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Open Problems in the Theory of Constructive Algebraic Systems

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Abstract. In this paper we concentrate on open problems in two directions in the development of the theory of constructive algebraic systems. The first direction deals with universal algebras whose positive open diagrams can be computably enumerated. These algebras are called positive algebras. Here we emphasize the interplay between universal algebra and computability theory. We propose a systematic study of positive algebras as a new direction in the development of the theory of constructive algebraic systems. The second direction concerns the traditional topics in constructive model theory. First we propose the study of constructive models of theories with few models such as countably categorical theories, uncountably categorical theories, and Ehrenfeucht theories. Next, we propose the study of computable isomorphisms and computable dimensions of such models. We also discuss issues related to the computability-theoretic complexity of relations in constructive algebraic systems.

1. Introduction

The use of constructive objects (e.g. numbers, symbols) has played a significant role in the development of mathematics. Greeks and Persians were fascinated with what can be constructed and studied using symbols. For example, they wanted to find explicit formulas for solutions of algebraic equations. This approach continued into the middle of the nineteenth century, when Kronecker used explicit formulas and algorithms in the study of algebra and geometry. Newton and Leibniz solved geometric and physical problems by translating them into symbolic form. In the 1930s the work of Church, Kleene and Turing formalized the notion of computable functions, that is, the functions that can be computed algorithmically. In the 1930s Church and Kleene applied the notion of a computable function to study the effective content of the theory of ordinals. In the late 1930s the algebraist van der Waerden considered and studied fields *given explicitly*, where "a field Δ is given

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explicitly if its elements are uniquely represented by distinguishable symbols with which addition, subtraction, multiplication and division can be performed by a finite number of operations". By the end of the 1940s the definition of a computable function and the Church-Turing thesis were widely accepted. In the 1950s Frölich and Shepherdson introduced and studied recursive fields by formalizing the notion of "fields given explicitly". Later Rabin continued research on recursive fields and provided examples of recursive fields that fail to have factorization algorithms. In the 1960s Malcev began a systematic study of interactions between algebra, model theory and computability theory. In the early 1970s fundamental work of Ershov and Nerode led to a vast amount of research in the former Soviet Union and the United States in the area. The area has now become known as the theory of constructive (effective, computable) algebraic systems. A goal of this theory is to study the effective content of the techniques, concepts and theorems in the theory of algebraic systems, in particular in model theory and universal algebra. The area is also devoted to the study of interactions between notions and concepts of computability theory and the theory of algebraic systems.

In this paper we discuss some open problems in the theory of constructive algebraic systems and suggest possible directions for further research in the area. We emphasize two directions. The first direction of research is quite general and proposes a systematic development of the theory of positive (or equivalently, computably enumerable) algebras. Generally speaking these are universal algebras whose positive atomic diagrams can be computably enumerated. For example, recursively presented algebras (e.g. finitely presented groups, rings, etc.) have computably enumerable (c.e.) positive diagrams. The Lindenbaum Boolean algebras of computably enumerable theories (e.g. Peano arithmetic) also have c.e. positive diagrams. We suggest the systematic development of the area of positive algebras and think that results and methods of computability theory can fruitfully be applied here. The second direction is more specific and concentrates on traditional topics in the theory of constructive algebraic systems. In particular, the paper is devoted to the study of constructive models of countably categorical theories, uncountably categorical theories, and Ehrenfeucht theories. In this direction we specify three areas of research and open problems. The first area is related to constructing models, the second area is related to the study of computable isomorphisms of constructive models, and finally the third area is related to the study of computability-theoretic properties of relations in constructive algebraic systems.

The paper consists of five sections including the introduction, conclusion, and references. In the next section we provide the basic notions about numbered algebraic systems and models. The section introduces positive universal algebras, contains some results about these algebras, and suggests a systematic development of the theory of positive algebras. The next section, Section 3, considers the questions related to finding constructive presentations (also known as computable presentations or constructivizations) of models of countably categorical, uncountably categorical, and Ehrenfeucht theories. Section 4 is devoted to the study of computable isomorphisms of models. As in the previous section, an emphasis is given to problems related to computable isomorphisms of models of countably categorical, uncountably categorical, and Ehrenfeucht theories. The section also discusses topics related to the dependency of computability-theoretic properties of relations on constructive presentations. Finally, the last section is a conclusion.

We assume that the reader knows some basic facts and concepts from model theory, universal algebra, and computability theory. Some knowledge of the first several chapters of the classic textbooks by Chang and Keisler on model theory [**4**], Grätzer on universal algebra [**15**], Soare on computability theory [**39**], and Malcev on general theory of algebraic systems [**31**] will suffice to follow the paper. We will assume that all algebraic systems considered in this paper are countable unless otherwise stated.

2. Numbered Algebraic Systems

The goal of this section is twofold. On the one hand, we give definitions of constructive and positive algebraic systems, the central notions of this paper. On the other hand, we propose a development of the theory of positive universal algebras, a relatively new area in the field of constructive algebraic systems. This section of the paper consists of five parts. The first part gives basic terminology about numberings. The second and third parts define constructive algebraic systems and positive universal algebras. The last two parts study some general properties of positive algebras.

2.1. Basics of Numeration Theory. In the theory of constructive algebraic systems the notion of numbering plays a central role. A **numbering** of a set A is a map $\nu : \omega \to A$ from the set ω of natural numbers onto the set A. The pair (A, ν) is then called a **numbered set**. For an element $a \in A$, if $\nu(n) = a$ then n is called a **ν-name of the element**. The basic idea in the definition of a numbering is to give constructive names to the elements of the set A, and thus, in some sense, to coordinate or represent elements of the set A by constructive means. Note that the same element $a \in A$ can have several ν-names. A natural set associated with the numbering ν is the equivalence relation E_ν on the set ω that identifies those numbers x, y that name the same element. Formally, we give the following definition:

Definition 2.1. Let (A, ν) be a numbered set. The **equivalence relation induced by** ν is the set $E_\nu = \{(x, y) | \nu(x) = \nu(y)\}$.

Given two numberings ν and μ of the set A, from a computational point of view, one is naturally interested in how these two numbering are related to each other. This leads one to the following natural notion of a reducibility between numberings of the set A. A numbering ν is **reducible** to a numbering μ, written $\nu \leq \mu$, if there exists a computable function from ω into ω such that $\nu(n) = \mu(f(n))$ for all $n \in \omega$. Informally, this means that there exists an effective procedure that applied to any ν-name of an element produces a μ-name of the same element. Two numberings ν and μ are **equivalent** if $\nu \leq \mu$ and $\mu \leq \nu$. Let $N(A)$ be the set of all equivalence classes of all numberings of A. The reducibility relation naturally induces a partial order, also denoted by $\leq$, on $N(A)$. Thus, we have a partially ordered set $(N(A), \leq)$ which constitutes one of the main objects of study in the theory of numerations [**5**]. Note that this partially ordered set is an uppersemilattice, where the join operation $\nu_1 \oplus \nu_2$ is defined as follows. For all $n \in \omega$ if $n = 2k$ then $\nu_1 \oplus \nu_2(n) = \nu_1(k)$; if $n = 2k+1$ then $\nu_1 \oplus \nu_2(n) = \nu_2(k)$. An interesting note is that when A consists of exactly two elements then the semilattice $(N(A), \leq)$ is isomorphic to the semilattice of all many-one reducibility degrees.

Depending on the set A, one considers certain types of numberings which can be of some interest. For example, A can be a set of some c.e. sets (or computable functions) in which case numberings known as computable numberings are of particular interest. A numbering ν of A is called **computable** if the set $\{(n,m) | n \in \nu(m)\}$ is a c.e. set. For example, the standard Kleene enumeration of all c.e. sets is a computable numbering. Informally, if A has a computable numbering then one can uniformly list all sets in A possibly with repetitions. This leads to considering the set $N_C(A)$ of all equivalence classes of computable numberings and studying the partially ordered set $(N_C(A), \leq)$. Computable numberings have been extensively studied, especially by the Novosibirsk school of logic. The paper by Badaev and Goncharov in this volume discusses issues related to the theory of numberings. We also refer the reader interested in the subject of the theory of numberings to the book by Ershov[**5**].

2.2. Numbered Algebraic Systems. We fix a language $L = < f_0^{n_0}, f_1^{n_1}, \ldots, P_0^{m_0}, P_1^{m_1}, \ldots, c_0, c_1, \ldots >$ for which the functions $i \to n_i$ and $j \to m_j$ are computable. Such languages are called **computable languages**. The symbols $f_i^{n_i}$ and $P_j^{m_j}$ are operation and predicate symbols, respectively. If the language contains no predicate symbols, then the algebraic systems of the language are called **universal algebras**, or for short **algebras**. We denote algebraic systems by letters $\mathcal{A}$, $\mathcal{B}$, etc. The domains are, respectively, denoted by A, B, etc. A **numbered algebraic system** is a pair $(\mathcal{A}, \nu)$, where ν is a numbering of the domain of $\mathcal{A}$.

