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# Halvings on Small Point Sets

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# Halvings on Small Point Sets

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## 1 Introduction

A conjecture of A. Hartman says that each admissible parameter set of an $LS[2](t,k,v)$ is realizable. Here admissible means that all $\binom{v-s}{k-s}$ are even for $s = 0,\ldots,t$. We present some new halvings and discuss the consequences for Hartman's conjecture for small $t$. So, small means $t \leq 5$ which seems to be already large compared to the cases considered so far. We show that for $t = 5,\ k = 6,7,8,9$, for $t = 4,\ k \leq 15$, and for $t = 3,\ k \leq 15$ the conjecture is true. The cases $t = 2,\ k \leq 15$ had already been settled before by S. Ajoodani-Namini and G. B. Khoshrovshahi [2].

Presently, theoretical methods are not yet powerful enough to find the large sets needed to settle Hartman's conjecture. We use a computer program DISCRETA[5] built in Bayreuth by Anton Betten and Alfred Wassermann and which is based on the Kramer-Mesner method. For a nice review of computer based methods in design theory and especially the Kramer-Mesner method, see[8]. We want to provide several large sets on small point sets as starting points for recursive constructions and to show where there is either still a lack of knowledge about further large sets on small point sets or of recursion theorems.

Some research presently is concerned with finding large sets on small point sets systematically, where the complete design usually is partitioned in more than only two $t$-designs[7]. On the other hand, restricting to halvings makes the construction problem easier. So, we could apply the methods that had been proven useful in finding $t$-designs with $t = 6,7,8$ recently. While in that research usually a small $\lambda$ is attractive, in the case of halvings $\lambda$ is as large as possible up to taking complements. It is a feature of the LLL-algorithm in its implementation by Alfred Wassermann[14] in DISCRETA that such large values of $\lambda$ can be handled. From the new halvings there result many infinite families of halvings. This is the technique that was used by Teirlinck in his famous Theorem that $t$-designs exist for all $t$,[13]. Since then, several further quite powerful recursive constructions for getting new large sets from old ones have been found by various authors. We use especially those by S. Ajoodani-Namini and G. B. Khoshrovshahi[1], [2]. From our results one can see where gaps still have to be closed because they are not yet covered by the presently known recursion methods. We present tables for the parameter sets for $2 < t \leq 6$ and $v < 63$ containing the known and undecided cases.

The paper relies on several new large sets, especially an $LS[2](7,24,10)$, which were found using DISCRETA[5], mostly,while the author was a guest of the Center for Discrete Mathematics and Theoretical Computer Science of the University of Auckland. The halving design with parameters 7-(24, 10, 340) found is thus a *Southern Hemisphere Design*. The author thanks Peter Gibbons and the Center for making available the computing power and for interesting discussions.

## 2 Non-Existence

A halving $t$-$(v,k,\lambda)$ of the complete design has just one half of all possible $k$-subsets as blocks. So,

$$\lambda \ = \ \frac{1}{2}\binom{v-t}{k-t}$$

is a natural number. Since a $t$-$(v,k,\lambda)$ design is also a $s$-$(v,k,\lambda_s)$ design for $0 <= s < t$, and of course still a halving, each of the $\lambda_s$ fulfill the corresponding equation for $s$ in place of $t$. So we have the well known necessary condition that all binomial coefficients

$$\binom{v-s}{k-s}$$

for $0 <= s < t$, must be even numbers.

The odd binomial coefficients can be determined from the Pascal triangle by determining its entries modulo 2, see also [15]. The recursion rule

$$\binom{n+1}{j+1} = \binom{n}{j} + \binom{n}{j+1}$$

can easily be used modulo 2 to generate this triangle. We denote the residue of $\binom{n}{k}$ modulo 2 by $\binom{n}{k}_2$.

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For $l$ a natural number and each $j \in \{0,\ldots,2^l-1\}$ the coefficient $\binom{2^l-1}{j}$ is odd. This is clear for $j = 0,1$. The general case follows by induction using the formula

$$\binom{2^l-1}{s+1} = \binom{2^l-1}{s}\frac{2^l-(s+1)}{s+1}$$

and the fact that $2^r|(2^l-k)$ if and only if $2^r|k$.

All entries in the next row of the triangle which are obtained from the recursion rule are even. Adding even numbers results in an even number such that all $\binom{2^l+r}{j}$ for $0 \le r < 2^l$ and $r < j < 2^l$ are even. Since $\binom{2^l+r}{0} = 1$ is odd, from the recursion rule all $\binom{2^l+r}{r}$ are odd. Thus, modulo 2, all entries in the triangle with the top entry $\binom{2^l}{0}$ depend only on the right and left borders which are equal to 1 everywhere. But this is also true for the whole triangle. Therefore

$$\binom{n}{j}_2 = \binom{2^l+n}{j}_2$$

for $0 \le n < 2^l$ and $0 \le j \le n$. As we noticed above, an $LS[2](t,k,v)$ can only exist if for $0 \le s < t$ each $\binom{v-s}{k-s}$ is even. This means that in the Pascal triangle modulo 2 a sequence of $t+1$ entries in a diagonal parallel to the right border must be 0. Each such sequence of 0-entries must lie between two parallel rows for some $n_1 = 2^l - 1$ and $n_2 = 2^{l+1}-1$, since in these rows all entries are 1. So, we obtain the following recursive description of admissable parameter sets for halvings.

**Theorem 1** *Let $l$ be a natural number and $2^l - 1 \le v < 2^{l+1} - 1$.*

- *If $2^l + t > v$ then $LS[2](t,k,v)$ is not admissable.*
- *If $2^l + t \leq v$ and $v - 2^l < k < 2^l$ then $LS[2](t,k,v)$ is admissable.*
- *If $2^l + t \leq v$ and $v - 2^l = k$ then $LS[2](t,k,v)$ is not admissable.*
- *If $2^l + t \leq v$ and $v - 2^l > k$ then $LS[2](t,k,v)$ is admissable if and only if $LS[2](t,k,v-2^l)$ is admissable.*

**Remark** *The smallest $(v,k)$ for which for a given $t$ a parameter set $LS[2](t,k,v)$ is admissable fulfills*

$$(2^{l+1}-2)-j \leq 2^{l+1}-2, 2^{l-1}-2 < t = (2^l-2)-j, k = t+1,$$

*where $j \in \{0,1,\ldots,2^l-1\}$.*

The smallest possible parameter sets for a halving for $t \leq 16$ are thus

- $LS[2](2,3,6)$,
- $LS[2](3,4,11)$, $LS[2](4,5,12)$, $LS[2](5,6,13)$, $LS[2](6,7,14)$,
- $LS[2](7,8,23)$, $LS[2](8,9,24)$, $LS[2](9,10,25)$, $LS[2](10,11,26)$, $LS[2](11,12,27)$, $LS[2](12,13,28)$, $LS[2](13,14,29)$, $LS[2](14,15,30)$,
- $LS[2](15,16,47)$, $LS[2](16,17,48)$.

## 3 Existence

An $LS[2](2,3,6)$ is well known and can easily be obtained from DISCRETA, describing the trivial automorphism group or $C_3$ acting semiregularly. We use a variety of recursive constructions to find large sets for other parameter sets. These rely on some starting points of which some are already known since some time. An $LS[2](6,7,14)$ has been constructed by Kreher and Radziszowski [11]. Further large sets used are $LS[2](6,7,22)$[13], $LS[2](6,8,22)$[12], $LS[2](6,12,24)$[6].

