=0}^{d} y^i p_i^j(x) + f_k(z)p_{d+1}^j(x)\right)\right| \\
&= \left|\lim_{k\to\infty} \phi\left(\sum_{j=0}^{d} y^i p_i^j(x) + f_k(z)p_{d+1}^j(x)\right)\right| \\
&\leq \| \lim_{k\to\infty} \sum_{i=0}^{d} \check{y}^i \check{p}_i^j + \check{f}_k \check{p}_{d+1}^j \|_E \\
&\leq \| \lim_{k\to\infty} \sum_{i=0}^{d} \check{y}^i \check{p}_i^j + \check{f}_k \check{p}_{d+1}^j \|_{M_{S(K)}} \\
&\leq \| \lim_{k\to\infty} \sum_{i=0}^{d} \check{y}^i \check{p}_i^j + \check{f}_k \check{p}_{d+1}^j \|_K \\
&= \|v_j\|_K.
\end{aligned}$$

Where we have used the weak-star continuity of ϕ. Using the same ideas as in Lemma 5.41 and repeating these arguments with v we deduce that $|\phi(v)| \leq \|v\|_K$ for all $\phi \in E$. Hence $\|\check{v}\|_E \leq \|v\|_K$. $\square$

Lemma 5.45. *If f is S-holomorphic on E as in the previous theorem, then there exists an S-holomorphic extension $\tilde{f}$ of f to $M_S(K)$ such that $\tilde{f}|_E = f$.*

Proof. We know that on E there exists a sequence $v_j(x,y) = \sum_{j=0}^{d} y^i p_i^j(x)$ which is uniformly convergence to f on E. But since $K \subset E$ we can use Lemma 5.41 to extend this convergence to $M_{S(K)}$. This gives the required extension of f. $\square$

Corollary 5.46. *In Theorem 5.44 we may take* $E = M_{S(K)}$.

Proof. In the proof of Theorem 5.44 we used the fact that

$$\left|\phi\left(\sum_{j=0}^{d} y^i p_i^j(x) + f_k(z)p_{d+1}^j(x)\right)\right| \leq \sup_{\phi \in M_{S(K)}} \left|\phi\left(\sum_{j=0}^{d} y^i p_i^j(x) + f_k(z)p_{d+1}^j(x)\right)\right|$$

which came from the definition of $M_{S(K)}$. It follows that $\phi \in M_{S(K)}$ is defined for this function, and moreover defined in the limit along k, and then j. $\square$

Remark 5.47. The proof for the analogous result for the uniform algebra is more involved due to the possibility that f^n is in the algebra for all $n \in \mathbb{N}$ meaning it's plausible that a new element would be introduced meaning a straight forward verification, as above, is insufficient.

Theorem 5.48. *Suppose that $\mathcal{V} \subset \mathbb{C}^2$ is an algebraic curve of degree 2 with Noether presentation (x,y). Let $K \subset \mathcal{V}$ be such that $K = \{(x, f(x)) : x \in U \subset \mathbb{C}\}$ for some $f \in S$ and U polynomially convex in $\mathbb{C}$. Then $M_{S(K)} = K$.*

Proof. $S(K) = \mathbb{C}[x] + y\mathbb{C}[x] \cong \mathbb{C}[x] + f(x)\mathbb{C}[x]$ for points in K. It follows that $M_{S(K)} \cong M_{[\mathbb{C}[x],f](K)} = M_{[\mathbb{C}[x]]}$ by the previous theorem, where the uniform closure in $M_{[\mathbb{C}[x]]}$ is taken with respect to U (which is the projection of K onto the x plane). Since U is polynomially convex is follows that $M_{[\mathbb{C}[x]]} = U$. So $M_{S(K)} = \{(x, f(x)) : x \in U\} = K$. $\square$

Notation 5.49. Write $K^{conv\, y}$ to denote the convex y-hull of a set $K \subset \mathbb{C}^2$, that is the set $\{(x,y) \in \mathbb{C}^2 : |L(y)| \leq \|L\|_K, L \text{ is a linear function of } y\}$. A set satisfying $K = K^{conv\, y}$ will be called y-convex.

Theorem 5.50. *Suppose that $\mathcal{V} \subset \mathbb{C}^2$ is an algebraic curve of degree 2 with Noether presentation (x,y). Let $K \subset \mathcal{V}$ be compact. Then $M_{S(K)} = M_{S(K)}^{conv\, y}$ i.e. $M_{S(K)}$ is y-convex.*

Proof. Clearly $M_{S(K)} \subset M_{S(K)}^{conv\, y}$ by definition of the y-convex hull. But note that any y-linear function has the form $L(y) = a + by$ for $a, b \in \mathbb{C}$ which implies that $L \in \mathbb{C}[x] + y\mathbb{C}[x] = S$. Hence $M_{S(K)}^{conv\, y} \subset \widehat{M_{S(K)}}_S = M_{S(K)}$ which completes the proof. $\square$

These two results show what the S-convex hull of K is when $\mathcal{V}$ is a quadratic curve. Given the comprehensive understanding of the convex hull in the literature we also know when $\hat{K}_S$ extends off of $\mathcal{V}$. The arbitrary hypersurface case has similar conditions on $\hat{K}_S$ but the lack of understanding of high-degree polynomial hulls prevents us from giving a description of whether there is an extension or not.