In order to motivate our next definitions we recall several notions from model theory. Let $\mathcal{A}$ be an algebraic system. When one is interested in properties of $\mathcal{A}$ from the predicate calculus point of view, e.g. the space of types of $\mathcal{A}$, elementary embeddings of $\mathcal{A}$, first order definable relations on $\mathcal{A}$, then the full diagram of $\mathcal{A}$, that is the set $FD(\mathcal{A}) = \{\phi(a_1, \ldots, a_n) \mid \phi(x_1, \ldots, x_n)$ is a formula, $\mathcal{A} \models \phi(a_1, \ldots, a_n)$, $a_1, \ldots, a_n \in A\}$, gives the necessary information about the properties. On the other hand, when one is interested in $\mathcal{A}$ from an algebraic point of view, e.g. subsystems of $\mathcal{A}$, embeddings of $\mathcal{A}$, homomorphisms of $\mathcal{A}$, then it is natural to consider the atomic diagram of $\mathcal{A}$, that is the set $AD(\mathcal{A}) = \{\phi(a_1, \ldots, a_n) \mid \phi(x_1, \ldots, x_n)$ is an atomic formula or a negation of an atomic formula, $\mathcal{A} \models \phi(a_1, \ldots, a_n), a_1, \ldots, a_n \in A\}$.

As we introduce numberings of the algebraic systems, one can consider the full and atomic diagrams of these systems under the numberings. Formally, let $(\mathcal{A}, \nu)$ be a numbered algebraic system. We expand the system by adding new constants a_i to the language L so that the value of each a_i is the element $\nu(i)$ for all $i \in \omega$. Let L_1 be the expanded language. The **full diagram of $\mathcal{A}$ under the numbering** ν is $FD_\nu(\mathcal{A}) = \{\phi(\nu(i_1), \ldots, \nu(i_n)) \mid \mathcal{A} \models \phi(\nu(i_1), \ldots, \nu(i_n))$, and $\phi(x_1, \ldots, x_n)$ is a formula of L_1, $i_1, \ldots, i_n, n \in \omega\}$. Similarly, the **atomic diagram of $\mathcal{A}$ under the numbering** ν is $AD_\nu(\mathcal{A}) = \{\phi(\nu(i_1), \ldots, \nu(i_n)) \mid \phi(x_1, \ldots, x_n)$ is an atomic formula or the negation of an atomic formula of L_1, $i_1, \ldots, i_n \in \omega$, and $\mathcal{A} \models \phi(\nu(i_1), \ldots, \nu(i_n))\}$. Here is a central definition of this paper.

DEFINITION 2.2. A pair $(\mathcal{A}, \nu)$ is a **strongly constructive algebraic system** if the set $FD_\nu(\mathcal{A})$ is a computable set. In this case, ν is called a **strong constructivization** of $\mathcal{A}$. Similarly, the system $(\mathcal{A}, \nu)$ is **constructive** if the set $AD_\nu(\mathcal{A})$ is a computable set. In this case, ν is called a **constructivization**, or equivalently a **constructive presentation** of $\mathcal{A}$.

Clearly every strongly constructive algebraic system is a constructive one. The converse does not always hold as $(\omega, +, \times, \leq)$ is an example of a constructive but not strongly constructive system whose constructivization is the identity mapping.

We say that two constructive models $(\mathcal{A}, \nu)$ and $(\mathcal{A}, \mu)$ are **computably isomorphic** if there exists an automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ and a computable function $f : \omega \to \omega$ such that $\alpha(\nu(n)) = \mu(f(n))$ for all n. In this case the constructivizations ν and μ are called **autoequivalent** constructivizations. In other words, autoequivalent constructivizations are those that are equivalent up to an automorphism of the system.

LEMMA 2.3. *Any constructive system $(\mathcal{A}, \nu)$ is computably isomorphic to a constructive system $(\mathcal{A}, \mu)$ such that μ is a one to one mapping.*

Proof. Since ν is a constructivization, the equivalence relation $E_\nu = \{(n, m) | \nu(n) = \nu(m)\}$ is a computable set. Let $k_0 < k_1 < k_2 < \ldots$ be an effective list of all minimal elements in the E_ν-equivalence classes. For every $n \in \omega$, set $\mu(n) = \nu(k_n)$. Clearly, μ is a one to one constructivization that is equivalent to ν. This proves the lemma.

In the literature there is an equivalent terminology for constructive models and constructivizations that does not refer to numberings. These are computable algebraic systems and computable presentations. An algebraic system is called **computable** if the domain of the system is ω and the atomic diagram is a computable set. Clearly, every computable system is a constructive system whose constructivization is the identity mapping from ω onto ω. An algebraic system is called **decidable** if the domain of the system is ω and the full diagram is a computable set. Clearly, every decidable system is a strongly constructive system. The lemma above shows the opposite, that is, every constructive (strongly constructive) algebraic system can be considered as a computable (decidable) algebraic system. Indeed, assume that $(\mathcal{A}, \nu)$ is a constructive algebraic system. Then by the lemma above, we can assume that ν is a one to one mapping. The numbering ν naturally induces an algebraic system with domain ω isomorphic to $\mathcal{A}$. If $(\mathcal{A}, \nu)$ is constructive then the system induced is computable. If $(\mathcal{A}, \nu)$ is strongly constructive then the system induced is decidable.

2.3. Numbered Algebras. Fix a computable language $L = < f_0^{n_0}, f_1^{n_1}, \ldots >$ with no predicate symbols. The algebraic systems of the language are now algebras. The study of numberings of algebras is of particular interest from a computability point of view because any algebra can be numbered in such a way that all the basic operations of the algebra become computable under the numbering. In other words, any algebra can, in some sense, be effectivized if it is numbered properly. To formalize this we give the following definition.

DEFINITION 2.4. A **numbered algebra** is a pair $(\mathcal{A}, \nu)$ for which there exists a computable sequence of computable functions $\psi_0^{n_0}, \psi_1^{n_1}, \ldots$ such that for all $i, t_1, \ldots, t_{n_i} \in \omega$ we have $f_i^{n_i}(\nu(t_1), \ldots, \nu(t_{n_i})) = \nu(\psi_i^{n_i}(t_1, \ldots, t_{n_i}))$. In this case ν is called a **numbering of the algebra**.

Thus, informally if ν is a numbering of an algebra $\mathcal{A}$ then all the basic operations of $\mathcal{A}$ can be carried out effectively under ν.

THEOREM 2.5. *Any algebra possesses a numbering.*

Proof. Indeed, let $\mathcal{A}$ be an infinite algebra. Consider the absolutely free algebra $\mathcal{F}$ of the language L whose set of generators is the domain A. Since A is a countable set, we can assume that $A = \omega$. Let $\nu : \omega \to F$ be a one to one numbering of the algebra so that the set $\{(t, n) \mid n \in \omega, t \in F, t = \nu(n)\}$ is computable. Then clearly $(\mathcal{F}, \nu)$ is a constructive algebra. Let h be a homomorphism from $\mathcal{F}$ onto $\mathcal{A}$ such that $h(i) = i$ for all $i \in \omega$. We note that such a homomorphism exists because $\mathcal{F}$ is absolutely free. Thus, the pair $(\mathcal{A}, \mu)$, where μ is defined by $\mu(n) = h(\nu(n))$, is a numbered algebra. This proves the theorem.

The theorem suggests that the complexity of a numbered algebra $(\mathcal{A}, \nu)$ can be identified with the complexity of the relation E_ν. We give the following definition.

Definition 2.6. A numbered algebra $(\mathcal{A}, \nu)$ is a Σ_n**-algebra** (Π_n**-algebra**) if the relation E_ν is a Σ_n-set (Π_n-set). If $(\mathcal{A}, \nu)$ is both a Σ_n-algebra and a Π_n-algebra then we call it a Δ_n**-algebra**.

Thus, Δ_1-algebras are exactly the class of all constructive algebras. Of course, there are natural examples of nonconstructive numbered algebras that have been intensively studied in computability theory. For example, the lattice $\mathcal{E} = (\{W_i\}_{i\in\omega}, \bigcup, \bigcap)$ of all computably enumerable sets is a Π_2-algebra, where $i \to W_i$ is a standard enumeration of all c.e. sets. Similarly, the algebra $\mathcal{E}^\star$, obtained from $\mathcal{E}$ by factoring it modulo finite sets is an example of a Σ_3-algebra.

For any Σ_1-algebra $(\mathcal{A}, \nu)$ the **positive diagram** of this algebra, that is, the set $\{\phi(\nu(t_1), \ldots, \nu(t_n)) \mid \mathcal{A} \models \phi(\nu(t_1), \ldots, \nu(t_n)), \ \phi(x_1, \ldots, x_n)$ is an atomic formula$\}$ is a c.e. set. Similarly, for a Π_1-algebra $(\mathcal{A}, \nu)$ the **negative diagram** of the algebra, that is, the set $\{\phi(\nu(t_1), \ldots, \nu(t_n)) \mid \mathcal{A} \models \phi(\nu(t_1), \ldots, \nu(t_n))$ $\phi(x_1, \ldots, x_n)$ is the negation of atomic formula$\}$ is a c.e. set. This observation suggests the following definition.

Definition 2.7. Any Σ_1-algebra is called a **positive algebra**. Any Π_1-algebra is called a **negative algebra**.

One of the general programs in the theory of constructive algebraic systems is the study of numbered algebras, in particular the study of Σ_n-algebras and/or Π_n-algebras. This is an open area of research where many results of universal algebra can be studied from a computability theory point of view. We pose this as an open problem for research:

Problem 1. Develop the general theory of Σ_n-algebras.

The following two sections give some interesting examples of results about positive algebras.

2.4. Positive Algebras. In this section we are interested in finding algebraic conditions for positive algebras to possess constructivizations. Recall that a **congruence** on an algebra $\mathcal{A}$ is an equivalence relation η on A such that, for every basic n-ary operation f and all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in A^n$, the condition $(x_1, y_1), \ldots, (x_n, y_n) \in \eta$ implies that the pair $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \eta$.