With the help of DISCRETA we found the following new large sets: An $LS[2](4,10,20)$ and an $LS[2](4,9,20)$ by prescribing $PGL(2,9)$ as an automorphism group with a simultaneous action on two copies of the natural set of 10 points. The Kramer-Mesner matrix has sizes $16 \times 272$ and $16 \times 316$ respectively. Adding one fixed point to this permutation representation of $PSL(2,9)$ then gives an $LS[2](5,9,21)$ from a Kramer-Mesner matrix of size $95 \times 950$. Here $PGL(2,9)$ has no solutions. An $LS[2](7,10,24)$ was found prescribing $PSL(2,23)$ as an automorphism group. In this case the Kramer-Mesner matrix has size $57 \times 356$.
In each but the case $LS[2](5,9,21)$ one corresponding $t$-$(v,k,\lambda)$ design is shown below in this article.

A first easy and also well known useful tool is the following result.

**Lemma 2** *Forming derived, residual, complementary and supplementary designs transforms a halving into a halving.*
**Proof** Let $D$ be a $t$-$(v,k,\lambda)$-design, where $\lambda = 1/2\binom{v-t}{k-t}$, i. e. a halving of the complete design. We call der$(D)$, res$(D)$, comp$(D)$, supp$(D)$ the derived, residual, complementary and supplementary design of $D$, respectively. Taking the complementary designs of a halving again partitions the set of $k$-subsets in a halving. For the supplementary designs we show that the new value of $\lambda$ is again half the maximal value. Since the new blocksize is $v-k$, half the maximal value of the new $\lambda$ is $\binom{v-t}{v-k-t.}$ On the other hand

$$\lambda_0^t = 1/2\binom{v-t}{k-t}\frac{\binom{v-t}{k}}{\binom{v-t}{k-t}} = (1/2)\binom{v-t}{k}.$$

Since there are just as many $k$-subsets as complements of $k$-subsets, both binomial expressions are equal.

Since $\lambda$ is unchanged when forming derived designs, we compare $\lambda$ to the required value for derived designs .

$$1/2\binom{(v-1)-(t-1)}{(k-1)-(t-1)} = 1/2\binom{v-t}{k-t} = \lambda.$$

In the case of residual designs this comparison is as follows.

$$1/2\binom{(v-1)-(t-1)}{k-(t-1)} = 1/2\binom{v-t}{k-t+1} =$$

$$1/2\binom{v-t}{k-t}\frac{(v-t)-(k-t)}{k-t+1} =$$

$$1/2\binom{v-t}{k-t}\frac{v-k}{k-t+1}.$$

and

$$\lambda_{t-1}-\lambda_t = \lambda_t(\frac{v-t+1}{k-t+1}-1) = \lambda_t\frac{v-k}{k-t+1} = 1/2\binom{v-t}{k-t}\frac{v-k}{k-t+1}.$$

□

Finding only one $LS[2](t-1+(t+2)/2, 2^l+(t+2)/2, 2^{l+1}+t-1+(t+2)/2)$ for each $l$ would suffice to get whole intervals of parameter sets by forming derived and residual designs iteratively with $k$ varying from $t+1$ to $2^l+(t+2)/2$. So, from the existence of an $LS[2](14,15,30)$ we could deduce the existence of all $LS[t,k,v)$ besides the $LS[2](2,3,6)$ for all admissible parameter sets.

Finding a large set for some value of $t$ generally allows to get intervals for some smaller values of $t$. So, finding an $LS[2](7,10,23)$ would suffice to get the interval of all $LS[2](t,k,21)$ with $8 \leq k \leq 13$. Then using the known $LS[2](6,7,22)$ would add the missing $LS[2](5,6,21)$ and $LS[2](5,7,21)$. Thus, by forming supplements the whole interval of admissible values for $v=21$ and $t=5$ would follow. But, we only could find an $LS[2](7,10,24)$, which leaves two cases even for for $t=4$ unsettled. Fortunately, we could find an $LS[2](4,20,10)$ and an $LS[2](5,9,21)$ directly. Thus the interval $LS[2](4,k,20)$ for $5 \leq k \leq 15$ is completely settled. Consequently, the corresponding intervals for $t \leq 3$ follow. For $t=5$ and $v=21$ only $k=10$ is unsettled.

We remark that the found 7-$(24,10,340)$-design giving the halving $LS[2](7,10,24)$ is not far from being an 8-$(24,10,60)$-design which would have the required 7-$(23,10,280)$ as a residual design.

Another well known tool is Tran van Trung's construction which from an interval $LS[2](t,k,v)$ , $a \leq k < b$, gives another interval $LS[2](t,k,v+1)$ , $a < k < b$, see also Khosrovshahi and Ajoodani-Namini [10]. So, iterating this process fills a triangle of parameter set values $LS[2](t,k',v+i)$ where $k+i \leq k' < b$ and $i \leq (b-a)$. This has been used by D. L. Kreher[12] to construct an $LS[2](6,8,23]$ from an $LS[2](6,8,22]$ and an $LS[2](6,7,22]$.

We combine these results with several recursion theorems.

**Theorem 3**(Khosrovshahi and Ajoodani-Namini) *If there are $LS[2](t,k,v)$ for all $k$ in an interval $[t+1,m]$ and there is an $LS[2](t,m,w)$ then there is also an $LS[2](t,m,v+w-t)$.*

This is important for the rows where $v=2^l+t$. Suppose such a row could be filled completely, as for example in the cases $t=4$ and $v=20$ or $t=3$ and $v=19$ above. Then applying this Theorem to $m=t+1$ up to $2^l-1$ fills the row for $v=2^{l+1}+t$ with $2^l-1-t$ values. Taking the supplementary designs leads to further $2^l-1-t$ values.

In other cases the following construction tool produces new intervals from old ones.

**Theorem 4(Ajoodani-Namini)**[1] *If an* $LS[2](t,k,v)$ *exists then there exist also an* $LS[2](t,2k,2v)$ *and an* $LS[2](t+1,2k+1,2(v+1))$.

**Corollary 5** *If an* $LS[2](t,k,v)$ *exists then there exist also* $LS[2](t,2k-1,2v)$, $LS[2](t,2k,2v)$, $LS[2](t,2k+1,2v)$, *and* $LS[2](t,2k,2v+1)$, $LS[2](t,2k+1,2v+1)$.

The first three parameter sets are obtained by the first part of Ajoodani-Namini's Theorem and by first constructing residual and derived designs and then applying the second part of Ajoodani-Namini's Theorem. The last two parameter sets are obtained by first constructing an $LS[2](t+1,2k+1,2(v+1))$ and then forming the residual and the derived designs.

Using Ajoodani-Namini's recursion we obtain from an $LS[2](5,k,v)$ an $LS[2](5,2k,2v)$ and an $LS[2](6,2k+1,2(v+1))$. Forming derived, residual designs and regarding the 6-design as a 5-design then produces $LS[2](5,2k+1,2(v+1))$, $LS[2](5,2k+1,2v+1)$, $LS[2](5,2k,2v+1)$ .

As an example consider an $LS[2](5,11,22)$. From this we obtain the following large sets:
$LS[2](5,21,44)$, $LS[2](5,22,44)$, $LS[2](5,23,44)$, $LS[2](5,22,45)$, $LS[2](5,23,45)$.

Intervals of values for $k$ for a fixed $v$ then give rise to intervals of double the size on $2v+1$ points.

**Corollary 6** *If* $LS[2](t,k,v)$ *exist for all* $a \leq k < b$ *then there exist also* $LS[2](t,k,2v+1)$ *for* $2a \leq k < 2b$ *and* $LS[2](t,k,2v)$ *for* $2a-1 \leq k < 2b-1$.

From an interval $LS[2](t,k,v)$ , $a \leq k < b$, one obtains by Tran van Trung's construction another interval $LS[2](t,k,v+1)$ , $a < k < b$, see also Khosrovshahi and Ajoodani-Namini [10].

So, from the above intervals of $LS[2](t,k,v)$ for $t \leq 5$ there result further intervals with increasing lower bound $a$, up to $a = 15$. The corresponding values of $k$ can be found in the table.

