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Abstract

We introduce effectiveness considerations into model theory of intuitionistic logic. We investigate effectiveness of completeness (by Kripke) results for intermediate logics such as for example, intuitionistic logic, classical logic, constant domain logic, directed frames logic, Dummett's logic, etc.

1 Motivation

The development of computable (equivalently, recursive) function theory made it possible to investigate the computational aspects of many mathematical notions and constructions within the context of classical mathematics.

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In the 1930's Kleene and Church investigated computability on the integers and in well–ordered sets and invented the notion of recursive ordinal. In the 1950's Fröelich and Shepherdson [9] investigated computability in fields. In the 1960's Rabin [18] and Malcev[12] initiated the study of computable algebra and computable model theory. In the 1970's, Ershov's school in Russia and Nerode's school in the United States began the systematic use of the priority methods from computability theory to determine under what conditions classical constructions in model theory or algebra can be made computable. Since then, theories of computable algebraic systems and computable models have been the subject of concentrated attention by many logicians. Computability has been investigated in such areas as vector spaces, orderings, Boolean algebras, Abelian groups, fields, rings, and lattices. We refer the reader to Downey [8], Nerode-Remmel [15], Hazarinov [10] and Millar [13] for surveys. Nowdays there are many papers in many other areas of mathematics which deal, in one or another sense, with computability in mathematical structures. For example, there are theories of computability in topological spaces, metric spaces, and Banach spaces. These latter all establish the relations between notions of computability on the one hand, and continuity on the other.

In this paper we investigate effectiveness of Kripke models for first order theories of *intermediate* logics, i.e. ones that lie between intuitionistic and classical predicate logic. How does one find an appropriate notion of effectiveness for these models and theories? We proceed by looking in detail at how one went from model theory of classical first order logic to model theory of intuitionistic logic. The completeness of classical predicate first order logic can be expressed by the assertion that if a theory , is consistent, then , has a classical model. Things are more complicated for the model theory of intuitionistic logic. There are several model theories for intuitionistic logic with quite different flavors. One is lambda calculus models, leading to the work of Girard and of Martin-Löf on typed lambda calculi, or, as Scott has observed, equivalently leading to closed cartesian categories (untyped lambda calculi). In such models existential quantifiers are interpreted as functionals (lambda terms). A second style of model is Kripke and/or Beth models. A third is the topological models as introduced by Rasiowa and Sikorski from prior work of Tarski, for their early 1950's proof of completeness of intuitionistic predicate logic within classical mathematics. All these classes of models are adequate to give classical proofs of completeness of intuitionistic predicate logic, although the literature is especially opaque when one looks for the equivalences and proofs of completeness (see the work of Läuchli and also of Scott). There is also a body of work on constructive proofs of completeness of predicate intuitionistic logic. These are based on a very careful choice of definition of model and a very careful formulation of the statement of completeness. These proofs use so-called feeble (in plain English, contradictory) models, see Troelstra and Van Dalen, volume 2 of [21]. In this paper we look only at Kripke models of intuitionistic predicate logic, leaving the others for other papers.

Thus, a formulation of completeness (by Kripke) of intuitionistic predicate logic can be expressed by the assertion that if , is consistent in intuitionistic predicate logic, then , has a (single) Kripke model $\mathcal{M}$ such that the sentences forced in $\mathcal{M}$ are exactly those intuitionistically provable from , . We call such models **adequate** models of , . The standard proof of this theorem can be thought of as generalizing to intuitionistic logic and its Kripke models the Henkin 1949 proof for classical predicate logic, see [7] or [19]. In that generalization the maximal filters of the Lindenbaum Boolean algebra of , are replaced by prime filters of the Lindenbaum Heyting algebra of , . Thus, a reasonable attempt to introduce effectiveness into model theory of intuitionistic logic is to begin by trying to understand the effective content of the completeness (by Kripke) theorem for intuitionistic predicate logic and more generally for intermediate logics. This leads us to investigate computable intuitionistic theories and computable adequate Kripke models.

Here we mention some previous results concerning computability of Kripke models and intuitionistic theories. Gabbay in [4] proved that for any decidable *finitely axiomatized* intuitionistic theory , and any sentence ϕ not intuitionistically derivable from , , there is a Kripke model (not necessarily adequate) of , which does not force ϕ, such that the underlying partially ordered set is a computable enumerable partial ordering, and such that forcing restricted to *atomic statements* is computably enumerable. In [11] a more sophisticated argument proves that any decidable intuitionistic theory , has an *adequate* Kripke model $\mathcal{M}$ with decidable forcing such that for all sentences ϕ, ϕ is an intuitionistic consequence of , if and only if $\mathcal{M}$ forces ϕ. This generalizes the theorem in classical computable model theory that a decidable theory has a decidable model. This generalizes Gabbay's assumption that , is finitely axiomatized, while his conclusion that forcing for atomic formulas is decidable is strengthened to the decidability of forcing in general, not merely atomic forcing, in the Kripke model constructed. However, the proof in [11] guarantees only that the underlying partial ordering is a Π^0_2–set, while here in section 5 we show that the underlying partial ordering can in fact be made computable.

This paper deals with the effective content of semantic completeness, with respect to Kripke models, of intuitionistic logic and some of its extensions such as classical logic **CPL**, constant domain logic **CD**, directed frames logic **QJ**, logic of frames with maximum elements **KJ**, logic of frames with maximum elements and constant domains **KJC**, Dummett's logic **DL**, etc. Our results for intermediate logics are new, whereas the result for intuitionistic logic refines [11] as mentioned above. The present paper is self–contained[1] and covers basic notions and terminology from intuitionistic model theory and computability theory. In the next section we briefly explain the material on Kripke models, forcing, computability, intermediate logics, and completeness of intermediate logics for Kripke models.

2 Basic Notions

In this section we summarize Kripke models, forcing, intuitionistic logic, some basic definitions from computability theory, intermediate logics, and Kripke completeness.

Kripke Frames and Models. Let $L =< P_0^{n_0}, \ldots, P_k^{n_k} \ldots, c_0, c_1, \ldots >$ be a countable first order language without function symbols. We suppose that the language L is computable, and that the set of constants $C = \{c_0, c_1, \ldots\}$ of the language and the function $k \to n_k$ are also computable. We denote the set of all sentences of L by $Sn(L)$.

A **frame** is a triple $F = (W, \leq, D)$ consisting of a non-empty set W, ("**states of knowledge"** or "**forcing conditions"**), a partial order $\leq$ on W, and a map D from W to a power set such that $v \leq w$ implies $D(v) \subseteq D(w)$. D is called the **domain function**.The partially ordered set $(W, \leq)$ is called the **base** of the frame.

We suppose that we are given a mapping V, called a **valuation ,** which assigns to each pair consisting of a $w \in W$ and an n–ary predicate symbol

[1]We assume that the reader is familiar with basics of computability theory. The last section uses Glivenko's theorem that states that a sentence α is provable in **CPL** if and only if $\neg\neg\alpha$ is provable in **KJ**.

P (constant c) from L , a n–ary relation on $D(w)$ (element of $D(w)$). Thus one can think that $V(w)$ is a classical L–structure which is associated with w.

Let $L(w)$ be the extension of the language L obtained by adding to L a constant (name) c_a for each element $a \in D(w)$. Let $A(w)$ be the set of all atomic sentences of language $L(w)$ classically true in $D(w)$ under the valuation V. Suppose that for all $v \leq w$ the set of all atomic sentences from $A(v)$ is a subset of $A(w)$. Then the 4–tuple $\mathcal{M} = (W, \leq, D, V)$ is called a **Kripke model (over frame F)**. Here is the definition of forcing in a Kripke model.

Definition 2.1 *Let $(W, \leq, D, V)$ be a Kripke model of language L, w be in W and ϕ be a sentence from $L(w)$. We give the definition of " w* **forces** *ϕ" by induction on the complexity of ϕ.*

1. *For atomic sentences ϕ, w forces ϕ iff $\phi \in A(w)$.*
2. *w forces $\phi \to \psi$ iff for all $v \geq w$, v forces ϕ implies v forces ψ.*
3. *w forces $\neg\phi$ iff for all $v \geq w$, v does not force ϕ.*
4. *w forces $\forall x \phi$ iff for all $v \geq w$ and all constants $c \in L(v)$, v forces $\phi(c)$.*
5. *w forces $\exists x \phi$ iff for some $c \in L(w)$, w forces $\phi(c)$.*
6. *w forces $\phi \vee \psi$ iff w forces ϕ or w forces ψ.*
7. *w forces $\phi \wedge \psi$ iff w forces ϕ and w forces ψ.*

We say that $\mathcal{M}$ **forces** a sentence ϕ of language L if every $w \in W$ forces ϕ. By induction on the length of sentences $\phi \in L(w)$, one can prove that if w forces ϕ and $v \geq w$, then v forces ϕ. If $(W, \leq, D, V)$ is a Kripke model whose base is antichain, then all sentences forced in $(W, \leq, D, V)$ coincides with the class of all sentences classically true in all structures $V(w)$, $w \in W$.

Let , be a subset of $Sn(L)$. The **closure** of , is the set of all sentences which are intuitionistically deducible from , . A set , of sentences is **consistent** if the closure of , does not contain falsehood $\perp$. Following the lines of Henkin's proof for classical logic, one can prove the classical (Kripke) completeness result of intuitionistic logic.

Theorem 2.1 *For any consistent set , of sentences of language L, there exists a Kripke model $\mathcal{M}$ such that for all ϕ, $\mathcal{M}$ forces ϕ if and only if ϕ is deducible from , .*

Full proofs of this theorem can be found in [7] or [19]. The proof proceeds by constructing "prime theories" containing , . These are consistent sets extending , in a language obtained by adding to the original language L infinitely many new constant symbols which are prime filters with the witness property in the Lindenbaum Heyting algebra defined by intuitionistic deducibility from , . The **base** of the Kripke model is the set of all these prime theories, the partial ordering is set–theoretic inclusion between two such prime theories. This theorem leads us to an important definition.

