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Abstract. In this paper we study some computability theoretic properties of two notions of randomness for finite strings: randomness based on the blank-endmarker complexity measure and Chaitin randomness based on the self-delimiting complexity measure. For example, we find the position of $RAND^K$ and $RAND^C$ at the *same* level in the scale of immunity notions by proving that both of them are not hyperimmune sets. Also we introduce a new notion of complex infinite sequences of finite strings. We call them K-*bounded* sequences.

Key Words: Random finite string, Chaitin complexity, Turing degree, computably enumerable set.

Category: F.1

1 Introduction

Our notation is standard, following that used by Calude [2], Chaitin [4] and Soare [13]. In particular, $\omega = \{0, 1, \ldots\}$ is the set of natural numbers and $\{W_e\}_{e\in\omega}$ is a standard enumeration of all computably enumerable (c. e.) sets, and $\{\varphi_e\}_{e\in\omega}$ is a Gödel numbering of partial computable functions. Let $\{0,1\}^*$ be the set of binary strings (also called programs). We will use the letters α, β, γ, δ to denote finite strings. We let $|\alpha|$ denote the length of α and let λ denote the empty string.

A tt–condition is a pair $< \{x_1, x_2, \ldots, x_n\}, \eta >$, where $x_1, x_2, \ldots, x_n$ are natural numbers and η is an n-ary Boolean function, $n \geq 1$. We assume an effective enumeration of all tt–conditions and we will denote the tt–condition with index k by tt_k.

Let $B \subseteq \omega$. We say that B *satisfies* the tt–condition tt_k and write $B \models_{tt} tt_k$, if $\eta((B(x_1), \ldots, B(x_n)) = 1$. If there exists a computable function f such that $x \in A \Leftrightarrow B \models_{tt} tt_{f(x)}$, for all x, then we say that A is *tt-reducible to* B and write $A \leq_{tt} B$. A set A is tt-complete if A is c. e. and every c. e. set is tt-reducible to A.

We shall work with Turing machines operating on strings. The *absolute program-size complexity* induced by a Turing machine φ (sometimes called *blank-endmarker computer*) is defined by $K_\varphi(\alpha) = \min\{|\beta| \mid \beta \in \{0,1\}^*, \varphi(\beta, \lambda) = \alpha\}$. A *Chaitin computer* is a Turing machine (operating on strings) which has a *prefix-free domain* (see Calude [2]). For a Chaitin computer C one associates the *absolute self-delimiting program-size complexity*, or *Chaitin complexity*, $H_C(\alpha) = \min\{|\beta| \mid \beta \in \{0,1\}^*, C(\beta) = \alpha\}$. The *Invariance Theorem* states the existence of a Turing machine ψ (Chaitin computer U) such that for every Turing machine φ (Chaitin computer C) there exists a constant *const* such that $K_\psi(\alpha) \leq K_\varphi(\alpha) + const$ ($H_U(\alpha) \leq H_C(\alpha) + const$) for all strings α.

For this paper we fix a universal Chaitin computer U and denote by H the induced program-size complexity. Also, we fix a universal blank-endmarker computer ψ and denote by K its induced program-size complexity.

A string α is *Kolmogorov random* (abbreviated, K-random), if

$$K(\alpha) \geq \max\{K(\beta) \mid |\alpha| = |\beta|\} = |\alpha| + \mathrm{O}(1).$$

A string α is *Chaitin random* (abbreviated, Ch-random), if

$$H(\alpha) \geq \max\{H(\beta) \mid |\alpha| = |\beta|\} = |\alpha| + \mathrm{O}(\log(|\alpha|)).$$

We will denote by $RAND^K$ and $RAND^C$ the sets of Kolmogorov and Chaitin random strings, respectively. It is known that Chaitin's definition of randomness is more demanding than Kolmogorov's one (see Calude [2]).

