Bearing strength surfaces implied in conventional bearing capacity calculations

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The classic bearing capacity equation for a shallow foundation determines the vertical load which will initiate bearing failure. The factor of safety is this failure load divided by the applied vertical load. However, a shallow foundation might also need to sustain moment and shear; in some situations the vertical load does not change significantly and the action taking the foundation towards bearing failure is moment, in which case the vertical load becomes a stabilising influence. Bearing strength calculation under general actions is thus more complex than under vertical load only. In these situations the standard estimation of the available reserve of bearing strength can be quite misleading and fails to indicate how rapidly bearing strength diminishes with increasing moment. A good visual understanding comes by way of the bearing strength surface (the locus of all possible combinations of vertical load, horizontal shear and moment that will induce bearing failure). The bearing strength of the foundation is then not a single number but the combination of the vertical load, horizontal shear force and moment where the action path (locus of the combinations of vertical load, horizontal shear and moment applied to the foundation) intersects the bearing strength surface.

KEYWORDS: bearing capacity; clays; footings/foundations; rafts; retaining walls; sands

INTRODUCTION

Evaluation of the bearing strength of a shallow foundation is obtained from the mobilisation of the shear strength of soil beneath and adjacent to the foundation. The classic way of estimating this (Terzaghi, 1943) is the summation of three distinct components – one accounting for the cohesive component of the shear strength of the soil, another for the surcharge pressure adjacent to the foundation, and a third accounting for frictional resistance of the soil beneath the foundation. Terzaghi (1943) considered a foundation subject to vertical load only, and treated a strip foundation at the ground surface. This can be extended to encompass rectangular foundations and also the effect of embedment by the incorporation of 'adjustment' factors.

Now, in addition to vertical load, the shallow foundation may be required to sustain horizontal shear, or moment, or combinations of these, and so bearing failure can be induced not only by vertical loading but also by combinations of these three types of action. The locus of all the combinations of vertical load, horizontal shear and moment which generate bearing strength failure forms a three-dimensional bearing strength surface (BSS), which affords a useful visual appreciation of the workings of conventional bearing strength calculations. When vertical actions are to be applied to the foundation, the usual terminology is to refer to the bearing capacity, which has units of pressure. When the vertical load is accompanied by shear and moment, it is helpful to speak of bearing strength, which has units of force rather than pressure; this terminology is used in the remainder of the paper.

The concept of the BSS is not new. Georgiadis (1985), Butterfield (1993) and Butterfield & Gottardi (1994) were among the first to point out the existence of the shallow foundation BSS and to emphasise that, for action combinations involving vertical load, horizontal shear and moment, consideration of foundation safety involved more than conventional estimation of the factor of safety. Since then the BSS has been used extensively in offshore geotechnical engineering (Dean *et al.*, 1997a, 1997b; Taiebat & Carter, 2000; Martin & Houlsby, 2001; Zdravkovic *et al.*, 2002; Muir Wood, 2004; Gourvenec, 2004; Randolph *et al.*, 2005; Butterfield, 2006; Randolph & Gourvenec, 2011; Arroyo *et al.*, 2013). Equations have been proposed for the BSS for offshore foundations based on the results of this experimental and numerical modelling; these are discussed below.

Furthermore, Salençon & Pecker (1994a, 1994b) and Pecker (1997) have developed a BSS that includes the effect of earthquake horizontal acceleration on the bearing strength of surface strip shallow foundations, an equation which is included in Eurocode 8, Part 5 (BSI, 2005).

Butterfield & Ticof (1979), Georgiadis (1985), Gottardi & Butterfield (1993) and Butterfield & Gottardi (1994) have pointed out repeatedly that BSSs have considerable relevance to onshore shallow foundation design. Butterfield (2012) has indicated how a bearing strength approach could be used in proportioning foundations for gravity retaining walls. Regrettably, despite these suggestions and demonstrations, BSS thinking has not been adopted by the onshore geotechnical community. The purpose of this paper is to take the expressions used for conventional bearing strength calculations and show how these imply BSSs having similar appearances to those proposed by the authors listed above. In this way it is hoped that the paper might serve as a bridge between the well-established BSS applications in offshore foundation engineering and the wider adoption of BSS thinking in onshore shallow foundation design.

The work reported in the paper makes apparent that the 'add-on' nature of the factors accounting for shape, embedment depth and horizontal shear in conventional bearing strength calculations lead to cumbersome equations for the shallow foundation BSSs. In other words, a case can be made

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for a new start in the formulation of shallow foundation bearing strength when general loading is applied. The aim of such work would be to develop expressions for BSSs that are simpler than those given in the paper which are implied by conventional bearing strength equations.

BEARING STRENGTH CALCULATIONS

A generalisation of the Terzaghi (1943) bearing strength equation that embraces embedded rectangular foundations rather than surface strip foundations, and the application of shear force and moment as well as vertical load, is given by

$$q_\mathrm{u} = c\lambda_\mathrm{cs}\lambda_\mathrm{cd}\lambda_\mathrm{ci}N_\mathrm{c} + q\lambda_\mathrm{qs}\lambda_\mathrm{qd}\lambda_\mathrm{qi}N_\mathrm{q} + \frac{1}{2}\gamma B\lambda_{\gamma\mathrm{s}}\lambda_{\gamma\mathrm{d}}\lambda_{\gamma\mathrm{i}}N_\gamma \tag{1}$$

where λ_{cs}, λ_{qs} and $\lambda_{\gamma s}$ are shape factors; λ_{cd}, λ_{qd} and $\lambda_{\gamma d}$ are depth factors; λ_{ci}, λ_{qi} and $\lambda_{\gamma i}$ are inclined load factors; and c, q, q_u, B, N_c, N_q and N_γ have their usual meanings (as defined in the notation list).

A set of shape, depth and inclined load factors is presented in Appendix 1. These are given in various places in the literature (e.g. Brinch Hansen, 1970; Vesic, 1975; Eurocode 7 (BSI, 2004); Salgado *et al*., 2004; Edwards *et al*., 2005; Salgado, 2008). The λ factors have evolved gradually since Terzaghi (1943) proposed that the bearing strength of a surface strip foundation can be estimated as the summation of three distinct components. The background to the λ factors consists of a few case histories of foundation failure, data obtained from field and laboratory testing at scales much below prototype, from some numerical analysis, and from reasoning about the expected foundation performance. As there is no overarching theoretical framework, the λ factors set out in Appendix 1 are derived from many separate sources; their acceptance is more a matter of professional consensus than theoretical rigour. Thus the λ factors are best considered as approximations that are thought to give reasonable indications of the effect of the various factors.

Given the origin of the λ factors one could question the appropriateness of those presented in Appendix 1. The justification for the particular expressions presented there is that they appear in a standard reference, Vesic (1975), and in Eurocode 7 (BSI, 2004) which was developed over a number of years and is the product of professional consensus. Similar queries can be raised about the bearing strength factors, N_c, N_q and N_γ; there are various ways of obtaining these as functions of the friction angle, those presented herein are the equations found in Eurocode 7 (BSI, 2004). However, it needs to be emphasised that equation (1) and the information found in Appendix 1 are not presented on the understanding that this is the definitive statement about shallow foundation bearing strength, but rather this set of equations is typical of what is used in geotechnical engineering practice. The purpose of this paper is to demonstrate that beneath equation (1) BSSs can be found; this is a necessary task because an important tool in understanding shallow foundation bearing strength under complex loading seems to have escaped the attention of much of the geotechnical community.

Use of equation (1) to estimate bearing strength is a simple calculation, but it gives little insight into what is actually happening; in other words the equation operates as something of a 'black box'. The locus of all the combinations of vertical load, horizontal shear and moment given by equation (1) forms a three-dimensional BSS, which affords a useful visual appreciation of the workings of equation (1).

With appropriate rearrangements of equation (1) there are two surfaces to consider: that for the undrained case and that for the drained case. A convenient way of presenting the surfaces is to use axes defined in terms of dimensionless foundation actions: one for vertical load, another for horizontal shear and a third for moment. The normalising parameter is the ultimate bearing strength of the foundation when subject to vertical load only. The suite of dimensionless actions used herein is

$$\mathrm{Vn} = \frac{V}{V_\mathrm{uo}}, \quad \mathrm{Hn} = \frac{H}{V_\mathrm{uo}}, \quad \mathrm{Mn} = \frac{M}{BV_\mathrm{uo}} \tag{2}$$

where V, H and M are actions applied to the foundation; Vn, Hn and Mn are dimensionless foundation actions; B is the width of the foundation; and V_{uo} is the ultimate vertical load capacity of the foundation in the absence of shear and moment.