Definition 2.2 *A Kripke model $\mathcal{M}$ is* **adequate for** , *if for all ϕ, $\mathcal{M}$ forces ϕ if and only if ϕ is deducible from , .*

Computability Theory. A function is *computable* if there is a Turing machine which computes it. We denote the set of all natural numbers by ω or N. A subset of the natural numbers is *computable* if its characteristic function is computable. A set of natural numbers is *computably enumerable* (c.e.) if it is the range of a computable function. We fix a standard effective enumeration $\Phi_0^X, \Phi_1^X, \ldots$ of all computable partial functions with oracle X. We call number n an **index** of Φ_n^X. A set A is called Σ_1^0 if there is a computable relation R such that $x \in A$ iff $\exists y R(x,y)$ holds. R is called Π_1^0 if $\omega - A$ is Σ_1^0. Finally we call A Σ_{n+1}^0 iff there is a Π_n^0 relation R such that $x \in A$ iff $\exists y R(x,y)$ holds (and similarly for Π_{n+1}^0). The Σ_n^0, Π_n^0 $(n \in \omega)$ sets form a proper hierarchy called the arithmetical hierarchy. We assume that the reader knows basic facts about the jump operator and Turing degrees. Briefly, for a set A, the set $A' = \{(i,j) | \phi_i(j)^A \text{is defined}\}$ is called the jump of A. Iterating the jump operation n times we get n–th jump of A denoted by A^n. $\mathbf{0}^n$ is the n–th jump of computable degree denoted by $\mathbf{0}$. $\mathbf{0}^\omega$ is the degree of the set $\{(x,i) | x \in \mathbf{0}^i\}$. The degree $\mathbf{0}^1$ is usually denoted by $\mathbf{0}'$. We refer to Soare [20] for the basic computability theory.

Intermediate Logics and Completeness. If we add the schema $\alpha \vee \neg\alpha$ to intuitionistic predicate logic **IPL**, then we obtain full classical predicate logic **CPL**. It is natural to ask what logics arise by adding schema to **IPL**

other than the law of the excluded middle. Here are some well–known intermediate logics. **Constant domain logic**, denoted by **CDL**, is obtained by adding the schema

$$\forall x(\alpha(x) \vee \beta) \rightarrow \forall x\alpha(x) \vee \beta,$$

where x is not free in β, to **IPL**. The logic denoted by **QJ** is obtained by adding the schema

$$\neg\alpha \vee \neg\neg\alpha$$

to **IPL**. The logic **KJ** is obtained by adding the schema

$$\forall x\neg\neg\alpha \rightarrow \neg\neg\forall x\alpha$$

to **QJ**. All of the five logics S above are closed under substitution. That is, if ϕ is intuitionistically deducible from S and ϕ' is obtained from ϕ by replacing any atomic subformula in ϕ by some formula, then ϕ' is also in S. Here is a formal definition of the notion of an intermediate logic.

Definition 2.3 *A set S of formulas provable in* **CPL** *is called an* **intermediate logic***, or briefly a* **logic***, if S is closed under intuitionistic deduction and substitution.*

¿From the definition it follows that **IPL**, **CPL**, **CD**, **QJ**, **KJ** are examples of intermediate logics. A natural semantical way to obtain intermediate logics is the following. Fix a class K of Kripke frames. Consider the set $S(K)$ of all sentences which are forced by all Kripke models over frames from K. Then $S(K)$ is an intermediate logic. Thus, one can consider logics of the type $S(K)$ for some natural classes of frames. We introduce several such classes. Let $F = (W, \leq, D)$ be a frame. F is **antichain** if for all $u, v \in W$, the condition $u \leq v$ implies $u = v$. F is a **tree frame** if its base is a tree. F is **constant domain frame** if for all $u, v \in W$, $D(u) = D(v)$. F is **linear frame** if $v \leq w$ or $w \leq v$ for all $v, w \in W$. F is **directed** if for all $v, w \in W$ there $z \in W$ such that $v \leq z$ and $w \leq z$. The frame $(W, \leq, D)$ is a **frame with maximum element** if there exists a $w \in W$ such that for all $v \in W$, $v \leq w$. These are all suggested naturally by algebra. Thus, we have logics of antichain frames, tree frames, constant domain frames, linear frames, etc.

Now we give one of the basic definitions in intuitionistic model theory.

Definition 2.4 *A logic S is* **complete** *for a class K of Kripke frames if the following two conditions hold:*

1. *All Kripke models over frames from K force all formulas from S.*

2. *For any $\alpha \in Sn(L)$ if α is not provable in S, then there is a Kripke model $\mathcal{M}$ over a Kripke frame in K such that $\mathcal{M}$ does not force α.*

Thus, if S is complete for a class K, then S coincides with $S(K)$.

The following completeness results are known from intuitionistic model theory: Classical predicate logic **CPL** is complete for the class of antichain frames; Intuitionistic predicate logic **IPL** is complete for the class of tree frames; The logic **CDL** is complete for the class of constant domain frames; The logic **QJ** is complete for the class of directed frames; The logic **KJ** is complete for the class of frames with maximum elements; Dummett's logic is complete for the class of linearly ordered frames, etc. For proofs of these results and surveys of the subject, see [3] [7] [19] [5] [6].

In sections 3 and 4 we define needed notions. We show that every computable theory can be extended to a so–called complete computable theory. We introduce saturated theories and show that every consistent computable theory can be extended to a computable saturated theory. In section 5 we define decidable Kripke models. Briefly, a Kripke model is decidable if its base and forcing are computable relations. We prove that every computable theory over **IPL** has an adequate decidable Kripke model. In section 6, we show that every decidable first order theory over **CPL** has a decidable Kripke model whose frame is antichain. In section 7, we introduce Henkin complete theories and prove that every computable theory over **CDL** has a decidable Kripke model over a constant domain frame. In the next two sections we investigate the computability of adequate models in logics **QJ** and **KJ**. We show, for example, that every computable theory in **QJ** has an adequate Kripke model decidable in $\mathbf{0}^{\omega}$ over a directed frame. The last section contains conclusions and acknowledgements. The proofs are based on recasting classical completeness proofs for intermediate logics to expose their effective content. This often requires substantial changes.

3 Theories and Their Extensions

We fix a language L and a logic S. When a sentence ϕ is intuitionistically deducible in logic S, we simply say that ϕ is deducible or S–deducible and write $\vdash_S \phi$. The following definition is the basic one for this paper, and stems from Stone's ideal and filter theory of distributive lattices and also from the theory of interpolants.

Definition 3.1 *1. A* **theory** *T is a pair $(,\ ,\Sigma)$, where , and Σ are sets of sentences. We set $lT =$, and $rT = \Sigma$.*

2. A Kripke model $\mathcal{M}$ is **adequate** *for T if for all sentences ϕ, ϕ is deducible from lT if and only if $\mathcal{M}$ forces ϕ.*

We say that a theory $T = (,\ ,\Sigma)$ is **inconsistent** if there exist $\alpha_1, \ldots, \alpha_n \in$, and $\beta_1, \ldots, \beta_m \in \Sigma$ such that $\alpha_1 \wedge \ldots \wedge \alpha_n \to \beta_1 \vee \ldots \vee \beta_m$ is S–deducible. A theory $T = (,\ ,\Sigma)$ is **consistent** if it is not inconsistent.

Proposition 3.1 *Let $T = (,\ ,\Sigma)$ be a consistent theory. Then there exists a theory $T' = (,\ ',\Sigma')$ such that*

1. $(,\ ',\Sigma')$ is consistent,

2. $,\ ' \bigcup \Sigma' = Sn(L)$,

3. $,\ \subset\ ,\ '$ and $\Sigma \subset \Sigma'$.

Proof. Let $\alpha_0, \alpha_1, \ldots$ be a list of all sentences of the language L. We construct a sequence $(,\ _0, \Sigma_0), (,\ _1, \Sigma_1), \ldots$ of theories such that

1. For all $i \in \omega$, $,\ _i \subset\ ,\ _{i+1}$, $\Sigma_i \subset \Sigma_{i+1}$,

2. For all $i \in \omega$, $(,\ _i, \Sigma_i)$ is consistent,

3. $\bigcup_i (,\ _i \bigcup \Sigma_i) = Sn(L)$.

We build this sequence by stages.

Stage 0. Put $(,\ _0, \Sigma_0) = T = (,\ ,\Sigma)$.

Stage $n+1$. Suppose that $T_n = (,\ _n, \Sigma_n)$ has been constructed. Take α_n. We have two cases.

Case 1. The theory $(,\ _n, \Sigma_n \bigcup \{\alpha_n\})$ is consistent. Then simply put $,\ _{n+1} =\ ,\ _n$ and $\Sigma_{n+1} = \Sigma_n \bigcup \{\alpha_n\}$.

Case 2. The theory $(,\ _n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent. Then put $,\ _{n+1} =\ ,\ _n \bigcup \{\alpha_n\}$ and $\Sigma_{n+1} = \Sigma_n$.

This ends the construction.

Put $,' = \bigcup_n ,_n$ and $\Sigma' = \bigcup_n \Sigma_n$. Since at each stage $\alpha_n \in ,_{n+1} \bigcup \Sigma_{n+1}$, we see that $Sn(L) = ,' \bigcup \Sigma'$.

We need to prove that $T = (,', \Sigma')$ is consistent. It suffices to show that for every n, the theory $(,_n, \Sigma_n)$ is consistent. We show it by induction on n. Clearly $(,_0, \Sigma_0) = T = (,, \Sigma)$ is consistent. Suppose that $T_n = (,_n, \Sigma_n)$ is consistent. Consider stage $n+1$. If the theory $(,_n, \Sigma_n \bigcup \{\alpha_n\})$ is consistent, then obviously $T_{n+1} = (,_{n+1}, \Sigma_{n+1})$ is consistent. Suppose that $(,_n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent. Consider $T_{n+1} = (,_{n+1}, \Sigma_{n+1})$ which is $(,_n \bigcup \{\alpha_n\}, \Sigma_n)$. Suppose that $T_{n+1} = (,_{n+1}, \Sigma_{n+1})$ is inconsistent. Then there exist $\alpha'_1, \ldots, \alpha'_k \in ,_{n+1}$ and $\beta_1, \ldots, \beta_m \in \Sigma_{n+1} = \Sigma_n$ such that

$$\alpha'_1 \wedge \ldots \wedge \alpha'_k \rightarrow \beta_1 \vee \ldots \vee \beta_m.$$

is deducible in S. By the induction hypothesis $T_n = (,_n, \Sigma_n)$ is consistent. Hence $\alpha_n \in \{\alpha'_1, \ldots, \alpha'_k\}$. Since $(,_n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent, there exist $\alpha''_1, \ldots, \alpha''_t \in ,_n$ and $\beta'_1, \ldots, \beta'_r \in \Sigma_{n+1} = \Sigma_n \bigcup \{\alpha_n\}$ such that

$$\alpha''_1 \wedge \ldots \wedge \alpha''_t \rightarrow \beta'_1 \vee \ldots \vee \beta'_r$$

is deducible in S. Hence, it is not hard to see that $T_n = (,_n, \Sigma_n)$ is inconsistent. This is a contradiction. It follows that $T = (,', \Sigma')$ is a consistent theory. The proposition is proved.

