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Embedding Quantum Universes into Classical Ones

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Embedding Quantum Universes into Classical Ones *

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Abstract

Do the partial order and lattice operations of a quantum logic correspond to the logical implication and connectives of classical logic? Re-phrased, how far might a classical understanding of quantum mechanics be, in principle, possible? A celebrated result by Kochen and Specker answers the above question in the negative. However, this answer is just one among different possible ones, not all negative. It is our aim to discuss the above question in terms of mappings of quantum worlds into classical ones, more specifically, in terms of embeddings of quantum logics into classical logics; depending upon the type of restrictions imposed on embeddings the question may get negative or positive answers.

1 Introduction

Quantum mechanics is a fantastically successful theory which appears to predict novel "mindboggling" phenomena (see Wheeler [39] Greenberger, Horne and Zeilinger [11]) even almost a century after its development, cf. Schrödinger [32], Jammer [14, 15]. Yet, it can be safely stated that quantum theory is not understood (Feynman [9]). Indeed, it appears that progress is fostered by abandoning long–held beliefs and concepts rather than by attempts to derive it from some classical basis, cf. Greenberg and YaSin [12].

But just how far might a classical understanding of quantum mechanics be, in principle, possible? We shall attempt an answer to this question in terms of mappings of quantum worlds into classical ones, more specifically, in terms of embeddings of quantum logics into classical logics.

One physical motivation for this approach is a result proven for the first time by Kochen and Specker [19] (cf. also Specker [33], Zierler and Schlessinger [41] and John Bell [2]; see reviews by Mermin [23], Svozil and Tkadlec [38], and a forthcoming monograph

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by Svozil [36]) stating the impossibility to "complete" quantum physics by introducing noncontextual hidden parameter models. Such a possible "completion" had been suggested, though in not very concrete terms, by Einstein, Podolsky and Rosen (EPR) [8]. These authors speculated that "elements of physical reality" exist irrespective of whether they are actually observed. Moreover, EPR conjectured, the quantum formalism can be "completed" or "embedded" into a larger theoretical framework which would reproduce the quantum theoretical results but would otherwise be classical and deterministic from an algebraic and logical point of view.

A proper formalization of the term "element of physical reality" suggested by EPR can be given in terms of two-valued states or valuations, which can take on only one of the two values 0 and 1, and which are interpretable as the classical logical truth assignments *false* and *true*, respectively. Kochen and Specker's results [19] state that for quantum systems representable by Hilbert spaces of dimension higher than two, there does not exist any such valuation $s : L \to \{0, 1\}$ designed on the set of closed linear subspaces of the space L (these subspaces are interpretable as quantum mechanical propositions) preserving the lattice operations and the orthocomplement, even if one restricts the attention to lattice operations carried out among commuting (orthogonal) elements. As a consequence, the set of truth assignments on quantum logics is not separating and not unital. That is, there exist different quantum propositions which cannot be distinguished by any classical truth assignment.

The Kochen and Specker result, it is commonly argued, e.g. by Peres [25] and Mermin [23], is directed against the noncontextual hidden parameter program envisaged by EPR. Indeed, if one takes into account the entire Hilbert logic (of dimension larger than two) and if one considers all states thereon, any truth value assignment to quantum propositions prior to the actual measurement yields a contradiction. This can be proven by finitistic means, that is, with a finite number of one-dimensional closed linear subspaces (generating an infinite set whose intersection with the unit sphere is dense; cf. Havlicek and Svozil [13]).

But, the Kochen Specker argument continues, it is always possible to prove the existence of separable valuations or truth assignments for classical propositional systems identifiable with Boolean algebras. Hence, there does not exist any injective morphism from a quantum logic into some Boolean algebra.

Since the previous reviews of the Kochen–Specker theorem by Peres [24, 25], Redhead [28], Clifton [5], Mermin [23], Svozil and Tkadlec [38], concentrated on the nonexistence of classical noncontextual elements of physical reality, we are going to discuss the options and aspects of embeddings in more detail.

Quantum logic, according to Birkhoff [4], Mackey [20], Jauch [16], Kalmbach [17], Pulmannova [27], identifies logical entities with Hilbert space entities. In particular, elementary propositions $p, q, \ldots$ are associated with closed linear subspaces of a Hilbert space through the origin (zero vector); the implication relation $\to$ is associated with the set theoretical subset relation $\subset$, and the logical *or* $\vee$, *and* $\wedge$, and *not* $'$ operations are associated with the set theoretic intersection $\cap$, with the linear span $\oplus$ of subspaces and the orthogonal subspace $\perp$, respectively. The trivial logical statement 1 which is always true is identified with the entire Hilbert space H, and its complement $\emptyset$ with the zero-dimensional subspace (zero vector). Two propositions p, q are co–measurable (commuting) if $p \to q'$.

Note that this is equivalent to $q \to p'$. The negation of $p \to q$ is denoted by $p \not\to q$.

2 Varieties of embeddings

One of the questions already raised in Specker's almost forgotten first article [33, in German] concerned an embedding of a quantum logical structure L of propositions into a classical universe represented by a Boolean algebra B. Thereby, it is taken as a matter of principle that such an embedding should preserve as much logico–algebraic structure as possible. An embedding of this kind can be formalized as a mapping $\varphi : L \to B$ with the following properties.[1] Let $p, q \in L$.

(i) *Injectivity*: two different quantum logical propositions are mapped into two different propositions of the Boolean algebra; i.e., if $p \neq q$, then $\varphi(p) \neq \varphi(q)$.

(ii) *Preservation of the order relation*: if $p \to q$, then $\varphi(p) \to \varphi(q)$.

(iii) *Preservation of lattice operations*, i.e. preservation of the

(ortho-)complement : $\varphi(p') = \varphi(p)'$,

or operation : $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$,

and operation : $\varphi(p \wedge q) = \varphi(p) \wedge \varphi(q)$.

As it turns out, we cannot have an embedding from the quantum to the classical universe satisfying all three requirements (i)–(iii). In particular, a head-on approach requiring (iii) is doomed to failure, since the nonpreservation of lattice operations among noncommuting (nonorthogonal) propositions is quite evident, given the nondistributive structure of quantum logics.

2.1 Injective lattice morphisms

Here we shall review the rather evident fact that there does not exist an injective lattice morphism from any nondistributive lattice into a Boolean algebra. Let us, for example, study a propositional structure encountered in the quantum mechanics of spin state measurements of a spin one-half particle along two different directions (mod π). It is the modular, orthocomplemented lattice MO_2 drawn in Figure 1 (where $p_- = (p_+)'$ and $q_- = (q_+)'$).

Clearly, MO_2 is a nondistributive lattice, since for instance,

$$p_- \wedge (q_- \vee q_+) = p_- \wedge 1 = p_-,$$

whereas

$$(p_- \wedge q_-) \vee (p_- \wedge q_+) = 0 \vee 0 = 0.$$

Hence,

$$p_- \wedge (q_- \vee q_+) \neq (p_- \wedge q_-) \vee (p_- \wedge q_+).$$

In fact, it is the smallest orthocomplemented nondistributive lattice.

[1] Specker had a modified notion of embedding in mind; see below.

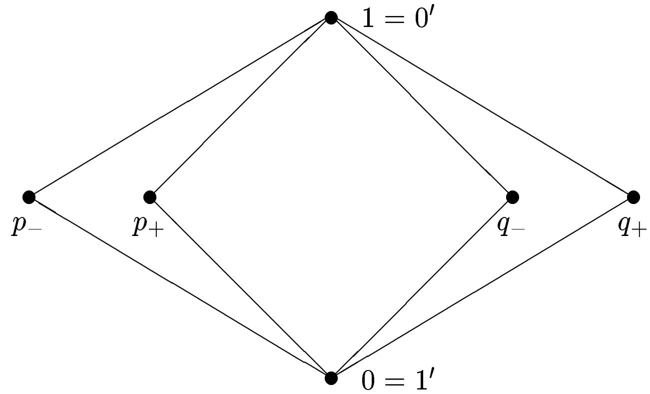

Figure 1: Hasse diagram of the “Chinese lantern” form of MO_2.

The requirement (iii) that the embedding φ preserves all lattice operations (even for non co–measurable propositions) would mean that $\varphi(p_-) \wedge (\varphi(q_-) \vee \varphi(q_+)) \neq (\varphi(p_-) \wedge \varphi(q_-)) \vee (\varphi(p_-) \wedge \varphi(q_+))$. That is, the argument implies that the distributive law is not satisfied for the range of φ. But since the range of φ is a subset of a Boolean algebra and for any Boolean algebra the distributive law is satisfied, this yields a contradiction.