Theorem 5.51. *Let $\mathcal{V} \subset \mathbb{C}^N$ be a hypersurface with Noether presentation (x, y) of degree $d+1$. Let π be the projection onto x. Suppose that $K \subset \mathcal{V}$ is compact, $\pi|_K$ is biholomorphic, and is contained in at most d branches of $\mathcal{V}$. Then $M_{S(K)} \subset \mathcal{V}$.*

Proof. Since $\pi|_K$ is biholomorphic it follows that each branch of $\mathcal{V}$ is representable by an analytic function $f_1(x), ..., f_d(x)$ i.e. that $\mathcal{V} \cap K = \{y - f_1(x), ..., y - f_d(x)\}$. So each f_i is in the uniform closure of $\mathbb{C}[x]$. Suppose that the $f_1, ..., f_j$ contains K, then $g(x, y) = (y - f_1(x))...(y - f_j(x)) \in S$ and is identically 0 on a subset of $\mathcal{V}$ containing K. By Corollary 5.33 we know that the Gelfand transform satisfies

$$\|\check{g}\|_{M_{S(K)}} \leq \|g\|_K = 0.$$

It then follows that $\phi \in M_{S(K)}$ implies $\phi(g) = 0$. So $M_{S(K)} \subset \{(x, y) \in \mathbb{C}^N : g(x, y) = 0\}$. But by construction, $\{g(x, y) = 0\} \subset \mathcal{V}$ so it follows that $M_{S(K)} \subset \mathcal{V}$. □

This shows that a necessary condition for the S-polynomial hull of K to extend off of $\mathcal{V}$ is that K does not lie in the zero set of an $S(K)$-holomorphic function.

Theorem 5.52. *Suppose that $\mathcal{V} \subset \mathbb{C}^N$ is a hypersurface of degree $d+1$ with Noether presentation (x, y). Let $K \subset \mathcal{V}$ be such that $K = \{(x, \pi_i^{-1}(x)) : x \in U \subset \mathbb{C}^{N-1}, 1 \leq i \leq j \leq d+1\}$ and U polynomially convex in $\mathbb{C}^{N-1}$ and K is a j-sheeted covering of U with π biholomorphic on branches (so $\pi_i^{-1} \in S$). Then $M_{S(K)} = K$ when $j \leq d$ and if $j = d+1$ then $M_{S(K)} \cap \{x \in C^{N-1}\} = U$ and $M_{S(K)} \supset K$.*

Proof. On any branch of K we can locally write $f \in S$ as $f_k(x) = \sum_{i=0}^{d} p_i(x)(\pi_k^{-1}(x))^i$ for some $k \in \{1, ..., j\}$. Then since $\pi_k^{-1}(x) \in S$ it follows that $(\pi_k(x))^i \in S$ for each i. Hence $f(x)$ is an element of the uniform completion of $\mathbb{C}[x]$. Suppose that $x \in U$, then $|f(x)| \leq \|f_k\|_U \leq \|f\|_K$. Since this argument is valid on any branch it follows that $\{(x, \pi_k^{-1}(x)) : x \in U, 1 \leq k \leq j\} \subset \hat{K}_S$. This proves the $j = d+1$ statement. If $j \leq d$ then by Theorem 5.51 there can be no other points contained in $\hat{K}_S$, i.e. $K = \hat{K}_S$. □

This shows that the S-polynomial hull restricted to $\mathcal{V}$ can be thought of as the inverse image of a polynomially convex set in $\mathbb{C}^{N-1}$, as one would expect. There may be extension in the y direction.

Notation 5.53. Write $K^{conv\, y^d}$ to denote the convex y^d-hull of a set $K \subset \mathbb{C}^N$, that is the set $\{(x, y) \in \mathbb{C}^N : |g(y)| \leq \|g\|_K, g$ is a polynomial in y of degree $d\}$. A set satisfying $K = K^{conv\, y^d}$ will be called y^d-convex.

Theorem 5.54. *Suppose that $\mathcal{V} \subset \mathbb{C}^N$ is a hypersurface of degree $d+1$ with Noether presentation (x, y). Let $K \subset \mathcal{V}$ be compact. Then $M_{S(K)} = M_{S(K)}^{conv\, y^d}$ i.e. $M_{S(K)}$ is y^d-convex.*

Proof. The proof is effectively the same as Theorem 5.50. Clearly $M_{S(K)} \subset M_{S(K)}^{conv\, y}$ by definition of the y-convex hull. But note that any y^d-polynomial has the form $g(y) = a_0 + ... + a_d y^d$ for

$a_0, ..., a_d \in \mathbb{C}$ which implies that $g \in \mathbb{C}[x] + ... + y^d\mathbb{C}[x] = S$. Hence $M_{S(K)}^{conv\, y^d} \subset \widehat{M_{S(K)}}_S = M_{S(K)}$ which completes the proof. $\square$

Corollary 5.55. *Suppose that $\mathcal{V} \subset \mathbb{C}^N$ is a hypersurface of degree $d+1$ with Noether presentation (x, y). Let $K \subset \mathcal{V}$ be compact. Then $M_{S(K)}$ is characterised by the polynomial hull of the x variables and the y^d-hull in the y variables.*

It follows that studying the y^d-hull may lead to an explicit extension of the extremal function to a function in $\mathcal{L}(\mathbb{C}^N)$.

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