Our aim is to study the computability theoretic properties of $RAND^K$ and $RAND^C$ in an attempt to estimate the computational difference between these two sets. It is known (see Calude [2], p. 92) that both $RAND^K$ and $RAND^C$ are effectively immune sets. Below we find the position of $RAND^K$ and $RAND^C$ at the *same* level in the scale of immunity notions by proving that both of them are not hyperimmune sets. This concept of a hyperimmune set turned out to have very interesting characterizations which were later shown to have important applications in many areas of computability and complexity theory. The characterization of hyperimmunity due to Medvedev and Uspensky (e. g. see Odifreddi [10], p. 272) states that for a hyperimmune set A there is no computable function f such that, for each n, $a_n \leq f(n)$, where a_n is the number coding the n-th string of A in an increasing order. Thus from the computability theoretic point of view both $RAND^K$ and $RAND^C$, being not hyperimmune, could be considered as not "meagre".

We would like to remark here that, nevertheless, we found some interesting differences between these two notions of randomness in terms of other computability theoretic hierarchies (see [1]).

In the following section on hyperimmunity and K-bounded sequences we will study a special kind of hyperimmune set and obtain results which justify the introduction of the concept of *complex* infinite sequences of finite strings. We call them *K-bounded* sequences.

2 Hypersimple sets and program-size complexity

Chaitin in his abstract on the information-theoretical aspects of Post's construction of a simple set (see Chaitin [5], p. 288) defines, for any integer $n \geq 0$, the following sets $P(n)$

and $Q(n)$ of finite binary strings:

$\alpha \in P(n)$ if and only if there is a program β with $|\alpha| > n + |\beta|$ and α is the first string computed by β.

$\alpha \in Q(n)$ if and only if $n + K(\alpha) < |\alpha|$.

Theorem 1 (Chaitin [5]) *There is a constant c such that for all n, $P(n+c)$ is contained in $Q(n)$, and $Q(n)$ is contained in $P(n)$.*

The set $P(n)$ is a version of Post's original construction of a simple set and, in particular, $P(n)$ is an effectively simple, not a hypersimple set. Consequently, all $Q(n)$, $n \geq 0$, are effectively simple and not hypersimple sets also.

Notice that $Q(0) = \{\alpha | \ \alpha \in \{0,1\}^*, K(\alpha) < |\alpha|\} = \overline{RAND^K}$. Therefore we can conclude with the following results.

Corollary 2 *The set of K-random strings is effectively immune, not a hyperimmune co-c. e. set.*

Note that the fact that $RAND^K$ is not hyperimmune follows also from a result with a quite involved proof due to Kummer (see [8]), according to which the set of Kolmogorov non random strings with respect to optimal numberings is tt-complete, and the result due to Post (see [11]) that a hypersimple set is not tt-complete.

Theorem 3 *The set of non K-random strings is wtt-complete.*

Proof. It follows from the theorem (see Odifreddi [10], p. 338) that every effectively simple, not hypersimple set is wtt-complete. □

Theorem 4 (Kummer [8]) *The set of non K-random strings can be tt-complete or non tt-complete, depending on the acceptable numbering.*

Proof. Lachlan has shown (see [9]) that Post's construction of a simple set can produce both tt-complete and not tt-complete effectively simple sets depending on which acceptable numbering of partial computable functions we are working with. Therefore we can now transfer this property to the set of K-random strings. □

In the following theorem we construct directly tt-complete supersets of Post's simple sets. In particular, this construction effectively approximates a proper subset of the set of K-random strings which is effectively immune, not hyperimmune and co-c. e.

Theorem 5 *Any coinfinite nonhypersimple c. e. set is a subset of a tt-complete set.*

Proof. Let S be a coinfinite, not hypersimple c. e. set. Then there exists a disjoint strong array $\{F_n\}_{n\in\omega}$ such that $F_n \cap \overline{S} \neq \emptyset$ for all n.

We will construct the desired c. e. superset S^* of S by meeting the following list of requirements:

$$\mathcal{R}_e : n \in W_e \qquad \Longleftrightarrow \qquad S^* \models_{tt} tt_{f(e,n)},$$

that is, $n \in W_e$ if and only if the tt-condition with the index $f(e,n)$ satisfies S^* and $f(e,n)$ is the computable function to be constructed.

Obviously, if we construct $S^* \supset S$ meeting all these requirements, then $W_e \leq_{tt} S^*$ for any e and the theorem will be proved.