Applying Ajoodani-Namini's Theorem we then obtain the next region of values for $42 \leq v < 63$ and $k < 11$ in the table. This recipe can be iteratively applied to cover an infinite series of regions. From the first fully settled interval for $v = 22$ in the case $t = 5$ we obtain the fully settled interval for $v = 44$ in the next region and generally for $v = n \cdot 24 - 1$ where $n$ is a natural number. So, in the interval $[2^l+5, 2^{l+1}-2]$ for the upper $2^{l-1}$ values of $v$ all values of $k$ for $2^l + v - 2^{l+1} \leq k \leq v/2$ belong to an $LS[2](5,k,v)$, that are $2^{l+1}-2$ parameter sets.

There are further recursive constructions where $k$ is not increased.

**Theorem 7**(Teirlinck[13], Khosrovshahi and Ajoodani-Namini [10]) *If there are an* $LS[2](t,t+1,v)$ *and an* $LS[2](t,t+1,w)$ *then there is also an* $LS[2](t,t+1,v+w-t)$.

**Theorem 8**(Qui-rong Wu[16]) *If there are* $LS[2](t-1,t+1,v)$, $LS[2](t-1,t,v-1)$, $LS[2](t-1,t+1,w)$, $LS[2](t-1,t,w-1)$, *then there is an* $LS[2](t-1,t+1,v+w-t)$.

Wu's construction gives an $LS[2](6,7,30)$ from the $LS[2](6,7,14)$. This results in the parameter sets $LS[2](5,6,29)$, $LS[2](5,7,29)$, $LS[2](5,7,30)$ for $t=5$.

**Theorem 9**(Alltop[3]) *If there is an* $LS[2](t,k,2k+1)$ *then there is an* $LS[2](t+1,k+1,2k+2)$.

The latter result can be applied to each of the parameter sets in the second column of our tables. Thus, there result many $LS[2](6,k+1,2k+2)$. Also, the $LS[2](5,11,22)$ is obtained from the $LS[2](4,11,21)$ by this Theorem.

There are further constructions of this type by Teirlinck, but these have very rapidly growing values of $v$ and are not used here.

There are already some known families of $LS[2](t,k,v)$ derived from some starting large sets by these rules. So, D. Kreher and S.P. Radziszowski found an $LS[2](6,7,14)$, Kreher found an $LS[2](6,8,22)$ and thus Teirlinck constructed $LS[2](6,7,6+8m)$, $LS[2](5,6,5+8m)$, and Kreher constructed $LS[2](6,8,23+16m)$, for natural $m$. Further series were derived from the recently found $LS[2](6,12,24)$[6].

The newly found $LS[2](7,10,24)$ now allows to add several new $LS[2](6,k,v)$ and $LS[2](5,k,v)'s$, which in combination with the mentioned results give a basis for many new series.

The $LS[2](5,6,5+8m)$, $LS[2](5,6,6+8m)$, $LS[2](5,7,6+8m)$ coming from the $LS[2](6,7,6+8m)$, and the $LS[2](5,8,23+16m)$ give the entries on the right border of the table. It is remarkable that these parameter sets just fill some isolated small areas of feasible parameter combinations completely.

Applying Ajoodani-Namini's constructions as above to the $LS[2](5,7,6+8m)$ yields 12 new entries each time. This occurs in the table for $v = 58, \ldots, 62$ and $k \leq 15$, where the basis is the $LS[2](6,7,30)$.

For $v \leq 36$ and $t \leq 5$ there is only one parameter set not yet decided: $LS[2](5,10,21)$. If this halving could be found the recursive constructions would allow to fill all regions where $k < 16$ and $t = 5$.

As can be seen from the table, there are many undecided cases. So, one should on one hand try to find new starting points like an $LS[2](6,11,23)$ and on the other hand look for further recursive constructions, filling for example the gap where $v = 62$ and $k = 23$.

# 4 Tables

We show the summarized results in some tables for $v \leq 62$ which are deduced from Pascal's triangle. The row number denotes $v$ and an entry $k$ means that an $LS[2](t,k,v)$ exists. An entry "-" means that the parameter set with the corresponding $k$ for an $LS[2](t,k,v)$ does not exist. Points mean undecided cases.

**Existence of $LS[2](6, k, v)$**

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| 58 | 29 || 28 || 27 || - || - || - || - || - || - || - || - || - || - || - || . || . || . || . || . || - || - || - || - ||||
| 59 || 29 || 28 || - || - || - || - || - || - || - || - || - || - || - || - || . || . || . || . || - || - || - || - || - |||
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**Existence of $LS[2](5, k, v)$**

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| 19 | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | |
| 20 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | | | |
| 21 | . | 9 | 8 | 7 | 6 | | | | | | | | | | | | | | | | | | | | | |
| 22 | 11 | 10 | 9 | 8 | 7 | - | | | | | | | | | | | | | | | | | | | | |
| 23 | 11 | 10 | 9 | 8 | - | - | | | | | | | | | | | | | | | | | | | | |
| 24 | 12 | 11 | 10 | 9 | - | - | - | | | | | | | | | | | | | | | | | | | |
| 25 | 12 | 11 | 10 | - | - | - | - | | | | | | | | | | | | | | | | | | | |
| 26 | 13 | 12 | 11 | - | - | - | - | - | | | | | | | | | | | | | | | | | | |
| 27 | 13 | 12 | - | - | - | - | - | - | | | | | | | | | | | | | | | | | | |
| 28 | 14 | 13 | - | - | - | - | - | - | - | | | | | | | | | | | | | | | | | |
| 29 | 14 | - | - | - | - | - | - | 7 | 6 | | | | | | | | | | | | | | | | | |
| 30 | 15 | - | - | - | - | - | - | - | 7 | - | | | | | | | | | | | | | | | | |
| 31 | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | | |
| 32 | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | |
| 33 | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | |
| 34 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | |
| 35 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | |
| 36 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | |
| 37 | . | . | . | . | . | . | . | . | . | 9 | 8 | 7 | 6 | | | | | | | | | | | | | |
| 38 | . | . | . | . | . | . | . | . | . | . | 9 | 8 | 7 | - | | | | | | | | | | | | |
| 39 | . | . | . | . | . | . | . | . | . | . | 9 | 8 | - | - | | | | | | | | | | | | |
| 40 | . | . | . | . | . | . | . | . | . | . | . | 9 | - | - | - | | | | | | | | | | | |
| 41 | . | . | . | . | . | . | . | . | . | . | . | - | - | - | - | | | | | | | | | | | |
| 42 | . | . | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | - | - | - | - | - | | | | | | | | | | |
| 43 | . | . | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | - | - | - | - | - | - | | | | | | | | | | |
| 44 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | - | - | - | - | - | - | - | | | | | | | | | |
| 45 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | - | - | - | - | - | - | 7 | 6 | | | | | | | | | |
| 46 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | - | - | - | - | - | - | - | 7 | - | | | | | | | | |
| 47 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | - | - | - | - | - | - | - | - | - | - | | | | | | | | |
| 48 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | - | - | - | - | - | - | - | - | - | - | - | | | | | | | |
| 49 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | |
| 50 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | |
| 51 | 25 | 24 | 23 | 22 | 21 | 20 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | |
| 52 | 26 | 25 | 24 | 23 | 22 | 21 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | |
| 53 | 26 | 25 | 24 | 23 | 22 | - | - | - | - | - | - | . | . | . | . | . | . | 9 | 8 | 7 | 6 | | | | | |
| 54 | 27 | 26 | 25 | 24 | 23 | - | - | - | - | - | - | - | . | . | . | . | . | . | 9 | 8 | 7 | - | | | | |
| 55 | 27 | 26 | 25 | 24 | - | - | - | - | - | - | - | - | . | . | . | . | . | . | 9 | 8 | - | - | | | | |
| 56 | 28 | 27 | 26 | 25 | - | - | - | - | - | - | - | - | - | . | . | . | . | . | . | 9 | - | - | - | | | |
| 57 | 28 | 27 | 26 | - | - | - | - | - | - | - | - | - | - | . | . | . | . | . | . | - | - | - | - | | | |
| 58 | 29 | 28 | 27 | - | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | - | - | - | - | - | | |
| 59 | 29 | 28 | - | - | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | - | - | - | - | - | - | | |
| 60 | 30 | 29 | - | - | - | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | - | - | - | - | - | - | - | |
| 61 | 30 | - | - | - | - | - | - | . | . | - | - | - | - | - | - | 15 | 14 | - | - | - | - | - | - | 7 | 6 | |
| 62 | 31 | - | - | - | - | - | - | - | . | - | - | - | - | - | - | - | 15 | - | - | - | - | - | - | - | 7 | - |