Thus we arrive at the conclusion that a lattice embedding in the form of an injective lattice morphism from Hilbert lattices into Boolean algebras is not possible even for two-dimensional Hilbert spaces. Could we still hope for a reasonable kind of embedding of a quantum universe into a classical one by weakening our requirements, most notably (iii)? In the next three sections we are going to give different answers to this question. In the first section we restrict the set of propositions among which we wish to preserve the three operations *complement* $'$, *or* $\vee$, and *and* $\wedge$. We will see that the Kochen Specker result gives a very strong negative answer even when the restriction is considerable. In the second section we analyze what happens if we try to preserve not all operations but just the complement. Here we will obtain a positive answer. In the third section we discuss a different embedding which preserves the order relation but no lattice operation.

2.2 Injective order morphisms preserving lattice operations among co–measurable propositions

Let us follow Zierler and Schlessinger [41] and Kochen and Specker [19] and weaken (iii) by requiring that the lattice operations need only to be preserved *among co–measurable, (commuting)* propositions. As shown by Kochen and Specker [19], this is equivalent to the requirement of separability of the set of valuations or two-valued probability measures or truth assignments on L. As a matter of fact, Kochen and Specker [19] proved nonseparability, but also much more—the *nonexistence* of valuations on Hilbert lattices associated with Hilbert spaces of dimension at least three. For related arguments and conjectures based upon a theorem by Gleason [10], see Zierler and Schlessinger [41] and John Bell [2].

Rather than rephrasing the Kochen and Specker argument [19] concerning nonexistence of valuations in three-dimensional Hilbert logics in its original form or in terms of less subspaces (cf. Peres [25], Mermin [23]), or of Greechie diagrams, which represent co–measurability (commutativity) very nicely (cf. Svozil and Tkadlec [38], Svozil [36]), we shall give two geometric arguments which are derived from proof methods for Gleason's theorem (see Piron [26], Cooke, Keene, and Moran [6], and Kalmbach [18]).

Let L be the lattice of closed linear subspaces of the three-dimensional real Hilbert space $\mathbb{R}^3$. A *two-valued probability measure* or *valuation* on L is a map $v : L \to \{0,1\}$ which maps the zero-dimensional subspace containing only the origin $(0,0,0)$ to 0, the full space $\mathbb{R}^3$ to 1, and which is additive on orthogonal subspaces. This means that for two orthogonal subspaces $s_1, s_2 \in L$ the sum of the values $v(s_1)$ and $v(s_2)$ is equal to the value of the linear span of s_1 and s_2. Hence, if $s_1, s_2, s_3 \in L$ are a tripod of pairwise orthogonal one-dimensional subspaces, then

$$v(s_1) + v(s_2) + v(s_3) = v(\mathbb{R}^3) = 1.$$

The valuation v must map one of these subspaces to 1 and the other two to 0. We will show that there is *no* such map. In fact, we show that there is no map v which is defined on all one-dimensional subspaces of $\mathbb{R}^3$ and maps *exactly one subspace out of each tripod of pairwise orthogonal one-dimensional subspaces to* 1 *and the other two to* 0.

In the following two geometric proofs we often identify a one-dimensional subspace of $\mathbb{R}^3$ with one of its two intersection points with the unit sphere

$$S^2 = \{x \in \mathbb{R}^3 \mid ||x|| = 1\}.$$

In the statements "a point (on the unit sphere) has value 0 (or value 1)" or that "two points (on the unit sphere) are orthogonal" we always mean the corresponding one-dimensional subspaces. Note also that the intersection of a two-dimensional subspace with the unit sphere is a great circle.

To start the first proof, let us assume that a function v satisfying the above condition exists. Let us consider an arbitrary tripod of orthogonal points and let us fix the point with value 1. By a rotation we can assume that it is the north pole with the coordinates $(0,0,1)$. Then, by the condition above, all points on the equator $\{(x,y,z) \in S^2 \mid z = 0\}$ must have value 0 since they are orthogonal to the north pole.

Let $q = (q_x, q_y, q_z)$ be a point in the northern hemisphere, but not equal to the north pole, that is $0 < q_z < 1$. Let $C(q)$ be the unique great circle which contains q and the points $\pm(q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ in the equator, which are orthogonal to q. Obviously, q is the northern-most point on $C(q)$. To see this, rotate the sphere around the z-axis so that q comes to lie in the $\{y = 0\}$-plane; see Figure 2. Then the two points in the equator orthogonal to q are just the points $\pm(0,1,0)$, and $C(q)$ is the intersection of the plane through q and $(0,1,0)$ with the unit sphere, hence

$$C(q) = \{p \in \mathbb{R}^3 \mid (\exists\, \alpha, \beta \in \mathbb{R})\ \alpha^2 + \beta^2 = 1 \text{ and } p = \alpha q + \beta(0,1,0)\}.$$

This shows that q has the largest z-coordinate among all points in $C(q)$.

Assume that q has value 0. We claim that then all points on $C(q)$ must have value 0. Indeed, since q has value 0 and the orthogonal point $(q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ on the equator

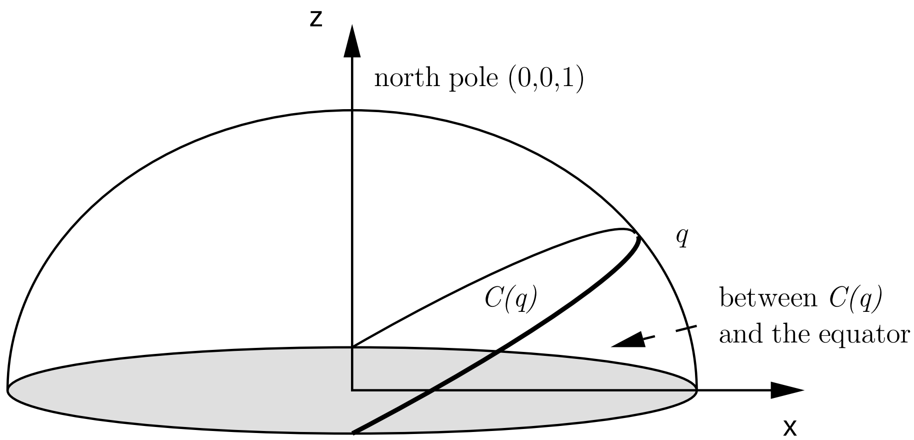

Figure 2: The great circle $C(q)$

also has value 0, the one-dimensional subspace orthogonal to both of them must have value 1. But this subspace is orthogonal to all points on $C(q)$. Hence all points on $C(q)$ must have value 0.

Now we can apply the same argument to any point $\tilde{q}$ on $C(q)$ and derive that all points on $C(\tilde{q})$ have value 0. The great circle $C(q)$ divides the northern hemisphere into two regions, one containing the north pole, the other consisting of the points below $C(q)$ or "lying between $C(q)$ and the equator", see Figure 2. The circles $C(\tilde{q})$ with $\tilde{q} \in C(q)$ certainly cover the region between $C(q)$ and the equator.[2] Hence any point in this region must have value 0.

But the circles $C(\tilde{q})$ cover also a part of the other region. In fact, we can iterate this process. We say that a point p in the northern hemisphere *can be reached* from a point q in the northern hemisphere, if there is a finite sequence of points $q = q_0, q_1, \ldots, q_{n-1}, q_n = p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$. Our consideration above shows that if q has value 0 and p can be reached from q, then also p has value 0.

The following geometric lemma due to Piron [26] (see also Cooke, Keane, and Moran [6] or Kalmbach [18]) is a consequence of the fact that the curve $C(q)$ is tangent to the horizontal plane through the point q.

> *If q and p are points in the northern hemisphere with $p_z < q_z$, then p can be reached from q.*

This lemma will be proved in an appendix. We conclude that, if a point q in the northern hemisphere has value 0, then every point p in the northern hemisphere with $p_z < q_z$ must have value 0 as well.

Consider the tripod $(1,0,0), (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Since $(1,0,0)$ (on the equator) has value 0, one of the two other points has value 0 and one has value 1. By the geometric lemma and our above considerations this implies that all points p in the northern

[2]This will be shown formally in the proof of the geometric lemma below.

hemisphere with $p_z < \frac{1}{\sqrt{2}}$ must have value 0 and all points p with $p_z > \frac{1}{\sqrt{2}}$ must have value 1. But now we can choose any point p' with $\frac{1}{\sqrt{2}} < p'_z < 1$ as our new north pole and deduce that the valuation must have the same form with respect to this pole. This is clearly impossible. Hence, we have proved our assertion that there is no mapping on the set of all one-dimensional subspaces of $\mathbb{R}^3$ which maps one space out of each tripod of pairwise orthogonal one-dimensional subspaces to 1 and the other two to 0.