We first effectively split the strong array $\{F_n\}_{n\in\omega}$ into the computable sequence of strong arrays $\{F(e,n)\}_{(e,n)\in\omega\times\omega}$ defining for all e, n: $F_{(e,n)} = F_{\langle e,n\rangle}$, so that we will connect the requirement $\mathcal{R}_e$ to the array $\{F(e,n)\}_{n\in\omega}$. (Here $\langle e,n\rangle = 1/2(e^2+2en+n^2+3e+n)$ denotes the standard pairing function from $\omega\times\omega$ onto ω, e. g. see Soare [13], page 3.)

We define (for fixed e and n) the value of the function $f(e,n)$ as follows. Let $F(e,n) = \{a_1, a_2, \ldots, a_k\}$. Then $f(e,n)$ is the index of the tt–condition $< \{a_1, a_2, \ldots, a_k\}, \eta >$, where η is the following Boolean function of k arguments:

$$\eta(x_1, x_2, \ldots, x_k) = 1 \text{ if and only if } x_1 = 1 \ \& \ x_2 = 1 \ \& \ \ldots \ \& \ x_k = 1.$$

Now let S^* be the following c. e. superset of S:

$$S^* = S \cup \{F(e,n)\}_{n\in W_e}.$$

If $n \notin W_e$, then by the construction, $F(e,n) \cap \overline{S^*} \neq \emptyset$. It follows that $\eta(S^*(a_1), S^*(a_2), \ldots,$ $S^*(a_k)) = 0$ and, therefore, the tt–condition $tt_{f(e,n)}$ is not satisfied by S^*. If $n \in W_e$ then $F(e,n) \subset S^*$, and

$$\eta(S^*(a_1), S^*(a_2), \ldots, S^*(a_k)) = 1.$$

Therefore, we have $n \in W_e$ if and only if the tt-condition $tt_{f(e,n)}$ satisfies S^*. □

Corollary 6 *There exists an effectively simple nonhypersimple tt-complete set.*

Proof. Let S be the simple set constructed by Post (see Soare [13], page 78). It is known that S is effectively simple and nonhypersimple. By theorems there is a tt-complete set $S^* \supseteq S$. Obviously, any coinfinite c. e. superset of an effectively simple set is again effectively simple. Therefore, S^* is an effectively simple tt-complete set. Since tt-complete sets cannot be hypersimple, the set S^* is not hypersimple. □

Proposition 7 *The set $RAND^C$ is not hyperimmune.*

Proof. Let $\{F_n\}_{n\in\omega}$ be the following strong array: $F_n = \{\alpha \mid |\alpha| = n\}$, for every $n \geq 0$. Obviously, the sequence $\{F_n\}_{n\in\omega}$ is a computable sequence of pairwise disjoint finite sets. It is easy to see that for each $n \geq 0$ there is a string α such that $|\alpha| = n$ and $\alpha \in RAND^C$. Therefore, $F_n \cap RAND^C \neq \emptyset$ for all n, and the set $RAND^C$ is not hyperimmune. □

Above we considered sets $P(n)$ and $Q(n)$ defined by Chaitin as sets which reflect information-theoretical aspects of Post's simple set. Generalizing his ideas to Dekker's construction of hypersimple sets and to known constructions of effectively hypersimple sets, we arrive at the following definition which we believe reflects information-theoretical aspects of these sets.

Below Φ denotes a Turing machine which is total and injective, i. e. it has the following properties.

1) Every program computes some string, i. e. $\Phi(u)$ converges for all programs u;

2) Different programs compute on Φ different strings: if $u \neq v$ then $\Phi(u) \neq \Phi(v)$.

Definition 8 *Let*

$$\mathbf{H}_{\mathbf{\Phi}} = \{\alpha | \Phi(\alpha) = \alpha_1 \Longrightarrow (\exists \beta)(|\beta| > |\alpha| \ \textit{and} \ \Phi(\beta) = \beta_1 \ \textit{and} \ |\alpha_1| > |\beta_1|)\}.$$

Thus, H_Φ is the set of all programs α such that if α computes a string α_1 then there exists a program β such that $|\beta| > |\alpha|$ and β computes a program β_1 with $|\alpha_1| > |\beta_1|$.