Definition 3.2 *A theory $T = (,, \Sigma)$ is* **complete** *if it is consistent and $Sn(L) = , \bigcup \Sigma$.*

We also say that $T = (,', \Sigma')$ **extends** $T = (,, \Sigma)$ if $, \subset ,'$ and $\Sigma \subset \Sigma'$. Thus, we have the following

Corollary 3.1 *Every consistent theory has a complete extension in the same language.* □

Definition 3.3 *A proper subset $,$ of $Sn(L)$ is* **prime** *if the following conditions are satisfied:*

1. *$,$ is closed under deduction in S.*
2. *For all $\alpha, \beta \in Sn(L)$ if $\alpha \vee \beta \in ,$, then either $\alpha \in ,$ or $\beta \in ,$.*

For any subset $X \subset Sn(L)$ let $\bar{X}$ be the complement of X in $Sn(L)$, that is $\bar{X} = Sn(L) \setminus X$.

Proposition 3.2 *A set* , $\subset Sn(L)$ *is prime if and only if the theory* $(, , \bar{ })$ *is complete.*

Proof. Suppose that , is prime. Clearly , $\bigcup \bar{,} = Sn(L)$. We need to show that $(, , \bar{ })$ is consistent. Suppose not. Then there exist $\alpha_1, \ldots, \alpha_k \in$, and $\beta_1, \ldots, \beta_m \in \bar{,}$ such that

$$\alpha_1 \wedge \ldots \wedge \alpha_k \to \beta_1 \vee \ldots \vee \beta_m.$$

is provable in S. Hence, $\beta_1 \vee \ldots \vee \beta_m \in$, since , is closed under deduction. Since , is prime a β_i belongs to , for some $i \leq n$. Contradiction.

Now suppose that $(, , \bar{ })$ is complete. If α is deducible from , in logic S, then $\alpha \in$, . Otherwise, $(, , \bar{ })$ would be inconsistent. Suppose that $\alpha \vee \beta \in$, but neither α nor β belongs to , . Hence $\alpha, \beta \in \bar{,}$ and $\alpha \vee \beta \to \alpha \vee \beta \in$, . Hence $(, , \bar{ })$ is inconsistent. Contradiction. The proposition is proved.

Definition 3.4 *We say that a set* , *of sentences is* Σ–**consistent** *if* $T = (, \Sigma)$ *is consistent. When* $\Sigma = \{\beta\}$*, then* Σ*–consistent set is called* β–**consistent**.

Thus, , $\subset Sn(L)$ is **consistent** if and only if , is $\perp$–consistent.

Definition 3.5 *A theory* $T = (, \Sigma)$ *is* **computable** *if the deductive closure of* , *in logic* S *and the set* Σ *are computable.*

We can relativize the above definition by saying that $T = (, \Sigma)$ is **computable in** X if the deductive closure of , in logic S and the set Σ are computable in X. From the proof of Proposition 3.1, we now have the following result.

Proposition 3.3 *Suppose that* $T = (, \Sigma)$ *is a computable consistent theory and* Σ *is finite. Then* T *has a complete computable extension.*

Proof. Let $T = (, , \Sigma)$ be a computable consistent theory with Σfinite. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a finite set of sentences. Then by the deduction theorem $, \bigcup \Delta$ proves ϕ if and only if $,$ proves $\wedge_{i=1}^{n} \alpha_i \rightarrow \phi$. It follows that the closure of $, \bigcup \Delta$ is also computable. Therefore for finite subsets $\Delta_1, \Delta_2 \subset Sn(L)$, the theory $(, \bigcup \Delta_1, \Sigma \bigcup \Delta_2)$ is computable. Since Σ is finite and the closure of $,$ is computable, the construction of the proof of Proposition 2.1 can be carried out effectively. We need to show that the extension $,'$ obtained in the construction is a computable set. Indeed take a sentence α. Then there is an n such that $\alpha = \alpha_n$. Then $\alpha \in ,'$ if and only if $\alpha_n \in ,_{n+1}$. Hence the theory $T = (,', \Sigma')$ obtained by applying the construction in Proposition 3.1 to the given theory $T = (, , \Sigma)$ is computable. The proposition is proved.

Corollary 3.2 *Any consistent theory $T = (, , \Sigma)$ computable in X with Σ finite has a complete extension computable in X.* $\Box$

4 Saturated Theories

In proving completeness theorems, constant expansions of the original language L play an important role. Thus, let L be a language and C be an infinite set of symbols, called **constants**, such that $L \bigcap C = \emptyset$. We put $L(C) = L \bigcup C$.

Definition 4.1 *Let L be a language. A theory $T = (, , \Sigma)$ is* **saturated** *if:*

1. *$T = (, , \Sigma)$ is consistent.*

2. *$,$ is prime.*

3. *For every formula $\exists x \phi(x)$, the condition $\exists x \phi(x) \in ,$ implies that there exists a constant $c \in L$ such that $\phi(c) \in ,$.*

Proposition 4.1 *Every consistent theory $T = (, , \Sigma)$ of the language L can be extended to a saturated theory $T = (,', \Sigma')$ of the language $L(C)$.*

Proof. Let $\alpha_0, \alpha_1, \ldots$ be a list of all sentences of the language $L(C)$. We construct a sequence $(,_0, \Sigma_0), (,_1, \Sigma_1), \ldots$ of theories such that

1. For all $i \in \omega$, $, _i \subset T_{i+1}$, $\Sigma_i \subset \Sigma_{i+1}$,

2. For all $i \in \omega$, $(, _i, \Sigma_i)$ is consistent,

3. $\bigcup_i (, _i \bigcup \Sigma_i) = Sn(L(C))$.

We construct this by stages.

Stage 0. Put $(, _0, \Sigma_0) = T = (, , \Sigma)$.

Stage $n+1$. Suppose that $T_n = (, _n, \Sigma_n)$ has been constructed. Take α_n. We have two cases.

Case 1. The theory $(, _n, \Sigma_n \bigcup \{\alpha_n\})$ is consistent. Then simply put $, _{n+1} = , _n$ and $\Sigma_{n+1} = \Sigma_n \bigcup \{\alpha_n\}$.

Case 2. The theory $(, _n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent and α_n is not of the form $\exists x \beta(x)$. Then put $, _{n+1} = , _n \bigcup \{\alpha_n\}$ and $\Sigma_{n+1} = \Sigma_n$.

Case 3. The theory $(, _n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent and α_n is of the form $\exists x \beta(x)$. Then put $, _{n+1} = , _n \bigcup \{\alpha_n, \beta(c)\}$ and $\Sigma_{n+1} = \Sigma_n$, where c is the first constant in C not used in the previous stages.

This ends the construction.

Put $, ' = \bigcup_n , _n$ and $\Sigma' = \bigcup_n \Sigma_n$. Since at each stage $\alpha_n \in , _{n+1} \bigcup \Sigma_{n+1}$, we see that $Sn(L(C)) = , ' \bigcup \Sigma'$.

Now we need to prove that $T' = (, ', \Sigma')$ is consistent. It suffices to show that for each n the theory $T_n = (, _n, \Sigma_n)$ is consistent. We show it by induction on n. The case $n = 0$ is trivial. Suppose that $T_n = (, _n, \Sigma_n)$ is consistent. If $T_{n+1} = (, _{n+1}, \Sigma_{n+1})$ is obtained from $T_n = (, _n, \Sigma_n)$ by either *Case 1* or *Case 2*, then we simply repeat the corresponding proof from Proposition 3.1. Suppose that $T_{n+1} = (, _{n+1}, \Sigma_{n+1})$ is obtained from $T_n = (, _n, \Sigma_n)$ by *Case 3*. Then $T_{n+1} = (, _{n+1}, \Sigma_{n+1})$ coincides with $(, _n \bigcup \{\alpha_n, \beta(c)\}, \Sigma_n)$. If $T_{n+1} = (, _{n+1}, \Sigma_{n+1})$ were inconsistent, then for some $\beta_1, \ldots, \beta_m$ from $\Sigma_{n+1} = \Sigma_n$ the sentence $\beta_1 \vee \ldots \vee \beta_m$ would belong to the closure of $, _n \bigcup \{\alpha_n, \beta(c)\}$. Since c does not occur in T_n and $(, _n, \Sigma_n \bigcup \{\alpha_n\})$, one can see that T_n is inconsistent. This contradicts with the inductive assumption.

Now we need to prove the last requirement for saturation. Suppose that $\exists x \beta(x)$ is in $, '$. Let n be such that $\alpha_n = \exists x \beta(x)$. Note that $(, _n, \Sigma_n \bigcup \{\alpha_n\})$ is inconsistent, otherwise, $\alpha_n \notin , '$. Hence by construction, at stage $n+1$ we have *Case 3*. It follows that $\beta(c) \in , '$ for some c. The proposition is proved.

An immediate corollary of this result is its effective version:

Proposition 4.2 *If $T = (, , \Sigma)$ is a computable consistent theory with finite Σ, then there exists a computable saturated extension $T' = (, ', \Sigma')$ of $T = (, , \Sigma)$ in the expansion $L(C)$.*

Proof. The proof follows from the facts that under the assumptions, all the stages in the construction of $(, ', \Sigma')$ from the previous proposition can be carried out effectively. Moreover, for any α_n, we have have $\alpha_n \in , '$ if and only if $\alpha_n \in , _n$. The proposition is proved.

Corollary 4.1 *If $T = (, , \Sigma)$ is computable in X and is a consistent theory with finite Σ, then there exists a computable in X saturated extension $T' = (, ', \Sigma')$ of $T = (, , \Sigma)$ in the expansion $L(C)$.* □

5 Decidable Adequate Models in IPL

For this section, S is **IPL**. We begin by defining the notion of decidable frame.

Definition 5.1 *Let X be a set of natural numbers. A frame $(W, \leq, D)$ is* **decidable in** *X if the relation*

$$w \in W \wedge w_1 \leq w_2 \wedge x \in D(w)$$

is computable in X. If X is computable, then the frame is called **decidable**.

Thus if F is a decidable frame, then from the definition it follows that we can assume that W is a computable subset of ω, or in fact that it is ω, that the order relation is a computable subset of W^2, and that the subsets $D(w)$, $w \in W$ are uniformly computable.

The next definition formalizes the notion of a decidable Kripke model. Informally, a Kripke model over a computable frame is decidable if the forcing in the model is a computable relation. Here is the definition.

Definition 5.2 *A Kripke model $(W, \leq, D, V)$ over a decidable in X frame $(W, \leq, D)$ is $X-$***decidable** *if the set*

$$\{(w, \alpha(c_1, \ldots, c_n)) | w \in W, \alpha(c_1, \ldots, c_n) \in Sn(L(w)),\ w \textit{ forces } \alpha(c_1, \ldots, c_n)\}$$

is computable in X. If X is computable, then the Kripke model is called **decidable**.

Now we are ready to prove an effective version of the model existence theorem of intuitionistic logic. We give a detailed proof of this theorem since we will refine this proof to obtain our further results.

Theorem 5.1 *Any computable theory* $(,\ ,\bot)$ *has a decidable model* $\mathcal{M}$ *such that for all* $\alpha \in Sn(L)$*,* α *is deducible from , if and only if* $\mathcal{M}$ *forces* α*.*

Proof. We set $L_0 = L$ and $L_{n+1} = L(C_{n+1})$, where $C_1, C_2, \ldots$ is an effective sequence of infinite, uniformly computable, and pairwise disjoint sets of constant symbols.