In the following we give a second topological and geometric proof for this fact. In this proof we shall not use the geometric lemma above.

Fix an arbitrary point on the unit sphere with value 0. The great circle consisting of points orthogonal to this point splits into two disjoint sets, the set of points with value 1, and the set of points orthogonal to these points. They have value 0. If one of these two sets were open, then the other had to be open as well. But this is impossible since the circle is connected and cannot be the union of two disjoint open sets. Hence the circle must contain a point p with value 1 and a sequence of points $q(n)$, $n = 1, 2, \ldots$ with value 0 converging to p. By a rotation we can assume that p is the north pole and the circle lies in the $\{y = 0\}$-plane. Furthermore we can assume that all points q_n have the same sign in the x-coordinate. Otherwise, choose an infinite subsequence of the sequence $q(n)$ with this property. In fact, by a rotation we can assume that all points $q(n)$ have positive x-coordinate (i.e. all points $q(n)$, $n = 1, 2, \ldots$ lie as the point q in Figure 2 and approach the northpole as n tends to infinity). All points on the equator have value 0. In the first proof we have seen that $v(q(n)) = 0$ implies that all points in the northern hemisphere which lie between $C(q(n))$ (the great circle through $q(n)$ and $\pm(0, 1, 0)$) and the equator must have value zero. Since $q(n)$ approaches the northpole, the union of the regions between $C(q(n))$ and the equator is equal to the open right half $\{q \in S^2 \mid q_z > 0, q_x > 0\}$ of the northern hemisphere. Hence all points in this set have value 0. Let q be a point in the left half $\{q \in S^2 \mid q_z > 0, q_x < 0\}$ of the northern hemisphere. It forms a tripod together with the point $(q_y, q_{-x}, 0)$ in the equator and the point $(-q_x, -q_y, \frac{q_x^2+q_y^2}{q_z})/||(-q_x, -q_y, \frac{q_x^2+q_y^2}{q_z})||$ in the right half. Since these two points have value 0, the point q must have value 1. Hence all points in the left half of the northern hemisphere must have value 1. But this leads to a contradiction because there are tripods with two points in the left half, for example the tripod $(-\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2})$, $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. This ends the second proof for the fact that there is no two-valued probability measure on the lattice of subspaces of the three-dimensional Euclidean space which preserves the lattice operations at least for orthogonal elements.

2.3 Injective morphisms preserving order and complementation

We have seen that we cannot hope to preserve the lattice operations, not even when we restrict ourselves to operations among co-measurable propositions.

An even stronger weakening of condition (iii) would be to require preservation of lattice operations merely among the center C, i.e., among those propositions which are co–measurable (commuting) with all other propositions. It is not difficult to prove that in the case of complete Hilbert lattices (and not mere subalgebras thereof), the center consists of just the least lower and the greatest upper bound $C = \{0, 1\}$ and thus is isomorphic to the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$. As it turns out, the requirement is trivially fulfilled and its implications are quite trivial as well.

Another weakening of (iii) is to restrict oneself to particular physical states and study the embeddability of quantum logics under these constraints; see Bell, Clifton [1].

In this section we analyze a completely different option: Is it possible to embed quantum logic into a Boolean algebra when one does not demand preservation of all lattice operations but merely of complementation?

This is indeed possible, as already Zierler and Schlessinger [41, Theorem 2.1] have claimed. We present their construction with a complete proof, filling a gap in the original proof. The construction is similar to the proof of the Stone representation theorem for Boolean algebras, cf. Stone [35]. The construction works for arbitrary sublattices (with 0 and 1) of the lattice of all closed linear subspaces of a Hilbert space. It is interesting to note that for a finite sublattice the constructed Boolean algebra will be finite as well. In fact the construction works even for an arbitrary orthocomplemented lattice $(L, \rightarrow, 0, 1, ')$.

An orthocomplemented lattice $(L, \rightarrow, 0, 1, ')$ is a set L which is endowed with a partial ordering $\rightarrow$, (i.e. a subset $\rightarrow$ of $L \times L$ satisfying 1) $p \rightarrow p$, 2) if $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow r$, 3) if $p \rightarrow q$ and $q \rightarrow p$, then $p = q$, for all $p, q, r \in L$). Furthermore, L contains distinguished elements 0 and 1 satisfying $0 \leq p$ and $p \leq 1$, for all $p \in L$. Finally, L is endowed with a function $'$ (*orthocomplementation*) from L to L satisfying the conditions 1) $p'' = p$, 2) if $p \rightarrow q$, then $q' \rightarrow p'$, 3) the least upper bound of p and p' exists and is 1, for all $p, q \in L$. Note that these conditions imply $0' = 1$, $1' = 0$, and that the greatest lower bound of p and p' exists and is 0, for all $p \in L$.

An arbitrary sublattice of the lattice of all closed linear subspaces of a Hilbert space is an orthocomplemented lattice, if it contains the subspace $\{0\}$ and the full Hilbert space. Namely, the subspace $\{0\}$ is the 0 in the lattice, the full Hilbert space is the 1, the set theoretic inclusion realizes the ordering $\rightarrow$, and the orthogonal complement is the orthocomplementation $'$.

In the rest of the section we always assume that L is an orthocomplemented lattice. We call a subset I of L a *maximal ideal* if for all $p, q \in L$

1. $p \in I$ iff $p' \notin I$,
2. if $p \rightarrow q$ and $q \in I$, then $p \in I$.

In other words, the maximal ideals are just the kernels of mappings from L to **2** which preserve the order relation and the complement.

Let $\mathcal{I}$ be the set of all maximal ideals in L, and let B be the power set of $\mathcal{I}$ considered as a Boolean algebra, i.e. B is the Boolean algebra which consists of all subsets of $\mathcal{I}$. The order relation in B is the set-theoretic inclusion, the lattice operations *complement*, *or*, and *and* are given by the set-theoretic complement, union, and intersection, and the elements 0 and 1 of the Boolean algebra are just the empty set and the full set $\mathcal{I}$. Consider the map

$$\varphi : L \rightarrow B$$

which maps each element $p \in L$ to the set

$$\varphi(p) = \{I \in \mathcal{I} \mid p \notin I\}$$

of all maximal ideals which do not contain p. We claim that the map φ

1) is injective,

2) preserves the order relation,

3) preserves complementation.

This provides an embedding of quantum logic into classical logic which preserves the implication relation and the negation.[3]

The rest of this section consists of the proof of the three claims. Let us start with claim 2). Assume that $p, q \in L$ satisfy $p \to q$. We have to show the inclusion

$$\varphi(p) \subseteq \varphi(q)\,.$$

Take a maximal ideal $I \in \varphi(p)$. Then $p \notin I$. If q were contained in I, then by the definition of a maximal ideal also p had to be contained in I. Hence $q \notin I$, thus proving that $I \in \varphi(q)$.

For claim 3) we have to show the relation:

$$\varphi(p') = \mathcal{I} \setminus \varphi(p)\,,$$

for all $p \in L$. This can be restated as

$$I \in \varphi(p') \text{ iff } I \notin \varphi(p)$$

for all $I \in \mathcal{I}$. But this means $p' \notin I$ iff $p \in I$, which is the first condition in the definition of a maximal ideal.

We proceed to claim 1). Assume that two different propositions $p, q \in L$ are given: $p \neq q$. Then we have $p \not\to q$ or $q \not\to p$. We can assume $p \not\to q$. We have to show that $\varphi(p)$ and $\varphi(q)$ are different as well. We will show that there is a maximal ideal $I \in \varphi(p) \setminus \varphi(q)$, that is, a maximal ideal I satisfying $p \notin I$ and $q \in I$. In order to obtain this maximal ideal we will apply Zorn's Lemma to a certain set of ideals. We call a subset K of L an *ideal* if for all $p, q \in L$:

1. if $p \in K$, then $p' \notin K$,

2. if $p \to q$ and $q \in K$, then $p \in K$.

Obviously, a maximal ideal is an ideal. Consider the set

$$\mathcal{I}_{p,q} = \{K \subseteq L \mid K \text{ is an ideal, } p \notin K,\ q \in K\}\,.$$

We have to show that among the elements of $\mathcal{I}_{p,q}$ there is a maximal ideal. Therefore we will use Zorn's Lemma. In order to apply it to $\mathcal{I}_{p,q}$ we have to show (a) that $\mathcal{I}_{p,q}$ is not empty, (b) that every chain in $\mathcal{I}_{p,q}$ has an upper bound.