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| 16 | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | | |
| 17 | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | | |
| 18 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | |
| 19 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | |
| 20 | 10 | 9 | 8 | 7 | 6 | 5 | | | | | | | | | | | | | | | | | | | | | |
| 21 | 10 | 9 | 8 | 7 | 6 | - | | | | | | | | | | | | | | | | | | | | | |
| 22 | 11 | 10 | 9 | 8 | 7 | - | - | | | | | | | | | | | | | | | | | | | | |
| 23 | 11 | 10 | 9 | 8 | - | - | - | | | | | | | | | | | | | | | | | | | | |
| 24 | 12 | 11 | 10 | 9 | - | - | - | - | | | | | | | | | | | | | | | | | | | |
| 25 | 12 | 11 | 10 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | |
| 26 | 13 | 12 | 11 | - | - | - | - | - | - | | | | | | | | | | | | | | | | | | |
| 27 | 13 | 12 | - | - | - | - | - | - | - | | | | | | | | | | | | | | | | | | |
| 28 | 14 | 13 | - | - | - | - | - | 7 | 6 | 5 | | | | | | | | | | | | | | | | | |
| 29 | 14 | - | - | - | - | - | - | 7 | 6 | - | | | | | | | | | | | | | | | | | |
| 30 | 15 | - | - | - | - | - | - | - | 7 | - | - | | | | | | | | | | | | | | | | |
| 31 | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | | |
| 32 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | |
| 33 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | |
| 34 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | |
| 35 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | |
| 36 | . | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | | | | | | | | | | | | | |
| 37 | . | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | - | | | | | | | | | | | | | |
| 38 | . | . | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | - | - | | | | | | | | | | | | |
| 39 | . | . | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | - | - | - | | | | | | | | | | | | |
| 40 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | - | - | - | - | | | | | | | | | | | |
| 41 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | - | - | - | - | - | | | | | | | | | | | |
| 42 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | - | - | - | - | - | - | | | | | | | | | | |
| 43 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | - | - | - | - | - | - | - | | | | | | | | | | |
| 44 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | - | - | - | - | - | 7 | 6 | 5 | | | | | | | | | |
| 45 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | - | - | - | - | - | - | 7 | 6 | - | | | | | | | | | |
| 46 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | - | - | - | - | - | - | - | 7 | - | - | | | | | | | | |
| 47 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | |
| 48 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | |
| 49 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | |
| 50 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | |
| 51 | 25 | 24 | 23 | 22 | 21 | 20 | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | | | | | | |
| 52 | 26 | 25 | 24 | 23 | 22 | 21 | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | - | | | | | |
| 53 | 26 | 25 | 24 | 23 | 22 | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | - | - | | | | | |
| 54 | 27 | 26 | 25 | 24 | 23 | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | - | - | - | | | | |
| 55 | 27 | 26 | 25 | 24 | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | - | - | - | - | | | | |
| 56 | 28 | 27 | 26 | 25 | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | - | - | - | - | - | | | |
| 57 | 28 | 27 | 26 | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | - | - | - | - | - | - | | | |
| 58 | 29 | 28 | 27 | - | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | - | - | - | - | - | - | - | | |
| 59 | 29 | 28 | - | - | - | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | - | - | - | - | - | - | - | | |
| 60 | 30 | 29 | - | - | - | - | - | . | . | . | - | - | - | - | - | 15 | 14 | - | - | - | - | - | - | 7 | 6 | 5 | |
| 61 | 30 | - | - | - | - | - | - | . | . | - | - | - | - | - | - | 15 | - | - | - | - | - | - | - | 7 | 6 | - | |
| 62 | 31 | - | - | - | - | - | - | - | . | - | - | - | - | - | - | - | - | - | - | - | - | - | - | 7 | - | - | |

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| 11 | 5 | 4 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| 12 | 6 | 5 | - | | | | | | | | | | | | | | | | | | | | | | | | | |
| 13 | 6 | - | - | | | | | | | | | | | | | | | | | | | | | | | | | |
| 14 | 7 | - | - | - | | | | | | | | | | | | | | | | | | | | | | | | |
| 15 | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | | | |
| 16 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | | |
| 17 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | | |
| 18 | - | - | - | - | - | - | | | | | | | | | | | | | | | | | | | | | | |
| 19 | 9 | 8 | 7 | 6 | 5 | 4 | | | | | | | | | | | | | | | | | | | | | | |
| 20 | 10 | 9 | 8 | 7 | 6 | 5 | - | | | | | | | | | | | | | | | | | | | | | |
| 21 | 10 | 9 | 8 | 7 | 6 | - | - | | | | | | | | | | | | | | | | | | | | | |
| 22 | 11 | 10 | 9 | 8 | 7 | - | - | - | | | | | | | | | | | | | | | | | | | | |
| 23 | 11 | 10 | 9 | 8 | - | - | - | - | | | | | | | | | | | | | | | | | | | | |
| 24 | 12 | 11 | 10 | 9 | - | - | - | - | - | | | | | | | | | | | | | | | | | | | |
| 25 | 12 | 11 | 10 | - | - | - | - | - | - | | | | | | | | | | | | | | | | | | | |
| 26 | 13 | 12 | 11 | - | - | - | - | - | - | - | | | | | | | | | | | | | | | | | | |
| 27 | 13 | 12 | - | - | - | - | 7 | 6 | 5 | 4 | | | | | | | | | | | | | | | | | | |
| 28 | 14 | 13 | - | - | - | - | - | 7 | 6 | 5 | - | | | | | | | | | | | | | | | | | |
| 29 | 14 | - | - | - | - | - | - | 7 | 6 | - | - | | | | | | | | | | | | | | | | | |
| 30 | 15 | - | - | - | - | - | - | - | 7 | - | - | - | | | | | | | | | | | | | | | | |
| 31 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | | |
| 32 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | |
| 33 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | | |
| 34 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | | | | | | | |
| 35 | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | | | | | | | | | | | | | | |
| 36 | . | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | - | | | | | | | | | | | | | |
| 37 | . | . | . | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | - | - | | | | | | | | | | | | | |
| 38 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | - | - | - | | | | | | | | | | | | |
| 39 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | - | - | - | - | | | | | | | | | | | | |
| 40 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | - | - | - | - | - | | | | | | | | | | | |
| 41 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | - | - | - | - | - | - | | | | | | | | | | | |
| 42 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | - | - | - | - | - | - | - | | | | | | | | | | |
| 43 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | - | - | - | - | 7 | 6 | 5 | 4 | | | | | | | | | | |
| 44 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | - | - | - | - | - | 7 | 6 | 5 | - | | | | | | | | | |
| 45 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | - | - | - | - | - | - | 7 | 6 | - | - | | | | | | | | | |
| 46 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | - | - | - | - | - | - | - | 7 | - | - | - | | | | | | | | |
| 47 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | | |
| 48 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | |
| 49 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | | |
| 50 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | | | | | | |
| 51 | 25 | 24 | 23 | 22 | 21 | 20 | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | | | | | | |
| 52 | 26 | 25 | 24 | 23 | 22 | 21 | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | - | | | | | |
| 53 | 26 | 25 | 24 | 23 | 22 | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | - | - | | | | | |
| 54 | 27 | 26 | 25 | 24 | 23 | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | - | - | - | | | | |
| 55 | 27 | 26 | 25 | 24 | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | - | - | - | - | | | | |
| 56 | 28 | 27 | 26 | 25 | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | 9 | - | - | - | - | - | | | |
| 57 | 28 | 27 | 26 | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | 10 | - | - | - | - | - | - | | | |
| 58 | 29 | 28 | 27 | - | - | - | - | - | - | - | - | - | - | - | 15 | 14 | 13 | 12 | 11 | - | - | - | - | - | - | - | | |
| 59 | 29 | 28 | - | - | - | - | . | . | . | . | - | - | - | - | 15 | 14 | 13 | 12 | - | - | - | - | 7 | 6 | 5 | 4 | | |
| 60 | 30 | 29 | - | - | - | - | - | . | . | . | - | - | - | - | - | 15 | 14 | 13 | - | - | - | - | - | 7 | 6 | 5 | - | |
| 61 | 30 | - | - | - | - | - | - | . | . | - | - | - | - | - | - | 15 | 14 | - | - | - | - | - | - | 7 | 6 | - | - | |
| 62 | 31 | - | - | - | - | - | - | - | . | - | - | - | - | - | - | - | 15 | - | - | - | - | - | - | - | 7 | - | - | - |