Lemma 5.1 *There exists an effective procedure* p *which for all* $x, i \in \omega$ *and all finite subsets* Δ*, if* x *is regarded as an index of a computable consistent theory* $(,\ ,\Delta)$ *of the language* L_i*, produces an index* $p(x, \Delta)$ *of a computable complete saturated theory* $(,\ (x,\Delta),\Sigma(x,\Delta))$ *in the language* L_{i+1} *extending* $(,\ ,\Delta)$*.*

Proof. The proof follows from the construction in the proof of Proposition 3.1. The proof shows that knowing an index of a consistent theory $(,\ ,\Delta)$ with finite given Δ, one can effectively build a saturated, complete and computable extentsion $(,\ ',\Sigma')$ of $(,\ ,\Delta)$. The theory $(,\ ',\Sigma')$ is a theory of the language L_{i+1}. Moreover, the construction provides an algorithm to decide $T = (,\ ',\Sigma')$. Hence the lemma is proved.

We want to define the base $(W, \leq)$ of the desired decidable adequate Kripke model for theory $(,\ ,\bot)$.

Let $\alpha_0 \ldots \alpha_n$ be a sequence of sentences with the following properties:

1. Every α_i belongs to $Sn(L_i)$.
2. Every α_i is either of the form $\beta \to \gamma$ or $\forall y \beta(y)$.

We define a procedure described below which depends on $\alpha_0 \ldots \alpha_n$ and consists of at most $n + 1$ steps.

Step 0. The step is **unsuccessful** if $(,\ ,\alpha_0)$ is inconsistent. If this happens we terminate the procedure. Otherwise, we consider two cases:

Case 1. α_0 is of the form $\beta \to \gamma$. In this case $(,\ \bigcup\{\beta\},\{\gamma\})$ is consistent. We effectively take an index x of this theory $(,\ \bigcup\{\beta\},\{\gamma\})$. Applying

Lemma 5.1, we get the theory $(, \bigcup\{\beta\}(x, \{\gamma\}), \Sigma(x, \{\gamma\}))$. We set $T(\alpha_0)$ to be $(, \bigcup\{\beta\}(x, \{\gamma\}), \Sigma(x, \{\gamma\}))$.

Case 2. α_0 is of the form $\forall y \beta(y)$. In this case there is a constant $c \in L_1$ such that $(, , \{\beta(c)\})$ is consistent. We effectively take an index x of this theory $(, , \{\beta(c)\})$. Applying Lemma 5.1, we get the theory $(, (x, \{\beta(c)\}), \Sigma(x, \{\beta(c)\}))$. We set $T(\alpha_0) = (, (x, \{\beta(c)\}), \Sigma(x, \{\beta(c)\}))$.

Step $i+1$**,** $i \leq n$. Suppose that $T(\alpha_0, \ldots, \alpha_i)$ has been constructed. Consider $lT(\alpha_1, \ldots, \alpha_i)$. The step is **unsuccessful** if $(lT(\alpha_1, \ldots, \alpha_i), \{\alpha_{i+1}\})$ is inconsistent. If this happens we terminate the procedure. Otherwise, consider two cases:

Case 1. α_{i+1} is $\beta \to \gamma$. In this case the theory $(lT(\alpha_1, \ldots, \alpha_i) \bigcup \{\beta\}, \{\gamma\})$ is consistent. We effectively take an index x of this theory. Applying Lemma 5.1, we get the theory $(lT(\alpha_1, \ldots, \alpha_i) \bigcup \{\beta\}(x, \{\gamma\}), \Sigma(x, \{\gamma\}))$. We set

$$T(\alpha_0, \ldots, \alpha_{i+1}) = (lT(\alpha_1, \ldots, \alpha_i) \bigcup \{\beta\}(x, \{\gamma\}, \Sigma(x, \{\gamma\})).$$

Case 2. α_{i+1} is of the form $\forall y \beta(y)$. In this case there is a constant $c \in L_{i+2}$ such that $(lT(\alpha_1, \ldots, \alpha_i), \{\beta(c)\})$ is consistent. We effectively compute an index x of this theory. Applying Lemma 5.1, we get the theory $(lT(\alpha_1, \ldots, \alpha_i)(x, \{\beta(c)\}), \Sigma(x, \{\beta(c)\}))$. We set

$$T(\alpha_1, \ldots, \alpha_{i+1}) = (lT(\alpha_1, \ldots, \alpha_i)(x, \{\beta(c)\}), \Sigma(x, \{\beta(c)\})).$$

This concludes the description of the procedure.

Definition 5.3 *We say that the sequence* $\alpha_0 \ldots \alpha_n$ *is* T–**ordered** *if the theory* $T(\alpha_0, \ldots, \alpha_n)$ *is defined.*

Let W be the set of all T–ordered sequences. Let w, v be elements of W. We put $w \leq v$ if and only if w is an initial segment of v, that is $v = w\alpha_k \ldots \alpha_m$ for some $\alpha_k, \ldots, \alpha_m \in L_{m+1}$. The relation $\leq$ is a computable relation, and is in fact a partial ordering of W. The next lemma follows from the definition of W and $\leq$.

Lemma 5.2 *The partially ordered set* $(W, \leq)$ *is computable. Moreover it is isomorphic to a disjoint union of countably many copies of an infinitely branching tree.* $\square$

We define a frame $(W, \leq, D)$ as follows, Let $w = \alpha_0 \dots \alpha_n$. Then,

$$D(w) = \text{the set of all constants of the language } L_{n+1}.$$

By Lemma 4.2 and the definition of D, the frame $(W, \leq, D)$ is computable. We define a valuation V on the frame as follows. Let $w = \alpha_0 \dots \alpha_n \in W$ and $P \in L$ be a predicate symbol. Then

$$P(c_1, \dots, c_n) \text{ is (classicaly) true iff } P(c_1, \dots, c_n) \text{ belongs to } lT(\alpha_0, \dots, \alpha_n).$$

Thus, we have a Kripke model $(W, \leq D, V)$. We need the following

Lemma 5.3 *Let $w = \alpha_o \dots \alpha_n$ be a "state of knowledge" from the Kripke model $\mathcal{M}$ defined above. Let ϕ and ϕ' be sentences of the language $L(w)$. Then:*

1. *$\phi \to \phi' \in lT(w)$ if, and only if, for all $v \geq w$, the condition $\phi \in lT(v)$ implies $\phi' \in lT(v)$.*
2. *$\neg\phi \in lT(w)$ if, and only if, for all $v \geq w$ we have $\phi \notin lT(v)$.*
3. *$\phi = \forall x \phi' \in lT(w)$ if, and only if, for all $v \geq w$ and $c \in V(v)$ we have $\phi'(c) \in lT(v)$.*
4. *$\phi \wedge \phi' \in lT(w)$ if, and only if, ϕ and ϕ' belong to $lT(w)$.*
5. *$\phi \vee \phi' \in lT(w)$ if, and only if, either ϕ or ϕ' belong to $lT(w)$.*

Proof. Let $T(w) = (, (w), \Sigma(w))$. We prove the lemma by induction on the length of sentences ϕ. If ϕ is atomic, then we have nothing to prove.

We prove part 1. If $\phi \to \phi' \in , (w)$, $\phi \in , (v)$ and $, (w) \subset , (v)$, then since $, (v)$ is closed under deduction we obtain that $\phi' \in , (v)$. Suppose that $\phi \to \phi' \notin , (w)$. It follows that $\phi \to \phi'$ is not intuitionistically deducible from $, (w)$. Hence, $w\phi \to \phi' \in W$ and $T(w\phi \to \phi')$ is a saturated consistent theory such that $, (w\phi \to \phi')$ contains ϕ but does not contain ϕ'. This proves Part 1. Parts 2 and 3 can be proved in a similar way.

To prove parts 4 and 5 note that if ϕ is $\phi' \wedge \phi''$ or $\phi' \vee \phi''$, then the proofs of these parts of the lemma follows from the facts that $, (w)$ is closed under deduction and is a prime theory.

¿From this lemma, again using induction on ϕ, we deduce that in the Kripke model $\mathcal{M} = (W, \leq, D, V)$, the state of knowledge w forces a sentence ϕ if and only if ϕ belongs to , (w). By the lemma above combined with Lemma 4.1, we conclude that the forcing in $\mathcal{M}$ is computable. Hence the model is decidable. Moreover, by the previous lemma we see that for any $\phi \in Sn(L)$, ϕ is deducible from T if and only if ϕ is forced in model $\mathcal{M}$. Hence $\mathcal{M}$ is adequate. The theorem is proved.

Corollary 5.1 *Any consistent theory* $(, , \perp)$ *computable in* X *has an* X*-decidable model* $\mathcal{M}$ *such that for all* $\alpha \in Sn(L)$*,* α *is deducible from , if and only if* $\mathcal{M}$ *forces* α*.*

Proof. Relativize the proof of the previous theorem. □

Definition 5.4 *We say that a theory* $T = (, , \perp)$ *is* **complete** *for a class* K *of Kripke models if for any* ϕ *not intuitionistically deducible from , there is a Kripke model* $\mathcal{M}$ *from* K *such that* $\mathcal{M}$ *is a model of , but not* ϕ*.*

The next result directly follows from Theorem 5.1 and the definition above.

Corollary 5.2 *Every computable intuitionistic theory* T *is complete for the class of decidable Kripke models* □

6 Decidable Adequate Models in CPL

In this section we assume that the logic S is the classical predicate logic **CPL**. Classically, we know that if $T = (, , \Sigma)$ is a theory, then there is a sequence $\mathcal{M}_0, \mathcal{M}_1, \ldots$ of classical models such that the set of all sentences classically true in all models from the sequence is exactly the set of all sentences deducible from , . We can transform the above sequence of models into a Kripke model $(W, \leq, D, V)$ as follows. We set $W = \omega$, $\leq = \{(i,i) | i \in \omega\}$, $D(i) = M_i$, $V(i) = \mathcal{M}_i$, where M_i is the domain of the model $\mathcal{M}_i$. In other words, frames $(W, \leq, D)$ such that $v \leq w$ implies $v = w$ characterize first order logic. We call such frames **antichains**. The main result of this section is the following theorem.

Theorem 6.1 *Let $T = (, , \bot)$ be a computable theory over* **CPL**. *Then T has a decidable adequate Kripke model over a computable antichain.*

Proof. Our proof follows the lines of a proof of [2] , used there for a different purpose. First, we expand the original language L to $L(C)$, where C is a computable infinite set of new constants. Let $\alpha_0, \alpha_1, \ldots$ be an effective sequence of all sentences of the language L not deducible from T. Let $\beta_0, \beta_1, \ldots$ be an effective sequence of all sentences in the expanded language. We present an effective procedure which. uniformly in i, $i \in \omega$, constructs a computable maximal consistent set $,_i$ of the language $L(C)$ such that $\neg\alpha_i \in T_i$. We proceed in stages.