Theorem 2.8. *A positive algebra $(\mathcal{A}, \nu)$ is constructive if and only if there is a c.e. set $S \subset \omega^2$ such that $\nu(x) \neq \nu(y)$ for all $(x, y) \in S$, and for any nonzero congruence relation η we have $(\nu(x), \nu(y)) \in \eta$ for some $(x, y) \in S$.*

Proof. We first show that there exists an effective procedure that given an index of a c.e. set $X \subset \omega^2$ produces an index of a c.e. set $Y \subset \omega^2$ such that the relation $\nu(Y) = \{(\nu(x), \nu(y)) | (x, y) \in Y\}$ is the smallest congruence relation containing $\nu(X)$. To prove this, note that any congruence relation η that contains $\nu(X)$ must satisfy the following three conditions: 1) $\{(a, a) | a \in A\} \subset \eta$; 2) $\nu(X) \subset \eta$; 3) For every basic operation f of arity n and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in A^n$ the condition $(x_1, y_1), \ldots, (x_n, y_n) \in \eta$ implies that $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n) \in \eta$. Thus, we can computably enumerate the set Y required in the lemma by satisfying the following three properties that correspond to the conditions 1), 2) and 3) above: 1) $E_\nu \subset Y$; 2) $X \subset Y$; 3) For every basic operation f of arity n, for all $s, t \in \omega$, $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \omega^n$ the conditions $(x_1, y_1), \ldots, (x_n, y_n) \in Y$, $\nu(t) = f(\nu(x_1), \ldots, \nu(x_n))$, and $\nu(s) = f(\nu(y_1), \ldots, \nu(y_n))$ imply that $(s, t) \in Y$.

Now in order to prove the theorem, we need to show that the equivalence relation E_ν is computable. Take $x, y \in \omega$. Consider the set Y such that $\nu(Y)$ is the minimal congruence relation that contains the pair $(\nu(x), \nu(y))$. Note that if $\nu(y) = \nu(x)$ then $Y = E_\nu$. Otherwise, there exists a pair $(n, m) \in S$ such that $(n, m) \in Y$. Thus, using the properties of S and that S, Y, and E_ν are c.e. sets, we can decide if $\nu(x) = \nu(y)$. Hence ν is a constructivization. The theorem is proved.

A corollary of this theorem is the following fact that gives an example of how algebraic properties may influence effective properties of numberings. An algebra is **quasisimple** if it has only a finite number of congruence relations.

Corollary 2.9. *Any positive and quasisimple algebra $(\mathcal{A}, \nu)$ is constructive.*

Proof. Let $\eta_0, \ldots, \eta_k$ be all nonzero congruences of $\mathcal{A}$. For each $i \leq k$ take (x_i, y_i) such that $\nu(x_i) \neq \nu(y_i)$ and $(\nu(x_i), \nu(y_i)) \in \eta_i$. The the set $S = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ satisfies the assumptions of Theorem 2.8.

We recall, before we state the next corollary, that the index of an equivalence relation is the number of its equivalence classes.

Corollary 2.10. *If $(\mathcal{A}, \nu)$ is a positive finitely generated algebra with nonzero congruences of finite index only and the language of $\mathcal{A}$ is finite then $(\mathcal{A}, \nu)$ is constructive.*

Proof. Let $c_1, \ldots, c_n$ be generators of $\mathcal{A}$. Consider the **ground terms** of the language expanded by constants $c_1, \ldots, c_n$, that is the terms that contain no variables. We inductively define the **height** of terms as follows. The height of each constant is 0. Let $t = f(t_1, \ldots, t_n)$ be a term. Let m be the maximum among all the height of $t_1, \ldots, t_n$. Then the height of t is $m + 1$. Note that for any given m we can effectively compute the number of terms of height m. For any finite algebra $\mathcal{B}$ whose generators are $c_1, \ldots, c_n$ there exists a number m with the following property. For any term t of height $m + 1$ there exists a term t' such that the equality $t = t'$ holds in $\mathcal{B}$ and the height of t' does not exceed m. Now for the positive algebra $(\mathcal{A}, \nu)$ we define the set S as follows. A pair (x, y) belongs to S if and only if the minimum congruence relation that contains the pair $(\nu(x), \nu(y))$ is of finite index. The set S is c.e. and satisfies the assumptions of the theorem above.

The next corollary is a result of McKinzey [**32**]. We need some preliminary notions. An algebra $\mathcal{A}$ is **residually finite in a class** K **of algebras** if, for any two distinct elements a, b of the algebra, there exists a homomorphism h of the algebra onto a finite algebra from K such that $h(a) \neq h(b)$. If K is the class of all finite algebras then $\mathcal{A}$ is called **residually finite**. A **conditional equation** is a universally quantified formula of the type $t_1 = q_1 \& \ldots \& t_n = q_n \to t = q$, where $t, q, t_1, q_1, \ldots, t_n, q_n$ are terms of the signature expanded by finitely many constants $c_0, \ldots, c_n$. Let C be a set of conditional equations. Then there exists the **free algebra**, denoted $\mathcal{F}_C$, that satisfies the following properties: 1) $\mathcal{F}_C$ is finitely generated with generators $c_0, c_1, \ldots, c_n$. 2) $\mathcal{F}_C$ satisfies C. 3) Any algebra with properties 1) and 2) is a homomorphic image of $\mathcal{F}_C$. 4) The set $\{t_1 = t_2 | \mathcal{A}$ satisfies the equality $t_1 = t_2$ and t_1, t_2 contain no variables $\}$ is c.e. in C. The first three properties define $\mathcal{F}_C$ up to isomorphism. If C is finite then the algebra $\mathcal{F}_C$ is called a **finitely presented algebra**.

Let ν be a one to one numbering of all ground terms of the language $L \bigcup \{c_0, \ldots, c_n\}$ such that the set $\{t = \nu(n) | n \in \omega, t$ is a ground term $\}$ is computable. The numbering ν induces a natural numbering ν_s of the algebra $\mathcal{F}_C$ (see Theorem 2.5). Then if C is a c.e. set then the numbered algebra $(\mathcal{F}_C, \nu_s)$ is a positive algebra.

Corollary 2.11. *If $\mathcal{F}_C$ is finitely presented and residually finite in the class of all algebras satisfying C then $(\mathcal{A}, \nu_s)$ is a constructive algebra.*

Proof. We can assume that C is finite. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots$ be an effective sequence of all finite algebras that satisfy C. Note that there exists a natural homomorphism h_i from $\mathcal{F}_C$ onto $\mathcal{A}_i$. Moreover the set $\{(i, x, y) | h_i(\nu_s(x)) = h_i(\nu_s(y))\}$ is computable. Consider the set $S \subset \omega^2$ such that $(x, y) \in S$ if and only if there exists an algebra $\mathcal{A}_i$ such that the images of $\nu_s(x)$ and $\nu_s(y)$ are distinct in $\mathcal{A}_i$ under the natural homomorphism h_i. The set S satisfies the assumptions of Theorem 2.8.

2.5. Positive Algebras with Countably Many Congruences. In this section we provide some interesting computability-theoretic and algebraic properties of positive algebras with a countable number of congruences. Positive quasi-simple algebras and algebras with congruences of finite index only are examples of algebras with countably many congruences. In this section we always assume that the language L is finite. All the results in this section were first obtained by N. Kasymov [**22**] [**23**]. All the algebras considered in this section are infinite.

For a positive algebra $(\mathcal{A}, \nu)$, the **characteristic transversal** $tr(\nu)$ of ν is the set $\{x | \forall y (y < x \to \nu(x) \neq \nu(y))\}$. For $x, y \in \omega$, let $\eta_\nu(x, y)$ be the smallest congruence containing the pair $(\nu(x), \nu(y))$. Note, $E_\nu \subset \nu^{-1}(\eta_\nu(x, y))$ and an index of $\nu^{-1}(\eta_\nu(x, y))$ can be obtained effectively given x and y.

Theorem 2.12. *For any positive algebra $(\mathcal{A}, \nu)$ with congruences of finite index only the set $tr(\nu)$ is either computable or hyperimmune.*

Proof. Assume that $tr(\nu)$ is not hyperimmune. We need to show that $tr(\nu)$ is computable. It suffices to show that the equivalence relation E_ν is a computable set. Since $tr(\nu)$ is not hyperimmune there must exist a strong array $S_0, S_1, \ldots$ of finite disjoint sets such that $S_i \bigcap tr(\nu) \neq \emptyset$ for all i. Hence for every S_m there exists a $z \in S_m$ such that for all $t < z$ we have $(t, z) \notin E_\nu$. Consider $\eta_\nu(x, y)$. If $\nu(x) \neq \nu(y)$ then $\eta_\nu(x, y)$ is of finite index. Hence there must exist an S_m such that for every $z \in S_m$ there exists a $t < z$ for which $(\nu(z), \nu(t)) \in \eta_\nu(x, y)$. Also, if $\nu(x) = \nu(y)$ then $E_\nu = \eta_\nu(x, y)$. These two are mutually exclusive cases which can be effectively checked given x and y. Hence E_ν is computable. The theorem is proved.

The theorem above implies the next result. The result shows that the assumption that $(\mathcal{A}, \nu)$ is a positive algebra, with congruences of finite index only, has not only computability-theoretic implications but also purely algebraic implications for the algebra $\mathcal{A}$.