The sign convention used for the actions applied to the foundation follows that of Butterfield *et al*. (1997).

When moment is applied about the length axis, the bearing strength is estimated by using a reduced foundation width as proposed by Meyerhof (1953)

$$B' = B - \frac{2M}{V} = B\left(1 - \frac{2M}{VB}\right) \tag{3}$$

An important assumption underlying equation (3) is that no tensile stress is possible between the underside of the foundation and the soil beneath, so conventional bearing strength calculations allow only compressive normal stresses at the interface between the foundation and the underlying soil.

BEARING STRENGTH SURFACES FOR SURFACE STRIP FOOTINGS

Undrained case

For a strip foundation at the surface of a clay layer, subject to vertical load, moment and shear, the undrained BSS derived from the relevant modification of equation (1) is given by

$$\begin{aligned} f_{\mathrm{strip-undrained}}(\mathrm{Vn},\mathrm{Hn},\mathrm{Mn}) &= \left[2\mathrm{Vn} - \left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]^2 \\ &- \left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2 + N_\mathrm{c}|\mathrm{Hn}|\left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right) = 0 \end{aligned} \tag{4}$$

This equation is developed in Appendix 2. A view of the upper half of this surface is given in Fig. 1. The $(1 - 2|\mathrm{Mn}|/\mathrm{Vn})$ terms in the above and following equations arise because moment acts about the longitudinal axis and so the width B' contributes to the bearing strength.

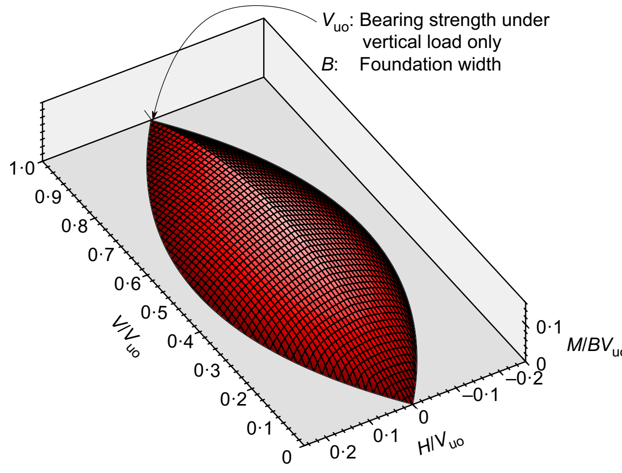

Fig. 1. Undrained BSS for a surface strip foundation

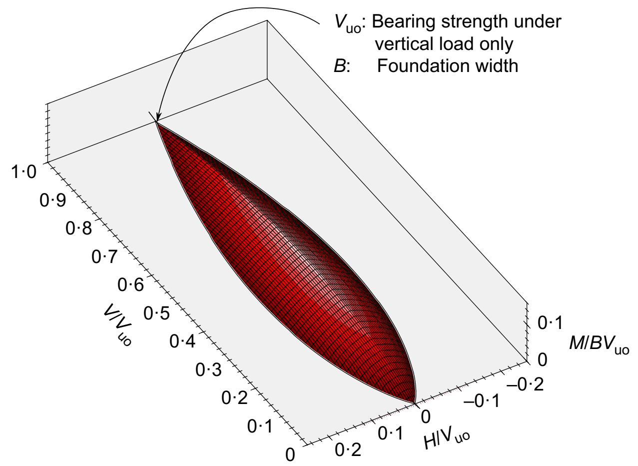

Fig. 2. Drained BSS for a surface strip foundation

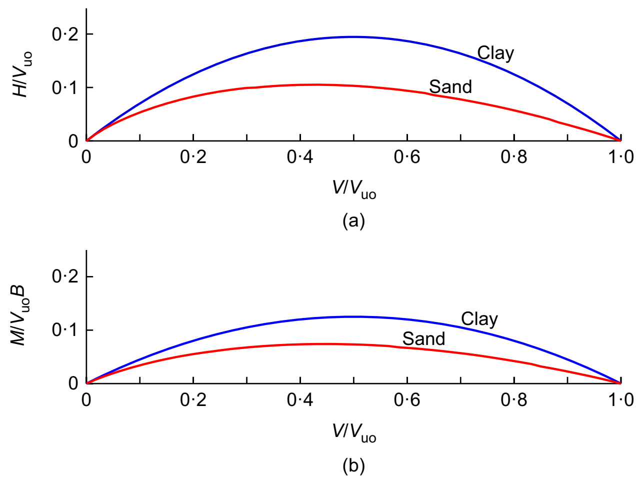

Fig. 3. Comparison of (a) *H–V* and (b) *M–V* sections of the BSSs for clay (undrained) and sand (drained)

Limiting cases, the Mn = 0 and Hn = 0 contours, are helpful in plotting views of the surface. The $\mathrm{Hn}_{(\mathrm{Mn}=0)}$ contour in the Hn–Vn plane is obtained as the solution to the following equation

$$N_c|\mathrm{Hn}| - 4\mathrm{Vn}(1 - \mathrm{Vn}) = 0 \tag{5}$$

The $\mathrm{Mn}_{(\mathrm{Hn}=0)}$ contour in the Mn–Vn plane is given by

$$\left[2\mathrm{Vn} - \left(1 - 2\frac{|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]^2 - \left(1 - 2\frac{|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2 = 0 \tag{6}$$

Drained case

For a strip foundation at the surface of a layer of cohesionless soil, subject to vertical load, horizontal shear and moment about the longitudinal axis, the drained static BSS derived from the relevant modification of equation (1) is given by

$$f_{\text{strip-drained}}(\mathrm{Vn}, \mathrm{Hn}, \mathrm{Mn}) = \mathrm{Vn} - \left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2 \times \left(1 - \frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^3 = 0 \tag{7}$$

This equation is developed in Appendix 2. A view of the upper half of the surface is given in Fig. 2.

Limiting cases, the Mn = 0 and Hn = 0 contours, are helpful in plotting views of the surface. The $\mathrm{Hn}_{(\mathrm{Mn}=0)}$ contour is given by

$$\mathrm{Vn} - \left(1 - \frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^3 = 0 \tag{8}$$

The $\mathrm{Mn}_{(\mathrm{Hn}=0)}$ contour is obtained as the solution to the following equation

$$\mathrm{Vn} - \left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2 = 0 \tag{9}$$

Figures 1 and 2 are drawn to the same scale to aid comparison. Note that the vertical load is restricted to positive values, whereas the moment and shear can be positive or negative. For vertical loads near V_{uo} only small values of shear and moment are possible, whereas the largest values of shear and moment can be sustained when Vn is about 0·5.

The BSSs show that the capacity of a shallow foundation is not a single number, but the combination of the vertical load, horizontal shear force and moment where the action path (locus of the combinations of vertical load, horizontal shear and moment applied to the foundation) intersects the BSS.

The appearance of the surface in Fig. 2 is similar to those proposed by Georgiadis (1985) and Butterfield & Gottardi (1994) with the exception that constant Vn sections of their surfaces are smooth curves (Fig. 11), whereas Fig. 2 has ridges along the Hn = 0 and Mn = 0 contours. The drained BSS for cohesionless soil in Fig. 2 also has a similar shape to the surface obtained from the equation given in Part V of Eurocode 8 (BSI, 2005) when the earthquake acceleration is set to zero.

Figure 3 compares the *H–V* and *M–V* sections of the two surfaces, equations (5), (6), (8) and (9). These show that, in terms of dimensionless actions, the shallow foundation on clay has greater shear and moment capacity than a foundation on sand. However, when the dimensionless action values are converted to actual actions this may not be the case, as V_{uo} for the foundation on sand may be greater than V_{uo} for a foundation of the same width on clay.

BEARING STRENGTH SURFACES FOR RECTANGULAR EMBEDDED FOUNDATIONS

An important function for the λ factors in equation (1) is to enable bearing strength calculations to approximate the effects of foundation shape and embedment. In this section equations (4) and (7) for surface strip foundations are extended to include these effects.