Stage 0. Put $,_{i,0} = , \bigcup\{\neg\alpha_i\}$.

Stage $n+1$. Suppose that $,_{i,n}$ has been constucted. Consider β_n. If $,_{i,n} \bigcup\{\beta_n\}$ is inconsistent, then set $,_{i,n+1} = ,_{i,n}$. Otherwise, we have two cases. If β_n is not of the form $\exists x \gamma(x)$, then $,_{i,n+1} = ,_{i,n} \bigcup\{\alpha_n\}$. If β_n is of the form $\exists x \gamma(x)$, then $,_{i,n+1} = ,_{i,n} \bigcup\{\alpha_n, \gamma(c)\}$, where c is the first new constant not appeared in $,_{i,n}$.

Put $,_i = \bigcup_n ,_{i,n}$. The following facts can be proved from the construction using induction.

1. Every $,_i$ is a consistent theory.

2. Every $,_i$ is a maximal consistent set.

Indeed, the first fact can be proved using induction on n. The second fact follows easily from the fact that a β_n belongs to $,_i$ if and only if β_n belongs to $,_{i,n+1}$. The construction also shows that the set

$$\{(i, \phi(c_1, \ldots, c_n)) | \phi(c_1, \ldots, c_n) \in L(C), \phi(c_1, \ldots, c_n) \in ,_i\}$$

is a computable set. Now define the antichain $(W, \leq)$ as follows: $W = \omega$, $\leq = \{(i,i) | i \in \omega\}$. Let $D(i)$ be the set of all constants of the expanded language. For each i define a valuation $V(i)$ as follows. Value $P(c_1, \ldots, c_n)$ true if and only if $P(c_1, \ldots, c_n) \in ,_i$. By induction on the complexity of sentences $\phi(c_1, \ldots, c_n)$, we can show that $\phi(c_1, \ldots, c_n)$ is classically true in $V(i)$ if and only if $\phi(c_1, \ldots, c_n) \in ,_i$. Since $(W, \leq)$ is an antichain, the forcing on every $w \in W$ coincides with classical truth on $V(w)$ [16]. Hence forcing is computable in the Kripke model $(W, \leq, D, V)$. It follows that $(W, \leq, D, V)$ is a decidable Kripke model.

By the construction of $,_i$, we also see that for every $\phi \in Sn(L)$, ϕ belongs to T if and only if ϕ belongs to $\bigcap_i ,_i$. Hence $(W, \leq, D, V)$ is an adequate model of T. The theorem is proved.

7 Decidable Adequate Models in CDL

A Kripke frame $(W, \leq, D)$ is a **constant domain frame** if for all $v, w \in W$, we have $D(v) = D(w)$. Thus, a frame $(W, \leq, D)$ is computable constant domain frame if it is a computable and constant domain frame. Let us recall that the constant domain logic denoted by**CDL** extends **IPL** by adding the following axiom schema:

$$\forall x(\phi(x) \vee \psi) \to \forall x \phi(x) \vee \psi,$$

where x is not free in ψ. One of the well–known results in model theory of intuitionistic logic states that**CDL** is complete for the class of constant domain Kripke frames [3] [17]. The goal of this section is to show that this result can be effectivized. Indeed, we prove that any computable theory over **CDL** is complete for the class of decidable constant domain frames. Thus, the main result of this section is the following theorem.

Theorem 7.1 *Let $T = (, , \bot)$ be a computable theory over* **CDL**. *Then the theory T possesses an adequate, decidable, constant domain Kripke model.*

Proof. Our proof is an effectivization of completeness proofs of **CDL** from [3] [17]. The proof also incorporates the ideas of the proof of Theorem 5.1. We will need several definitions and lemmas. We begin our proof by giving the following definition.

Definition 7.1 *A theory $(, , \Sigma)$ is* **Henkin complete** *if it is complete and the following conditions hold:*

1. *For all $\alpha(x)$, if $\alpha(c) \in ,$ for every constant $c \in L$, then $\forall x \alpha(x) \in ,$.*
2. *For every formula $\exists x \alpha(x)$, if $\exists x \alpha(x) \in ,$, then $\alpha(c) \in ,$ for some $c \in L$.*

It is clear that every Henkin complete theory is saturated as well. We now prove the following proposition whose proof is similar to the proofs of Propositions 3.1 and 4.1 but is more delicate.

Proposition 7.1 *Every consistent theory* $(, , \Sigma)$ *over language* L *can be extended to a Henkin complete theory over the expanded language* $L(C)$.

Proof of the Proposition. We first expand the language L to $L(C)$. Now consider the following three sequences of sentences. The first sequence

$$\forall x\alpha_0(x), \forall x\alpha_1(x), \forall x\alpha_2(x), \ldots$$

contains all universal sentences in $L(C)$. The second sequence

$$\exists x\beta_0(x), \exists x\beta_1(x), \exists x\beta_2(x) \ldots$$

contains all existential sentences in $L(C)$. The third sequence

$$\gamma_0, \gamma_1, \gamma_2 \cdots$$

contains all sentences in $L(C)$. We construct a sequence $(, {}_0, \Sigma_0), (, {}_1, \Sigma_1), \ldots$ of theories such that each $(, {}_i, \Sigma_i)$ extends $(, {}_{i-1}, \Sigma_{i-1})$. The construction will guarantee that the union $(\bigcup_i , {}_i, \bigcup_i \Sigma_i)$ is the desired extension.

Stage -1. We put $(, {}_{-1}, \Sigma_{-1}) = (, , \Sigma)$.

Stage $n+1$. We assume that $(, {}_n, \Sigma_n)$ has been constructed and is consistent.

Suppose that $n+1 = 3k$. We have two cases.

Case 1a. The theory $(, {}_n, \Sigma_n \bigcup \{\forall x\alpha_k(x)\})$ is consistent. Then put $, {}_{n+1} = , {}_n$ and $\Sigma_{n+1} = \Sigma_n \bigcup \{\forall x\alpha_k(x), \alpha(c)\}$, where c is the first constant not used in the previous stages.

Case 2a. The theory $(, {}_n, \Sigma_n \bigcup \{\forall x\alpha_k(x)\})$ is inconsistent. Then set $, {}_{n+1} = , {}_n \bigcup \{\forall x\alpha_k(x)\}$ and $\Sigma_{n+1} = \Sigma_n$.

Suppose that $n+1 = 3k+1$. We have two cases.

Case 1b. The theory $(, {}_n, \Sigma_n \bigcup \{\exists x\beta_k(x)\})$ is consistent. Then put $, {}_{n+1} = , {}_n$ and $\Sigma_{n+1} = \Sigma_n \bigcup \{\exists x\beta_k(x)\}$.

Case 2b. The theory $(, {}_n, \Sigma_n \bigcup \{\exists x\beta_k(x)\})$ is inconsistent. Then set $\Sigma_{n+1} = \Sigma_n$ and $, {}_{n+1} = , {}_n \bigcup \{\exists x\beta_k(x), \beta_k(c)\}$, where c is the first constant not used in the previous stages.

Suppose that $n+1 = 3k+2$.

Case 1c. The theory $(,_n, \Sigma_n \bigcup\{\gamma_k\})$ is consistent. Then put $,_{n+1} = ,_n$ and $\Sigma_{n+1} = \Sigma_n \bigcup\{\gamma_k\}$.

Case 2c. The theory $(,_n, \Sigma_n \bigcup\{\gamma_k\})$ is inconsistent. Then set $\Sigma_{n+1} = \Sigma_n$ and $,_{n+1} = ,_n \bigcup\{\gamma_k\}$.

This ends the construction at stage $n+1$.

Now put $\Sigma' = \bigcup_i \Sigma_i$ and $,' = \bigcup_i ,_i$. Using similar ideas as in the proofs of propositions 2.1 or 3.1, one can see that $(,', \Sigma')$ is a consistent and complete theory. We need to prove that $(,', \Sigma')$ is Henkin complete. Indeed, suppose that $\exists x \phi(x) \in ,'$. Let n be such that $\exists x \beta_n(x) = \exists x \phi(x)$. It follows that at stage $3n+1$, we have *Case 2b*. Hence $\phi(c) \in ,'$. Suppose that $\alpha(c) \in ,'$ for all c but $\forall x \alpha(x) \not\in ,'$. Let $\forall x \alpha_n(x)$ be equal to $\forall x \alpha(x)$. It follows that at stage $3n$, we have *Case 1a*. Then $\alpha_n(c) \in \Sigma_{n+1}$ for some constant c. It follows that $\alpha(c) \in \Sigma'$. Contradiction. Thus, we have proved the proposition.

An immedate corollary of this proposition is the following result.

Corollary 7.1 *If* $(,, \Sigma)$ *is a consistent and computable theory with finite* Σ*, then one can effectively extend the theory to a Henkin complete computable theory* $(,', \Sigma')$ *, computing an index of a computable characteristic function for* $(,', \Sigma')$*.* $\Box$

We give the following important definition.

Definition 7.2 *Let* $,$ *be a consistent set of sentences of a language* L*.*

1. $,$ *is* **strongly universal** *if for all* $\forall x \beta(x)$ *and finite* $\Delta \subset Sn(L)$*, the condition* $(,, \{\forall x \beta(x)\} \bigcup \Delta)$ *is consistent implies that* $(,, \{\beta(c)\} \bigcup \Delta)$ *is consistent for some* c *in* L*.*

2. $,$ **strongly existential** *if for all* $\exists x \beta(x)$ *and finite* $\Delta \subset Sn(L)$*, the condition* $(, \bigcup\{\exists x \beta(x)\}, \Delta)$ *is consistent implies that* $(, \bigcup\{\beta(c)\}, \Delta)$ *is consistent for some* c *in* L*.*

The following follows from Proposition 7.1.

Corollary 7.2 *For every Henkin complete theory* $(,', \Sigma')$ *the set* $,'$ *is strongly universal and strongly existential. Hence every consistent theory* $(,, \Sigma)$ *can be extended to a theory* $(,', \Sigma')$ *such that* $,'$ *is strongly universal and strongly existential.*

Proof. The proof follows from Proposition 7.1 and the above definition. □

Now we prove one of the basic lemmas. We borrow the proof from [3] [17].