## 5 A 7-$(24, 10, 340)$ design

The 7-$(24, 10, 340)$ design has 980628 blocks and automorphism group $PSL(2, 23)$. It is constructed by the Kramer-Mesner method using the implementation in the program system DISCRETA developed at the University of Bayreuth[5]. We do not know the number of isomorphism types of designs with this parameter set. The solution found is not isomorphic to its complement, since it has the same automorphism group but has a different number of orbits. So, we have at least 2 different isomorphism types of designs, which together form only one isomorphism type of an $LS[2](7, 10, 24)$. The automorphism group $PSL(2, 23)$ is prescribed by the following permutation presentation.

$$\begin{aligned} G &= \langle (1\,3\,14\,10\,8\,16\,6\,12\,5\,20\,9\,23\,4\,18\,7\,22\,15\,21\,11\,19\,17\,13\,24), \\ &\quad\; (1\,22\,11\,15\,17\,9\,19\,13\,20\,5\,16\,2\,21\,7\,18\,3\,10\,4\,14\,6\,8\,12\,24), \\ &\rangle \end{aligned}$$

The order is $|G| = 6072$
The group has 356 orbits on 10-sets.

The design is formed by the orbits with the following representatives, and the appended stabilizer orders.

**basic 10-orbits:**

$\{1,2,3,4,5,6,7,8,9,10\}_1$
$\{1,2,3,4,5,6,7,8,12,22\}_1$
$\{1,2,3,4,5,6,7,8,19,20\}_1$
$\{1,2,3,4,5,6,7,8,21,22\}_1$
$\{1,2,3,4,5,6,7,10,11,12\}_1$
$\{1,2,3,4,5,6,7,8,12,13\}_2$
$\{1,2,3,4,5,6,7,8,13,14\}_1$
$\{1,2,3,4,5,6,7,9,11,12\}_1$
$\{1,2,3,4,5,6,7,8,9,12\}_1$
$\{1,2,3,4,5,6,7,8,9,14\}_1$
$\{1,2,3,4,5,6,7,8,9,17\}_1$
$\{1,2,3,4,5,6,7,8,9,18\}_1$
$\{1,2,3,4,5,6,7,8,9,19\}_1$
$\{1,2,3,4,5,6,7,8,9,23\}_1$
$\{1,2,3,4,5,6,7,9,11,13\}_1$
$\{1,2,3,4,5,6,7,9,13,16\}_1$
$\{1,2,3,4,5,6,7,9,13,18\}_1$
$\{1,2,3,4,5,6,7,8,10,14\}_1$
$\{1,2,3,4,5,6,7,8,17,20\}_1$
$\{1,2,3,4,5,6,7,12,18,19\}_1$
$\{1,2,3,4,5,6,7,12,14,19\}_1$
$\{1,2,3,4,5,6,7,8,12,19\}_1$
$\{1,2,3,4,5,6,7,10,12,20\}_1$
$\{1,2,3,4,5,6,7,10,14,20\}_1$
$\{1,2,3,4,5,6,7,8,13,21\}_1$
$\{1,2,3,4,5,6,7,9,12,22\}_1$
$\{1,2,3,4,5,6,7,8,18,20\}_1$
$\{1,2,3,4,5,6,7,8,20,22\}_1$
$\{1,2,3,4,5,6,7,8,11,20\}_1$
$\{1,2,3,4,5,6,7,8,19,21\}_1$
$\{1,2,3,4,5,6,7,8,21,23\}_1$
$\{1,2,3,4,5,6,7,8,13,22\}_1$
$\{1,2,3,4,5,6,7,8,16,21\}_1$
$\{1,2,3,4,5,6,7,10,12,18\}_1$
$\{1,2,3,4,5,6,7,12,14,18\}_2$
$\{1,2,3,4,5,6,7,12,15,18\}_1$
$\{1,2,3,4,5,6,7,10,12,15\}_1$
$\{1,2,3,4,5,6,7,9,10,12\}_1$
$\{1,2,3,4,5,6,7,8,12,14\}_1$
$\{1,2,3,4,5,6,7,8,13,15\}_1$
$\{1,2,3,4,5,6,7,8,12,15\}_1$
$\{1,2,3,4,5,6,7,8,10,13\}_1$
$\{1,2,3,4,5,6,7,8,10,16\}_1$
$\{1,2,3,4,5,6,7,8,10,18\}_1$
$\{1,2,3,4,5,6,7,8,11,14\}_1$
$\{1,2,3,4,5,6,7,8,11,16\}_1$
$\{1,2,3,4,5,6,7,8,11,22\}_2$
$\{1,2,3,4,5,6,7,8,11,23\}_1$
$\{1,2,3,4,5,6,7,8,14,15\}_1$
$\{1,2,3,4,5,6,7,8,14,18\}_1$
$\{1,2,3,4,5,6,7,8,14,23\}_1$
$\{1,2,3,4,5,6,7,8,15,16\}_1$
$\{1,2,3,4,5,6,7,8,15,17\}_1$
$\{1,2,3,4,5,6,7,8,15,18\}_2$
$\{1,2,3,4,5,6,7,8,15,22\}_1$
$\{1,2,3,4,5,6,7,8,15,23\}_1$
$\{1,2,3,4,5,6,7,8,16,19\}_1$
$\{1,2,3,4,5,6,7,8,16,23\}_1$
$\{1,2,3,4,5,6,7,8,17,19\}_1$
$\{1,2,3,4,5,6,7,8,18,22\}_2$
$\{1,2,3,4,5,6,7,8,18,23\}_2$
$\{1,2,3,4,5,6,7,8,19,22\}_1$
$\{1,2,3,4,5,6,7,10,11,14\}_1$
$\{1,2,3,4,5,6,7,9,10,18\}_1$
$\{1,2,3,4,5,6,7,11,13,18\}_1$
$\{1,2,3,4,5,6,7,10,11,13\}_1$
$\{1,2,3,4,5,6,7,11,13,14\}_2$
$\{1,2,3,4,5,6,7,13,16,17\}_2$
$\{1,2,3,4,5,6,7,13,17,18\}_1$
$\{1,2,3,4,5,6,7,10,13,17\}_1$
$\{1,2,3,4,5,6,7,13,14,17\}_1$
$\{1,2,3,4,5,6,7,13,17,22\}_1$
$\{1,2,3,4,5,6,7,13,18,19\}_1$
$\{1,2,3,4,5,6,7,10,13,19\}_1$
$\{1,2,3,4,5,6,7,13,14,19\}_1$
$\{1,2,3,4,5,6,7,9,21,22\}_1$
$\{1,2,3,4,5,6,7,13,16,18\}_2$
$\{1,2,3,4,5,6,7,13,14,16\}_1$
$\{1,2,3,4,5,6,7,13,16,22\}_1$
$\{1,2,3,4,5,6,7,9,15,21\}_1$
$\{1,2,3,4,5,6,7,13,21,22\}_2$
$\{1,2,3,4,5,6,7,13,18,22\}_1$
$\{1,2,3,4,5,6,7,10,13,14\}_1$
$\{1,2,3,4,5,6,7,10,13,22\}_1$
$\{1,2,3,4,5,6,7,10,13,15\}_1$
$\{1,2,3,4,5,6,7,13,14,22\}_1$
$\{1,2,3,4,5,7,9,13,14,21\}_2$
$\{1,2,3,4,5,7,8,9,14,16\}_1$
$\{1,2,3,4,5,6,7,15,16,19\}_2$
$\{1,2,3,4,5,6,9,13,17,18\}_1$
$\{1,2,3,4,5,6,9,10,13,17\}_1$
$\{1,2,3,4,5,7,8,9,10,18\}_1$
$\{1,2,3,4,5,6,7,15,16,18\}_1$
$\{1,2,3,4,5,7,8,9,13,20\}_1$
$\{1,2,3,4,5,6,9,12,13,18\}_1$
$\{1,2,3,4,5,6,8,10,14,17\}_1$
$\{1,2,3,4,5,6,8,9,13,15\}_1$
$\{1,2,3,4,5,6,9,17,19,21\}_1$
$\{1,2,3,4,5,7,8,10,11,13\}_2$
$\{1,2,3,4,5,6,7,18,19,22\}_1$
$\{1,2,3,4,5,6,9,18,19,21\}_1$
$\{1,2,3,4,5,7,8,10,11,17\}_1$
$\{1,2,3,4,5,7,8,13,16,17\}_2$