The first condition is proved to be true by the *principal ideal* of q:

$$\langle q\rangle = \{s \in L \mid s \to q\}\,.$$

[3] Note that for a finite lattice L the Boolean algebra B is finite as well. Indeed, if L is finite, then it has only finitely many subsets, especially only finitely many maximal ideals. Hence $\mathcal{I}$ is finite, and thus also its power set B is finite.

Note that this set is an ideal. Indeed, $r \to s$ and $s \to q$ imply $r \to q$. Our assumption $p \not\to q$ implies $q \neq 1$; hence, if $r \to q$, then $r' \not\to q$. Furthermore $p \not\to q$ implies $p \notin \langle q \rangle$. Finally $q \in \langle q \rangle$ is clear. Thus, the set $\langle q \rangle$ is an element of the set $\mathcal{I}_{p,q}$. Hence $\mathcal{I}_{p,q}$ is not empty. Now we are going to show the second condition, namely that each chain in $\mathcal{I}_{p,q}$ has an upper bound in $\mathcal{I}_{p,q}$. This means that, given a subset (*chain*) $\mathcal{C}$ of $\mathcal{I}_{p,q}$ with the property

$$\text{for all } J, K \in \mathcal{C} \text{ one has } J \subseteq K \text{ or } K \subseteq J\,,$$

we have to show that there is an element (*upper bound*) $U \in \mathcal{I}_{p,q}$ with $K \subseteq U$ for all $K \in \mathcal{C}$. The union

$$U_{\mathcal{C}} = \bigcup_{K \in \mathcal{C}} K$$

of all ideals $K \in \mathcal{C}$ is the required upper bound! It is clear that all $K \in \mathcal{C}$ are subsets of $U_{\mathcal{C}}$. We have to show that $U_{\mathcal{C}}$ is an element of $\mathcal{I}_{p,q}$. Since $p \notin K$ for all $K \in \mathcal{C}$ we also have $p \notin U_{\mathcal{C}}$. Similarly, since $q \in K$ for some (even all) $K \in \mathcal{C}$, we have $q \in U_{\mathcal{C}}$. We still have to show that $U_{\mathcal{C}}$ is an ideal. Given two propositions r, s with $r \to s$ and $s \in U_{\mathcal{C}}$ we conclude that s must be contained in one of the ideals $K \in \mathcal{C}$. Hence also $r \in K \subseteq U_{\mathcal{C}}$. Now assume $r \in U_{\mathcal{C}}$. Is it possible that the complement r' belongs to $U_{\mathcal{C}}$? The answer is negative, since otherwise $r \in J$ and $r' \in K$, for some ideals $J, K \in \mathcal{C}$. But since $\mathcal{C}$ is a chain we have $J \subseteq K$ or $K \subseteq J$, hence $r, r' \in K$ in the first case and $r, r' \in J$ in the second case. Both cases contradict the fact that J and K are ideals. Hence, $U_{\mathcal{C}}$ is an ideal and thus an element of $\mathcal{I}_{p,q}$. We have proved that $\mathcal{I}_{p,q}$ is not empty and that each chain in $\mathcal{I}_{p,q}$ has an upper bound in $\mathcal{I}_{p,q}$.

Consequently, we can apply Zorn's Lemma to $\mathcal{I}_{p,q}$ and obtain a maximal element I in the ordered set $\mathcal{I}_{p,q}$, i.e. an element $I \in \mathcal{I}_{p,q}$ such that:

$$\text{if } K \in \mathcal{I}_{p,q} \text{ and } I \subseteq K, \text{ then } K = I\,.$$

We claim that I is a maximal ideal. Since I, as an element of $\mathcal{I}_{p,q}$, is already an ideal, we have to show only that $r' \notin I$ implies $r \in I$, for any $r \in L$. In other words, we have to show only that I contains at least one proposition out of each pair r, r'. Assume this is not the case and there is a proposition r with $r \notin I$ and $r' \notin I$. Since $p \neq 0$ (remember our assumption $p \not\to q$) we get $p \not\to r$ or $p \not\to r'$. Without loss of generality we can assume $p \not\to r$. We will show that under this assumption the set

$$J = I \cup \{s \in L \mid s \to r\}$$

is larger than I and an element of $\mathcal{I}_{p,q}$, contradicting the assumption that I is a maximal element of $\mathcal{I}_{p,q}$.

First, we show that J is an ideal. If s, t are propositions in L with $s \to t$ and $t \in J$, then we must have $t \in I$ or $t \to r$. In the first case we conclude $s \in I \subseteq J$ and in the second case $s \to r$, hence $s \in J$ as well. Can we have $s \in J$ and $s' \in J$, for some $s \in L$? The answer is again negative, since otherwise one of the following four cases must be true: (1) $s, s' \in I$, (2) $s \to r$ and $s' \to r$, (3) $s \in I$, $s' \to r$, (4) $s \to r$, $s' \in I$. The first case is impossible since I is an ideal. The second case is ruled out by the fact that $r \neq 1$ (namely, $r = 1$ would imply $r' = 0$, which would contradict our assumption $r' \notin I$). The third case is impossible since $s' \to r$ would imply $r' \to s$, which, combined with $s \in I$ would imply

$r' \in I$, contrary to our assumption. Finally, the fourth case is nothing but a reformulation of the third case with s and s' interchanged. Thus, we have proved that J is an ideal. Since $p \notin I$ and $p \not\to r$ we have $p \notin J$. On the other hand, $q \in I \subseteq J$. This completes the proof that J is an element of $\mathcal{I}_{p,q}$.

But J, containing r, is a larger element of $\mathcal{I}_{p,q}$ than I! This contradicts the assumption that I is a maximal element of $\mathcal{I}_{p,q}$. Hence, our assumption that there is a proposition r with $r \notin I$ and $r' \notin I$ must be false. We conclude that I is a maximal ideal. We have reached our aim to construct a maximal ideal I with $p \notin I$ and $q \in I$. This ends the proof of claim 3), the claim that the map φ is injective.

We have shown: any sublattice of the lattice of all closed linear subspaces of a Hilbert space can be embedded into a Boolean algebra where the embedding preserves the order relation and the complementation.

2.4 Injective order preserving morphisms

In this section we analyze a different embedding suggested by Malhas [21, 22].

As in the last section we consider an orthocomplemented lattice $(L, \leq, 0, 1, ')$, i.e. a lattice $(L, \leq, 0, 1)$ with $0 \leq x \leq 1$ for all $x \in L$, with orthocomplementation, that is with a mapping $' : L \to L$ satisfying the following three properties: a) $x'' = x$, b) if $x \leq y$, then $y' \leq x'$, c) $x \cdot x' = 0$ and $y \vee y' = 1$. Here $x \cdot y = \mathrm{glb}(x, y)$ and $x \vee y = \mathrm{lub}(x, y)$.

Furthermore, we will assume that L is atomic[4] and satisfies the following additional property:

$$\text{for all } \ x, y \in L, x \leq y \ \text{ iff for every atom } \ a \in L, a \leq x \ \text{ implies } a \leq y. \tag{1}$$

Every atomic Boolean algebra and the lattice of closed subspaces of a separable Hilbert space satisfy the above conditions.

Consider next a set U[5] and let $W(U)$ be the smallest set of words over the alphabet $U \cup \{', \to\}$ which contains U and is closed under negation (if $A \in W(U)$, then $A' \in W(U)$) and implication (if $A, B \in W(U)$, then $A \to B \in W(U)$).[6] The elements of U are called *simple propositions* and the elements of $W(U)$ are called *(compound) propositions.*

A *valuation* is a mapping

$$t : W(U) \to \mathbf{2}$$

such that $t(A) \neq t(A')$ and $t(A \to B) = 0$ iff $t(A) = 1$ and $t(B) = 0$. Clearly, every assignment $s : U \to \mathbf{2}$ can be extended to a unique valuation t_s.

A *tautology* is a proposition A which is true under every possible valuation, i.e., $t(A) = 1$, for every valuation t. A set $\mathcal{K} \subseteq W(U)$ is *consistent* if there is a valuation making true every proposition in $\mathcal{K}$. Let $A \in W(U)$ and $\mathcal{K} \subseteq W(U)$. We say that A *derives* from $\mathcal{K}$, and write $\mathcal{K} \models A$ in case $t(A) = 1$ for each valuation t, which makes true every proposition in $\mathcal{K}$ (that is, $t(B) = 1$, for all $B \in \mathcal{K}$). Let

$$Con(\mathcal{K}) = \{A \in W(U) \mid \mathcal{K} \models A\}.$$

[4] For every $x \in L \setminus \{0\}$, there is an atom $a \in L$ such that $a \leq x$. An atom is an element $a \in L$ with the property that if $0 \leq y \leq a$, then $y = 0$ or $y = a$.