Obviously, this definition can be considered as a version of Dekker's original hypersimple set (see [6]). The following theorem about H_Φ holds true.

Theorem 9 *For any Turing machine Φ, let A be the c. e. set of all strings which are computed by Φ. If the set A is not computable then H_Φ is hypersimple.*

Proof. The proof has been motivated by the original proof of Dekker's theorem. Obviously, the set H_Φ is c. e. and, since Φ computes different strings for different programs, the set $\{0,1\}^* \Leftrightarrow H_\Phi$ is infinite. Let $\{0,1\}^* \Leftrightarrow H_\Phi = \{\beta_0 < \beta_1 < \ldots\}$. Then, by the definition of H_Φ, we have for any β that

$$\beta \in A \Longleftrightarrow \beta \in \{\alpha_0, \alpha_1, \ldots, \alpha_{b_\beta}\}.$$

Now, if $\{0,1\}^* \Leftrightarrow H_\Phi$ is majorized by a computable function g, then it follows that

$$\beta \in A \Longleftrightarrow \beta \in \{\alpha_0, \alpha_1, \ldots, \alpha_{g(\beta)}\},$$

which means that A is computable. This is a contradiction. □

Corollary 10 *For any Turing c. e. degree* $\mathbf{a} > \mathbf{0}$ *there exists a Turing machine Φ such that H_Φ is a hypersimple set of degree* $\mathbf{a}$.

Proof. It is easy to see that in Theorem 9 the set H_Φ has the same Turing degree as the set A. □

3 Hyperimmunity and K-bounded sequences

In this section we study a special kind of hyperimmune set. Basing on this notion we introduce a new notion of complexity for infinite sequences of finite strings.

Definition 11 *A sequence $\{F_n\}_{n\in\omega}$ of finite sets is a* **disjoint strong (and singular) array** *if there is a computable function f such that:*

- *$F_n = D_{f(n)}$ for all n;*
- *$n \neq m \Rightarrow D_{f(n)} \neq D_{f(m)}$ for all n, m;*
- *(and $|D_{f(n)}| = 1$ for all n).*

In the early forties, Post introduced a *hyperimmune* set with computably enumerable complement in order to solve *Post's Problem* (see Soare [13] or Post [11]) for tt-reducibility. The intuition which led to the definition of a hyperimmune set was to strengthen the notion of simple set, which solved Post's Problem for m-reducibility, but did not solve Post's Problem for tt-reducibility. The idea was to consider in the definition of immune set A infinite c. e. sets as disjoint strong singular arrays (i. e. with members all are singular sets) intersecting $\overline{A}$ and to weaken this condition by replacing singular sets to finite, so that each F_n contains some $x \in \overline{A}$ but we cannot explicitly compute which $x \in F_n$ has this property.

Definition 12 *A set A is* **hyperimmune** *if it is infinite, and there is no disjoint strong array with members all intersecting it, i. e. $F_n \cap A \neq \emptyset$ for all n.*

Later on, in the early fifties, the great Russian mathematician Kolmogorov presented to the participants of the Moscow's seminar "Recursive Arithmetic" the problem (see Uspensky [14]) of *what nonmajorizable by computable functions sets are*. Medvedev and Uspensky had shown independently that those sets are exactly the hyperimmune ones in the original sense of Post. Nowadays this beautiful characterization of hyperimmune sets by means of nonexistence of any majorizing computable function is often adopted as a definition of those sets (e. g. see Rogers [12]).

Definition 13 *If f and g are total functions, f* **majorizes** *g if $f(n) \geq g(n)$ for all n, and f* **dominates** *a partial function φ if $f(n) \geq \varphi(n)$ for all but finitely many n such that $\varphi(n)$ is defined.*

Definition 14 *If $A = \{a_0 < a_1 < a_2 \ldots\}$ is an infinite set, the* **principal function** *of A is p_A, where $p_A = a_n$.*

Definition 15 *A function f majorizes (dominates) an infinite set A if f majorizes (dominates) p_A.*

Theorem 16 (Medvedev [7], Uspensky [15]) *An infinite set A is hyperimmune if and only if no computable function f majorizes A.*

The effectively hyperimmune set was defined as a natural effectivization of the definition of the hyperimmune set (see Odifreddi [10], p. 277).