$\{1,2,3,4,5,7,10,11,16,17\}_2$
$\{1,2,3,4,5,6,7,15,18,19\}_1$
$\{1,2,3,4,5,6,7,14,15,18\}_1$
$\{1,2,3,4,5,6,7,10,11,17\}_1$
$\{1,2,3,4,5,6,7,10,14,16\}_1$
$\{1,2,3,4,5,6,7,16,18,22\}_1$
$\{1,2,3,4,5,6,10,12,14,17\}_1$
$\{1,2,3,4,5,6,9,10,14,17\}_1$
$\{1,2,3,4,5,6,7,14,16,19\}_2$
$\{1,2,3,4,5,6,7,9,10,17\}_1$
$\{1,2,3,4,5,6,7,9,10,14\}_2$
$\{1,2,3,4,5,6,7,9,10,23\}_1$
$\{1,2,3,4,5,6,7,9,11,18\}_1$
$\{1,2,3,4,5,6,7,9,14,21\}_1$
$\{1,2,3,4,5,6,7,9,14,22\}_1$
$\{1,2,3,4,5,6,7,9,15,17\}_2$
$\{1,2,3,4,5,6,7,9,15,18\}_1$
$\{1,2,3,4,5,6,7,9,15,22\}_1$
$\{1,2,3,4,5,6,7,9,15,23\}_1$
$\{1,2,3,4,5,6,7,9,17,22\}_1$
$\{1,2,3,4,5,6,7,9,18,22\}_1$
$\{1,2,3,4,5,6,7,10,14,17\}_1$
$\{1,2,3,4,5,6,7,10,15,16\}_1$
$\{1,2,3,4,5,6,7,10,15,17\}_1$
$\{1,2,3,4,5,6,7,10,15,18\}_1$
$\{1,2,3,4,5,6,7,10,15,22\}_1$
$\{1,2,3,4,5,6,7,10,16,17\}_1$
$\{1,2,3,4,5,6,7,10,17,22\}_1$
$\{1,2,3,4,5,6,7,11,17,18\}_2$
$\{1,2,3,4,5,6,7,14,15,21\}_2$
$\{1,2,3,4,5,6,7,15,16,22\}_1$
$\{1,2,3,4,5,6,7,15,17,22\}_1$
$\{1,2,3,4,5,6,7,16,17,22\}_1$
$\{1,2,3,4,5,6,9,17,18,21\}_1$
$\{1,2,3,4,5,6,9,12,13,16\}_1$
$\{1,2,3,4,5,6,8,9,15,23\}_1$
$\{1,2,3,4,5,6,8,9,15,16\}_1$
$\{1,2,3,4,5,6,8,14,17,21\}_1$
$\{1,2,3,4,5,6,9,13,14,19\}_1$
$\{1,2,3,4,5,6,8,14,21,22\}_1$
$\{1,2,3,4,5,6,8,9,10,17\}_1$
$\{1,2,3,4,5,6,12,13,17,18\}_2$
$\{1,2,3,4,5,6,8,14,22,23\}_2$
$\{1,2,3,4,5,6,8,9,10,18\}_2$
$\{1,2,3,4,5,6,8,9,10,19\}_1$
$\{1,2,3,4,5,6,8,9,12,15\}_2$
$\{1,2,3,4,5,6,8,9,12,18\}_1$
$\{1,2,3,4,5,6,8,9,13,16\}_1$
$\{1,2,3,4,5,6,8,9,15,22\}_2$
$\{1,2,3,4,5,6,8,9,17,18\}_1$
$\{1,2,3,4,5,6,8,9,17,22\}_1$
$\{1,2,3,4,5,6,8,10,13,16\}_1$
$\{1,2,3,4,5,6,8,10,16,17\}_1$
$\{1,2,3,4,5,6,8,10,16,21\}_1$
$\{1,2,3,4,5,6,8,10,17,22\}_1$
$\{1,2,3,4,5,6,8,12,15,21\}_2$
$\{1,2,3,4,5,6,8,12,16,17\}_1$
$\{1,2,3,4,5,6,8,12,16,21\}_2$
$\{1,2,3,4,5,6,8,12,16,22\}_1$
$\{1,2,3,4,5,6,8,17,18,22\}_2$
$\{1,2,3,4,5,6,9,10,12,18\}_2$
$\{1,2,3,4,5,6,9,10,13,14\}_1$
$\{1,2,3,4,5,6,9,10,13,19\}_2$
$\{1,2,3,4,5,6,9,10,13,22\}_1$
$\{1,2,3,4,5,6,9,10,13,23\}_1$
$\{1,2,3,4,5,6,9,10,14,19\}_1$
$\{1,2,3,4,5,6,9,10,14,22\}_1$
$\{1,2,3,4,5,6,9,10,19,23\}_1$
$\{1,2,3,4,5,6,9,12,14,17\}_2$
$\{1,2,3,4,5,6,9,12,14,21\}_2$
$\{1,2,3,4,5,6,9,12,16,17\}_1$
$\{1,2,3,4,5,6,9,12,17,18\}_1$
$\{1,2,3,4,5,6,9,12,18,21\}_1$
$\{1,2,3,4,5,6,10,13,18,19\}_2$
$\{1,2,3,4,5,6,12,13,18,21\}_2$
$\{1,2,3,4,5,6,12,17,18,21\}_2$
$\{1,2,3,4,5,7,8,11,16,17\}_2$
$\{1,2,3,4,5,7,8,9,16,17\}_2$

## 6 A $4$-$(20,9,2184)$ and a $4$-$(20,10,4004)$ design

The prescribed automorphism group is $PGL(2,9)$ acting on two copies of the natural underlying set of 10 points.

$$\begin{aligned} G \; = \; & \langle (1\,5\,7)(2\,6\,8)(9\,17\,11)(10\,18\,12)(13\,15\,19)(14\,16\,20), \\ & (3\,5\,7)(4\,6\,8)(9\,11\,13)(10\,12\,14)(15\,17\,19)(16\,18\,20), \\ & (5\,9\,17\,19\,7\,15\,13\,11)(6\,10\,18\,20\,8\,16\,14\,12)\rangle \end{aligned}$$

The order is $|G| = 720$.