THEOREM 2.13. *If $(\mathcal{A}, \nu)$ is a positive but not constructive algebra with congruences of finite index only, then the algebra $\mathcal{A}$ is locally finite, residually finite and the language of $\mathcal{A}$ contains at least one binary function symbol.*

Proof. Recall that an algebra is **locally finite** if every finitely generated subalgebra of the algebra is finite. Assume that $(\mathcal{A}, \nu)$ is not locally finite. Hence there exist finitely many elements $a_1, \ldots, a_n \in A$ such that the subalgebra $\mathcal{B}$ generated by the elements is infinite. Note that $\nu^{-1}(B)$ is a c.e. set. Since $\mathcal{B}$ is infinite and finitely generated there exists a computable sequence $B_0 \subset B_1 \subset \ldots$ of subsets of ω such that $B = \bigcup_i B_i$, the function $i \to |B_i|$ is computable, and $\nu(B_{i+1}) \setminus \nu(B_i) \neq \emptyset$. Hence the function $i \to max(B_i)$ majorizes the characteristic transversal $tr(\nu)$. But by the previous theorem $tr(\nu)$ is hyperimmune since, by assumption, $(\mathcal{A}, \nu)$ is not a constructive algebra. This is a contradiction. Hence $\mathcal{A}$ is locally finite.

Now we prove that $\mathcal{A}$ is residually finite. Let $\nu(x)$ and $\nu(y)$ be counterexamples to the fact that $\mathcal{A}$ is residually finite. Hence for all $n, m \in \omega$ if $\nu(n) \neq \nu(m)$ then, since every congruence is of finite index, we have $(\nu(x), \nu(y)) \in \eta_\nu(n, m)$. We conclude that the equality $\nu(n) = \nu(m)$ can be checked effectively which implies that $(\mathcal{A}, \nu)$ is a constructive algebra. This contradicts with the assumption.

Now we show that the language of $\mathcal{A}$ contains at least one binary operation symbol. Assume that all basic operations of $\mathcal{A}$ are unary. Take an element $b \in A$. Consider the subalgebra $\mathcal{B}$ generated by b. The subalgebra $\mathcal{B}$ is finite. Consider the equivalence relation $\{(a, a) \mid a \in A\} \bigcup B^2$. The equivalence relation is a congruence relation on $\mathcal{A}$ because all the basic operations are unary. However the index of the congruence relation is not finite. This is a contradiction with the assumption. The theorem is proved.

By Corollary 2.10 every positive finitely generated algebra with congruences of finite index only is a constructive algebra. On the other hand, Kassimov provided examples of positive algebras without constructivizations every congruence of which is of finite index [**22**].

The last example of results about positive algebras is the next theorem that has an implication on the cardinality of the congruence lattice of a given algebra. It gives a sufficient condition for a positive algebra not to have countably many congruence relations. We need one notion. We say that a numbered algebra $(\mathcal{A}, \nu)$ is **effectively infinite** if there exists an infinite c.e. set X such that $\nu(x) \neq \nu(y)$ for all distinct $x, y \in X$.

THEOREM 2.14. *Every positive finitely generated noneffectively infinite algebra $(\mathcal{A}, \nu)$ has a continuum number of congruence relations.*

Proof. Let $a_0, \ldots, a_n$ be the generators of the algebra $\mathcal{A}$. Consider the ground terms of the language of $\mathcal{A}$ expanded by the constants $a_0, \ldots, a_n$. Each ground term t naturally defines an element of $\mathcal{A}$ which we also denote by t. Let $t_1, q_1, \ldots, t_s, q_s$ be a set of ground terms such that in the algebra $\mathcal{A}$ the set of inequalities $I = \{t_i \neq q_i \mid i = 1, \ldots, s\}$ holds.

Claim 1. There exist ground terms p, q and s, t such that $\eta_\nu(p,q)$ and $\eta_\nu(s,t)$ are each of infinite index, not comparable, and inequalities in I are true in any quotient algebra with respect to these congruence relations.

Proof. Let F be the set of all pairs of ground terms (t_1,t_2) such that $\eta_\nu(t_1,t_2)$ is of finite index. Let V be the set of all pairs (t_1,t_2) of ground terms such that the quotient algebra with respect to $\eta_\nu(t_1,t_2)$ does not satisfy I. Now note that F and V are computably enumerable sets. Hence if for all (t_1,t_2) either $(t_1,t_2)\in E_\nu$ or $(t_1,t_2)\in F$ or $(t_1,t_2)\in V$ then ν would be a constructivization. Therefore, there exists a pair p,q such that $p\neq q$, $\eta_\nu(p,q)$ has an infinite index, and the quotient algebra with respect to $\eta_\nu(p,q)$ satisfies I. We now need to show that there exists a pair s,t that satisfies the claim. Assume that such s and t do not exist. Then for all ground terms s,t for which $(s,t)\notin\eta_\nu(p,q)$ either $\eta_\nu(s,t)$ is of finite index or the quotient algebra with respect to $\eta_\nu(s,t)$ does not satisfy I or $(p,q)\in\eta_\nu(s,t)$. Using this fact, we now show that the algebra $(\mathcal{A},\nu)$ is effectively infinite. We construct a c.e. infinite sequence $r_0, r_1, \ldots$ of ground terms as follows.

Stage 0. Let r_0 be any ground term.

Stage n+1. Assume that the sequence $r_0,\ldots,r_n$ has been constructed and $r_i\neq r_j$ holds true in the quotient $\mathcal{A}$ with respect to $\eta_\nu(p,q)$ for all $i\neq j$. Note that there exists an r such that $r\neq r_i$ holds in the quotient $\mathcal{A}$ with respect to $\eta_\nu(p,q)$. Find an r such that for all $i\leq n$ either $\eta_\nu(r,r_i)$ is of finite index or the quotient algebra with respect to $\eta_\nu(r,r_i)$ does not satisfy I or $(p,q)\in\eta_\nu(r,r_i)$. Note that such an r exists and can be found effectively. Let r_{n+1} be the first such r found.

Thus, we have contradicted with the assumption that $(\mathcal{A},\nu)$ is not effectively infinite. Hence the claimed pairs $(p,q),(s,t)$ exist. The claim is proved.

Now it is easy to embed the set of paths through the binary tree into the lattice of congruences of $\mathcal{A}$. We send the root of the tree into $\eta_\emptyset=\{(a,a)\mid a\in A\}$. We set $I_\emptyset=\emptyset$. Assume that the congruence relation $\eta_{i_0\ldots i_n}$ and the finite set $I_{i_0\ldots i_n}$ have been defined, where $i_0,\ldots,i_n\in\{0,1\}$. Consider the algebra $\mathcal{B}$ obtained by factoring $\mathcal{A}$ with respect to $\eta_{i_0\ldots i_n}$. The algebra is positive and is not effectively infinite by the induction hypothesis. By the claim there exists $(p,q),(s,t)$ such that $\eta_\nu(p,q)$ and $\eta_\nu(s,t)$ are each of infinite index, not comparable, and inequalities in $I_{i_0\ldots i_n}$ hold in any quotient algebra with respect to these congruence relations. Map $i_0\ldots i_n0$ into $\eta'_\nu(p,q)$, $i_0\ldots i_n1$ onto $\eta'_\nu(s,t)$, where $\eta'_\nu(r_1,r_2)$ denotes the smallest congruence of $\mathcal{B}$ that contains r_1 and r_2. Set $I_{i_0\ldots i_n0}=I_{i_0\ldots i_n}\bigcup\{s\neq t\}$ and $I_{i_0\ldots i_n1}=I_{i_0\ldots i_n}\bigcup\{p\neq q\}$. Clearly each path corresponds to a congruence relation and the correspondence is one to one. The theorem is proved.

Corollary 2.15. *Every finitely generated infinite positive algebra with countably many congruence relations is effectively infinite.*

We end this section by proposing the following research program.

Problem 2. Develop the theory of positive algebras.

3. Constructive Models of Theories

In contrast to the previous section, this section and the next are devoted to specific open problems related to constructive models of theories and computable isomorphisms. Many of the problems have been open and known for many years and, perhaps, new ideas, constructions, and concepts will be needed to solve these problems. This section consists of three parts. The first part is devoted to constructive models of countably categorical theories. The second part discusses the issues related to constructivizations of models of uncountably categorical theories. The last part will deal with constructive models of Ehrenfeucht theories.

3.1. Constructive Countably Categorical Models. Before we discuss the issues related to constructive models of countably categorical theories, we recall some basic general facts about constructive models of theories. By a theory we always mean a set of sentences closed under deduction. So, let T be a consistent theory. The Completeness Theorem states that T has a model. A proof of this result can be based on a Henkin type construction. If, in addition, T is a decidable theory then the construction can be carried out effectively. Hence the full diagram of the model constructed is decidable. Thus, we have the following fundamental theorem, known as The Effective Completeness Theorem:

THEOREM 3.1. *If T is decidable and consistent theory then T has a strongly constructive model.*

The Effective Completeness Theorem suggests several fundamental questions about models of theories: Which models of T have strong constructivizations? If T has a prime model then does the prime model have a strong constructivization? If T has a saturated model then does the saturated model have a strong constructivization? When does a given homogeneous model of T have a strong constructivization? How many strongly constructive models can T have? There has been extensive research in the study of these question. Basically all these questions have been answered. The following theorem proved in [**14**] and [**18**] and other places characterizes the theories with strongly constructive prime models:

THEOREM 3.2. *A decidable complete theory has a strongly constructive prime model if and only if there exists an algorithm that, for any formula consistent with T, produces a principal type of the theory that contains the formula.*

The following theorem proved in [**14**] and [**34**] characterizes all the theories whose saturated models have strong constructivizations.

THEOREM 3.3. *A decidable complete theory T has a strongly constructive saturated model if and only if the set of all types of T has a computable numbering.*

For a survey of results related to strongly constructive models of theories we refer the reader to papers in [**6**], especially the paper by Harizanov. Now we concentrate our attention on constructivizations of **countably categorical models**, that is the models of countably categorical theories. Recall the following definition.