Lemma 7.1 *Suppose that* $T = (\ ,\ , \Sigma)$ *is a theory in a language* L *with strongly universal and strongly existential* , . *Let* S *be a logic containing* **CDL**. *Then for every sentence* $\alpha \in Sn(L)$, *the extension* , $\bigcup\{\alpha\}$ *is also strongly universal and strongly existentional.*

Proof. We first prove that , $\bigcup\{\alpha\}$ is strongly universal. Take any finite set $\Delta \subset Sn(L)$ and a formula $\forall x\beta(x)$. Suppose that $(\ ,\ \bigcup\{\alpha\}, \Delta \bigcup\{\forall x\beta(x)\})$ is consistent. Suppose that for every constant $c \in L$, $(\ ,\ \bigcup\{\alpha\}, \Delta \bigcup\{\beta(c)\})$ is inconsistent. Hence, for every c there is a $\gamma_1, \ldots, \gamma_n \in$, such that

$$\vdash_S \wedge_{i=1}^n \gamma_i \wedge \alpha \rightarrow \vee\Delta \vee \beta(c),$$

where $\vee\Delta$ is the disjunction of all sentences from Δ. It follows that

$$,\ \vdash_S \alpha \rightarrow \vee\Delta \vee \beta(c)$$

for all c. Hence $(\ ,\ , \{\alpha \rightarrow \vee\Delta \vee \beta(c)\})$ is inconsistent for all c. It follows that $(\ ,\ , \{\forall x(\alpha \rightarrow \vee\Delta \vee \beta(x))\})$ is inconsistent. Indeed, otherwise since , is strongly universal there would exist a constant c such that $(\ ,\ , \{\alpha \rightarrow \vee\Delta \vee \beta(c)\})$ is consistent. This would be a contradiction. Now using the fact that S contains **CDL**, we get that $(\ ,\ , \{\alpha \rightarrow \vee\Delta \vee \forall x\beta(x)\})$ is inconsistent. But this can not happen since $(\ ,\ \bigcup\{\alpha\}, \{\forall\beta(x)\} \bigcup \Delta)$ is consistent, a contradication. Thus, the first part of the lemma is proved.

Now we prove the second part of the lemma. We need to show that , $\bigcup\{\alpha\}$ is strongly existential. Suppose that $(\ ,\ \bigcup\{\exists x\beta(x)\} \bigcup\{\alpha\}, \Delta)$ is consistent. Suppose that $(\ ,\ , \{\beta(c) \rightarrow \vee\Delta\})$ is inconsistent for every constant c. Then by the first part $(\ ,\ \bigcup\{\alpha\}, \{\forall x(\beta(x) \rightarrow \vee\Delta)\})$ is inconsistent. Hence $(\ ,\ \bigcup\{\alpha\}, \{\exists x\beta(x) \rightarrow \vee\Delta\})$ is inconsistent by intuitionistic logic. Hence we have a contradiction with the original assumption that $(\ ,\ \bigcup\{\alpha\}, \{\exists x\beta(x) \rightarrow \vee\Delta\})$ is consistent. The lemma is proved.

Now we can repeat the proof of Theorem 5.1. We set $L_0 = L$ and $L_1 = L(C_1)$, and $L_{n+1} = L_n$, where C_1 is infinite and computable set of constant symbols. We can define the base W in the same way as we did in the proof of Theorem 5.1. The point is that the lemmas above allow us to proceed constructing Henkin complete theories corresponding to the nodes w of W without expanding the language L_1. Therefore for all $w \in W$, the set $D(w)$ is the set of all constants of the language L_1. Hence there exists a decidable, adequate, constant domain Kripke model for $(,\ ,\bot)$. This proves the theorem.

Definition 7.3 *An adequate Kripke model of a theory* $(,\ ,\Sigma)$ *is a* **Henkin model** *if for every* w *and every sentence* $\forall x\alpha(x) \in L(w)$ *the condition* w *does not force* $\forall x\alpha(x)$ *implies that there is a* $c \in D(w)$ *such that* w *does not force* $\alpha(c)$.

Corollary 7.3 *Every theory possesses a Henkin model. Moreover every computable theory possesses a decidable Henkin model.*

Proof. The proof follows from the Proposition 7.1 and Theorem 5.1 □

8 Computability of Adequate Models in QJ

Let S be a logic complete for a class K of frames. The results of the previous sections suggest the following natural question:

If T is a computable theory, then what can be said about computability of adequate models of T over frames from K?

¿From the previous sections we see that in the case when S is either **IPL** or **CPL** or **CDL**), then computability of T implies that T possesses decidable adequate Kripke models over the class of tree frames, antichain frames, constant domain frames, respectively. But it is not possible in general to contsruct decidable Kripke models for computable theories if we consider Kripke Models over a fixed class of frames K, even when S is complete for K. In this and the next section we show that proofs of completeness results for logic **QJ** as well as **KJ** do not necessarily produce decidable adequate Kripke models for computable theories.

We fix the logic **QJ** and begin with the investigation of computability of adequate models for computable theories over logic **QJ**. We follow ideas of the completeness proof of **QJ** from [5]. The completness result for **QJ** states that **QJ** is complete for the class of directed Kripke frames. The goal of the section is to prove the following theorem. Our proof is an efectivization of the proof from [5].

Theorem 8.1 *Let $T = (, , \perp)$ be a computable saturated theory over logic* **QJ**. *Then T possesses an adequate Kripke model which is decidable in $\mathbf{0}^\omega$ and whose base is a directed frame.*

In proving this theorem we provide several definitions and lemmas which can be of independent interest. We begin with considering the partially ordered set $(N^\star, \leq)$, where $N^\star$ is the set of all finite words over natural numbers, and $\leq$ is defined as follows. For $v, w \in N^\star$ $v \leq w$ iff w is an extension of v, that is, there exists a $z \in N^\star$ such that $v = wz$. λ denotes the empty word. Hence λ is the least element of $(N^\star, \leq)$. This partially ordered set is in fact isomorphic to an infinitely branching tree. We fix a computable theory $T = (, , \perp)$ *with only one assumption,* that , is saturated. Now we give a definition which carries all the information needed to construct an adequate model of T.

Definition 8.1 *A* **subordination model for** , *is a triple* $(N^\star, \leq, \bar{,})$ *which satisfies the following properties.*

1. $\bar{,}$ *is a mapping which assigns to every $w \in N^\star$ a saturated theory $\bar{,}(w)$ of the language $L(w) = L + C(w)$, where $C(w)$ is an inifnite set of constants.*

2. *For all $v \leq w$, $L(v) \subset L(w)$ and $\bar{,}(w) \subset \bar{,}(w)$.*

3. *If $w_1 = wnv_1$, $w_2 = wkv_2$, and $n \neq k$, then $(C(w_1) \setminus C(w)) \bigcap (C(w_2) \setminus C(w)) = \emptyset$.*

4. *If $\alpha \to \beta \notin \bar{,}(w)$, then there exists an n such that $\alpha \in \bar{,}(wn)$ and $\beta \notin \bar{,}(wn)$.*

5. *If $\forall x \alpha(x) \notin \bar{,}(w)$, then there exists an n such that $\alpha(c) \notin \bar{,}(wn)$ for some $c \in C(wn)$.*

6. $\bar{,}(\lambda) = ,$.

Here is the lemma which shows that a subordination model for , carries all the information needed to construct an adequate model of T.

Lemma 8.1 *Let $T = (,\ ,\bot)$ be a saturated theory. Every subordination model $(N^\star, \leq, \bar{,}\,)$ for , can be transformed into an adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(N^\star, \leq)$.*

Proof. We define the domain function D and the valuation V in the following natural way. For every w, we set $D(w)$ to be the set of all constants of the language $L(w)$. For every w and predicate P, we set $P(c_1, \ldots, c_n)$ to be true iff $P(c_1, \ldots, c_n) \in \bar{,}\,(w)$. Now one can check (see for example Lemma 5.3) that the Kripke model constructed is the desired one. The lemma is proved. □

Definition 8.2 *We say that a subordination model $(N^\star, \leq, \bar{,}\,)$ for , is X–* **decidable** *if the set $\{(\phi, w) | \phi \in Sn(L_w) \wedge \phi \in \bar{,}\,(w)\}$ is computable in X. If X is a computable set, then the X–decidable subordination model is called* **decidable**.

Lemma 8.2

1. *For every saturated theory $T = (,\ ,\bot)$ computable in X, there exists an X–decidable subordination model for , .*
2. *Every X–decidable subordination model $(N^\star, \leq, \bar{,}\,)$ for , can be transformed into an X–decidable adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(N^\star, \leq)$.*

Proof. Slightly modifying the proof of Theorem 5.1, one can see that every computable in X theory $T = (,\ ,\bot)$ possesses an X–decidable subordination model for , [2]. The proof of the second part follows from the fact that if subordination model $(N^\star, \leq, \bar{,}\,)$ is X–decidable, then the adequate Kripke model constructed in the previous lemma is X–decidable as well. □

The next lemma, first proved in [5], uses the schema of the logic **QJ** and the definition of subordination model in an essential way.

Lemma 8.3 *Let $T = (,\ ,\bot)$ be a saturated theory and let $(N^\star, \leq, \bar{,}\,)$ be a subordination model for , . Then the set $,(\infty) = \bigcup_{w \in N^\star} \bar{,}\,(w)$ is $\bot$–consistent.*

[2]To see this, in the proof of Theorem 5.1 for every $w \in W$ and all immediate extensions $w\alpha$ of w introduce iniformly computable sequence $C_{w\alpha}$ of pairwise disjoint sets of new constants

Proof. It suffices to prove that for every $m \in \omega$, the set $\Gamma_m = \bigcup_{|w|=m} \bar{\Gamma}_w$ is $\bot$–consistent, where $|w|$ is the length of w. Suppose that there exists an m such that Γ_m is not $\bot$–consistent. We prove that in this case Γ_{m-1} is also not $\bot$–consistent. In this way by induction, one can show that Γ is not $\bot$-consistent.

Since Γ_m is not $\bot$–consistent there exist finite words $w_1k_1, \ldots, w_nk_n$ of length m such that $\bar{\Gamma}(w_1k_1) \bigcup \ldots \bigcup \bar{\Gamma}(w_nk_n)$ is not $\bot$–consistent. Hence, there exist sentences $\beta_1(\bar{a}_1, \bar{b}_1) \in \bar{\Gamma}(w_1k_1)$, ..., $\beta_n(\bar{a}_n, \bar{b}_n) \in \bar{\Gamma}(w_nk_n)$ such that

$$\vdash_{\mathbf{QJ}} \beta_1(\bar{a}_1, \bar{b}_1) \wedge \ldots \wedge \beta_n(\bar{a}_n, \bar{b}_n) \to \bot,$$

where $b_i \in C(w_i)$, $a_i \in C(w_ik_i) \setminus C(w_i)$ for all i, $1 \leq i \leq n$. By the definition of subordination model we have $(C(w_ik_i) \setminus C(w_i)) \bigcap (C(w_jk_j) \setminus C(w_j)) = \emptyset$. Therefore, from intuitionistic logic we obtain

$$\vdash_{\mathbf{QJ}} \exists \bar{x}_1 \beta_1(\bar{x}_1, \bar{b}_1) \wedge \ldots \wedge \exists \bar{x}_n \beta_n(\bar{x}_n, \bar{b}_n) \to \bot\ .$$

Again from intuitionistic logic it also follows that

$$\vdash_{\mathbf{QJ}} \neg\neg\exists \bar{x}_1 \beta_1(\bar{x}_1, \bar{b}_1) \wedge \ldots \wedge \neg\neg\exists \bar{x}_n \beta_n(\bar{x}_n, \bar{b}_n) \to \bot\ .$$

Note that $\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in Sn(L(w_i))$. From the the fact that the logic is **QJ**, we see that

$$\bar{\Gamma}(w_i) \vdash_{\mathbf{QJ}} \neg\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \vee \neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i).$$

Since $\bar{\Gamma}(w_i)$ is prime we get that $\neg\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in \bar{\Gamma}(w_i)$ or $\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in \bar{\Gamma}(w_i)$. It follows that $\neg\neg\exists \bar{x}_i \beta_i(\bar{x}_i, \bar{b}_i) \in \bar{\Gamma}(w_i)$. Consequently $\bigcup_{|w|=m-1} \bar{\Gamma}(w)$ is not $\bot$–consistent. Continuing this reasoning, we obtain that Γ is not $\bot$–conistent. This is a contradiction. This proves the lemma.