[5] Not containing the logical symbols $\cup, ', \to$.

[6] Define in a natural way $A \cup B = A' \to B$, $A \cap B = (A \to B')'$, $A \leftrightarrow B = (A \to B) \cap (B \to A)$.

Finally, a set $\mathcal{K}$ is a *theory* if $\mathcal{K}$ is a fixed-point of the operator Con:

$$Con(\mathcal{K}) = \mathcal{K}.$$

It is easy to see that Con is in fact a finitary closure operator, i.e., it satisfies the following four properties:

- $\mathcal{K} \subseteq Con(\mathcal{K})$,
- if $\mathcal{K} \subseteq \tilde{\mathcal{K}}$, then $Con(\mathcal{K}) \subseteq Con(\tilde{\mathcal{K}})$,
- $Con(Con(\mathcal{K})) = Con(\mathcal{K})$,
- $Con(\mathcal{K}) = \bigcup_{\{X \subseteq \mathcal{K}, X \text{ finite}\}} Con(X)$.

The main example of a theory can be obtained by taking a set X of valuations and constructing the set of all propositions true under all valuations in X:

$$Th(X) = \{A \in W(U) \mid t(A) = 1, \text{ for all } t \in X\}.$$

In fact, every theory is of the above form, that is, *for every theory $\mathcal{K}$ there exists a set of valuations X (depending upon $\mathcal{K}$) such that $\mathcal{K} = Th(X)$.* Indeed, take

$$X_{\mathcal{K}} = \{t : W(U) \to \mathbf{2} \mid t \text{ valuation with } t(A) = 1, \text{ for all } A \in \mathcal{K}\},$$

and notice that

$$\begin{aligned} Th(X_{\mathcal{K}}) &= \{B \in W(U) \mid t(B) = 1, \text{ for all } t \in X_{\mathcal{K}}\} \\ &= \{B \in W(U) \mid t(B) = 1, \text{ for every valuation with } t(A) = 1, \\ &\quad \text{for all } A \in \mathcal{K}\} \\ &= Con(\mathcal{K}) = \mathcal{K}. \end{aligned}$$

In other words, *theories are those sets of propositions which are true under a certain set of valuations (interpretations).*

Let now $\mathcal{T}$ be a theory. Two elements $p, q \in U$ are $\mathcal{T}$-equivalent, written $p \equiv_{\mathcal{T}} q$, in case $p \leftrightarrow q \in \mathcal{T}$. The relation $\equiv_{\mathcal{T}}$ is an equivalence relation. The equivalence class of p is $[p]_{\mathcal{T}} = \{q \in U \mid p \equiv_{\mathcal{T}} q\}$ and the factor set is denoted by $U_{\equiv_{\mathcal{T}}}$; for brevity, we will sometimes write $[p]$ instead of $[p]_{\mathcal{T}}$. The factor set comes with a natural partial order:

$$[p] \leq [q] \text{ if } p \to q \in \mathcal{T}.$$

Note that in general, $(U_{\equiv_{\mathcal{T}}}, \leq)$ is not a Boolean algebra.[7]

In a similar way we can define the $\equiv_{\mathcal{T}}$-equivalence of two propositions:

$$A \equiv_{\mathcal{T}} B \text{ if } A \leftrightarrow B \in \mathcal{T}.$$

Denote by $[[A]]_{\mathcal{T}}$ (shortly, $[[A]]$) the equivalence class of A and note that for every $p \in U$,

$$[p] = [[p]] \cap U.$$

[7]For instance, in case $\mathcal{T} = Con(\{p\})$, for some $p \in U$.

The resulting Boolean algebra $W(U)_{\equiv_{\mathcal{T}}}$ is the Lindenbaum algebra of $\mathcal{T}$.

Fix now an atomic orthocomplemented lattice $(L, \leq, 0, 1, ')$ satisfying (1). Let U be a set of cardinality greater or equal to L and fix a surjective mapping $f : U \to L$. For every atom $a \in L$, let $s_a : U \to \mathbf{2}$ be the assignment defined by $s_a(p) = 1$ if $a \leq f(p)$. Take

$$X = \{t_{s_a} \mid a \text{ is an atom of } L\}^8 \text{ and } \mathcal{T} = Th(X).$$

Malhas [21, 22] has proven that the *lattice* $(U_{\equiv_{\mathcal{T}}}, \leq)$ *is orthocomplemented,* and, in fact, *isomorphic to* L. Here is the argument. Note first that there exist two elements $\underline{0}, \underline{1}$ in U such that $f(\underline{0}) = 0$, $f(\underline{1}) = 1$. Clearly, $\underline{0} \notin \mathcal{T}$, but $\underline{1} \in \mathcal{T}$. Indeed, for every atom a, $a \leq f(\underline{1}) = 1$, so $s_a(\underline{1}) = 1$, a.s.o.

Secondly, for every $p, q \in U$,

$$p \to q \in \mathcal{T} \text{ iff } f(p) \leq f(q).$$

If $p \to q \notin \mathcal{T}$, then there exists an atom $a \in L$ such that $t_{s_a}(p \to q) = 0$, so $s_a(p) = t_{s_a}(p) = 1$, $s_a(q) = t_{s_a}(q) = 0$, which — according to the definition of s_a — mean $a \leq f(p)$, but $a \not\leq f(q)$. If $f(p) \leq f(q)$, then $a \leq f(q)$, a contradiction. Conversely, if $f(p) \not\leq f(q)$, then by (1) there exists an atom a such that $a \leq f(p)$ and $a \not\leq f(q)$. So, $s_a(p) = t_{s_a}(p) = 1$, $s_a(q) = t_{s_a}(q) = 0$, i.e., $(p \to q) \notin \mathcal{T}$.

As immediate consequences we deduce the validity of the following three relations: for all $p, q \in U$,

- $f(p) \leq f(q)$ iff $[p] \leq [q]$,
- $f(p) = f(q)$ iff $[p] = [q]$,
- $[\underline{0}] \leq [p] \leq [\underline{1}]$,

Two simple propositions $p, q \in U$ are *conjugate* in case $f(p)' = f(q)$.[9] Define now the operation $^* : U_{\mathcal{T}} \to U_{\mathcal{T}}$ as follows: $[p]^* = [q]$ in case q is a conjugate of p. It is not difficult to see that the operation * is well-defined and actually is an orthocomplementation. It follows that $(U_{\mathcal{T}}, \leq_{\mathcal{T}}, ^*)$ is an orthocomplemented lattice.

To finish the argument we will show that this lattice is *isomorphic* with L. The isomorphism is given by the mapping $\varphi : U_{\mathcal{T}} \to L$ defined by the formula $\varphi([p]) = f(p)$. This is a well-defined function (because $f(p) = f(q)$ iff $[p] = [q]$), which is bijective ($\varphi([p]) = \varphi([q])$ implies $f(p) = f(q)$, and surjective because f is onto). If $[p] \leq [q]$, then $f(p) \leq f(q)$, i.e. $\varphi([p]) \leq \varphi([q])$. Finally, if q is a conjugate of p, then

$$\varphi([p]^*) = \varphi([q]) = f(q) = f(p)' = \varphi([p])'.$$

In particular, *there exists a theory with orthocomplementation whose induced ortho partial order is isomorphic to the lattice of all closed subspaces of a separable Hilbert space.* How does this relate to the Kochen-Specker theorem? *The natural embedding*

$$\Gamma : U_{\equiv_{\mathcal{T}}} \to W(U)_{\equiv_{\mathcal{T}}}, \ \Gamma([p]) = [[p]]$$

[8] Recall that t_s is the unique valuation extending s.

[9] Of course, this relation is symmetrical.

	A	B	C	D	E	F	G	H
s_{p_-}	0	1	1	0	0	0	1	1
s_{p_+}	0	0	0	1	0	0	1	1
s_{q_-}	0	0	0	0	1	0	1	1
s_{q_+}	0	0	0	0	0	1	1	1

Table 1: Truth assignments on U corresponding to atoms $p_-, p_+, q_-, q_+ \in MO_2$.

is order preserving and one-to-one, but in general it does not preserve orthocomplementation, i.e. in general $\Gamma([p]^*) \neq \Gamma([p])'$. We always have $\Gamma([p]^*) \leq \Gamma([p])'$, but sometimes $\Gamma([p])' \not\leq \Gamma([p]^*)$. The reason is that for every pair of conjugate simple propositions p, q one has $(p \to q') \in \mathcal{T}$, but the converse is not true.