DEFINITION 3.4. A theory T is **countably categorical** if T has exactly one countable model up to isomorphism.

¿From a model theory point of view, countably categorical theories and their models have been very well-studied and understood. Clearly, if T is countably categorical then we have the following result that follows from The Effective Completeness Theorem:

THEOREM 3.5. *A countably categorical theory T is decidable if and only if all models of T have strong constructivizations if and only if T has a strongly constructive model.*

Thus, the question of the existence of strongly constructive models for countably categorical categorical theories is answered via their decidability.

The Ryll-Nardzewski Theorem characterizes countably categorical theories in terms of types. The theorem states that a theory T is countably categorical if and only if for every n the number of n-types of T is finite. This theorem leads us to define the type function of the theory T, denoted by $type_T$, that associates with each $n \geq 1$ the number of n-types of the theory. Note that if T is decidable then $type_T$ is a Δ^0_2-function. A natural question arises as to what can be said about the computability-theoretic complexity of the type function $type_T$ when T is decidable. The following theorem, proved independently by Herrmann, Schmerl and Venning, answers the question.

THEOREM 3.6. [**38**] *For any c.e. degree* $\mathbf{x}$ *there exists a decidable countably categorical theory* T *such that the type function* $type_T$ *is of degree* $\mathbf{x}$.

These two theorems basically answer the questions related to the existence of strongly constructive models for countably categorical decidable theories. However, the theorems and their proofs do not give a clear picture about the existence of constructive models of countably categorical theories when one omits the assumption of decidability. We pose the following problem:

PROBLEM 3. Characterize the countably categorical theories that have constructive models.

We note that there has been some research on this problem. Lerman and Schmerl in [**29**] give some sufficient conditions for countably categorical *arithmetic* theories to have a constructive model. More precisely, they have shown that if T is a countably categorical arithmetical theory such that the set of all sentences beginning with an existential quantifier and having $n+1$ alternations of quantifiers is Σ^0_{n+1} for each n, then T has a constructive model. However, this result cannot be considered as a solution to the problem. We do not even know of any example that satisfies the conditions of this results for sufficiently large n. Hence one of the ways to approach the problem above is to actually build such theories:

CONJECTURE 1. For every $n \geq 1$, there exists a countably categorical theory of Turing degree $\mathbf{0}^{(n)}$ that has a constructive model.

It is not hard to see that if a theory T has a constructive model then T is computable in $\mathbf{0}^{(\omega)}$, the degree of the ω-jump of a computable set. In fact, this bound is sharp as there exist theories (the theory of $(\omega, +, \times, \leq)$ for example) with computable models which are Turing equivalent to $\mathbf{0}^{(\omega)}$. We now end this section with a formulation of the next conjecture that states that a nonarithmetical countably categorical theory with constructive models exists.

CONJECTURE 2. There exists a countably categorical theory of Turing degree $\mathbf{0}^{(\omega)}$ that has a constructive model.

3.2. Constructive Uncountably Categorical Models. Another class of theories well-studied in model theory is the class of uncountably categorical theories. In this section we deal with models of uncountably categorical theories. Throughout this section, we assume that the theories considered are *not* countably categorical. We recall the definition.

DEFINITION 3.7. A theory T is **uncountably categorical** if any two models of T of cardinality ω_1 are isomorphic. Models of uncountably categorical theories are called **uncountably categorical models**.

Typical examples of uncountably categorical theories are the following: the theory of algebraically closed fields of fixed characteristic, the theory of vector spaces over a fixed countable field, the theory of the structure (ω, S), where S is the successor function on ω. Roughly speaking, all the countable models of each of these theories can be listed in an $\omega + 1$ chain: the first element of the chain is the prime model, the last element of the chain is the saturated model; moreover, for any two models of the theory one can be elementarily embedded into the other. It turns out these are one of the basic structural properties of the class of models of an uncountably categorical theory. More precisely, in [**3**] Baldwin and Laclan showed that all models of any uncountably categorical theory T can be listed into the following chain, denoted by $chain(T)$, of elementary embeddings:

$$\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots \mathcal{A}_\omega,$$

where $\mathcal{A}_0$ is the prime model of T, $\mathcal{A}_\omega$ is the saturated model of T, and each $\mathcal{A}_{i+1}$ is a minimal proper elementary extension of $\mathcal{A}_i$. As we are interested in constructive models of T the following definition is central.

DEFINITION 3.8. Let T be an uncountably categorical theory. **The spectrum of constructive models of** T, denoted by $SCM(T)$, is the set

$$\{i \mid \text{the model } \mathcal{A}_i \text{ of } T \text{ has a constructivization}\}.$$

Thus a natural open problem about constructive models of uncountably categorical theories is the following:

PROBLEM 4. Characterize all the subsets X of $\omega \bigcup \{\omega\}$ for which there exist uncountably categorical theories T such that $SCM(T) = X$.

An initial step in the study of this problem is to assume that the theory T is decidable. Clearly, in general, the decidability of a theory T (independently of whether or not T is uncountably categorical) does not imply that all the models of T have strong constructivizations. However, one of the important results in the study of (strongly) constructive models of a given theory T is the following result of Harrington [**18**] and Khisamiev [**27**]:

THEOREM 3.9. *Let T be an uncountably categorical theory. Then T is decidable if and only if T has a decidable model if and only if all models of T have decidable presentations.*

The proof of this theorem consists of two steps. The first step uses a purely model theoretic fact that states that every model $\mathcal{A}$ of an uncountably categorical theory T can be considered as the prime model of a uncountably categorical theory T' with a strongly minimal formula. Moreover, if T is decidable then so is T'. The second step of the proof shows that for any decidable uncountably categorical theory there exists an algorithm that, given a formula consistent with the theory, produces a principal type that contains the formula. Then Theorem 3.2 is applied to show that the model $\mathcal{A}$ has a strong constructivization.

Again, we have a situation similar to the one for countably categorical theories. The theorem above basically answers the question related to the existence of strongly constructive models for uncountably categorical decidable theories. However, neither the theorem nor the proof of the theorem gives a clear picture for building constructive models of uncountably categorical theories when one omits the assumption of decidability. We pose the following problem:

Problem 5. Characterize uncountably categorical theories that have constructive models.

The situation for uncountably categorical theories is also complicated by the following fact. In general, the existence of a constructive model for an uncountably categorical theory T does not imply that all models of T have constructivizations. Indeed, in [**7**] Goncharov showed that there exists an uncountably categorical theory T for which $SCM(T) = \{0\}$, that is, the only model of T which has a constructivization is the prime model of T. Kudaibergenov extended this result by showing that for every $n \geq 0$ there exists an uncountably categorical T such that $SCM(T) = \{0, 1 \ldots, n\}$ [**28**]. This theory T basically codes a noncomputable, computably enumerable set X in such a way that from the open diagram of the model $\mathcal{A}_{n+1}$ in $chain(T)$ one can decode the set X. More complicated codings are needed to prove the following theorem:

Theorem 3.10. [**26**] *There exist uncountably categorical theories T_1 and T_2 such that $SCM(T_1) = \omega$, and $SCM(T_2) = \omega \bigcup \{\omega\} \setminus \{0\}$.*

The constructions of the theory T_1 is based on a coding of a Σ_2-set, while the construction of the theory T_2 is based on a coding of a Π_2-set. This theorem together with the theorem by Khissamiev and Harrington are the only known results so far about the spectra of constructive models of uncountably categorical theories. We also note that the theories T_1 and T_2 of the theorem above have infinite languages. Recently Herwig, Lempp and Ziegler [**19**] have constructed an uncountably categorical theory T of a finite language such that $SCM(T) = \{0\}$.

It is interesting to note that all the known uncountably categorical theories that have constructive models are computable in $\mathbf{0}''$. Khoussainov, Lempp, and Solomon have recently constructed examples (the paper is in preparation) of uncountably categorical theories that compute $\mathbf{0}^{(n)}$, $n > 2$, and all of whose models have constructivizations. In connection with this result Lempp has asked the following question:

Question 6. If an uncountably categorical theory has a constructive model then must the theory be arithmetical?

We note that the theorem of Harrington and Khissamiev can be relativized to show that if T is uncountably categorical and arithmetical then all models of T have arithmetical numberings. Hence a positive answer to the question of Lempp implies that all models of an uncountably categorical theory with constructive models have arithmetical numberings. However, we state the following hypotheses which, if correct, would negatively answer the question of Lempp:

Conjecture 3. There exists a nonarithmetical uncountably categorical theory with a constructive model.

One of the ways to approach this conjecture is, for example, to answer to the following question:

QUESTION 7. Does there exist an uncountably categorical theory T whose models are $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \ldots \preceq \mathcal{A}_\omega$ such that $\mathcal{A}_0$ has a constructivization, and each $\mathcal{A}_{(i+1)}$, $i \in \omega$, has a constructivization computable in $\mathbf{0}^{(i+1)}$ but does not have constructivizations computable in $\mathbf{0}^{(i)}$?

As we have already stated, there are examples of uncountably categorical theories T with models which have no constructivizations but with $SCM(T) \neq \emptyset$. On the other hand, we also know that in some cases all models of T have constructivizations. We single out such theories in the following definition.

DEFINITION 3.11. A theory T has **constructively complete** if all countable models of T have constructivizations.

Decidable countably categorical and decidable uncountably categorical theories are examples of theories which are constructively complete. There are also examples of undecidable constructively complete theories. In general, constructively complete theories do not need to be complete. For example, the theory of all dense linearly ordered sets is constructively complete. We end this section by posing the following question:

QUESTION 8. When is an uncountably categorical theory T constructively complete?