Undrained case

Equation (4) for the undrained case, when extended to include shape and depth factors, becomes for gross actions

$$f_{\text{full-undrained}}(\mathrm{Vn}, \mathrm{Hn}, \mathrm{Mn}) = \mathbb{S}^2 - 2\mathbb{S} + \frac{\Gamma|\mathrm{Hn}|N_c}{1 - 2|\mathrm{Mn}|/\mathrm{Vn}} \tag{10}$$

where

$$\mathbb{S} = \frac{2\Gamma\mathrm{Vn}}{\left[(1 - 2|\mathrm{Mn}|/\mathrm{Vn})^{0\cdot5} + \alpha_1\chi(1 - 2|\mathrm{Mn}|/\mathrm{Vn})^{1\cdot5} + \alpha_2\eta^{0\cdot5}\right]\left[(1 - 2|\mathrm{Mn}|/\mathrm{Vn})^{0\cdot5} + \beta_1\eta^{0\cdot5}\right] + \omega(1 - 2|\mathrm{Mn}|/\mathrm{Vn})}$$

and χ is B/L, η is (D_f/B), ω is $\gamma D_f/s_u N_c$, Γ is $(1+\alpha_1\chi+\alpha_2\eta^{0\cdot5})(1+\beta_1\eta^{0\cdot5})+\omega$ and, for calculating normalised foundation actions

$$V_{uo} = s_u N_c \Gamma BL \tag{11}$$

The values for α_1, α_2 and β_1 are given by Salgado *et al.* (2004) as 0·12, 0·17 and 0·27, respectively. These values of α_1 and α_2 are for $1 \leq L/B \leq 5$ and $0 \leq D_f/B \leq 1$; for other configurations the values are slightly different and can be obtained from tables in the Salgado *et al.* (2004) paper. Reference to Appendix 1 indicates that the term $(1+\alpha_1\chi+\alpha_2\eta^{0\cdot5})$ in equation (10) is the shape factor, λ_{cs}, when the shallow foundation is subject only to vertical load; similarly $(1+\beta_1\eta^{0\cdot5})$ is the depth factor, λ_{cd}, when the foundation is subject only to vertical load. (The paper by Edwards *et al.* (2005) provides further insight into values for the depth factor λ_{cd}.)

Equation (10) is developed in Appendix 3. When ω is zero, the above surface applies to net foundation actions. Equation (10) gives the bearing strength derived only from the base of the foundation; no capacity is mobilised from the embedded sides of the foundation.

Limiting cases, the Mn = 0 and Hn = 0 contours, are helpful in plotting views of the surface. The $\mathrm{Hn}_{(\mathrm{Mn}=0)}$ contour is obtained as the solution to the following equation

$$N_c \Gamma \mathrm{Hn} - 4\mathrm{Vn}(1-\mathrm{Vn}) = 0 \tag{12}$$

The $\mathrm{Mn}_{(\mathrm{Hn}=0)}$ contour is given by

$$\Gamma \mathrm{Vn} - \left[\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{0\cdot5} + \alpha_1\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{1\cdot5} + \alpha_2\eta^{0\cdot5}\right] \times \left[\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{0\cdot5} + \beta_1\eta^{0\cdot5}\right] - \omega\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right) = 0 \tag{13}$$

In developing equation (10) for the BSS for an embedded rectangular foundation in saturated clay, difficulties were encountered with the term arising from the surcharge pressure q. The λ_{qi} parameter used in developing equation (10) is usually assigned a value of 1·0; Salgado (2008) gives a justification. Using this value the Hn–Vn boundary in the Mn = 0 plane becomes negative at very small values of Vn. Clearly this is not possible; the problem was overcome by assuming that $\lambda_{qi} = \lambda_{ci}$ rather than $\lambda_{qi} = 1\cdot0$. The reason is apparent from equation (12); the horizontal shear capacity needs to be zero when Vn is 0 and 1, which is ensured by the term Vn(1–Vn). However, if λ_{qi} is set to 1·0 then equation (12) becomes more complex and does not return zero when Vn = 0·0; setting $\lambda_{qi} = \lambda_{ci}$ was found to eliminate the problem.

Drained case

Equation (7) for the drained case (zero cohesion), when extended to include shape and depth factors, becomes

$$f_{\text{full-drained}}(\mathrm{Vn},\mathrm{Hn},\mathrm{Mn}) = \zeta\mathrm{Vn} - \left[1+\chi\sin(\phi)\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}+\psi\eta\right)\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^m + \frac{\kappa}{2\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2\left[1-0\cdot3\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^{m+1} \tag{14}$$

where χ and η have the same meanings as for equation (10), ϕ = angle of shearing resistance, $\kappa = N_\gamma/N_q$, $m = (2+\chi)/(1+\chi)$, $\psi = 2\tan(\phi)[1-\sin(\phi)]^2$, $\zeta = [1+\chi\sin(\phi)](1+\psi\eta)+\kappa(1-0\cdot3\chi)/2\eta$, and, for calculating normalised foundation actions

$$V_{uo} = \left\{[1+\chi\sin(\phi)](1+\psi\eta)+\frac{\kappa}{2\eta}(1-0\cdot3\chi)\right\}N_q\gamma D_f BL \tag{15}$$

Equation (14) is developed in Appendix 3. Equation (14) gives the bearing strength derived only from the base of the foundation; no capacity is mobilised from the embedded sides of the foundation.

Limiting cases, the Mn = 0 and Hn = 0 contours, are helpful in plotting views of the surface.

The $\mathrm{Hn}_{(\mathrm{Mn}=0)}$ contour is obtained as the solution to the following equation

$$\zeta Vn - [1+\chi\sin(\phi)](1+\psi\eta)\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^m - \frac{\kappa}{2\eta}(1-0\cdot3\chi)\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^{m+1} = 0 \tag{16}$$

The $\mathrm{Mn}_{(\mathrm{Hn}=0)}$ contour is obtained as the solution to the following equation

$$\zeta Vn - \left[1+\chi\sin(\phi)\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}+\psi\eta\right) - \frac{\kappa}{2\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2\left[1-0\cdot3\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right] = 0 \tag{17}$$

To the knowledge of the author this is the first time the BSSs implied by conventional bearing strength calculations, including the effects of foundation shape and depth– that is, using equation (1) – have been presented. Much of the work cited in the introduction was restricted to surface foundations.

COMPARISON OF BSS SHAPES FOR SURFACE STRIP AND EMBEDDED RECTANGULAR FOUNDATIONS

Equations (4) and (7) are relatively simple, whereas equations (10) and (14) are significantly more complex; consequently the purpose of this section is to compare the shapes of sections of the BSS to see if the surfaces for the surface strip foundation are related to those for embedded rectangular foundations. For the embedded rectangular foundations the magnitudes of V_{uo}, equations (11) and (15), are larger than those of the surface strip foundations. Clearly, shape and embedment effects increase the bearing strength under vertical load only, so, in calculating dimensionless foundation actions, this enhanced value is used for V_{uo}.

Undrained case

Figures 4 and 5 compare the shapes of the BSS for a surface strip foundation on clay with those for an embedded rectangular foundation. The Hn = 0 boundary is almost identical to that for the surface strip foundation, Fig. 4(a). However, the Mn = 0 boundary is not the same as that for the

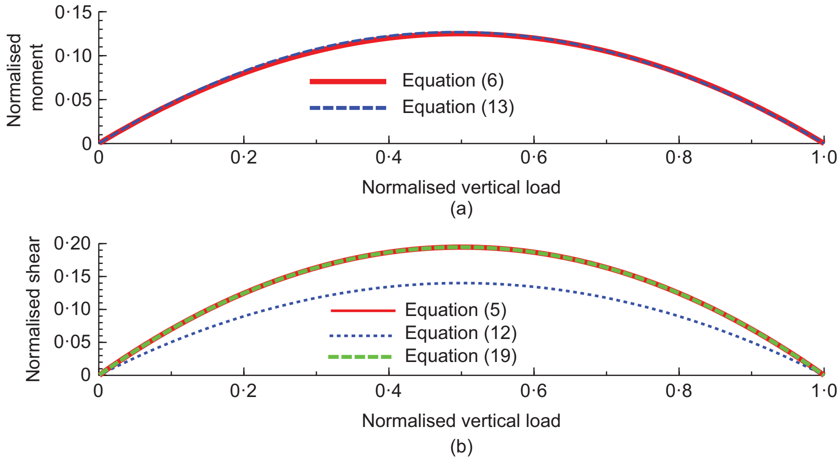

Fig. 4. Longitudinal sections of the clay BSS. (a) Hn = 0 sections showing that the boundary of the surface strip BSS is close to that for the embedded rectangular BSS. (b) Mn = 0 sections showing that the boundary for the surface strip foundation (equation (5)) is different from that for the embedded rectangular foundation (equation (12)), but using a different definition of Hn, equation (19) gives boundaries which are the same