The group has 272 orbits on 9-sets. The group has 316 orbits on 10-sets. The designs are formed by the orbits of the listed representatives.

**9-orbits:**

$\{12,13,14,15,16,17,18,19,20\}_1$
$\{5,13,14,15,16,17,18,19,20\}_2$
$\{3,1,14,15,16,17,18,19,20\}_1$
$\{7,1,14,15,16,17,18,19,20\}_1$
$\{8,1,14,15,16,17,18,19,20\}_2$
$\{11,1,14,15,16,17,18,19,20\}_1$
$\{4,2,14,15,16,17,18,19,20\}_2$
$\{7,2,14,15,16,17,18,19,20\}_1$
$\{8,2,14,15,16,17,18,19,20\}_2$
$\{10,2,14,15,16,17,18,19,20\}_1$
$\{11,2,14,15,16,17,18,19,20\}_1$
$\{5,3,14,15,16,17,18,19,20\}_1$
$\{8,3,14,15,16,17,18,19,20\}_1$
$\{9,3,14,15,16,17,18,19,20\}_1$
$\{12,3,14,15,16,17,18,19,20\}_1$
$\{6,4,14,15,16,17,18,19,20\}_1$
$\{7,5,14,15,16,17,18,19,20\}_1$
$\{8,5,14,15,16,17,18,19,20\}_1$
$\{12,5,14,15,16,17,18,19,20\}_1$
$\{9,7,14,15,16,17,18,19,20\}_1$
$\{11,7,14,15,16,17,18,19,20\}_1$
$\{12,9,14,15,16,17,18,19,20\}_1$
$\{12,10,14,15,16,17,18,19,20\}_3$
$\{3,2,1,15,16,17,18,19,20\}_4$
$\{11,3,1,15,16,17,18,19,20\}_2$
$\{5,3,2,15,16,17,18,19,20\}_1$
$\{11,3,2,15,16,17,18,19,20\}_2$
$\{13,3,2,15,16,17,18,19,20\}_2$
$\{9,5,3,15,16,17,18,19,20\}_1$
$\{11,5,3,15,16,17,18,19,20\}_1$
$\{10,5,3,1,16,17,18,19,20\}_1$
$\{12,5,3,1,16,17,18,19,20\}_1$
$\{13,5,3,1,16,17,18,19,20\}_1$
$\{7,6,3,1,16,17,18,19,20\}_1$
$\{11,6,3,1,16,17,18,19,20\}_1$
$\{12,6,3,1,16,17,18,19,20\}_1$
$\{10,7,3,1,16,17,18,19,20\}_1$
$\{12,7,3,1,16,17,18,19,20\}_2$
$\{11,8,3,1,16,17,18,19,20\}_1$
$\{12,8,3,1,16,17,18,19,20\}_1$
$\{13,8,3,1,16,17,18,19,20\}_1$
$\{13,10,3,1,16,17,18,19,20\}_1$
$\{13,11,3,1,16,17,18,19,20\}_1$
$\{13,12,3,1,16,17,18,19,20\}_1$
$\{14,12,3,1,16,17,18,19,20\}_1$
$\{8,5,4,1,16,17,18,19,20\}_1$
$\{8,6,4,1,16,17,18,19,20\}_1$
$\{10,6,4,1,16,17,18,19,20\}_1$
$\{12,6,4,1,16,17,18,19,20\}_1$
$\{13,6,4,1,16,17,18,19,20\}_1$
$\{14,6,4,1,16,17,18,19,20\}_1$
$\{10,7,4,1,16,17,18,19,20\}_1$
$\{11,7,4,1,16,17,18,19,20\}_1$
$\{13,7,4,1,16,17,18,19,20\}_1$
$\{10,8,4,1,16,17,18,19,20\}_1$
$\{14,8,4,1,16,17,18,19,20\}_1$
$\{13,11,4,1,16,17,18,19,20\}_1$
$\{14,12,4,1,16,17,18,19,20\}_1$
$\{11,7,5,1,16,17,18,19,20\}_1$
$\{13,7,5,1,16,17,18,19,20\}_1$
$\{12,8,5,1,16,17,18,19,20\}_1$
$\{14,8,5,1,16,17,18,19,20\}_1$
$\{14,11,5,1,16,17,18,19,20\}_1$
$\{14,12,5,1,16,17,18,19,20\}_1$
$\{12,7,6,1,16,17,18,19,20\}_2$
$\{14,7,6,1,16,17,18,19,20\}_1$
$\{7,5,3,2,16,17,18,19,20\}_1$
$\{10,7,3,2,16,17,18,19,20\}_1$
$\{11,7,3,2,16,17,18,19,20\}_1$
$\{14,7,3,2,16,17,18,19,20\}_1$
$\{9,8,3,2,16,17,18,19,20\}_1$
$\{14,10,3,2,16,17,18,19,20\}_1$
$\{13,11,3,2,16,17,18,19,20\}_2$
$\{13,12,3,2,16,17,18,19,20\}_2$
$\{14,12,3,2,16,17,18,19,20\}_1$
$\{11,7,4,2,16,17,18,19,20\}_1$
$\{10,8,4,2,16,17,18,19,20\}_1$
$\{14,8,4,2,16,17,18,19,20\}_2$
$\{14,11,4,2,16,17,18,19,20\}_2$
$\{9,7,5,3,16,17,18,19,20\}_1$
$\{14,7,5,3,16,17,18,19,20\}_1$
$\{13,8,5,3,16,17,18,19,20\}_2$
$\{14,8,5,3,16,17,18,19,20\}_2$
$\{11,9,5,3,16,17,18,19,20\}_1$
$\{13,9,5,3,16,17,18,19,20\}_2$
$\{11,10,5,3,16,17,18,19,20\}_1$
$\{12,10,5,3,16,17,18,19,20\}_1$
$\{11,9,7,3,16,17,18,19,20\}_2$
$\{13,11,7,3,16,17,18,19,20\}_1$
$\{11,7,5,3,1,17,18,19,20\}_1$
$\{13,9,5,3,1,17,18,19,20\}_2$
$\{13,11,5,3,1,17,18,19,20\}_2$
$\{11,9,7,5,3,1,18,19,20\}_1$
$\{12,9,7,5,3,1,18,19,20\}_1$
$\{15,9,7,5,3,1,18,19,20\}_1$

$\{12,10,7,5,3,1,18,19,20\}_1$
$\{13,10,7,5,3,1,18,19,20\}_1$
$\{14,10,7,5,3,1,18,19,20\}_2$
$\{15,10,7,5,3,1,18,19,20\}_1$
$\{16,11,7,5,3,1,18,19,20\}_1$
$\{14,12,7,5,3,1,18,19,20\}_1$
$\{11,9,8,5,3,1,18,19,20\}_1$
$\{14,9,8,5,3,1,18,19,20\}_1$
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$\{13,10,8,5,3,1,18,19,20\}_1$
$\{16,10,8,5,3,1,18,19,20\}_1$
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$\{15,12,8,5,3,1,18,19,20\}_2$
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$\{15,14,8,5,3,1,18,19,20\}_1$
$\{16,14,8,5,3,1,18,19,20\}_1$
$\{16,11,9,5,3,1,18,19,20\}_2$
$\{16,12,9,5,3,1,18,19,20\}_1$
$\{14,12,10,5,3,1,18,19,20\}_1$
$\{16,12,10,5,3,1,18,19,20\}_1$
$\{16,14,12,5,3,1,18,19,20\}_1$
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$\{14,12,8,6,3,1,18,19,20\}_2$
$\{15,12,8,6,3,1,18,19,20\}_1$
$\{16,12,8,6,3,1,18,19,20\}_1$
$\{16,12,10,6,3,1,18,19,20\}_1$
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$\{16,14,12,6,3,1,18,19,20\}_1$
$\{14,10,8,5,4,1,18,19,20\}_2$
$\{16,10,8,6,4,1,18,19,20\}_1$
$\{16,12,10,6,4,1,18,19,20\}_2$
$\{12,10,8,6,4,2,18,19,20\}_2$
$\{15,13,11,9,7,5,3,1,20\}_8$
$\{16,13,11,9,7,5,3,1,20\}_2$
$\{15,14,11,10,7,5,3,1,20\}_6$
$\{16,14,12,10,7,5,3,1,20\}_4$
$\{16,14,12,9,8,5,3,1,20\}_2$
$\{18,14,12,9,8,5,3,1,20\}_1$
$\{16,14,12,10,8,5,3,1,20\}_1$
$\{17,15,13,11,9,7,5,3,1\}_{72}$