Now we introduce another partially ordered set and define the notion of n–subordination model, where $n \in \omega$. We consider the partially ordered set $(\{0, 1, \ldots, n\} \times N^{\star}, \leq)$, where $\leq$ is defined as follows: $(k, w) \leq (m, v)$ if and only if either $k < m \leq n$ or if $k = m$, then v extends w. Informally, this partially ordered set can be thought as a disjoint union of infinitely branching tees A_0, A_1, A_2, ..., A_n such that every element in tree A_i is greater than all elements in A_{i-1}.

Definition 8.3 *Let* $T = (\Gamma, \bot)$ *be a theory.* **An** n**–subordination model for** Γ *is a triple* $(\{0, \ldots, n\} \times N^{\star}, \leq, \bar{\Gamma})$ *which satisfies the following properties.*

1. Γ *is a mapping which assigns to every* $w \in \{0,1,\ldots,n\} \times N^\star$ *a saturated theory* Γ_w *of the language* $L(w) = L + C(w)$.

2. *For every* $k \leq n$, *the triple* $(\{k\} \times N^\star, \leq^n, \Gamma^n)$ *is a subordination model for* $\Gamma((k,\lambda))$, *where* $\leq^k, \Gamma^k$ *are restrictions of* $\leq, \Gamma$ *to* $\{k\} \times N^\star$.

3. *For all* $k \leq n$, $\bigcup_{w<(k,\lambda)} \Gamma(w) \subset \Gamma((k,\lambda))$ *and* $\bigcup_{w<(k,\lambda)} C(w) \subset C((k,\lambda))$.

4. *For all* $k \leq n$, $(\Gamma((k,\lambda)), \perp)$ *is a saturated theory.*

5. $\Gamma((0,\lambda)) = \Gamma$.

A standard technique developed in the previous lemmas shows that the following lemma is true.

Lemma 8.4 *Let* $T = (\Gamma, \perp)$ *be a theory. Every* n*–subordination model* $(\{0,1,\ldots,n\} \times N^\star, \leq, \Gamma)$ *for* Γ *can be transformed into an adequate Kripke model* $\mathcal{M}$ *for* T. *Moreover the base of* $\mathcal{M}$ *is* $(\{0,1,\ldots,n\} \times N^\star, \leq)$. $\Box$

Theorem 8.2 *For every computable saturated theory* $T = (\Gamma, \perp)$, *there exists an adequate Kripke model* $\mathcal{M}$ *with the following properties:*

1. *The base of* $\mathcal{M}$ *is* $(\{0,1,\ldots,n\} \times N^\star, \leq)$.

2. *The model* $\mathcal{M}$ *is decidable in* $\mathbf{0}^n$.

Proof. From Lemma 8.2, we see that every computable theory $T = (\Gamma, \perp)$ possesses a decidable subordination model $(N^\star, \leq, \Gamma)$ for Γ. Consider the theory $T = (\bigcup_{w\in N^\star} \Gamma(w), \perp)$. This theory is computably enumerable. It follows that the deductive closure of $\bigcup_{w\in N^\star} \Gamma(w)$ is computable in $\mathbf{0}'$. We can extend this theory to a saturated theory $T' = (\Gamma', \perp)$ over an expanded language such that T' is computable in $\mathbf{0}'$. Now we can develop a subordination model $(N^\star, \leq, \Gamma')$ for Γ' in a such way that the set $\{(w,\phi) | \phi \in L_w \wedge \phi \in \Gamma'(w)\}$ is computable in $\mathbf{0}'$. This shows that we can construct a 1–subordination model for Γ for which the set $\{(w,\phi) | \phi \in L(w) \wedge \phi \in \Gamma'(w)\}$ is computable in $\mathbf{0}'$. Hence by the previous lemma we can transform this 1–subordination model into an adequate model of T which is decidable in $\mathbf{0}'$. Iterating this procedure $n-1$ times and using Lemma 8.2, we see that Γ has an n–subordination model $\{0,1,\ldots,n\} \times N^\star, \leq, \Gamma^{(n)}$ for which the set $\{(w,\phi) | \phi \in L(w) \wedge \phi \in \Gamma^{(n)}(w)\}$ is computable in $\mathbf{0}^n$. This proves Theorem 8.2.

Definition 8.4 *Let $T = (\Gamma, \perp)$ be a saturated theory. An ω–***subordination model for** Γ *is a triple $(\omega \times N^\star, \leq, \bar{\Gamma})$ such that.*

1. *$\bar{\Gamma}$ is a mapping which assigns to every $w \in \omega \times N^\star$ a saturated theory $\bar{\Gamma}_w$ of the language $L(w) = L + C(w)$.*

2. *For every $n \in \omega$, the triple $(\{n\} \times N^\star, \leq^n, \bar{\Gamma}^n)$ is a subordination model for $\bar{\Gamma}((n, \lambda))$, where $\leq^n, \bar{\Gamma}^n$ are restrictions of $\leq, \bar{\Gamma}$ to $\{n\} \times N^\star$.*

3. *For all $n \in \omega$, $\bigcup_{w<(n,\lambda)} \bar{\Gamma}(w) \subset \bar{\Gamma}((n, \lambda))$ and $\bigcup_{w<(n,\lambda)} C(w) \subset C((n, \lambda))$.*

4. *For all $n \in \omega$, $(\bar{\Gamma}((n, \lambda)), \perp)$ is a saturated theory.*

5. *$\bar{\Gamma}((0, \lambda)) = \Gamma$.*

The following lemma is immediate

Lemma 8.5 *Let $T = (\Gamma, \perp)$ be a saturated theory. Every ω–subordination model $(\omega \times N^\star, \leq, \bar{\Gamma})$ for Γ can be transformed into an adequate Kripke model $\mathcal{M}$ for T. Moreover the base of $\mathcal{M}$ is $(\omega \times N^\star, \leq)$.* □

Note that $(\omega \times N^\star, \leq)$ is a directed partially ordered set. Now we are ready to prove the main theorem of this subsection.

Proof of Theorem 8.1. Iterating the proof of Theorem 8.2 countably many times define a triple $(\omega \times N^\star, \leq, \bar{\Gamma})$ such that:

1. $(\{0\} \times N^\star, \leq^0, \bar{\Gamma}^0)$ is a decidable subordination model for Γ, where $\leq^0, \bar{\Gamma}^0$ are restrictions of $\leq$ and $\bar{\Gamma}$ to $\{0\} \times N^\star$

2. $\bar{\Gamma}(n, \lambda)$ is a saturated extension of $\bigcup_{w<(n,\lambda)} \bar{\Gamma}(w)$.

3. For every $n \in \omega$, $(\{0, 1, \ldots, n\} \times N^\star, \leq^n, \bar{\Gamma}^n)$ is a n–subordination model for Γ computable in $\mathbf{0}^n$, where $\leq^n$, $\bar{\Gamma}^n$ are restrictions of $\leq$ and $\bar{\Gamma}$ to $\{0, 1, \ldots, n\} \times N^\star$

4. The set $\{(\phi, w) | w \in \omega \times N^\star, \phi \in L(w), \phi \in \bar{\Gamma}(w)\}$ is computable in $\mathbf{0}^\omega$.

By Lemma 8.2 and 8.3 above we see that the triple $(\omega \times N^\star, \leq, \bar{\Gamma})$ is an ω–subordination model for T. Hence this subordination model defines an adequate Kripke model $\mathcal{M}$ for T by Lemma 8.5. By the last item of the properties $\bar{\Gamma}$ we see that $\mathcal{M}$ is decidable in $\mathbf{0}^\omega$. This proves Theorem 8.1.

9 Almost Decidable Adequate Models

In this subsection we investigate computability of adequate models in logic **KJ**. All the deductions are in **KJ**. We also assume that the given theory $T = (, , \bot)$ is saturated. The completness result for this logic states that **KJ** is complete for the class of frames with maximum elements. Our basic definition is the following.

Definition 9.1 *Kripke model* $(W, \leq, D, V)$ *is* **almost decidable** *if there is a finite subset* F *of* W *such that*

1. *The Kripke model* $(W \setminus F, \leq, D, V)$ *is decidable.*
2. *The Kripke model* $(W \setminus F, \leq, D, V)$ *and the Kripke model* $(W, \leq, D, V)$ *force the same sentences.*

The goal of this subsection is to prove the following theorem.

Theorem 9.1 *Let* $T = (, , \bot)$ *be a computable saturated theory. Then* T *possesses an adequate model* $\mathcal{M}$ *with the following properties:*

1. *The frame of* $\mathcal{M}$ *is a frame with maximum element.*
2. $\mathcal{M}$ *is decidable in* $\mathbf{0}'$.
3. $\mathcal{M}$ *is almost decidable.*

Our proof is an effectivization of the completeness proof from [5]. In our proof we use the following result, known as Glivenko's theorem. For the proof of this theorem, see for example [3].

Theorem (Glivenko) *For any sentence* α*,* $\neg\neg\alpha$ *is provable in* **KJ** *if and only if* α *is provable in* **CPL**.

The proof of Theorem 9.1 is based on the technique developed in the previous section. Indeed, we use a modification of the notion of subordination model. We extend the partially ordered set $(N^\star, \leq)$ to the partially ordered set $(N^\star \bigcup \{\infty\}, \leq)$, where for all $w \in N^\star$ we declare $w < \infty$.

Definition 9.2 *A* **subordination model with maximum element for** $,$ *is a triple* $(N^\star \bigcup\{\infty\}, \leq, \bar{,})$ *which satisfies:*

1. $(N^\star, \leq, \bar{,})$ *is a subordination model form* $,$.
2. *For all* $w \in N^\star$, $\bar{,}(w) \subset \bar{,}(\infty)$.
3. $(\bar{,}(\infty), \perp)$ *is Henkin complete.*
4. *If* $\alpha \to \beta \notin \bar{,}(\infty)$, *then* $\alpha \in \bar{,}(\infty)$ *and* $\beta \notin \bar{,}(\infty)$.

