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Small Trivalent Graphs of Large Girth

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SMALL TRIVALENT GRAPHS OF LARGE GIRTH

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Abstract

Definitions are given for seven trivalent Cayley graphs, of girths $17, 18, 20, 21, 22, 23$ and 24. At the time of writing (June 1997) each of these is the smallest known trivalent graph of the corresponding girth.

1 Introduction

The girth $g = g(\Gamma)$ of a graph Γ is the length of its shortest circuit. When Γ is regular, say of degree k, counting the number of vertices at distance up to $g/2$ from any given vertex provides a lower bound on the number of vertices in Γ, known as the *Moore* bound:

$$|V\Gamma| \leq \begin{cases} 1 + k + k(k-1) + \ldots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd} \\ 1 + k + k(k-1) + \ldots + k(k-1)^{(g-4)/2} + (k-1)^{(g-2)/2} & \text{if } g \text{ is even.} \end{cases}$$

Graphs which meet this lower bound are relatively scarce: they include the simple circuit graphs C_n (of degree 2 and girth n), complete graphs K_{k+1} (of girth 3), complete bipartite graphs $K_{k,k}$ (of girth 4), the Petersen graph (on 10 vertices, of degree 3 and girth 5), the Hoffman-Singleton graph (on 50 vertices, of degree 7 and girth 5), and generalised polygons (of girth $6, 8$ or 12 and restricted degree).

More generally, any k-regular graph of girth g with the minimum possible number of vertices is known as a (k, g)-cage, or simply a *cage*. Further information on cages may be found in a number of articles and books ([4], [6], [8], [11]), and also in a database maintained by Gordon Royle (currently available on the world-wide web at `http://www.cs.uwa.edu.au/~gordon`).

In the trivalent case ($k = 3$), cages have been found for girth up to 12, as shown in the table below:

Girth	*Moore bound*	*Cage(s)*	*No. of vertices*
3	4	K_4	4
4	6	$K_{3,3}$	6
5	10	Petersen	10
6	14	Heawood graph	14
7	22	McGee graph	24
8	30	Tutte's 8-cage	30
9	46	Several examples [5]	58
10	62	Several examples [10]	70
11	94	One example known [1]	112
12	126	Generalized hexagon	126

Trivalent cages of girth at most 12

Examples of small trivalent graphs of larger girth have been described by several authors, with the amount by which the number of vertices differs from the Moore bound increasing dramatically with the girth (see [1], [2], [3], [5], [7], [10] for examples).

Many of the smallest known examples appear to be Cayley graphs (associated with special types of generating sets) for small finite groups. In this paper we provide seven new examples of small trivalent Cayley graphs of relatively large girths, namely $17, 18, 20, 21, 22, 23$ and 24. At the time of writing (June 1997) each of these is the smallest known trivalent graph of the corresponding girth.

2 The graphs

The graphs presented below were obtained as a result of a systematic search for appropriate generating pairs and triples in the projective linear groups PSL$(2, q)$, PGL$(2, q)$ and PΓL$(2, q)$, for prime powers q such that $q \leq 53$. This search was carried out by computer using the MAGMA package ([9]).

In each case, three involutory permutations a, b and c are given, and the graph is the Cayley graph $\Gamma = \text{Cay}(G, X)$ where G is the group generated by the permutations in the set $X = \{a, b, c\}$: vertices of Γ may be taken as the elements of G, and edges are of the form $h — hx$ for all $h \in G$ and $x \in X$. As the three elements of X are involutions, the graph is trivalent.

Further, since the group G acts naturally by left multiplication on the Cayley graph $\text{Cay}(G, X)$, the graph is vertex-transitive, and so its girth may be calculated simply by counting the numbers of vertices at increasing distances from the identity element until a shortest circuit (based at the identity element) is found.

Also in each case a presentation is given for the group G in terms of the generators in X, with defining relations of the form a^2, b^2, c^2 and a number of words of length g or more (where g is the girth). Again these defining relations were obtained with the help of the MAGMA package ([9]).