**10-orbits:**

$\{1,2,3,4,5,6,7,8,9,10\}_4$
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$\{1,2,3,4,5,6,8,9,10,14\}_2$
$\{1,2,3,4,5,6,8,9,10,11\}_1$
$\{1,2,3,4,5,6,8,9,10,12\}_1$
$\{1,2,3,4,5,6,8,9,10,20\}_2$
$\{1,2,3,4,5,6,7,9,10,12\}_1$
$\{1,2,3,4,5,6,7,9,10,17\}_1$
$\{1,2,3,4,5,6,7,9,10,15\}_2$
$\{1,2,3,4,5,6,9,10,11,17\}_4$
$\{1,2,3,4,5,6,7,8,9,12\}_1$
$\{1,2,3,4,5,6,7,8,10,12\}_2$
$\{1,2,3,4,5,6,7,9,13,18\}_1$
$\{1,2,3,4,5,6,7,9,16,20\}_1$
$\{1,2,3,4,5,6,7,9,11,16\}_1$
$\{1,2,3,4,5,6,7,9,12,20\}_1$
$\{1,2,3,4,5,6,7,9,12,16\}_1$
$\{1,2,3,4,5,6,7,10,14,18\}_1$
$\{1,2,3,4,5,6,7,9,12,17\}_1$
$\{1,2,3,4,5,6,7,10,12,14\}_1$
$\{1,2,3,4,5,6,7,9,16,19\}_1$
$\{1,2,3,4,5,6,7,9,13,16\}_1$
$\{1,2,3,4,5,6,8,9,12,15\}_1$
$\{1,2,3,4,5,6,8,9,16,20\}_1$
$\{1,2,3,4,5,6,8,9,18,20\}_1$
$\{1,2,3,4,5,6,8,9,11,16\}_1$
$\{1,2,3,4,5,6,8,10,12,16\}_1$
$\{1,2,3,4,5,6,8,9,14,18\}_1$
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$\{1,2,3,4,5,6,9,12,14,15\}_1$
$\{1,2,3,4,5,6,9,13,16,18\}_1$
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$\{1,2,3,4,5,6,9,14,18,20\}_1$
$\{1,2,3,4,5,6,9,11,14,20\}_1$
$\{1,2,3,4,5,6,9,11,13,18\}_1$
$\{1,2,3,4,5,6,9,12,13,18\}_1$
$\{1,2,3,4,5,6,9,12,14,17\}_1$
$\{1,2,3,4,5,6,9,16,18,20\}_1$
$\{1,2,3,4,5,6,9,11,16,20\}_2$
$\{1,2,3,4,5,6,9,11,14,16\}_1$
$\{1,2,3,4,5,6,9,12,18,20\}_1$
$\{1,2,3,4,5,6,9,12,13,16\}_1$
$\{1,2,3,4,5,6,9,12,14,16\}_1$
$\{1,2,3,4,5,6,9,12,16,19\}_1$
$\{1,2,3,4,5,6,9,12,16,17\}_2$
$\{1,2,3,4,5,6,10,12,14,16\}_1$
$\{1,2,3,4,5,6,9,11,14,18\}_2$
$\{1,2,3,4,5,6,9,12,14,18\}_1$
$\{1,2,3,4,5,6,10,12,14,18\}_2$
$\{1,2,3,4,5,6,9,12,13,17\}_1$
$\{1,2,3,4,5,6,9,12,14,19\}_2$
$\{1,2,3,4,5,6,9,12,17,19\}_1$
$\{1,2,3,4,5,6,9,11,14,15\}_1$
$\{1,2,3,4,5,6,7,8,9,19\}_8$
$\{1,2,3,4,5,6,7,9,13,15\}_3$
$\{1,2,3,4,5,6,7,9,11,15\}_1$

$\{1,2,3,4,5,6,8,9,13,15\}_3$
$\{1,2,3,4,5,6,8,9,11,13\}_1$
$\{1,2,3,4,5,6,8,9,11,15\}_1$
$\{1,2,3,4,5,6,9,11,13,15\}_1$
$\{1,2,3,4,5,6,9,11,15,19\}_2$
$\{1,2,3,4,5,6,9,11,13,19\}_2$
$\{1,2,3,4,5,7,9,11,13,20\}_1$
$\{1,2,3,4,5,7,9,11,15,20\}_1$
$\{1,2,3,4,5,7,9,11,16,20\}_2$
$\{1,2,3,4,5,7,9,11,14,16\}_1$
$\{1,2,3,4,5,7,9,12,16,17\}_1$
$\{1,2,3,4,5,7,9,12,16,19\}_2$
$\{1,2,3,4,5,7,9,12,17,19\}_1$
$\{1,2,3,4,5,7,9,16,17,20\}_1$
$\{1,2,3,4,5,7,9,12,18,19\}_1$
$\{1,2,3,4,5,7,9,13,16,18\}_1$
$\{1,2,3,4,5,7,9,12,13,19\}_1$
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$\{1,2,3,4,5,8,9,11,13,20\}_2$
$\{1,2,3,4,5,8,9,11,14,20\}_1$
$\{1,2,3,4,5,8,9,12,16,18\}_2$
$\{1,2,3,4,5,8,9,12,16,19\}_1$
$\{1,2,3,4,5,8,9,13,16,19\}_2$
$\{1,2,3,4,5,8,9,13,16,20\}_1$
$\{1,2,3,4,5,7,9,14,17,19\}_2$
$\{1,2,3,4,5,8,9,11,13,18\}_1$
$\{1,2,3,4,5,7,9,14,17,20\}_1$
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$\{1,2,3,4,5,8,9,12,18,19\}_1$
$\{1,2,3,4,5,8,9,12,18,20\}_1$
$\{1,2,3,4,5,8,9,13,18,19\}_1$
$\{1,2,3,4,5,8,9,12,14,19\}_1$
$\{1,2,3,4,5,8,9,12,14,20\}_1$
$\{1,2,3,4,5,8,10,12,13,19\}_1$
$\{1,2,3,4,5,8,10,12,15,19\}_2$
$\{1,2,3,4,5,8,10,15,17,20\}_1$
$\{1,2,3,4,5,8,10,12,14,19\}_1$
$\{1,2,3,4,5,8,10,12,15,20\}_1$
$\{1,2,3,4,5,8,10,15,18,20\}_2$
$\{1,2,3,4,5,8,10,12,14,15\}_2$
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$\{1,2,3,4,5,8,10,16,17,19\}_1$
$\{1,2,3,4,5,8,10,16,17,20\}_1$
$\{1,2,3,4,5,8,10,13,16,18\}_2$
$\{1,2,3,4,5,8,10,14,16,18\}_1$
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$\{1,3,5,7,10,12,14,16,18,20\}_{24}$
$\{1,3,5,8,9,11,14,16,18,20\}_{10}$
$\{1,3,5,8,10,12,14,16,18,20\}_6$
$\{1,3,6,8,10,12,14,16,18,20\}_{16}$
$\{1,4,6,8,10,12,14,16,18,20\}_{72}$
$\{2,4,6,8,10,12,14,16,18,20\}_{720}$

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