The following lemma is immediate:

Lemma 9.1 *Let* $T = (, , \perp)$ *be a saturated theory. Every subordination model with maximum element* $(N^\star \bigcup\{\infty\}, \leq, \bar{,})$ *for* $,$ *can be transformed into an adequate Kripke model* $\mathcal{M}$ *for* T*, Moreover the base of* $\mathcal{M}$ *is* $(N^\star \bigcup\{\infty\}, \leq)$. $\Box$

We need another lemma about extensions of consistent theories. Say that $(, , \{\phi\})$ is ϕ–**maximal** if for any sentence α of the language of $,$ if $(, \bigcup\{\alpha\}, \{\phi\})$ is consistent, then $\alpha \in ,$.

Lemma 9.2 *Suppose that* $T = (, , \{\psi_0\})$ *is a consistent theory in language* L . *Let* $L(C) = L \bigcup C$ *where* C *is an infinite set of constants such that* $L \bigcap C = \emptyset$. *Then there is a* ψ_0*–maximal saturated theory* $T' = (, ', \{\psi_0\})$ *of the language* $L(C)$ *extending* $(, , \{\psi_0\})$.

Proof. Let ϕ_0, ϕ_1, ϕ_2, ... be a computable sequence of all sentences of the language $\bar{L}$ in which every sentence appears infinitely many times. We construct $,'$ by stages. At stage $t+1$ we define $,_{t+1}$ such that $,_t \subseteq ,_{t+1}$. At the end we put $,' = \bigcup_t ,_t$. At each stage $t+1$ we treat the sentence ϕ_t. If we do not put ϕ_t into $,_{t+1}$, then ϕ_t will not belong to $,'$. Since the procedure is effective, $,'$ will be computable.

Stage 0. $,_0 = ,$.

Stage $t+1$. Suppose that $,_t$ has been constructed. Take ϕ_t. We have three cases.

Case 1. ϕ_t is $A \vee B$. If ψ_0 is not deducible from $,_t \bigcup\{A\}$, then we define $,_{t+1} = ,_t \bigcup\{A\}$. Suppose that ψ_0 is deducible from $,_t \bigcup\{A\}$. Then if ψ_0 is not deducible from $,_t \bigcup\{B\}$, we define $,_{t+1}$ to be $,_t \bigcup\{B\}$. Otherwise, we define $,_{t+1} = ,_t$.

Case 2. ϕ_t is $\exists x \phi(x)$. If ψ_0 is not deducible from $,_t \bigcup\{\phi_t\}$, then we define $,_{t+1}$ to be $,_t \bigcup\{\phi_t, \phi(c)\}$, where c is the first constant not belonging to $,_t$. Otherwise, we define $,_{t+1} = ,_t$.

Case 3. Suppose that neither of the previous cases holds. If ψ_0 is not deducible from $,_t \bigcup\{\phi_t\}$, then we set $,_{t+1} = ,_t \bigcup\{\phi_t\}$. Otherwise, we define $,_{t+1} = ,_t$.

This ends the construction.

Define $,'$ to be $\bigcup_t ,_t$. We prove that $(,', \{\psi_0\})$ is a ψ_0–maximal theory.

First, we show that ψ_0 is not intuitionistically deducible from $,$. Suppose otherwise. Then there exists a t such that ψ_0 is deducible from $,_{t+1}$. We prove by induction on k that ψ_0 is not deducible from $,_k$. Clearly, ψ_0 is not deducible from $,_0$. Suppose that ψ_0 is not deducible from $,_t$.

Suppose that Case 1 of stage $t+1$ holds. Then $,_{t+1}$ properly extends $,_t$ since by inductive hypothesis ψ_0 is not deducible from $,_t$. It follows that $\phi_t = A \vee B$ and that either $,_{t+1}$ is $,_t \bigcup\{A\}$ or $,_t \bigcup\{B\}$. If $,_{t+1}$ is $,_t \bigcup\{A\}$, then by the definition of $,_{t+1}$, ψ_0 is not deducible from $,_{i+1} \bigcup\{A\}$. Similarly, if $,_{t+1}$ is $,_t \bigcup\{B\}$, then ψ_0 is not deducible from $,_{t+1}$. This is a contradiction.

Suppose that Case 2 holds. Then $,_{t+1}$ is $,_t \bigcup\{\phi_t, \phi(c)\}$. Then ψ_0 is deducible from $,_t \bigcup\{\phi_t, \phi(c)\}$, hence ψ_0 is deducible from $,_t \bigcup\{\phi_t\}$.

Suppose that Case 3 holds, then ψ_0 is not deducible from $,_{t+1}$.

It follows that ψ_0 is not deducible from $,'$.

We need to show that $,'$ is closed under deduction. Suppose that ϕ is deducible from $,'$. There is a t such that $\phi = \phi_t$. It follows that ψ_0 is not deducible from $,_t \bigcup\{\phi\}$. So, by the definition of $,_{t+1}$, ϕ belongs to $,_{t+1}$.

Suppose that $\phi \vee \psi \in ,'$. There is a t such that $\phi \vee \psi \in ,_{t+1}$. Since every sentence ϕ appears infinitely many time in the sequence $\phi_0, \phi_1, \ldots$, we see that there is a $k > t$ such that $\phi_k = \phi \vee \psi$. Hence at stage $k+1$ either ϕ or ψ enters $,'$.

Suppose that $\exists x \phi(x) \in ,'$. There is a k such that $\phi_k = \exists x \phi(x)$. At stage $k+1$, $\phi(c)$ enters $,'$ for some c by the definition of the stage.

Now we prove that if ψ_0 is not a consequence of $,' \bigcup\{\phi\}$, then $\phi \in ,'$. There is a t such that $\phi_t = \phi \vee \phi$. Then at stage $t+1$, ϕ enters $,'$. The lemma is proved.

¿From the proof of this lemma we get the following corollary.

Corollary 9.1 *Consider a saturated theory* $(\,,\,,\bot)$ *computable in* X. *There is a maximal saturated extension* $(\,,\,',\bot)$ *of* $(\,,\,,\bot)$ *which is computable in* X. $\Box$

Lemma 9.3 *Suppose that* $(\,,\,,\bot)$ *is a maximal and saturated theory. Then* $(\,,\,,\bot)$ *is Henkin complete. Moreover, if* $\alpha \to \beta \notin\,,$ *, then* $\alpha \in\,,$ *and* $\beta \notin\,,$ *.*

Proof. Recall that the logic is **KJ**. Also note that since $(\,,\,,\bot)$ is maximal, for every sentence γ either γ belongs to , or $\neg\gamma$ belongs to , . We need to prove that , is Henkin complete. Suppose that $\forall x\alpha(x) \notin\,,$ but for all constants of the language of , , $\alpha(c) \in\,,$. Since $\forall x\alpha(x) \notin\,,$ it follows that $\neg\forall x\alpha(x) \in\,,$. We know that $\neg\forall x\alpha(x) \to \exists x\neg\alpha(x)$ is classically true. Hence, by Glivenko's theorem we have that $\neg(\neg\forall x\alpha(x) \to \exists x\neg\alpha(x))$ belongs to , . Therefore $(\neg\forall x\alpha(x) \to \exists x\neg\alpha(x))$ belongs to , . It follows that $\exists x\neg\alpha(x) \in\,,$. Since $(\,,\,,\bot)$ is saturated, we see that $\neg\alpha(c) \in\,,$. This is a contradiction.

Now we prove the second part of the lemma. Suppose that $\alpha \to \beta \notin\,,$. Then $\neg(\alpha \to \beta) \in\,,$. By intuitionistic logic $\neg\neg\alpha \wedge \neg\beta \in\,,$. It follows that $\neg\neg\alpha \in\,,$ and $\neg\beta \in\,,$. Hence $\alpha \in\,,$,but $\beta \notin\,,$. The lemma is proved.

Proof of the Theorem 9.1. Consider a computable saturated theory $T = (\,,\,,\bot)$. Develop a decidable subordination model $(N^\star, \leq\,,\bar{\,})$. This defines a decidable Kripke model $\mathcal{M}_1$ with base $(N^\star, \leq)$. By Lemma 8.3, the theory $\bigcup_{w\in N^\star}\,,\bar{\,}(w)$ is consistent. Note that $\bigcup_{w\in N^\star}\,,\bar{\,}(w)$ is computably enumerable. Hence its deductive closure is computable in $\mathbf{0}'$. By Lemma 9.2 we can extend the theory $(\bigcup_{w\in N^\star}\,,\bar{\,}(w), \bot)$ to a maximal theory $(\,,\bar{\,}(\infty), \bot)$. Thus, we have a subordination model with maximum element $(N^\star \bigcup\{\infty\}, \leq\,,\,,\bar{\,})$ for , . By Lemma 9.1 we can transform this subordination model to a Kripke model $\mathcal{M}_2$ for T. Note that by Corollary 9.1 this model is decidable in $\mathbf{0}'$. The base of this model is the frame $(N^\star \bigcup\{\infty\}, \leq)$. Clearly $\mathcal{M}_2$ is almost decidable since $\mathcal{M}_1$ is decidable. This proves the theorem. $\Box$

10 Conclusion

The reader can see that results similar to the results of the previous sections can be obtained for many other intermediate logics. We state without proofs two other results for two intermediate logics which are known to be complete.

The first logic is **Dummett's logic,** denoted by **DL**. This logic is obtained by adding the schema

$$(\alpha \to \beta) \vee (\beta \to \alpha)$$

to **IPL**. The second logic, denoted by **KJC**, is obtained by adding the schema

$$\forall x \neg\neg \alpha(x) \to \neg\neg \forall x \alpha(x)$$

to **QJ + CD**.

Dummett's logic **DL** is complete for the class of frames $(W, \leq, D)$ such that for $u, w \in W$ either $u \leq w$ or $w \leq w$. These frames are called **linear frames**. The proof of this result is in [6]. A careful checking of this proof shows that the following theorem is true.

Theorem 10.1 *Every computable theory over logic* **DL** *possesses a decidable Kripke model whose frame is a linear frame.* $\square$

The logic **KJC** is complete for the class of frames with maximum elements and with constant domain. A proof of this fact is in [5]. One can check that the proof of this completeness result leads to the following

Theorem 10.2 *Every computable theory over logic* **KJC** *possesses an almost decidable Kripke model whose base is a frame with a maximum element and constant domains. Moreover the model is decidable in* $\mathbf{0}'$. $\square$

Further investigation of computable model theory of intuitionistic logic looks to be fruitful and interesting. We believe that this kind of computability theory brings new ideas and insight into the understanding of Kripke models of intuitionistic theories.

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