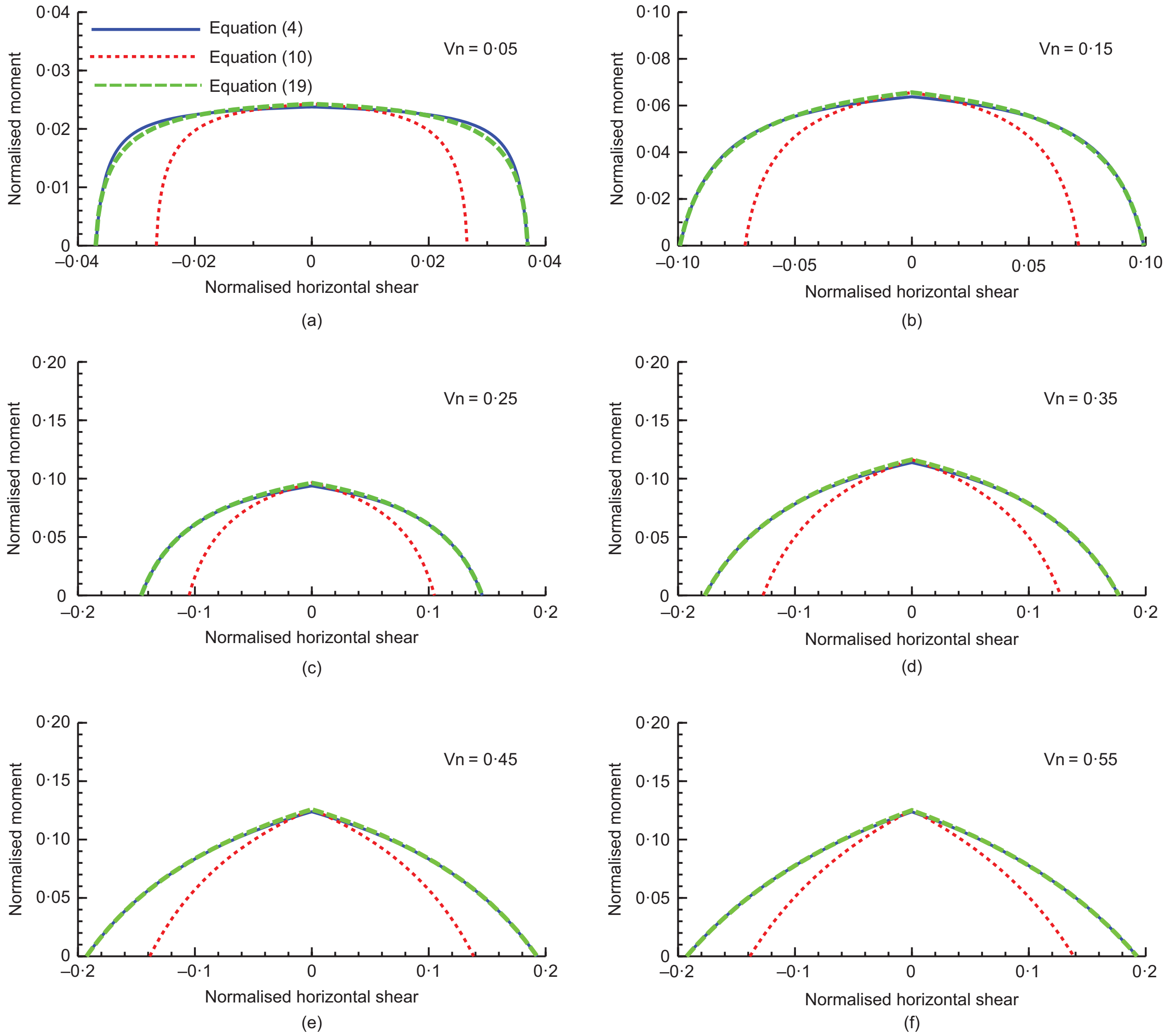

Fig. 5. Cross-sections of the clay BSS for Vn between 0·05 and 0·55. Equation (10) is for the embedded rectangular foundation and equation (4) for the surface strip foundation. Using the modified Hn parameter, Hn′ equation (19), the embedded rectangular BSS comes close to that for the surface strip foundation

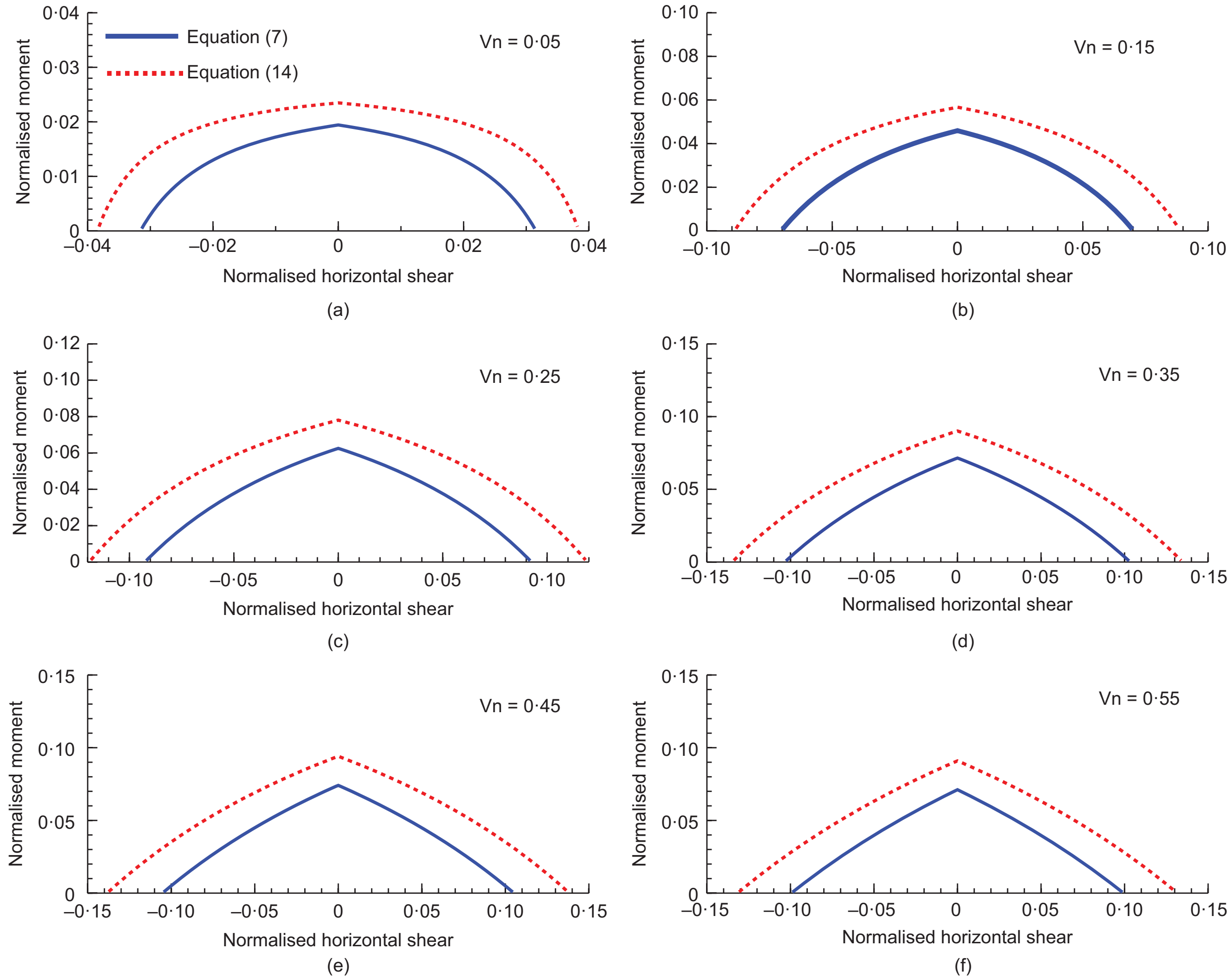

Fig. 6. Cross-sections of the drained surface strip BSS for a sand having a friction angle of 35° with Vn between 0·05 and 0·55. Equation (7) for the surface strip foundation and equation (14) for the embedded rectangular foundation

surface strip foundation, Fig. 4(b). A possible reason is that the λ_{ci} term is not changed by increased vertical load (apart from the effect this has on B'), that is the limiting shear capacity of the foundation–soil interface, being controlled by s_u, is independent of the vertical load and shape and embedment effects.

Comparison of equation (5), surface strip Mn = 0 contour, and equation (12), embedded rectangular Mn = 0 contour, shows that the only difference between the two contours is the parameter Γ. If a modified Hn parameter is defined

$$\mathrm{Hn}' = \mathrm{Hn}\Gamma \tag{18}$$

it is found that the Mn = 0 contour for the embedded rectangular foundation, Hn′ against Vn, is identical to the Mn = 0 contour for the surface strip foundation when Hn is plotted against Vn, as shown in Fig. 4(b).