Note that for large girth, the orders of the products ab, bc and ac need to be moderately large, an observation which reduces the search space. Our search also considered possible generating sets of the form $X = \{u, v, v^{-1}\}$ where u is an involution and v is an element of moderately large order, and in some cases this produced examples of the same order and girth as those given below, but none better.

2.1 Girth 17:

Let Γ be the Cayley graph constructed using the three involutions

$a =$ (1,9)(2,8)(3,7)(4,6)(10,17)(11,16)(12,15)(13,14),

$b =$ (1,14)(2,16)(3,6)(4,8)(5,12)(7,9)(10,17)(13,15),

$c =$ (1,12)(2,13)(3,17)(4,5)(6,16)(8,15)(9,10)(11,14).

These permutations generate a subgroup of the symmetric group S_{17} isomorphic to the projective special linear group PSL$(2, 16)$, of order 4080, and satisfy the defining relations

$$a^2 = b^2 = c^2 = ababcabcababcbcac = abababcbcbababacac$$

$$= ababcabacbabacbabc = abacbacbcacabcbcbc = 1.$$

The elements ab, bc and ca have orders $15, 17$ and 17 respectively, the graph has girth 17 and diameter 14, and its automorphism group has order 8160. This graph improves on the previously smallest known trivalent graph of girth 17 (a Cayley graph on 6072 vertices described in [3]).

2.2 Girth 18:

Let Γ be the Cayley graph constructed using the three involutions

$$a = (1,9)(2,8)(3,7)(4,6)(10,17)(11,16)(12,15)(13,14),$$

$$b = (1,11)(2,5)(3,8)(4,14)(6,15)(7,12)(9,17)(10,13),$$

$$c = (1,2)(3,13)(5,12)(6,7)(8,11)(9,15)(10,16)(14,17).$$

Again these permutations generate a subgroup of the symmetric group S_{17} isomorphic to the projective special linear group PSL$(2, 16)$, of order 4080, but this time satisfy the defining relations

$$a^2 = b^2 = c^2 = abababcbcbababacac = ababacacacbcbcacac$$

$$= abacabacbacabacabc = abcabcbabcbacbabcb = abcacbcacbacbcacbc$$

$$= ababababcababcbcbac = 1.$$

The elements ab, bc and ca all have order 17, the graph has girth 18 and diameter 16, and its automorphism group has order 24480. (In fact it is 2-arc-transitive.) This graph is smaller that the previously smallest known trivalent graph of girth 18 (the hexagon graph $H(37)$ on 4218 vertices described in [7]).

2.3 Girth 20:

Let Γ be the Cayley graph constructed using the three involutions

$$a = (1,7)(2,17)(3,27)(4,25)(5,13)(6,20)(8,26)(9,30)(10,15)(12,16)$$
$$(14,28)(18,22)(19,24)(21,29),$$
$$b = (1,5)(2,24)(3,17)(4,29)(6,21)(7,11)(8,13)(9,25)(10,18)(14,23)$$
$$(15,27)(16,22)(19,28)(20,26),$$
$$c = (1,16)(2,13)(4,22)(5,7)(6,14)(8,15)(9,23)(10,28)(11,20)(12,21)$$
$$(17,24)(18,26)(19,30)(25,27).$$

These permutations generate a subgroup of the symmetric group S_{30} isomorphic to the projective special linear group $\mathrm{PSL}(2,29)$, of order 12180, and satisfy the defining relations

$$a^2 = b^2 = c^2 = (ababacbabc)^2 = (ababcacbac)^2 = (ababcbabac)^2$$
$$= (abac)^5 = (abcacbcbac)^2 = (abcbacbcbc)^2 = (abcbcacbcb)^2 = (acbc)^5$$
$$= abababacabacbabcabc = abababacbabcbacbcbcbc = 1.$$

The elements ab, bc and ca all have order 15, the graph has girth 20 and diameter 15, and its automorphism group has order 24360. This graph is smaller that the previously smallest known trivalent graph of girth 20 (the sextet graph $S(71)$ on 14910 vertices described in [3]).