3.3. Constructive Models of Ehrenfeucht Theories. Another class of theories that has been well studied and has attracted considerable attention is the class of Ehrenfeucht theories. Here is an exact definition.

DEFINITION 3.12. A theory is an **Ehrenfeucht theory** if it has finitely many models.

Vaught proved that no complete theory has exactly two models. On the other hand, for each $n \geq 3$ there exists a theory with exactly n models. For example, the theory of the model $(Q, \leq, c_0, c_1, \ldots)$, where $(Q, \leq)$ is the natural ordering of rationals and $c_0 < c_1 < c_2 < \ldots$, has exactly three models. This example can be generalized to give theories with exactly $n \geq 4$ models. Inspired by Theorem 3.9 Nerode posed the following question: If an Ehrenfeucht theory T is decidable then do all models of T have strong constructivizations? It turns out that models of decidable Ehrenfeucht theories are not as well behaved as decidable uncountably or decidable countably categorical theories. For instance, the following theorem is true.

THEOREM 3.13. [**34**] [**36**] *For each $n \geq 3$ there exists a decidable Ehrenfeucht theory T_0 that admits elimination of quantifiers, has exactly n models and exactly one model with a strong constructivization.*

In relation to this theorem we make the following comments. First of all we note that the prime model of any decidable Ehrenfeucht theory must have a strong constructivization. This follows from an effective version of the Omitting Types Theorem for decidable theories [**33**] which is not discussed in this paper. Hence the strongly constructive model of the theory T_0 in the theorem above is a prime model. Secondly, the reason that not all models of T_0 have strong constructivizations is that T_0 has a noncomputable type. Based on this Morley asked the following question that has become known as Morley's problem:

Question 9. If all types of an Ehrenfeucht theory T are computable then do all models of T have strong constructivizations?

This is an open problem which has been attempted by many with no success. Ash and Millar obtained several interesting results in the study of this question. One of the results is the following. We say that an Ehrenfeucht theory T is **persistently Ehrenfeucht** if any complete extension of T with finitely many new constants is also an Ehrenfeucht theory. Here is the theorem:

Theorem 3.14. [**2**] *If T is persistently Ehrenfeucht all of whose types are arithmetical then all models of T have arithmetic presentations.*

In relation to this theorem and Morley's problem, it is interesting to note that the following question, asked by Goncharov and Millar, is still open:

Question 10. If T is an arithmetic Ehrenfeucht theory whose types are arithmetical, do then all models of T have arithmetic presentations?

We now briefly discuss the problem of existence of constructive models of Ehrenfeucht theories. As for categorical theories, there has not been much research about finding constructive models for (undecidable) Ehrenfeucht theories. We note that the results in finding constructive models for undecidable Ehrenfeucht theories can be quite different from those about decidable Ehrenfeucht theories. We give an example. If all types of an Ehrenfeucht theory T are computable then T must have at least three strongly constructive models (a proof of this can, for example, be found in [**25**]). Therefore for any decidable Ehrenfeucht theory T with exactly three models, the saturated model of T has a strong constructivization if and only if all models of T have strong constructivizations. We also recall that the prime model of every decidable Ehrenfeucht theory has a strong constructivization. In contrast to this, in [**26**] the following theorem is proved:

Theorem 3.15. *There exists an Ehrenfeucht theory with exactly three models of which only the saturated one has a constructivization.*

We conclude this section with the following research proposal.

Problem 11. Work towards characterizing strongly constructive or/and constructive models of Ehrenfeucht theories.

4. Computable Isomorphisms

There has been extensive research on computable isomorphisms of constructive algebraic systems. Many researchers have worked on problems and research directions discussed in this section. These are still in the center of research interest and play a significant role in the creation of new ideas, theorems and concepts. This section consists of four parts. The first part introduces the basic concepts about computable isomorphisms between constructive algebraic systems. In the second, we discuss computable isomorphisms of countably categorical models. The third part is devoted to computable isomorphisms of uncountably categorical models. In the final part we deal with the problem of the dependency of computability-theoretic properties of relations on constructivizations.

4.1. Basic Notions. A fundamental concept in the theory of constructive models is the notion of a computable isomorphism. Informally, the notion of a computable isomorphism allows one to compare constructivizations of a given model, and to tell whether or not two constructivizations have the same computability-theoretic properties. We recall the definition.

Definition 4.1. Two constructive algebraic systems $(\mathcal{A}, \nu)$ and $(\mathcal{A}, \mu)$ are **computably isomorphic** if there exists an automorphism α of $\mathcal{A}$ and a computable function f such that $\alpha\nu(n) = \mu(f(n))$ for all $n \in \omega$. In this case we also say that ν and μ are **autoequivalent**.

One of the fundamental properties of computably isomorphic structures is that they cannot be distinguished in terms of computability-theoretic properties of definable relations. This means that for any relation R invariant under the automorphisms of $\mathcal{A}$, the Turing degrees of R under the constructivizations ν and μ are equivalent, that is, $\nu^{-1}(R)$ and $\mu^{-1}(R)$ have the same Turing degree. In addition, if ν and μ are one to one, then $\nu^{-1}(R)$ and $\mu^{-1}(R)$ are computably isomorphic. One of the important concepts introduced in the study of computable isomorphisms is Goncharov's notion of dimension. Here is the definition.

Definition 4.2. The **computable dimension** of an algebraic system $\mathcal{A}$, denoted $dim(\mathcal{A})$, is the maximal number of its nonautoequivalent constructivizations.

Informally, the computable dimension tells us as how many effective realizations the algebraic system $\mathcal{A}$ possesses. In computability-theoretic terms the computable dimension of a given algebraic system can be thought of as the number of its computable isomorphism types. Thus, if the dimension of $\mathcal{A}$ is 1 then $\mathcal{A}$ has exactly one effective realization. We single out the algebraic systems of dimension 1, and give the following definition first introduced by Malcev.

Definition 4.3. An algebraic system $\mathcal{A}$ is **autostable** if $dim(\mathcal{A}) = 1$. The system $\mathcal{A}$ is **strongly autostable** if all its strong constructivizations are autoequivalent.

An important notion introduced by Goncharov in the study of computable isomorphisms is the notion of an effectively infinite algebraic system. We say that a sequence $(\mathcal{A}_0, \nu_0), (\mathcal{A}_1, \nu_1), \ldots$ of constructive models is **effective** if the set $\{(i, \phi) | \phi \in AD_{\nu_i}(\mathcal{A}_i)\}$ is computably enumerable. Informally, an effective sequence of constructive models is one which can be constructed in a uniform manner.

Definition 4.4. An algebraic system $\mathcal{A}$ is **effectively infinite** if there exists an algorithm that applied to any index of an effective sequence of constructive systems $(\mathcal{A}, \nu_0), (\mathcal{A}, \nu_1), \ldots$ produces a constructive algebraic system $(\mathcal{A}, \nu)$ such that $(\mathcal{A}, \nu)$ is not computably isomorphic to $(\mathcal{A}, \nu_i)$ for $i \in \omega$.

Thus, if an algebraic system $\mathcal{A}$ is effectively infinite then it has infinite computable dimension. One of the first important results in the study of autostable models is the following result of Nurtazin [**35**] that basically characterizes strongly autostable algebraic systems:

Theorem 4.5. *A strongly constructive algebraic system $(\mathcal{A}, \nu)$ is strongly autostable if and only if there exists a finite number $a_0, \ldots, a_n \in A$ of elements such that the following properties hold:*

1. *The set of all complete formulas of the theory T of the algebraic system $(\mathcal{A}, a_0, \dots, a_n)$ is computable.*
2. *The system $(\mathcal{A}, a_0, \dots, a_n)$ is the prime model of its own theory T.*

Moreover, if $(\mathcal{A}, \nu)$ is not strongly autostable then there exists an algorithm that applied to any index of an effective sequence of strongly constructive systems $(\mathcal{A}, \nu_0), (A, \nu_1), \dots$ produces a strongly constructive algebraic system $(\mathcal{A}, \nu)$ such that $(\mathcal{A}, \nu)$ is not computably isomorphic to $(\mathcal{A}, \nu_i)$ for any $i \in \omega$.

Thus, by this theorem, the computable dimension of any non-strongly autostable strongly constructive algebraic system is infinite.

4.2. Isomorphisms of Countably Categorical Models. Let $\mathcal{A}$ be an algebraic system. Natural questions that arise about the computable isomorphisms of $\mathcal{A}$ are the following. Is $\mathcal{A}$ autostable? If $\mathcal{A}$ is autostable then why is it so? If $\mathcal{A}$ is not autostable then why is it not? What is the dimension of $\mathcal{A}$? Can $\mathcal{A}$ have infinite dimension? Can $\mathcal{A}$ have finite dimension?, etc.

These and related questions have been extensively studied with respect to known classes of algebraic systems such as linearly ordered sets, Boolean algebras, Abelian groups, rings, groups, partially ordered sets, fields, vector spaces, etc. One of the first results obtained in the study of these questions is the following theorem proved independently by Goncharov and Remmel.

THEOREM 4.6. *A linearly ordered set is autostable if and only if the set of adjacent pairs of the linearly ordered set is finite. Similarly, a Boolean algebra is autostable if and only if the set of all atoms of the algebra is finite. Moreover, nonautostable linearly ordered sets and Boolean algebras are effectively infinite.*

An immediate corollary of this theorem is the following result which has not been explicitly stated in the literature.

COROLLARY 4.7. *A linearly ordered set is autostable if and only if the linearly ordered set is countably categorical. Similarly, a Boolean algebra is autostable if and only if the algebra is countably categorical.*

This corollary suggests the study of computable dimensions of those algebraic systems whose theories and/or algebraic, model-theoretic properties are well-understood. Thus, one can study the computable dimensions of countably categorical models.