This means that, instead of using equation (10) for the undrained embedded rectangular BSS, equation (4) for the surface strip foundation can be used with Hn′ instead of Hn.

$$f_{\text{full-undrained-approx}}(\mathrm{Vn},\mathrm{Hn}',\mathrm{Mn}) \approx \left[2\mathrm{Vn} - \left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]^2 - \left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2 + N_c|\mathrm{Hn}'|\left(1 - \frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right) = 0 \tag{19}$$

Figure 5 compares the shapes of Hn–Mn cross-sections of the undrained BSS surface for Vn between 0·05 and 0·55; it is clear that the cross-sections obtained using equation (19) are very close to those obtained from equation (10). The above cross-sections were calculated for $L/B = 1$; other calculations showed that the effect of the L/B ratio on the comparison is small.

Drained case

Figure 6 compares the shapes of the BSS for a surface strip foundation on sand having a friction angle of 35° with that for an embedded rectangular foundation. Note that the V_{uo} values used are different: V_{uo} for the surface strip foundation is less than the value per unit length for the embedded rectangular foundation. Even with the larger value of V_{uo} the size of the BSS for the embedded rectangular foundation is larger than that for the surface strip foundation. However, it is found that the shapes of the surfaces are approximately similar. The similarity is revealed in Fig. 7 by applying a scaling parameter, denoted herein as Ω, to cross-sections of the equation (7) surface; this parameter has a value greater than unity. A brief parameter study covering the range of friction angles between 25 and 45°, length to breadth ratios between 1 and 5, and embedment ratios between 0 and 1, showed that the values of Ω are in the range 1·2–1·7; they decrease as the friction angle increases and as the length to breadth ratio increases, and increase with increasing depth to breadth ratio. Fig. 7 shows, for a friction angle of 35°, a length to breadth ratio of 2, and a depth to breadth ratio of 1/3, that a value of 1·29 for Ω gives reasonably good matching

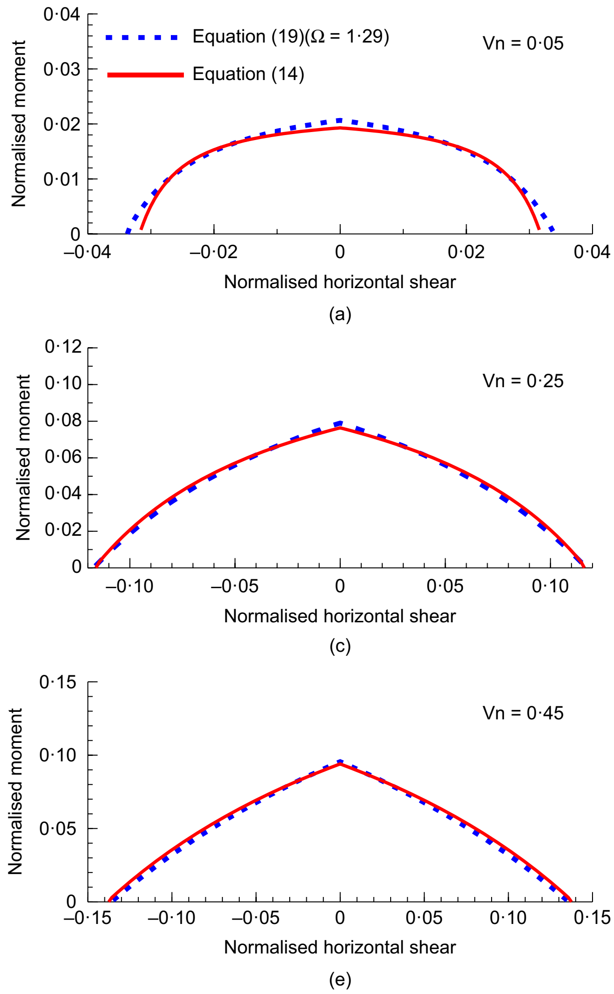

Fig. 7. Cross-sections of the drained embedded rectangular BSS for a sand having a friction angle of 35° with Vn between 0·05 and 0·55. Multiplying Mn and Hn in equation (14) for the surface strip BSS by Ω (≈ 1·29) means that the BSS for the embedded rectangular foundation can be described using the BSS for the surface strip foundation

of the cross-sections for Vn between 0·05 and 0·55 (it also works well for values of Vn beyond 0·55; however, this part of the surface is beyond the range of interest in practical applications). Closer examination reveals that for friction angles of 35° and greater a single value of Ω gives a reasonable match for all values of Vn, but at a friction angle of 25° a single value of Ω is not satisfactory. Given that equation (14) is much more complex than equation (7), the above suggests that the shape of the drained BSS, including the effect of foundation shape and embedment, can be approximated by using V_{uo} for the embedded rectangular foundation (obtained from equation (15)) and then using equation (7) with Mn and Hn multiplied by an appropriate value for the scaling parameter Ω. So, with this small modification of the simpler equation (7), an approximation to the surface specified by equation (14) can be obtained from

$$f_{\text{full-drained-approx}}(\mathrm{Vn}, \mathrm{Hn}, \mathrm{Mn}) \approx \mathrm{Vn} - \left(1 - \frac{2|\mathrm{Mn}|\Omega}{\mathrm{Vn}}\right)^2 \left(1 - \frac{|\mathrm{Hn}|\Omega}{\mathrm{Vn}}\right)^3 = 0 \qquad (20)$$

As explained above, initially it was thought that Ω was a function of the tangent of the friction angle, length to breadth ratio, and the ratio of embedment to breadth of the foundation; subsequently it was realised that the normalised vertical load, Vn, is a further factor at friction angles less than 35°. It should be possible to develop an equation by non-linear regression analysis expressing Ω in terms of the values for these four variables; this is not pursued herein.

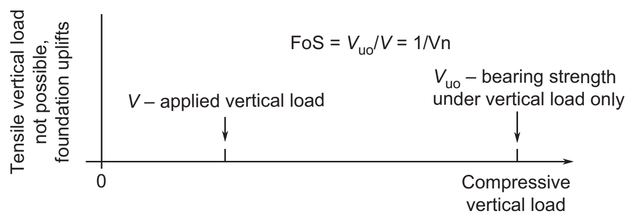

Fig. 8. Diagrammatic representation of the definition of the factor of safety (FoS) for a foundation subject only to vertical load

USE OF THE BSS SURFACES

Possibly the most important insight gained from BSS thinking is an understanding of the meaning of the standard definition of the bearing strength factor of safety. This is defined in terms of vertical loads (or bearing pressures) – the ratio of the vertical load that would cause bearing failure to the applied load. Fig. 8 shows how, for a foundation subject

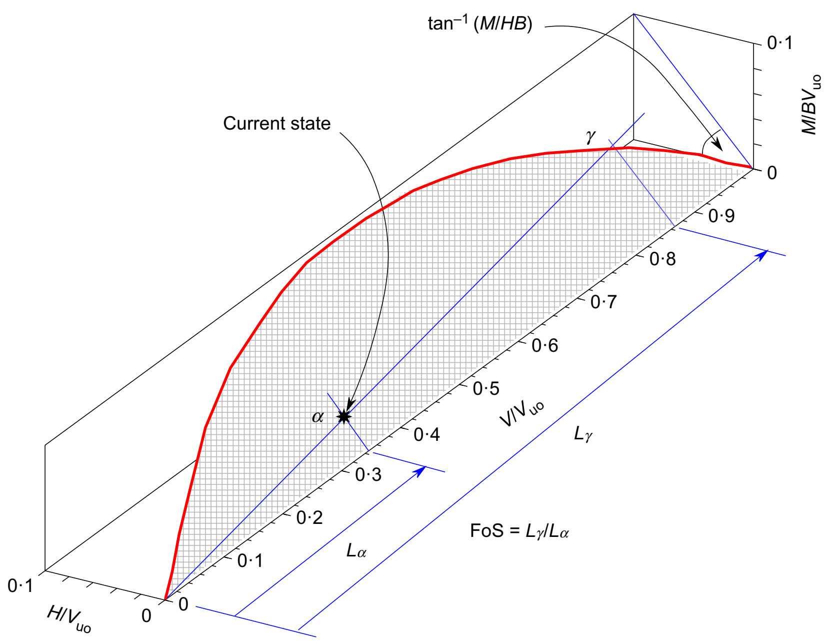

Fig. 9. Interpretation of the bearing strength factor of safety (FoS) in terms of the BSS for a foundation subject to vertical load, horizontal shear and moment

to vertical load only, this is the ratio of distances along the Vn axis. It is instructive to seek an interpretation of this factor in terms of the BSS when the applied actions on the foundation consist of vertical load, moment and shear force. One approach is to assume that B' remains constant while the vertical load is increased; Fig. 9 shows how the definition has a very specific and limited meaning associated with only one point on the surface (or four points if one takes account of the symmetry of the BSS). No insight is given for the myriad of other load, or action, paths that are possible from the current state in Fig. 9. This is particularly important when it is possible that increases in actions other than vertical load might occur. The current normalised vertical load in Fig. 9 will, typically, be somewhere in the range 0·1–0·3, so a large increase in vertical load would be needed to induce a bearing strength failure, but a much smaller increase in horizontal shear or moment; Pender (2015) gives worked examples illustrating this. In this sense the bearing strength factor of safety defined in terms of vertical load may be misleading and engender a false notion of security.