2.4 Girth 21:

Let Γ be the Cayley graph constructed using the three involutions

$$a = (1,36)(2,4)(3,37)(5,10)(6,29)(8,34)(9,14)(11,23)(13,28)(15,17)$$
$$(16,20)(18,21)(19,30)(22,32)(24,31)(25,35)(26,33)(27,38),$$
$$b = (1,38)(2,11)(3,7)(4,28)(5,30)(6,37)(8,13)(9,25)(10,15)(12,21)$$
$$(14,36)(16,20)(17,24)(18,31)(19,33)(22,23)(26,29)(32,35),$$
$$c = (1,27)(2,34)(3,32)(4,38)(5,8)(6,9)(7,26)(10,14)(11,20)(12,18)$$
$$(13,25)(15,37)(16,33)(17,28)(19,36)(21,29)(23,31)(24,35).$$

These permutations generate a subgroup of the symmetric group S_{38} isomorphic to the projective special linear group PSL$(2,37)$, of order 25308, and satisfy the defining relations

$$a^2 = b^2 = c^2 = abababcacacabcbacabac = abababcbabcbacabcbcbc$$

$$= abacbacabcbacbabcacbc = abacabacbacbabcbabcabc$$

$$= abacabcacacbabcbacbcbc = 1.$$

The elements ab, bc and ca have orders $18, 19$ and 19 respectively, the graph has girth 21 and diameter 17, and its automorphism group has order 50616. This graph is believed to be the smallest known trivalent graph of girth 21.

2.5 Girth 22:

Let Γ be the Cayley graph constructed using the three involutions

$$a = (1,19)(2,18)(3,17)(4,16)(5,15)(6,14)(7,13)(8,12)(9,11)(20,33)$$
$$(21,32)(22,31)(23,30)(24,29)(25,28)(26,27),$$

$$b = (1,33)(2,21)(3,27)(4,6)(5,15)(8,10)(9,32)(11,20)(12,26)(13,14)$$
$$(16,23)(17,28)(18,22)(19,30)(24,31)(25,29),$$

$$c = (1,23)(2,6)(3,7)(4,14)(5,28)(8,19)(9,22)(10,16)(11,18)(12,27)$$
$$(13,25)(15,30)(17,29)(20,33)(24,31)(26,32).$$

These permutations generate a subgroup of the symmetric group S_{33} isomorphic to the projective special linear group PSL$(2,32)$, of order 32736, and satisfy the defining relations

$$a^2 = b^2 = c^2 = abababacbacbababab cabc = ababacbcabacbcbcabacbc$$

$$= ababcababcacbcbacbcbac = ababcababcbacacbacacbc$$

$$= ababcacbabcacacbabcacb = abcabcacacacbacbacacac$$

$$= abcabcbcbcbacbacbcbcbc = abcbacabcbcbacabcbacac$$

$$= ababacabacbcabacbacbcbc = 1.$$

The elements ab, bc and ca all have order 31, the graph has girth 22 and diameter 20, and its automorphism group has order 196416. (In fact it is 2-arc-transitive.) This graph is believed to be the smallest known trivalent graph of girth 22.

2.6 Girth 23:

Let Γ be the Cayley graph constructed using the three involutions

$$a = (1,20)(2,36)(3,39)(4,51)(5,35)(6,54)(7,18)(8,44)(9,34)(10,25)$$
$$(11,45)(12,42)(13,38)(14,19)(15,17)(16,48)(21,24)(22,37)$$
$$(23,26)(27,46)(28,33)(29,40)(30,32)(31,53)(41,47)(43,50),$$

$$b = (1,42)(2,38)(3,29)(4,54)(5,30)(6,46)(7,47)(8,25)(9,12)(10,31)$$
$$(11,52)(13,33)(14,37)(15,51)(16,39)(17,19)(18,27)(20,40)$$
$$(22,43)(23,48)(24,50)(26,35)(28,45)(34,36)(41,44)(49,53),$$

$$c = (1,6)(2,36)(3,54)(4,9)(5,32)(7,10)(8,28)(11,51)(12,19)(13,53)$$
$$(14,50)(15,43)(16,39)(18,30)(20,33)(21,49)(22,38)(23,24)$$
$$(25,48)(26,42)(27,29)(31,44)(34,46)(35,37)(40,41)(45,52).$$