Definition 17 *An infinite set $A = \{a_0, a_1, \ldots\}$ is effectively hyperimmune if and only if there is a computable function f such that for any e,*

$$\varphi_e \text{ total } \Longrightarrow (\exists n \leq f(e))(a_n > \varphi_e(n)).$$

Knowing an index e of a total computable fuction φ_e, we effectively find the interval $\{0, 1, \ldots, f(e)\}$ such that the function φ_e does not dominate A via a witness n from this interval.

The notion of effectively hyperimmune set naturally suggests to study the following notion of complexity for infinite sequences of finite strings.

Let $U(e, x)$ be a universal Turing machine defined on $\omega \times \omega$, i. e. $U(e, x) = \varphi_e(x)$ for every e and φ_e is a Turing program with the Gödel number e. Let $A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be an infinite sequence of binary strings.

Definition 18 *We will say that the infinite sequence A of binary strings is φ_e–**bounded** for a fixed φ_e if the following three conditions hold true:*

a) $(\forall i, j)\,(i < j \Longrightarrow K(\alpha_i) < K(\alpha_j))$,

b) $(\forall i \geq 0)\,(\alpha_i \subset \alpha_{i+1})$,

(Here and below we write $\alpha \subset \beta$ if $\alpha * \gamma = \beta$ for some $\gamma \neq \emptyset$.)

c) $(\exists n > 0)\,(\forall m > n)\,(\varphi_e(m) < \infty$ *and* $\varphi_e(m) < K(\alpha_m))$.

We can easily see that for any total computable function φ_e there exist φ_e–bounded sequences. Indeed, let us fix an enumeration of all binary strings and define the following sequence.

$\alpha_0 = \mu\beta\{\varphi_e(0) < K(\beta)\}$, and for $n \geq 0$,

$\alpha_{n+1} = \mu\beta\{\alpha_n \subset \beta \,\&\, \varphi_e(n+1) < K(\beta) \,\&\, K(\alpha_n) < K(\beta)\}$.

It is obvious that the sequence $\{\alpha_0, \alpha_1, \ldots\}$ is φ_e–bounded.

Definition 19 *We say that a sequence A is K-**bounded** if it is φ–bounded for every total computable function φ.*

Again, it is easy to see, that there are K-bounded sequences. Indeed, let $A = \{a_0, a_1, \ldots\}$ be an infinite set which majorizes all partial computable functions. Now we define the sequence $\{\alpha_0, \alpha_1, \ldots\}$ as follows:

$\alpha_0 = \mu\beta\{K(\beta) > a_0\}$, and for $n \geq 0$,

$$\alpha_{n+1} = \mu\beta\{\alpha_n \subset \beta \,\&\, K(\beta) > a_{n+1} \,\&\, K(\beta) < K(\alpha_n)\}.$$

It is obvious that the sequence $\{\alpha_0, \alpha_1, \ldots\}$ is K-bounded.

In the following main theorem we prove that in the definition of the K-bounded sequence we can change the condition

$$(\exists n > 0)\,(\forall m > n)\,(\varphi_e(m) < \infty \text{ and } \varphi_e(m) < K(\alpha_m))$$

to a much weaker condition

$$(\exists f \leq_T \emptyset)(\forall e)(\exists y)(\varphi_e\, total \Longrightarrow y \leq f(e) \,\&\, K(\alpha_y) \geq \varphi_e(y)).$$

Theorem 20 *Let A be an infinite sequence of binary strings $\alpha_0, \alpha_1, \ldots$ such that the following properties hold true:*

- *for any i and j, $i < j \Longrightarrow K(\alpha_i) < K(\alpha_j)$,*
- *for every $i \geq 0$, $\alpha_i \subset \alpha_{i+1}$,*
- *there is a computable function f such that for any e, if $U(e, x)$ is total then for some $y \leq f(e)$, $K(\alpha_y) > U(e, y)$.*

Then A is a K-bounded sequence.