The above comments add weight to the point made by Butterfield (1993) and Georgiadis (1985) that the action path is important in assessing the bearing strength of a shallow foundation. In principle the BSS provides an effective visual means of appreciating what will happen under a given action path and hence gives a method for assessing combinations of ultimate actions. Having an expression for the surface it is then possible to plot the section of the surface in which the action path lies.

Frequently the vertical load will remain fixed while the moment and shear increase (for example the foundation for a gravity retaining wall, a wind turbine with a shallow foundation, or shallow foundation supporting a structure subject to earthquake). In these cases shear and moment are the actions driving instability and the vertical load acts as a stabilising influence. Usually the shear and moment are related as the shear applied some distance above foundation level generates the moment. Fig. 10 shows how the bearing strength factor of safety with respect to foundation moment (and shear) might be defined as the ratio of the distances OA and OB. Depending on the situation, the section of the BSS

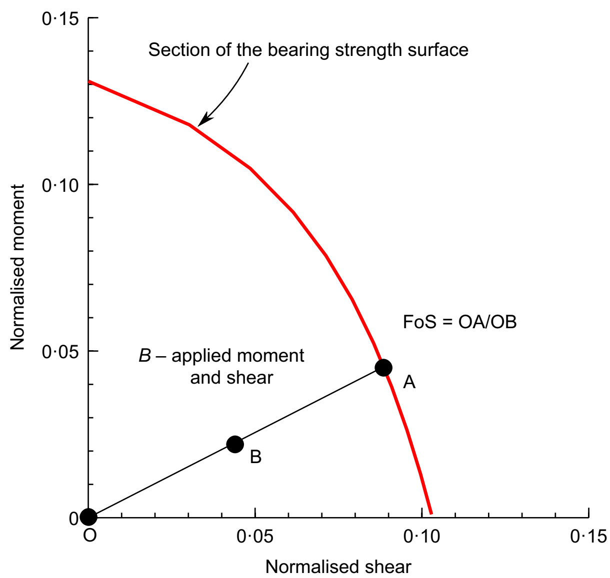

Fig. 10. Use of the BSS for estimation of the factor of safety (FoS) with respect to moment for foundation subject to shear and moment at constant vertical load

can be calculated using equations (19) or (20) rather than the more complex equations (10) and (14).

SHALLOW FOUNDATION BEARING STRENGTH FROM NUMERICAL ANALYSIS AND PHYSICAL MODELLING

The references cited in the introduction derived BSSs from three-dimensional non-linear finite-element analyses or with data obtained from physical modelling. A consistent conclusion from these papers is that constant Vn sections of the BSS are not symmetrical about the moment axis. Fig. 11 has an example, from Butterfield & Gottardi (1994), which shows sections of the surface skewed towards the quadrant

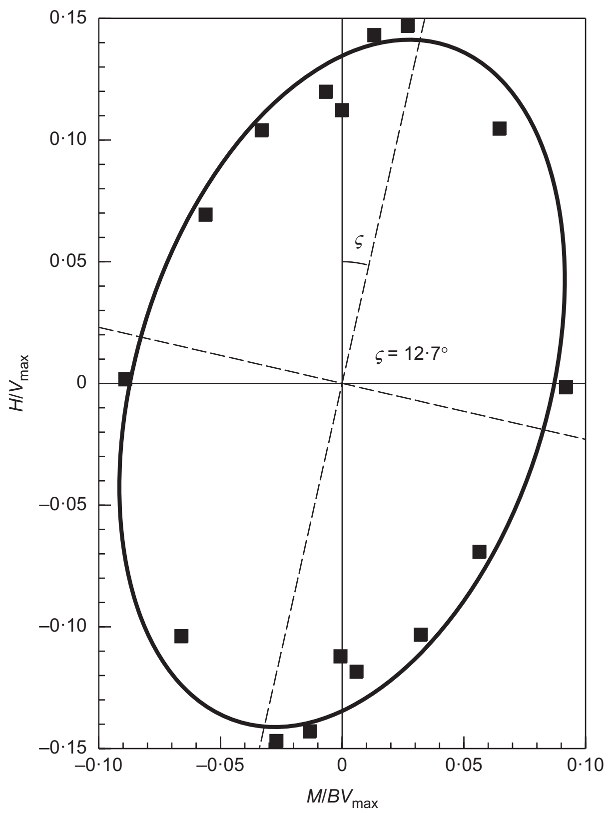

Fig. 11. Shallow foundation bearing strength data fitted to a Vn = 0·5 cross-section of a BSS in the form of a skewed ellipse – points are the results of physical model tests (after Butterfield & Gottardi (1994))

with positive moment and positive shear force and the quadrant with negative moment and negative shear force. Taiebat & Carter (2000), Bransby & Randolph (1998), Martin & Houlsby (2001) and Gourvenec (2008) also show asymmetry about the moment axis for Mn–Hn sections of the BSS. The BSSs derived from equation (1) are not capable of representing this aspect of shallow foundation behaviour.

Several of these research studies have proposed expressions for a BSS much simpler than equations (10) and (14). Because they are considering applications to offshore shallow foundations they accept tensile contact stresses between the underside of the foundation and the underlying soil. This is justified as a consequence of suction generated at the interface, something not usually considered in onshore shallow foundations. Gourvenec (2008) and Zhang *et al.* (2012) considered a circular foundation embedded in saturated clay subject to shear and moment as well as vertical load. Using non-linear finite-element analysis they found that the interactions along the sides of the foundation have a significant effect on the capacity, a factor which is not considered in the BSSs discussed herein.

The above comments suggest that interpretation of data from sophisticated finite-element modelling, which includes interaction along the foundation sides but does not allow tensile interaction with the surrounding soil, might lead to BSS equations simpler than equations (10) and (14). Perhaps the time has come for a new look at shallow foundation bearing strength under general loading for onshore applications. Building on the work of Butterfield, Geordiadis, Gottardi and Pecker there is a need to develop an approach that discards the ‘add-on’ λ terms set out in Appendix 1, but incorporates: (*a*) not only shape and embedment effects, but also the contribution to bearing strength which comes from compressive contact stresses along the foundation sides, and (*b*) the skew between the principal axes of the BSS and the moment and shear axes (illustrated in Fig. 11). If this can be achieved, BSS thinking will be attractive to those practising onshore geotechnical engineering.

CONCLUSIONS

The following conclusions are reached.

(*a*) The most important conclusion from the paper is that behind conventional shallow foundation bearing strength calculations there are BSSs which show that the capacity of a shallow foundation is not a single number but the combination of the vertical load, horizontal shear force and moment where the action path (locus of the combinations of vertical load, horizontal shear and moment applied to the foundation) intersects the BSS.

(*b*) The classical definition of the bearing strength factor of safety, focused on vertical loads, provides limited and possibly misleading understanding for foundations subject to vertical load, horizontal shear and moment (Fig. 9).

(*c*) The relatively simple equations (4) and (7) give the BSSs for surface strip foundations. Fig. 1 gives the surface for the undrained clay case, and Fig. 2 that for the drained cohesionless case. Figs 1 and 2, even without calculations, clarify, in a qualitative manner, shallow foundation bearing strength under general loading.

(*d*) When the effects of shape and embedment are included, the resulting BSS equations (10) and (14) are cumbersome.

(*e*) Using a modified definition of Hn, equation (18), the equation for the undrained surface strip BSS, approximates closely the undrained BSS for an embedded rectangular foundation, equation (19) and Figs 4 and 5.

(*f*) Figure 7 shows how a single value for a scaling factor, Ω, can transform the drained surface strip BSS to a close approximation to that for an embedded rectangular foundation, equation (20).

(*g*) Neither of the surfaces described by equations (10) and (14) includes contributions to the bearing strength from resistance developed along the embedded sides of the foundations.

(*h*) Figure 10 shows how a section of the BSS can be used to define the factor of safety with respect to moment for a foundation under constant vertical load.