Using the result of Nurtazin mentioned at the end of the previous section, the following theorem characterizes all strongly autostable countably categorical models.

THEOREM 4.8. *A strongly constructive model $(\mathcal{A}, \nu)$ of a countably categorical theory T is strongly autostable if and only if the type function $type_T$ of the theory T is computable.*

Proof. Assume that the type function $type_T$ is a computable function. Let $(\mathcal{A}, \mu)$ be a strongly constructive model. We want to show that ν and μ are autoequivalent. We claim that for any m-tuple $b_1, \dots, b_m \in A$ with $\mu(n_1) = b_1, \dots, \mu(n_m) = b_m$ we can effectively find a complete formula of the type determined by $(b_1, \dots, b_m)$. Indeed, to do this we compute $type_T(m)$ and then find consistent with T formulas

$$\phi_1(x_1, \dots, x_m), \dots, \phi_{t(m)}(x_1, \dots, x_m)$$

with exactly m number of variables such that

$$\phi_i(x_1, \dots, x_m) \to \phi_j(x_1, \dots, x_m) \notin T$$

for all $1 \leq i \neq j \leq t(m) = type_T(m)$. These formulas can be found effectively because $type_T$ and T are computable. Now, using a back and forth, it is not hard to see that one can construct a computable isomorphism between $(\mathcal{A}, \nu)$ and $(\mathcal{A}, \mu)$.

Now we assume that $\mathcal{A}$ is strongly autostable. We need to show that the type function $type_T$ is computable. Since $\mathcal{A}$ is strongly autostable, by the theorem of Nurtazin there exists a finite sequence $a_0, \dots, a_n$ of elements of $\mathcal{A}$ such that $(\mathcal{A}, a_0, \dots, a_n)$ is the prime model of the theory T' of $(\mathcal{A}, a_0, \dots, a_n)$ and the set of complete formulas of T' is computable. We claim that the set of complete formulas of T is also a computable set. Indeed, take a formula $\phi(\bar{x})$ of the language of T which is consistent with T. Find a complete formula $\psi(\bar{x}, a_0, \dots, a_n)$ of T' such that $T' \vdash \psi(\bar{x}, a_0, \dots, a_n) \to \phi(\bar{x})$. Then $\phi(\bar{x})$ is a complete formula of T if and only if

$$(\phi(\bar{x}) \leftrightarrow \exists y_1 \dots \exists y_n \psi(\bar{x}, y_1, \dots, y_n)) \in T.$$

This shows that the set of complete formulas of T is a computable set. Now $type_T(n) = m$ if and only if there exist exactly m formulas $\phi_1, \dots, \phi_m$ with exactly n variables such that all of these formulas are complete formulas of T and

$$\forall x_1 \dots \forall x_n (\phi_1 \bigvee \dots \bigvee \phi_m) \in T.$$

This shows that the type function $type_T$ of T is computable. The theorem is proved.

The following is a corollary of the theorem.

Corollary 4.9. *Let $\mathcal{A}$ be a model of a countably categorical theory T that admits effective elimination of quantifiers. Then the following are equivalent:*

1. *The dimension of $\mathcal{A}$ is 1.*
2. *There exists a finite sequence $a_0, \dots, a_n$ of elements of $\mathcal{A}$ such that $(\mathcal{A}, a_0, \dots, a_n)$ is the prime model of the theory T' of $(\mathcal{A}, a_0, \dots, a_n)$ and the set of atoms of T' is computable.*
3. *The type function t_T is computable.*

Proof. Since T admits effective elimination of quantifiers any constructivization of $\mathcal{A}$ is also a strong constructivization. The corollary is proved.

We note that there exists a strongly autostable countably categorical but not autostable model. Indeed, consider the structure (A, E), where E is an equivalence relation on A such that every E-equivalence class has either one or two elements. Clearly the system is countably categorical. Moreover, it is strongly autostable. However, it is not hard to prove that if E contains E has infinitely many equivalence classes of each size one and two then $\mathcal{A}$ is not autostable. One can guess that the noncomputability of the type function $type_T$ for a countably categorical theory may imply that the model of T has dimension greater than 1. However, the following result (which is in preparation) recently proved by Khoussainov, Lempp and Solomon gives a counterexample.

Theorem 4.10. *There exists a countably categorical theory T with a noncomputable type function such that the model of T is autostable.*

Note that if T is countably categorical and has a constructive model then the type function of T is computable in $\mathbf{0}^{(\omega)}$. Hence the results above lead us to the following open question:

QUESTION 12. Does there exist a countably categorical theory T such that the type function of T computes $\mathbf{0}^{(\omega)}$ and T has a constructive autostable model?

The results that construct nonautostable algebraic systems of finite dimension do not control model-theoretic properties of the structures constructed. For example, all structures constructed by Goncharov (see [**9**] and [**10**]), Cholak, Goncharov, Khoussainov, Shore (see [**8**]), and Khoussainov and Shore (see [**24**]) have theories without prime models. Moreover all the known countably categorical models have dimensions equal to either 1 or ω. Hence the following question arises naturally.

QUESTION 13. If a countably categorical model is not autostable then is the model effectively infinite?

A positive answer to this question would show that countably categorical and nonautostable models do not have finite dimensions. Hence one may approach the question above by trying to give a counterexample. We pose this as the next question:

QUESTION 14. Does there exist, for a given $n > 1$, a countably categorical model of dimension n?

We finish this section with a comment followed by an open question. Using the theorem of Nurtazin and Theorem 4.8 we see that if a countably categorical decidable theory T has a noncomputable type function then the model of T has infinite dimension. One can ask whether this result can be generalized assuming that T is computable in $\mathbf{0}^{(n)}$ but the type function $type_T$ is not computable in $\mathbf{0}^{(n)}$. We pose this as the following question:

QUESTION 15. Assume that a countably categorical theory T has a constructive model. If T is computable in $\mathbf{0}^n$ and the type function $type_T$ is not computable in $\mathbf{0}^n$, then can we conclude that the model of T is not autostable?

4.3. Isomorphisms of Uncountably Categorical Models. Now we turn to the study of computable dimensions of uncountably categorical models. To provide some intuition, we present some examples of uncountably categorical models and their dimensions.

Let us consider the algebraic system (ω, S). The theory T of this system is uncountably categorical. The isomorphism type of a model $\mathcal{A}$ of T is determined by the number of its components. The saturated model of T has infinitely many components. All nonsaturated models of T are autostable. One can prove that the saturated model of T is not autostable, and is, in fact, effectively infinite.

Consider a second example. Let V be a vector space over a given infinite computable field F. Then the theory T of V (in the language that consists of $+$ for vector addition and unary operations f, $f \in F$, for multiplication by f) is uncountably categorical. It is a well known fact that the isomorphism type of a model $\mathcal{A}$ of T is characterized by the dimension of $\mathcal{A}$. The saturated model of T is the one of infinite dimension. Similarly to the example above, every finite dimensional vector space over F is autostable; the saturated model of T is not autostable, and is, in fact, effectively infinite.

Along the lines of the examples above, one can prove the following theorem about the models of the theory T of algebraically closed fields of a fixed characteristic which is uncountably categorical.

THEOREM 4.11. *Let T be the theory of algebraically closed fields of a fixed characteristic. Then a model $\mathcal{A}$ of T is autostable if and only if it has finite transcendence degree over its prime field.*

Note that all the theories in these examples are decidable and admit elimination of quantifiers. Hence one might expect that the nonsaturated models of such theories are autostable. Here we give a counterexample.

PROPOSITION 4.12. *There exists a decidable uncountably categorical theory T that admits elimination of quantifiers such that the prime mode of T is not autostable.*

Proof. To prove the proposition we provide an uncountably categorical theory T with prime model $\mathcal{A}$ such that the set of complete formulas of the theory of the model $(\mathcal{A}, a_1, \dots, a_n)$ is not computable for all $a_1, \dots, a_n \in A$. By the theorem of Nurtazin this will give the desired result. The theory T is basically the one constructed in [**25**] (page 204). The language of T consists of infinitely many unary predicates R_i. Each R_i contains exactly two elements. Distinct R_i and R_j are pairwise disjoint except for **designated triples** $< i, j, k >$ such that R_k consists of one element from each R_i and R_j. Moreover, for all designated distinct triples $< i, j, k >$ and $< i', j', k' >$ we have $(R_i \bigcup R_j \bigcup R_k) \bigcap (R_{i'} \bigcup R_{j'} \bigcup R_{k'}) = \emptyset$. So the theory is essentially determined by the list of designated triples and is uncountably categorical. If the list of designated triples is computable then the theory is decidable. Now consider the formula $R_i(x)$. This formula is a complete formula if and only if i is not a part of any designated triple. Assume that $< i, j, k >$ is a designated triple. Then each of the formulas $R_i(x) \& R_k(x)$ and $R_i(x) \& \neg R_k(x)$ is a complete formula. The list of designated triples is effectively enumerated in increasing order (and so is computable) by waiting to diagonalize each computable partial function ϕ_i at the formula R_{2i}. If $\phi_i(R_{2i}(x))$ converges at stage s we choose j, k so that $< 2i, 2j+1, 2k+1 >$ is bigger than all pairs already and declare the triple $< 2i, 2j+1, 2k+1 >$ to be designated. Hence no computable partial function ϕ can decide the set of complete formulas of one variable of the theory constructed. Now one notes that for the prime model $\mathcal{A}$ of the theory constructed and all $a_1, \dots, a_n \in A$, the set of complete formulas of the theory of the model $(\mathcal{A}, a_1, \dots, a_n)$ is not computable. The proposition is proved.