The above construction of Malhas gives us another method how *to embed any quantum logic into a Boolean logic in case we require that only the order is preserved.*[10]

Next we shall give a simple example of a Malhas type embedding $\varphi : MO_2 \to \mathbf{2}^4$. Consider again the finite quantum logic MO_2 represented in Figure 1. Let us choose

$$U = \{A, B, C, D, E, F, G, H\}.$$

Since U contains more elements than MO_2, we can map U surjectively onto MO_2; e.g.,

$$\begin{aligned} f(A) &= 0, \\ f(B) &= p_-, \\ f(C) &= p_-, \\ f(D) &= p_+, \\ f(E) &= q_-, \\ f(F) &= q_+, \\ f(G) &= 1, \\ f(H) &= 1. \end{aligned}$$

For every atom $a \in MO_2$, let us introduce the truth assignment $s_a : U \to \mathbf{2} = \{0, 1\}$ as defined above (i.e. $s_a(r) = 1$ iff $a \leq f(r)$) and thus a valuation on $W(U)$ separating it from the rest of the atoms of MO_2. That is, for instance, associate with $p_- \in MO_2$ the function s_{p_-} as follows:

$$\begin{aligned} s_{p_-}(A) = s_{p_-}(D) = s_{p_-}(E) = s_{p_-}(F) = 0, \\ s_{p_-}(B) = s_{p_-}(C) = s_{p_-}(G) = s_{p_-}(H) = 1. \end{aligned}$$

The truth assignments associated with all the atoms are listed in Table 1.

The theory $\mathcal{T}$ we are thus dealing with is determined by the union of all the truth assignments; i.e.,

$$X = \{t_{s_{p_-}}, t_{s_{p_+}}, t_{s_{q_-}}, t_{s_{q_+}}\} \text{ and } \mathcal{T} = Th(X).$$

[10] In the last section we saw that it is possible to embed quantum logic into a Boolean logic preserving the order and the complement.

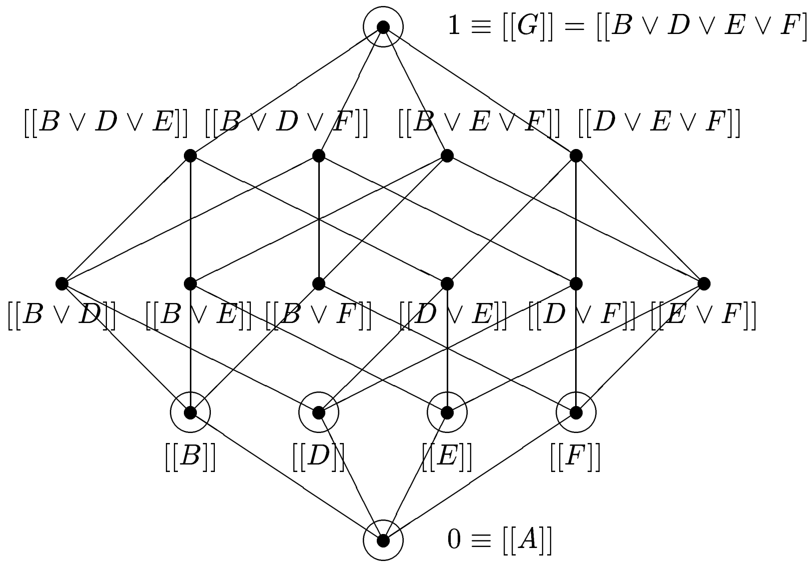

Figure 3: Hasse diagram of an embedding of the quantum logic MO_2 represented by Figure 1. Concentric circles indicate the embedding.

The way it was constructed, U splits into six equivalence classes with respect to the theory $\mathcal{T}$; i.e.,

$$U_{\equiv_\mathcal{T}} = \{[A], [B], [D], [E], [F], [G]\}.$$

Since $[p] \rightarrow [q]$ if and only if $(p \rightarrow q) \in \mathcal{T}$, we obtain a partial order on $U_{\equiv_\mathcal{T}}$ induced by T which isomorphically reflects the original quantum logic MO_2; in particular, we obtain

$$\begin{aligned}
\varphi(0) &= [A],\\
\varphi(p_-) &= [B],\\
\varphi(p_+) &= [D],\\
\varphi(q_-) &= [E],\\
\varphi(q_+) &= [F],\\
\varphi(1) &= [G].
\end{aligned}$$

The Boolean Lindenbaum algebra $W(U)_{\equiv_\mathcal{T}} = \mathbf{2}^4$ is obtained by forming all the compound propositions of U and imposing a partial order with respect to $\mathcal{T}$. It is represented in Figure 3. The embedding is order–preserving but does not preserve operations such as the complement. Although, in this particular example, $f(B) = (f(D))'$ implies $(B \rightarrow D') \in \mathcal{T}$, the converse is not true in general. For example, there is no $s \in X$ for which $s(B) = s(E) = 1$. Thus, $(B \rightarrow E') \in T$, but $f(B) \neq (f(E))'$.

One needs not be afraid of order-preserving embeddings which are no lattice morphisms, after all. Even automaton logics [37, Chapter 11] and [29, 30, 31, 7] can be embedded in this way. Take again the lattice MO_2 depicted in Figure 1. A partition (automaton) logic realization is, for instance,

$$\{\{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}\},$$

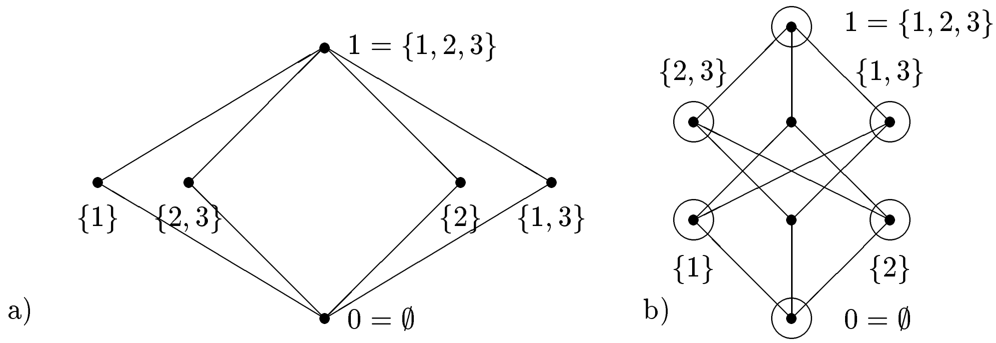

Figure 4: Hasse diagram of an embedding of MO_2 drawn in a) into $\mathbf{2}^3$ drawn in b). Again, concentric circles indicate points of $\mathbf{2}^3$ included in MO_2.

with

$$\begin{aligned} \{1\} &\equiv p_-, \\ \{2,3\} &\equiv p_+, \\ \{2\} &\equiv q_-, \\ \{1,3\} &\equiv q_+, \end{aligned}$$

respectively. If we take $\{1\}$,$\{2\}$ and $\{3\}$ as atoms, then the Boolean algebra $\mathbf{2}^3$ generated by all subsets of $\{1,2,3\}$ with the set theoretic inclusion as order relation suggests itself as a candidate for an embedding. The embedding is quite trivially given by

$$\varphi(p) = p \in \mathbf{2}^3.$$

The particular example considered above is represented in Figure 4. It is not difficult to check that the embedding satisfies the requirements (i) and (ii); that is, it is injective and order preserving.

It is important to realize at that point that, although different automaton partition logical structures may be isomorphic from a logical point of view (one-to-one translatable elements, order relations and operations), they may be very different with respect to their embeddability. Indeed, any two distinct partition logics correspond to two distinct embeddings.

It should also be pointed out that in the case of an automaton partition logic and for all finite subalgebras of the Hilbert lattice of two-dimensional Hilbert space, it is always possible to find an embedding corresponding to a logically equivalent partition logic which is a lattice morphism for co–measurable elements (modified requirement (iii)). This is due to the fact that partition logics and MO_n have a separating set of valuations. In the MO_2 case, this is, for instance

$$\{\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\}\},$$

with

$$\{1,2\} \equiv p_-,$$

	p_-	p_+	q_-	q_+
s_1	1	0	1	0
s_2	1	0	0	1
s_3	0	1	1	0
s_4	0	1	0	1

Table 2: The four valuations s_1, s_2, s_3, s_4 on MO_2 take on the values listed in the rows.