(*i*) None of the BSSs described by equations (4), (7), (10) and (14) has principal axes skewed in the Mn–Hn plane like the section from the experimentally determined surface shown in Fig. 11.

APPENDIX 1. SHALLOW FOUNDATION BEARING STRENGTH λ FACTORS

The general ultimate bearing capacity equation (1)

$$q_u = c\lambda_{cs}\lambda_{cd}\lambda_{ci}N_c + q\lambda_{qs}\lambda_{qd}\lambda_{qi}N_q + \frac{1}{2}\gamma B\lambda_{\gamma s}\lambda_{\gamma d}\lambda_{\gamma i}N_\gamma \tag{21}$$

where λ_{cs}, λ_{qs} and $\lambda_{\gamma s}$ are the shape factors; λ_{cd}, λ_{qd} and $\lambda_{\gamma d}$ are the depth factors; and λ_{ci}, λ_{qi} and $\lambda_{\gamma i}$ are the inclined load factors.

Shape factors: λ_{cs}, λ_{qs} and $\lambda_{\gamma s}$
for $\phi = 0$

$$\lambda_{cs} = 1 + 0{\cdot}12\left(\frac{B}{L}\right) + 0{\cdot}17\sqrt{\frac{D_f}{B}} \quad \lambda_{qs} = 1 \tag{22}$$

for $\phi > 0$

$$\lambda_{cs} = \frac{\lambda_{qs}N_q - 1}{N_q - 1} \tag{23}$$

$$\lambda_{qs} = 1 + \left(\frac{B}{L}\right)\sin\phi \tag{24}$$

$$\lambda_{\gamma s} = 1 - 0{\cdot}3\left(\frac{B}{L}\right) \tag{25}$$

Depth factors: λ_{cd}, λ_{qd} and $\lambda_{\gamma d}$ are considered in turn below.
λ_{cd}

$$\text{for } \phi = 0: \quad \lambda_{cd} = 1 + 0{\cdot}27\sqrt{\frac{D_f}{B}} \tag{26}$$

$$\text{for } \phi > 0: \quad \lambda_{cd} = \lambda_{qd} - \frac{(1-\lambda_{qd})}{N_q \tan\phi} \tag{27}$$

λ_{qd}

$$\text{for } \frac{D_f}{B} \leq 1: \quad \lambda_{qd} = 1 + 2\tan\phi(1-\sin\phi)^2\left(\frac{D_f}{B}\right) \tag{28}$$

$$\text{for } D_f/B > 1: \quad \lambda_{qd} = 1 + 2\tan\phi(1-\sin\phi)^2\tan^{-1}\left(\frac{D_f}{B}\right) \tag{29}$$

$\lambda_{\gamma d}$

$$\lambda_{\lambda d} = 1 \tag{30}$$

Inclined load factors: λ_{ci}, λ_{qi} and $\lambda_{\gamma i}$
for $\phi = 0$

$$\lambda_{ci} = 0{\cdot}5\left(1 + \sqrt{1 - H/As_u}\right) \quad \lambda_{qi} = 1 \tag{31}$$

for $\phi > 0$

$$\lambda_{ci} = \lambda_{qi} - \frac{1 - \lambda_{qi}}{N_c \tan\phi} \tag{32}$$

$$\lambda_{qi} = \left(1 - \frac{H}{V + Ac\cot\phi}\right)^m \tag{33}$$

$$\lambda_{\gamma i} = \left(1 - \frac{H}{V + Ac\cot\phi}\right)^{m+1} \tag{34}$$

where

$$m = m_B = \frac{2 + B/L}{1 + B/L} \quad \text{when } H \text{ acts in the direction of } B \tag{35}$$

$$m = m_L = \frac{2 + L/B}{1 + L/B} \quad \text{when } H \text{ acts in the direction of } L \tag{36}$$

In cases where the horizontal load acts in a direction forming an angle θ with the direction of L, m may be calculated from

$$m = m_L\cos^2\theta + m_B\sin^2\theta \tag{37}$$

The applied actions must be expressed as dimensionless actions

$$\frac{H}{V} = \frac{\text{Hn}}{\text{Vn}}; \quad \text{Vn}\,V_{uo} = V; \quad \frac{\text{Mn}}{\text{Vn}} = \frac{M}{V_{uo}B}\frac{1}{V/V_{uo}} = \frac{M}{VB} \quad \frac{M}{V} = \frac{\text{Mn}B}{\text{Vn}}$$

If the foundation is subject to moment loading then B' is used, given by

$$B' = \left(B - \frac{2M}{V}\right) = B\left(1 - \frac{2\text{Mn}}{\text{Vn}}\right)$$

For equation (4) and Fig. 1 – surface strip foundation on saturated clay

$$q_u = s_u N_c \lambda_{ci} = \frac{1}{2}s_u N_c\left(1 + \sqrt{1 - \frac{|H|}{B's_u}}\right) \quad \text{and} \quad V_{uo} = s_u N_c B$$

Convert the ultimate bearing pressure to ultimate vertical load per unit length by multiplying by B'

$$V = \frac{1}{2}s_u N_c B'\left(1 + \sqrt{1 - \frac{|H|}{B's_u}}\right)$$

Convert to dimensionless actions by dividing both sides by V_{uo}

$$\text{Vn} = \frac{1}{2}\frac{s_u N_c B'}{s_u N_c B}\left(1 + \sqrt{1 - \frac{|H|}{B's_u}}\right)$$

Substitute for B'

$$\text{Vn} = \frac{1}{2}\left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)\left[1 + \sqrt{1 - \frac{|H|}{(B - 2|M|/V)s_u}}\right]$$

Convert the term inside the square root bracket to dimensionless form

$$\text{Vn} = \frac{1}{2}\left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)\left[1 + \sqrt{1 - \frac{|\text{Hn}|N_c}{(1 - 2|\text{Mn}|/\text{Vn})}}\right]$$

Rearrange and eliminate the square root

$$\left[2\text{Vn} - \left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)\right]^2 = \left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)^2\left[1 - \frac{|\text{Hn}|N_c}{(1 - 2|\text{Mn}|/\text{Vn})}\right]$$

Final rearranging

$$\left[2\text{Vn} - \left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)\right]^2 - \left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)^2 + N_c|\text{Hn}|\left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right) = 0$$

For equation (7) and Fig. 2 – surface strip foundation on sand (for a strip foundation the parameter m takes the value 3 (equation (36))

$$q_u = \frac{1}{2}B'\gamma N_\gamma\left(1 - \frac{|H|}{V}\right)^3 \quad \text{and} \quad V_{uo} = \frac{1}{2}B^2\gamma N_\gamma$$

Convert q_u to the vertical load per unit length of the strip foundation

$$V = \frac{1}{2}B'^2\gamma N_\gamma\left(1 - \frac{|H|}{V}\right)^3$$

Convert to dimensionless actions by dividing by V_{uo}

$$\text{Vn} = \left(\frac{B'}{B}\right)^2\left(1 - \frac{|\text{Hn}|}{\text{Vn}}\right)^3$$

Substitute for (B'/B) and rearrange

$$\text{Vn} - \left(1 - \frac{2|\text{Mn}|}{\text{Vn}}\right)^2\left(1 - \frac{|\text{Hn}|}{\text{Vn}}\right)^3 = 0$$

APPENDIX 2. BEARING STRENGTH SURFACES FOR SURFACE STRIP FOUNDATIONS

This appendix considers the BSSs for strip foundations at the ground surface, subject to vertical load, shear and moment about the length axis (equations (4) and (5)).

APPENDIX 3. BEARING STRENGTH SURFACES FOR EMBEDDED RECTANGULAR FOUNDATIONS

This appendix considers development of the BSSs for embedded rectangular foundations, subject to vertical load, shear and moment about the length axis (equations (6) and (10)).

The shear force is applied in the direction of the foundation width (see equation (35) for the definition of m). This means that the moment is applied about the length axis.