Thus, the following question and conjecture arise naturally:

QUESTION 16. Let T be a decidable uncountably categorical theory which has a strongly autostable prime model. Is every nonsaturated model of T strongly autostable?

CONJECTURE 4. The saturated model of any decidable uncountably categorical theory is effectively infinite.

If we omit the assumption of decidability for an uncountably categorical theory, then the situation becomes more complex. There has not been any research done on computable isomorphisms and dimensions of constructive models of uncountably categorical theories. For example, we do not know the spectra of dimensions of uncountably categorical models. Recall that all the models of an uncountably categorical theory T can be listed in a $\omega+1$ chain of models:

$$chain(T): \qquad \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots \preceq \mathcal{A}_\omega.$$

We formulate the following problem.

PROBLEM 17. Consider the model $\mathcal{A}_i$ of an uncountably categorical theory T in the $chain(T)$. Give necessary and sufficient conditions for $\mathcal{A}_i$ to be autostable.

In the study of this problem it would be interesting to see if one can control the dimension of uncountably categorical models. In particular, we ask the following question:

QUESTION 18. Does there exist an uncountably categorical nonautostable model of finite dimension?

Our comment about this question is as follows. As we noted in the previous section, Goncharov constructed nonautostable algebraic systems of finite computable dimension. In [**9**] [**10**] [**12**] [**24**] [**8**] [**20**] nonautostable algebraic systems of finite computable dimension have also been constructed to answer some open problems in the theory of constructive models. None of these algebraic systems are prime models of their own theories. In other words, the type structure of the theories of these algebraic systems is quite complicated. Therefore it is natural to ask whether it is possible to construct algebraic systems of finite computable dimension greater than 1 whose theories belong to a class of well-understood theories, e.g. uncountably categorical theories. We do not know the answer to this question. We state, however, that Hirschfeldt and Khoussainov in [**21**] noted that the construction in [**24**] can be modified to build a noncomputably categorical prime model of finite computable computable dimension:

THEOREM 4.13. *For every natural number $n > 1$ there exists an algebraic system $\mathcal{A}$ of computable dimension n such that $\mathcal{A}$ is the prime model of its own theory.*

We now make some comments about saturated models using terminology from dimension theory developed for uncountably categorical theories. Roughly speaking, the saturated model for a given uncountably categorical theory T is the most complicated one because the model has infinite dimension while all other models have finite dimension. Moreover, all models of T are elementarily embedded into the saturated model. In this sense one may suggest that the saturated model can also be complex from the computability-theoretic point of view. One way to show this would be to prove that the computable dimension of the saturated model is always infinite. All the known examples of uncountably categorical saturated models have, in fact, infinite computable dimension. Hence we naturally ask the following question:

QUESTION 19. Does there exist an uncountably categorical theory whose saturated model is autostable?

4.4. The Degree Spectra of Relations. Another central topic in the theory of constructive algebraic systems concerns the dependence of computability-theoretic properties of relations on constructivizations. The topic is closely related to autostability because of the following simple fact. Assume that R is an invariant relation on $\mathcal{A}$ (that is R is closed under automorphisms of $\mathcal{A}$) such that for two constructivizations ν and μ of $\mathcal{A}$ the sets $\nu^{-1}(R)$ and $\mu^{-1}(R)$ have different degrees. Then $\mathcal{A}$ is not autostable. Consider for example the linearly ordered set $(\omega, \leq)$ whose constructivization is the identity mapping. In this constructivization the successor function is computable. On the other hand, $(\omega, \leq)$ has a constructivization under which the successor function is not computable. In [**1**] Ash and Nerode singled out those relations whose computability is invariant with respect to all constructivization. Here is a definition.

DEFINITION 4.14. A relation R on a model $\mathcal{A}$ is **intrinsically computable (intrinsically c. e.)** if for all constructivizations ν of $\mathcal{A}$ the set $\nu^{-1}(R)$ is computable (c.e.).

Thus, for example in $(\omega, \leq)$ the successor function is not intrinsically computable. On the other hand, any computable, invariant relation in any autostable algebraic system is intrinsically computable. Thus, for autostable algebraic systems computability or computable enumerability of the invariant relations is independent on constructivizations.

One of the programs in the theory of constructive algebraic systems is to study the intrinsically computable (c.e.) relations. It turns out that for a large class of algebraic systems which have certain decidability properties one can characterize intrinsically c.e. relations. We define the following notion given by Ash and Nerode in [**1**].

DEFINITION 4.15. An n-ary relation R on a model $\mathcal{A}$ is **formally c.e.** if R is equivalent to a disjunction $\bigvee_i \phi_i(x_1, \dots, x_n, \bar{a})$ of a computable sequence of existential formulas ϕ_i with free variables $x_1, \dots, x_n$, where $\bar{a}$ is a finite sequence of elements from $\mathcal{A}$.

If R is formally c.e. then R is intrinsically c.e. To state the next theorem proved by Ash and Nerode we need to introduce the following notion. We say that $(\mathcal{A}, \nu)$ is 1**-decidable** if the set of all existential formulas true in the expansion $(\mathcal{A}, \nu(0), \nu(1), \dots)$ is computable. Here is the theorem:

THEOREM 4.16. [**1**] *Let $(\mathcal{A}, \nu)$ be a* 1*-decidable algebraic system. Then for any R, the relation R is intrinsically c.e. if and only if R is formally c.e.*

Two natural problems are suggested by this theorem. One is to investigate c.e. intrinsic relations in constructive algebraic systems which are not 1-decidable. The other problem suggests to study those relations R which are not intrinsically c.e. In particular, one can be interested in computability-theoretic complexity of R under different constructivizations. An approach to these problems is suggested by the following definition.

DEFINITION 4.17. For a relation R on an algebraic system $\mathcal{A}$, the **degree spectrum of** R, $DgSp(R)$, is the set of all Turing degrees of $\nu^{-1}(R)$ under all constructivizations ν of $\mathcal{A}$.

There are a number of results that give conditions under which $DgSp(R)$ coincides with a given set of Turing degrees, e.g. the set of all c.e. degrees or of all degrees. Here we concentrate on the issue of finding conditions under which $DgSp(R)$ is finite. This search is basically motivated by our interest in whether we are able to control computable dimensions and the degree spectra of relations in building constructivizations of algebraic systems. Recasting the theorem of Goncharov in [**9**], Harizanov in [**16**] provided an example of a relation R in a system of computable dimension 2 such that $DgSp(R) = \{0, c\}$, where c is noncomputable and Δ_2^0. Later, the authors showed that there exists a relation R in a system of computable dimension 2 such that $DgSp(R) = \{0, c\}$, where c is the degree of a c.e. set [**12**]. This result was independently generalized by Khoussainov and Shore in [**24**]. In particular, Khoussainov and Shore proved that for any finite partially ordered set P there exists an intrinsically c.e. relation R in a system of computable dimension $|P|$ such that $DgSp(R)$ is isomorphic to P, where the ordering of $DgSp(R)$ is given by Turing reducibility. However, the following question has remained opened: Which finite sets $\{a_1, \dots, a_n\}$ of computably enumerable degrees coincide with $DgSp(R)$, where R is a relation in an algebraic system of computable dimension n? Recently Khoussainov and Shore, and independently Hirschfeldt by different methods in [**20**], have been able to provide the following answer to this question.

THEOREM 4.18. *For any finite set $\{a_1, \dots, a_n\}$ of computably enumerable degrees there exists an algebraic system $\mathcal{B}$ of computable dimension n and an intrinsically c.e. relation R in it such that $DgSp(R) = \{a_1, \dots, a_n\}$.*

In light of these results the following question remains open:

QUESTION 20. Let $\{a_1, \dots, a_n\}$ be a finite set of Turing degrees of Σ_n-sets. Does there exists a relation R in an algebraic system of computable dimension n such that $DgSp(R) = \{a_1, \dots, a_n\}$?

A weaker version of this question not asking to control the dimension of the system is the following.

QUESTION 21. For a given finite collection $\{a_1, \dots, a_n\}$ of Turing degrees of Σ_n-sets, does there exist a relation R in an algebraic system such that $DgSp(R) = \{a_1, \dots, a_n\}$?

We end this section with the following question related to computable dimension and the two questions asked above. It is known that all constructed nonautostable models of finite algorithmic dimensions are Δ_3^0-autostable. Hence the following question, posed by Khoussainov and Shore, arises:

QUESTION 22. Is it true that for any $n \geq 3$ there exists a non Δ_n^0-autostable but Δ_{n+1}^0-autostable model of dimension 2?

5. Conclusion

In this paper we concentrated on open problems in two directions in the development of the theory of constructive algebraic systems. The first direction deals with positive algebras. In our discussion of positive algebras an emphasis was on showing the interplay between universal algebra and computability theory. We think that systematic development of the theory of positive algebras can bring fruitful results and deeper understanding of interactions between fundamental concepts of universal algebra and computability theory. Hence we proposed a systematic study of positive algebras as a new direction in the development of constructive algebraic systems. The second direction concerns the traditional topics in constructive model theory. First we proposed the study of constructive models of theories with few models such as countably categorical theories, uncountably categorical theories, and Ehrenfeucht theories. Next, we proposed the study of computable isomorphisms and computable dimensions of such models. We also discussed issues related to the computability-theoretic complexity of relations in constructive algebraic systems. We stress that we have not discussed many other equally important topics and open problems in other parts of this area. For a comprehensive survey of results and directions in this area we refer the reader to the papers in the Handbook of Recursive Mathematics [**6**].

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