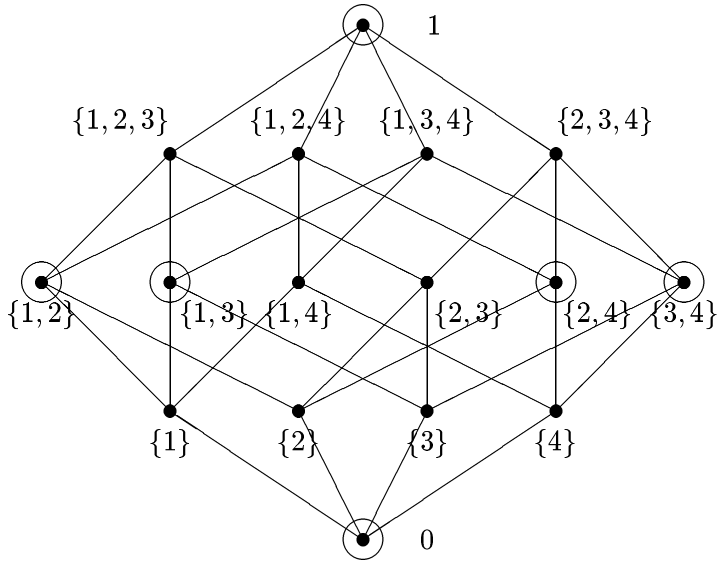

Figure 5: Hasse diagram of an embedding of the partition logic $\{\{\{1,2\},\{3,4\}\},\{\{1,3\},\{2,4\}\}\}$ into $\mathbf{2}^4$ preserving lattice operations among co–measurable propositions. Concentric circles indicate the embedding.

$$
\begin{aligned}
\{3,4\} &\equiv p_+, \\
\{1,3\} &\equiv q_-, \\
\{2,4\} &\equiv q_+,
\end{aligned}
$$

respectively. This embedding is based upon the set of all valuations listed in Table 2. These are exactly the mappings from MO_2 to $\mathbf{2}$ preserving the order relation and the complementation. They correspond to the maximal ideals considered in Section 2.3. In this special case the embedding is just the embedding one would obtain by applying the construction of Section 2.3, which had been suggested by Zierler and Schlessinger [41, Theorem 2.1]. The embedding is drawn in Figure 5.

3 Surjective extensions?

The original proposal put forward by EPR [8] in the last paragraph of their paper was some form of completion of quantum mechanics. Clearly, the first type of candidate for such a completion is the sort of embedding reviewed above. The physical intuition behind an embedding is that the "actual physics" is a classical one, but because of some yet unknown reason, some of this "hidden arena" becomes observable while others remain hidden.

Nevertheless, there exists at least another alternative to complete quantum mechanics. This is best described by a *surjective map* $\phi : B \to L$ of a classical Boolean algebra onto a quantum logic, such that $|B| \geq |L|$.

Platos cage metaphor applies to both approaches, in that observations are mere shadows of some more fundamental entities.

Appendix: Proof of the geometric lemma

In this appendix we are going to prove the geometric lemma due to Piron [26] which was formulated in Section 2.2. First let us restate it. Consider a point q in the northern hemisphere of the unit sphere $S^2 = \{p \in \mathbb{R}^3 \mid ||p|| = 1\}$. By $C(q)$ we denote the unique great circle which contains q and the points $\pm(q_y, -q_x, 0)/\sqrt{q_x^2 + q_y^2}$ in the equator, which are orthogonal to q, compare Figure 2. We say that a point p in the northern hemisphere *can be reached* from a point q in the northern hemisphere, if there is a finite sequence of points $q = q_0, q_1, \ldots, q_{n-1}, q_n = p$ in the northern hemisphere such that $q_i \in C(q_{i-1})$ for $i = 1, \ldots, n$. The lemma states:

> *If q and p are points in the northern hemisphere with $p_z < q_z$, then p can be reached from q.*

For the proof we follow Cooke, Keane, and Moran [6] and Kalmbach [18]). We consider the tangent plane $H = \{p \in \mathbb{R}^3 \mid p_z = 1\}$ of the unit sphere in the north pole and the projection h from the northern hemisphere onto this plane which maps each point q in the northern hemisphere to the intersection $h(q)$ of the line through the origin and p with the plane H. This map h is a bijection. The north pole $(0, 0, 1)$ is mapped to itself. For each q in the northern hemisphere (not equal to the north pole) the image $h(C(q))$ of the great circle $C(q)$ is the line in H which goes through $h(q)$ and is orthogonal to the line through the north pole and through $h(q)$. Note that $C(q)$ is the intersection of a plane with S^2, and $h(C(q))$ is the intersection of the same plane with H; see Figure 6. The line $h(C(q))$ divides H into two half planes. The half plane not containing the north pole is the image of the region in the northern hemisphere between $C(q)$ and the equator. Furthermore note that $q_z > p_z$ for two points in the northern hemisphere if and only if $h(p)$ is further away from the north pole than $h(q)$.

First, we show that, if p and q are points in the northern hemisphere and p lies in the region between $C(q)$ and the equator, then p can be reached from q. In fact, we show that there is a point $\tilde{q}$ on $C(q)$ such that p lies on $C(\tilde{q})$. Therefore we consider the images of q and p in the plane H; see Figure 7. The point $h(p)$ lies in the half plane bounded by $h(C(q))$ not containing the north pole. Among all points $h(q')$ on the line $h(C(q))$ we set

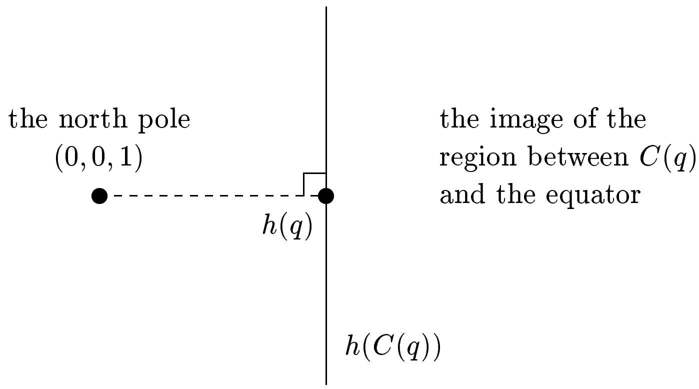

Figure 6: The plane H viewed from above

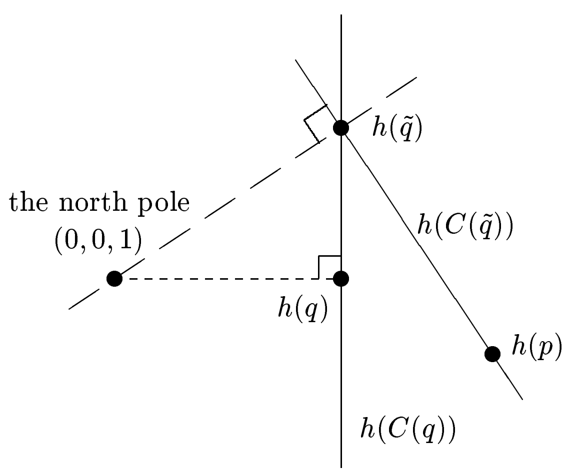

Figure 7: The point p can be reached from q

$\tilde{q}$ to be one of the two points such that the line trough the north pole and $h(q')$ and the line through $h(q')$ and $h(p)$ are orthogonal. Then this last line is the image of $C(\tilde{q})$, and $C(\tilde{q})$ contains the point p. Hence p can be reached from q. Our first claim is proved.

Fix a point q in the northern hemisphere. Starting from q we can wander around the northern hemisphere along great circles of the form $C(p)$ for points p in the following way: for $n \geq 5$ we define a sequence $q_0, q_1, \ldots, q_n$ by setting $q_0 = q$ and by choosing q_{i+1} to be that point on the great circle $C(q_i)$ such that the angle between $h(q_{i+1})$ and $h(q_i)$ is $2\pi/n$. The image in H of this configuration is a shell where $h(q_n)$ is the point furthest away from the north pole; see Figure 8. First, we claim that any point p on the unit sphere with