These permutations generate a subgroup of the symmetric group S_{54} isomorphic to the projective special linear group PSL$(2,53)$, of order 74412, and satisfy the defining relations

$$a^2 = b^2 = c^2 = abababcabcacbacbcacbcac = abababcbcacbcacbacbcabc$$
$$= abacabcbacbacbacabcbabc = ababcabcacbcbabacbacbcac$$
$$= (abacacacacbc)^2 = abacacbcbacabcacbcacbabc = 1.$$

The elements ab, bc and ca have orders $27, 13$ and 13 respectively, the graph has girth 23 and diameter 20, and its automorphism group has order 148824. This graph is believed to be the smallest known trivalent graph of girth 23.

2.7 Girth 24:

Let Γ be the Cayley graph constructed using the three involutions

$$a = (1,36)(2,32)(3,42)(4,31)(5,9)(6,43)(7,35)(8,12)(10,26)(11,18)$$
$$(13,30)(14,19)(15,29)(16,25)(17,37)(20,39)(21,34)(22,41)$$
$$(23,38)(24,44)(27,40),$$

$$b = (1,34)(2,6)(3,19)(4,22)(5,10)(7,21)(8,35)(9,14)(11,28)(12,42)$$
$$(13,17)(15,41)(16,44)(18,29)(20,30)(23,38)(24,31)(25,40)$$
$$(26,27)(32,39)(33,43)(36,37),$$

$c = (1,29)(2,28)(3,27)(4,26)(5,25)(6,24)(7,23)(8,22)(9,21)(10,20)$
$(11,19)(12,18)(13,17)(14,16)(30,44)(31,43)(32,42)(33,41)$
$(34,40)(35,39)(36,38)$.

These permutations generate a subgroup of the symmetric group S_{44} isomorphic to the projective general linear group PGL(2, 43), of order 79464, and satisfy the defining relations

$$a^2 = b^2 = c^2 = abababababacbcbabcacbabcbc = abababcabcbcacbabcabacac$$
$$= ababacabcabcbcbacacababc = ababacacababcbcbacabcabc$$
$$= ababcabacbcbcabacbababcb = ababcabcbabcacacabcbcabc$$
$$= (ababcacabcbc)^2 = ababcbacbabcacacacbacbcb$$
$$= abacabcabcacabacbcbcbabc = (abacacbcacbc)^2$$
$$= (acbcbcbcbcbc)^2 = abababcacbacababacbac = 1.$$

The elements ab, bc and ca have orders 42, 42 and 22 respectively, the graph has girth 24 and diameter 19, and its automorphism group has order 79464. This graph is believed to be the smallest known trivalent graph of girth 24.

3 Summary

We conclude this paper with a summary of the currently best known candidates for trivalent cages of girth girth between 13 and 24: see the table below. (The cages of smaller girth were given in a similar table in the Introduction.)

Subsequent improvements to the table below are expected to appear on the "cubic cages" page in the combinatorial database maintained by Gordon Royle (on the world-wide web at `http://www.cs.uwa.edu.au/~gordon`).

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Girth	*Moore bound*	*No. of vertices*	*Reference*
13	190	272	[3]
14	254	406	[3]
15	382	620	[3]
16	510	990	[3]
17	766	4080	§2.1
18	1022	4080	§2.2
19	1534	4324	[7]
20	2046	12180	§2.3
21	3070	25308	§2.4
22	4094	32736	§2.5
23	6142	74412	§2.6
24	8190	79464	§2.7

Smallest known trivalent graphs of girth between 13 *and* 24

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