Remark 21 *It follows from the Theorem that the set A is even effectively K-bounded in the sense that for any φ_e we can effectively compute the place $n(e)$ (which obviously depends on e) from where the φ_e–boundicity of the sequence holds true: $(\forall m > n(e))\,(\varphi_e(m) < \infty \Longrightarrow \varphi_e(m) < K(\alpha_m))$.*

The previous example gives a K-bounded sequence with $n(e) \leq e$:

$$(\forall m > e)\,(\varphi_e(m) < \infty \Longrightarrow \varphi_e(m) < K(\alpha_m)).$$

Indeed, if for infinitely many e,

$$(\exists m > e)(\varphi_e(m) < \infty \Longrightarrow \varphi_e(m) \geq K(\alpha_m)),$$

then A does not majorize the following partial computable function f: for all e, $f(e) = \varphi_e(e)$ if $\varphi_e(e) < \infty$, and $f(e)$ be undefined otherwise. This contradicts the choice of A.

Proof of Theorem 20. The proof will immediately follow from Theorem 23, which is interesting on its own and based on the following Lemma 22.

Lemma 22 *Let g be an increasing computable function. Then there exists a increasing computable function α such that for any x:*

1. $\varphi_{\alpha(x)}$ is a non-decreasing computable function;

2. $\varphi_{\alpha(x)}(0) \geq g(\alpha(x+1))$ for all $x \geq x_0$, where x_0 is some fixed number. Further, there exists a computable procedure which given this number x_0 produces the index of g.

Proof of the Lemma. Let β be an increasing computable function such that $\beta(0) = 1$, and

$$\varphi_{\beta(e)}(n) = g(\varphi_e \varphi_e(n))$$

Let for some b, $\beta = \varphi_b$. Then,

$$\varphi_{\beta(b)}(n) = g(\varphi_b \varphi_b(n)) = g\beta\beta(n) = g(\beta)^2(n);$$

$$\varphi_{\beta\beta(b)}(n) = g\varphi_{\beta(b)}\varphi_{\beta(b)}(n) = gg(\beta)^2 g(\beta)^2(n) \geq g(\beta)^4(n);$$

And for any $x > 0$,

$$\varphi_{\beta^x(b)}(n) \geq g(\beta^{2x}(n)).$$

Let $\alpha(x) = \beta^x(b)$, if $x > 0$, and $\alpha(0) = b$.

Then we have

$$\varphi_{\alpha(x)}(0) \geq g(\beta^{2x})(0) \geq$$

(for $x \geq b+1$)

$$\geq g\beta^{x+b+1}(0) \geq g\beta^{x+1}\beta^b(0) \geq g\beta^{x+1}(b) = g(\alpha(x+1)).$$

□

Theorem 23 *Let $A = \{a_0 < a_1 < a_2 \ldots\}$ be an effectively hyperimmune set and h be a computable function such that*

$$(\forall x)(\varphi_x \ \textit{ is total } \implies (\exists y \leq h(x))(\varphi_x(y) < a_y)).$$

Let f be an arbitrary increasing computable function. Then for some x_0, which can be computed from the index of f, and for any $x > x_0$, we have $a_x > f(x)$.

Proof. Without loss of generality, we assume along, that given set A is effectively hyperimmune with an increasing computable function h. Then by Lemma 22 we get for the function $g = fh$ the computable function α and the integer x_0. For any $n \geq h(\alpha(x_0))$ there exists $x \geq x_0$ such that

$$h(\alpha(x)) \leq n < h(\alpha(x+1)).$$

Since A is effectively hyperimmune, we have

$$(\exists t \leq h(\alpha(x)))[a_t > \varphi_{\alpha(x)}(t) \geq \varphi_{\alpha(x)}(0)].$$

So, we have for $x \geq x_0$,

$$a_n \geq a_{h(\alpha(x))} > \varphi_{\alpha(x)}(0) \geq fh(\alpha(x+1)) > f(n).$$

□

Thus, for any effectively hyperimmune set we can get a K-bounded sequence as in the previous examples.

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