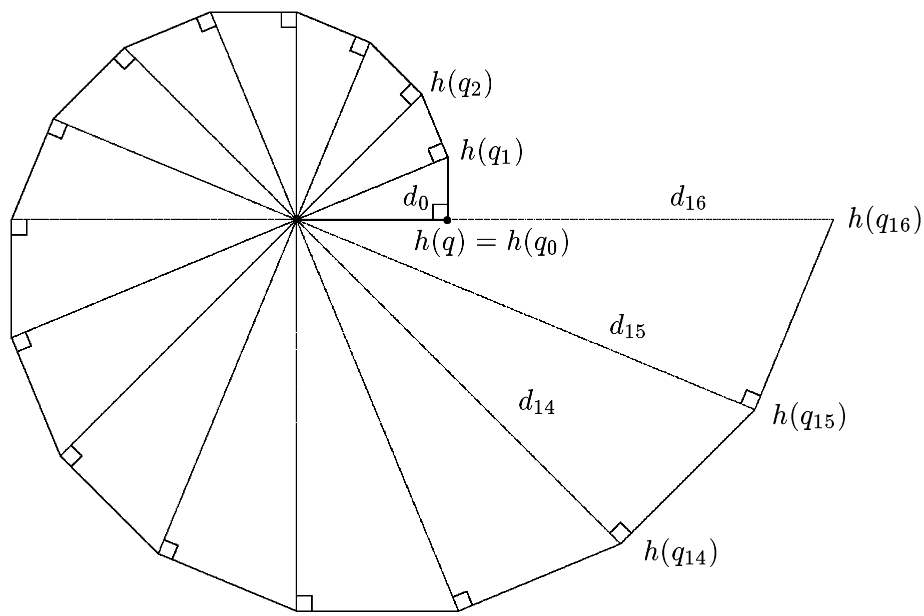

Figure 8: The shell in the plane H for $n = 16$

$p_z < q_{n_z}$ can be reached from q. Indeed, such a point corresponds to a point $h(p)$ which is further away from the north pole than $h(q_n)$. There is an index i such that $h(p)$ lies in the half plane bounded by $h(C(q_i))$ and not containing the north pole, hence such that p lies in the region between $C(q_i)$ and the equator. Then, as we have already seen, p can be reached from q_i and hence also from q. Secondly, we claim that q_n approaches q as n tends to infinity. This is equivalent to showing that the distance of $h(q_n)$ from $(0,0,1)$ approaches the distance of $h(q)$ from $(0,0,1)$. Let d_i denote the distance of $h(q_i)$ from $(0,0,1)$ for $i = 0, \ldots, n$. Then $d_i/d_{i+1} = \cos(2\pi/n)$, see Figure 8. Hence $d_n = d_0 \cdot \cos(2\pi/n)^{-n}$. That d_n approaches d_0 as n tends to infinity follows immediately from the fact that $\cos(2\pi/n)^n$

approaches 1 as n tends to infinity. For completeness sake[11] we prove it by proving the equivalent statement that $\log(\cos(2\pi/n)^n)$ tends to 0 as n tends to infinity. Namely, for small x we know the formulae $\cos(x) = 1 - x^2/2 + \mathcal{O}(x^4)$ and $\log(1+x) = x + \mathcal{O}(x^2)$. Hence, for large n,

$$
\begin{aligned}
\log(\cos(2\pi/n)^n) &= n \cdot \log(1 - 2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})) \\
&= n \cdot (-2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})) \\
&= -\frac{2\pi^2}{n} + \mathcal{O}(n^{-3}) .
\end{aligned}
$$

This ends the proof of the geometric lemma.

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References

[1] John L. Bell and Robert K. Clifton. Quasiboolean algebras and simultaneously definite properties in quantum mechanics. *International Journal of Theoretical Physics*, 43(12):2409–2421, 1995.

[2] John S. Bell. On the problem of hidden variables in quantum mechanics. *Reviews of Modern Physics*, 38:447–452, 1966. Reprinted in [3, pp. 1-13].

[3] John S. Bell. *Speakable and Unspeakable in Quantum Mechanics.* Cambridge University Press, Cambridge, 1987.

[4] Garrett Birkhoff and John von Neumann. The logic of quantum mechanics. *Annals of Mathematics*, 37(4):823–843, 1936.

[5] Rob Clifton. Getting contextual and nonlocal elements–of–reality the easy way. *American Journal of Physics*, 61:443–447, 1993.

[6] R. Cooke, M. Keane, and W. Moran. An elementary proof of gleason's theorem. *Math. Proc. Camb. Soc.*, 98:117–128, 1985.

[7] Anatolij Dvurečenskij, Sylvia Pulmannová, and Karl Svozil. Partition logics, orthoalgebras and automata. *Helvetica Physica Acta*, 68:407–428, 1995.

[8] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47:777–780, 1935. Reprinted in [40, pp. 138-141].

[9] Richard P. Feynman. *The Character of Physical Law.* MIT Press, Cambridge, MA, 1965.

[10] Andrew M. Gleason. Measures on a closed subspaces of a Hilbert space. *Journal of Mathematics and Mechanics*, 6:885–893, 1957.

[11] Actually, this is an exercise in elementary analysis.

[11] D. B. Greenberger, M. Horne, and A. Zeilinger. Multiparticle interferometry and the superposition principle. *Physics Today*, 46:22–29, August 1993.

[12] Daniel B. Greenberger and A. YaSin. "Haunted" measurements in quantum theory. *Foundation of Physics*, 19(6):679–704, 1989.

[13] Hans Havlicek and Karl Svozil. Density conditions for quantum propositions. *Journal of Mathematical Physics*, 37(11):5337–5341, November 1996.

[14] Max Jammer. *The Conceptual Development of Quantum Mechanics*. McGraw-Hill Book Company, New York, 1966.

[15] Max Jammer. *The Philosophy of Quantum Mechanics*. John Wiley & Sons, New York, 1974.

[16] J. M. Jauch. *Foundations of Quantum Mechanics*. Addison-Wesley, Reading, MA., 1968.

[17] Gudrun Kalmbach. *Orthomodular Lattices*. Academic Press, New York, 1983.

[18] Gudrun Kalmbach. *Measures and Hilbert Lattices*. World Scientific, Singapore, 1986.

[19] Simon Kochen and Ernst P. Specker. The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics*, 17(1):59–87, 1967. Reprinted in [34, pp. 235–263].

[20] G. W. Mackey. Quantum mechanics and Hilbert space. *Amer. Math. Monthly, Supplement*, 64:45–57, 1957.

[21] Othman Quasim Malhas. Quantum theory as a theory in a classical propositional calculus. *Journal of Symbolic Logic*, 52(3):834–841, September 1987.

[22] Othman Quasim Malhas. Quantum logic and the classical propositional calculus. *International Journal of Theoretical Physics*, 31(9):1699–1712, 1992.

[23] N. D. Mermin. Hidden variables and the two theorems of John Bell. *Reviews of Modern Physics*, 65:803–815, 1993.

[24] Asher Peres. Two simple proofs of the Kochen–Specker theorem. *Journal of Physics*, A24:L175–L178, 1991. Cf. [25, pp. 186-200].

[25] Asher Peres. *Quantum Theory: Concepts and Methods*. Kluwer Academic Publishers, Dordrecht, 1993.

[26] C. Piron. *Foundations of Quantum Physics*. W. A. Benjamin, Reading, MA, 1976.

[27] Pavel Pták and Sylvia Pulmannová. *Orthomodular Structures as Quantum Logics*. Kluwer Academic Publishers, Dordrecht, 1991.

[28] Michael Redhead. *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics*. Clarendon Press, Oxford, 1990.

[29] Martin Schaller and Karl Svozil. Partition logics of automata. *Il Nuovo Cimento*, 109B:167–176, 1994.

[30] Martin Schaller and Karl Svozil. Automaton partition logic versus quantum logic. *International Journal of Theoretical Physics*, 34(8):1741–1750, August 1995.

[31] Martin Schaller and Karl Svozil. Automaton logic. *International Journal of Theoretical Physics*, 35(5):911–940, May 1996.

[32] Erwin Schrödinger. Die gegenwärtige Situation in der Quantenmechanik. *Naturwissenschaften*, 23:807–812, 823–828, 844–849, 1935. English translation in [40, pp. 152-167].

[33] Ernst Specker. Die Logik nicht gleichzeitig entscheidbarer Aussagen. *Dialectica*, 14:175–182, 1960. Reprinted in [34, pp. 175–182].

[34] Ernst Specker. *Selecta*. Birkhäuser Verlag, Basel, 1990.

[35] M. H. Stone. The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.

[36] K. Svozil. Introduction to quantum logic. in preparation.

[37] Karl Svozil. *Randomness & Undecidability in Physics*. World Scientific, Singapore, 1993.

[38] Karl Svozil and Josef Tkadlec. Greechie diagrams, nonexistence of measures in quantum logics and Kochen–Specker type constructions. *Journal of Mathematical Physics*, 37(11):5380–5401, November 1996.

[39] John A. Wheeler. Law without law. In John A. Wheeler and W. H. Zurek, editors, *Quantum Theory and Measurement*, pages 182–213. Princeton University Press, Princeton, 1983. [40].

[40] John Archibald Wheeler and Wojciech Hubert Zurek. *Quantum Theory and Measurement*. Princeton University Press, Princeton, 1983.

[41] Neal Zierler and Michael Schlessinger. Boolean embeddings of orthomodular sets and quantum logic. *Duke Mathematical Journal*, 32:251–262, 1965.