Basic definitions are given below.

$$\chi = \frac{B}{L}; \quad \eta = \frac{D_f}{B}; \quad \kappa = \frac{N_\gamma}{N_q}; \quad m = \frac{2+\chi}{1+\chi};$$

$$\psi = 2\tan(\phi)[1-\sin(\phi)]^2; \quad \omega = \frac{\gamma D_f}{s_u N_c}$$

$$\zeta = [1+\chi\sin(\phi)](1+\psi\eta) + \frac{\kappa}{2\eta}(1-0{\cdot}3\chi)$$

$$\frac{H}{V} = \frac{\mathrm{Hn}}{\mathrm{Vn}}; \quad \mathrm{Vn}V_{uo} = V; \quad \frac{\mathrm{Mn}}{\mathrm{Vn}} = \frac{M}{V_{uo}B}\frac{1}{V/V_{uo}} = \frac{M}{VB}$$

$$B' = B - \frac{2M}{V} = B\left(1-\frac{2M}{VB}\right) = B\left(1-\frac{2\mathrm{Mn}}{\mathrm{Vn}}\right);$$

$$\frac{B'}{L} = \chi\left(1-\frac{2\mathrm{Mn}}{\mathrm{Vn}}\right)$$

Equation (10) – shallow foundation on saturated clay

$$\lambda_{cs} = 1 + \alpha_1\left(\frac{B}{L}\right) + \alpha_2\sqrt{\frac{D_f}{B}}$$

$$\lambda_{cd} = 1 + \beta_1\sqrt{\frac{D_f}{B}}$$

$V_{uo} = (s_u N_c \lambda_{csB}\lambda_{cdB} + \gamma D_f)BL = \Gamma s_u \ N_c BL$ where λ_{csB} and λ_{cdB} are the values when the moment is zero, and where $\Gamma = \lambda_{csB}\lambda_{cdB} + \omega = [1+\alpha_1\chi + \alpha_2(\eta)^{1/2}][1+\beta_1(\eta)^{1/2}] + \omega$

$$H = \mathrm{Hn}V_{uo} = \mathrm{Hn}\Gamma s_u N_c BL \quad \frac{H}{s_u BL} = \mathrm{Hn}\Gamma N_c$$

Equation (14) – shallow foundation on sand

$$\begin{aligned} V_{uo} &= \left[D_f N_q(1+\chi\sin(\phi))(1+\psi\eta) + \frac{1}{2}BN_\gamma(1-0{\cdot}3\chi)\right]\gamma BL \\ &= \left[(1+\chi\sin(\phi))(1+\psi\eta) + \frac{\kappa}{2\eta}(1-0{\cdot}3\chi)\right]N_q D_f\gamma BL \\ &= \zeta N_q D_f \gamma BL \end{aligned}$$

$$\begin{aligned} V_u = &\left\{D_f N_q\left[1+\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\sin(\phi)\right]\left(1+\frac{\psi\eta}{1-2|\mathrm{Mn}|/\mathrm{Vn}}\right)\right. \\ &\times\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^m + \frac{1}{2}B\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)N_\gamma\left[1-0{\cdot}3\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right] \\ &\times\left.\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^{m+1}\right\}\gamma\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)BL \end{aligned}$$

$$\begin{aligned} V_u = &\left\{\left[1+\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\sin(\phi)\right]\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}+\psi\eta\right)\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^m\right. \\ &+\left.\frac{\kappa}{2\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2\left[1-0{\cdot}3\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^{m+1}\right\} \\ &\times N_q D_f \gamma BL \end{aligned}$$

Now divide both sides by V_{uo} and simplify, to obtain

$$\begin{aligned} \zeta\mathrm{Vn} = &\left[1+\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\sin(\phi)\right]\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}+\psi\eta\right)\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^m \\ &+\frac{\kappa}{2\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^2\left[1-0{\cdot}3\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)\right]\left(1-\frac{|\mathrm{Hn}|}{\mathrm{Vn}}\right)^{m+1} \end{aligned}$$

$$\begin{aligned} V_u = s_u N_c &\left\{\left[1+\alpha_1\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)+\alpha_2\sqrt{\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{-0{\cdot}5}\right]\left[1+\beta_1\sqrt{\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{-0{\cdot}5}\right]+\omega\right\}x \\ &\times\frac{1}{2}\left[1+\sqrt{1-\frac{|H|}{s_u BL(1-2|\mathrm{Mn}|/\mathrm{Vn})}}\right]\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)BL \end{aligned}$$

$$\begin{aligned} \mathrm{Vn} = &\left\{\left[1+\alpha_1\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)+\alpha_2\sqrt{\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{-0{\cdot}5}\right]\left[1+\beta_1\sqrt{\eta}\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{-0{\cdot}5}+\omega\right]\right\}x \\ &\times\frac{1}{2}\left[1+\sqrt{1-\frac{|\mathrm{Hn}|}{s_u BL(1-2|\mathrm{Mn}|/\mathrm{Vn})}}\right]\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)s_u N_c BL/s_u N_c \Gamma BL \end{aligned}$$

$$\begin{aligned} 2\mathrm{Vn}\Gamma = &\left\{\left[\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{0{\cdot}5}+\alpha_1\chi\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{1{\cdot}5}+\alpha_2\sqrt{\eta}\right]\left[\left(1-\frac{2|\mathrm{Mn}|}{\mathrm{Vn}}\right)^{0{\cdot}5}+\beta_1\sqrt{\eta}\right]+\omega\right\}x \\ &\times\left[1+\sqrt{1-\frac{|\mathrm{Hn}|}{s_u BL(1-2|\mathrm{Mn}|/\mathrm{Vn})}}\right] \end{aligned}$$

$$\begin{aligned} &\frac{2\mathrm{Vn}\Gamma}{\left[(1-2|\mathrm{Mn}|/\mathrm{Vn})^{0{\cdot}5}+\alpha_1\chi(1-2|\mathrm{Mn}|/\mathrm{Vn})^{1{\cdot}5}+\alpha_2\sqrt{\eta}\right]\left[(1-2|\mathrm{Mn}|/\mathrm{Vn})^{0{\cdot}5}+\beta_1\sqrt{\eta}\right]+\omega(1-2|\mathrm{Mn}|/\mathrm{Vn})}-1 \\ &\quad=\sqrt{1-\frac{|\mathrm{Hn}|\Gamma N_c}{(1-2|\mathrm{Mn}|/\mathrm{Vn})}} \end{aligned}$$

Squaring both sides and simplifying gives

$$\mathbb{S}^2 - 2\mathbb{S} + \frac{|\mathrm{Hn}|\Gamma N_c}{1-2|\mathrm{Mn}|/\mathrm{Vn}} = 0$$

where

$$\mathbb{S} = \frac{2\Gamma\mathrm{Vn}}{\left[(1-2|\mathrm{Mn}|/\mathrm{Vn})^{0{\cdot}5}+\alpha_1\chi(1-2|\mathrm{Mn}|/\mathrm{Vn})^{1{\cdot}5}+\alpha_2\sqrt{\eta}\right]\left[(1-2|\mathrm{Mn}|/\mathrm{Vn})^{0{\cdot}5}+\beta_1\sqrt{\eta}\right]+\omega(1-2|\mathrm{Mn}|/\mathrm{Vn})}$$

NOTATION

B	foundation width
B'	reduced foundation width in the presence of moment about the length axis
c	cohesive component of shear strength
D_f	depth to underside of the foundation
Hn'	$= \Gamma\mathrm{Hn}$, modified value of Hn used for approximating undrained BSS including shape and depth effects from that for a surface strip foundation
$\mathrm{Hn}_{(\mathrm{Mn}=0)}$	Hn–Vn contour when $\mathrm{Mn} = 0$
L	foundation length
L'	reduced foundation length in the presence of moment about the breadth axis
$\mathrm{Mn}_{(\mathrm{Hn}=0)}$	Mn–Vn contour when $\mathrm{Hn} = 0$
m	$= (2 + \chi)/(1 + \chi)$
N_c, N_q, N_γ	bearing capacity factors
q	surcharge pressure adjacent to the foundation
q_u	ultimate gross bearing pressure
s_u	undrained shear strength of saturated clay
V, H, M	actions applied to the foundation
Vn, Hn, Mn	dimensionless foundation actions
V_{uo}	ultimate vertical load capacity of the foundation in the absence of shear and moment
α_1, α_2, β_1	coefficients used in evaluating λ_{cs}
γ	soil unit weight
ζ	$= (1 + \chi\sin(\phi))(1 + \psi\eta) + \kappa(1–0{\cdot}3\chi)/2\eta$
η	$= (D_f/B)$
κ	$= N_\gamma/N_q$
Γ	$= (1 + \alpha_1\chi + \alpha_2\eta^{0{\cdot}5})(1 + \beta_1\eta^{0{\cdot}5}) + \omega$
λ_{cd}, λ_{qd}, $\lambda_{\gamma d}$	depth factors
λ_{ci}, λ_{qi}, $\lambda_{\gamma i}$	inclined load factors
λ_{cs}, λ_{qs}, $\lambda_{\gamma s}$	shape factors
ϕ	angle of shearing resistance
χ	$= B/L$
ψ	$= 2\tan(\phi)(1–\sin(\phi))^2$
Ω	scaling parameter for approximating the drained BSS including shape and depth effects from that for a surface strip foundation
ω	$= \gamma D_f/s_u N_c$

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