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Small Cones of Oriented Semi-Metrics

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Abstract

We consider polyhedral cones, associated with quasi-semi-metrics (oriented distances), in particular, with oriented multi-cuts, on n points. We computed the number of facets and of extreme rays, their adjacencies, and incidences of the cones $QMET_n$ and $OMCUT_n$ for $n = 3, 4, 5$ (see Table 1) and, partially, for $n = 6$. Some results for general n are also given.

1 Introduction and basic notions

The notions of directed distances, quasi-metrics and oriented multi-cuts are generalizations of the notions of distances, metrics and cuts. The notions of distances and metrics are central objects in Graph Theory and Combinatorial Optimization. Quasi-metrics are used in the Semantics of Computation (see, for example, [Se97]) and in Computational Geometry (see, for example, [ACLM98]). Oriented distances have been used already in [Ha14], pages 145–146. Any quasi-semi-metric on n points embeds isometrically into $\mathbb{R}^n$ with the directed norm $||.||^{or}_{\infty}$ (see proposition 1).

The knowledge of extreme rays of $QMET_n$ (cone of all quasi-semi-metrics on n points) for small n will allow to build a theory of multi-commodity flows on oriented graphs, as well as it was done for non-oriented graphs using dual MET_n (cone of all semi-metrics on n points). For general theory of quasi-metrics see, for example, [Fr06], [Ha14], [Wi31]; for general theory of metrics see [Bl53], [DeLa97].

Set $V_n = \{1, \dots, n\}$. Define a *quasi-semi-metric* on V_n as a function $d : V_n^2 \to R$, with $d_{ii} = 0$, satisfying

- the *oriented triangle inequality* $OT_{ij,k} := d_{ik} + d_{kj} - d_{ij} \geq 0$;
- the *non-negativity inequality* $NN_{ij} := d_{ij} \geq 0$.

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If all non-negativity inequalities are strict, then d is called a *quasi-metric*. If for all $i,j \in V_n^2$, $d_{ij} = d_{ji}$, then d is called a *semi-metric*.

The convex cone $QMET_n$ is defined by $n(n-1)(n-2)$ oriented triangle inequalities $OT_{ij,k}$ and $n(n-1)$ non-negativity inequalities NN_{ij} on n points; it is a full-dimensional cone in $R^{n(n-1)}$.

Note, that in the semi-metric case, the triangle inequalities imply non-negativity of the distance; it is not the case in our oriented case.

Through all paper we will represent vectors in $R^{n(n-1)}$ as square matrices of order n with zeroes on the main diagonal. It is easy to see that

$$MET_n = \{a + a^T, \text{ with } a \in QMET_n\}$$

where a^T denote the transpose matrix. Moreover, any extreme ray e of MET_n has a form $e = g + g^T$, where g is an extreme ray of $QMET_n$ (in fact, e is a sum of extreme rays of $QMET_n$ with non-negative coefficients, and $e = (e + e^T)/2$).

Consider now the notions of oriented cut and oriented multi-cut quasi-semi-metrics. Given an ordered partition $(S_1, \dots, S_q)$ $(q \geq 2)$ of V_n, the quasi-semi-metric $\delta'(S_1, \dots, S_q)$ is called an *oriented multi-cut*, if $\delta'(S_1, \dots, S_q)_{ij} = 1$ for $i \in S_\alpha$, $j \in S_\beta$, $\alpha < \beta$ and $\delta'(S_1, \dots, S_q)_{ij} = 0$, otherwise. This notion was considered, for example, in [ShLi95]. Given a subset S of V_n, the quasi-semi-metric $\delta'(S)$ is called *oriented cut*, if $\delta'(S)_{ij} = 1$ for $i \in S, j \notin S$ and $\delta(S)_{ij} = 0$, otherwise. Clearly, an oriented cut is the case $q = 2$ of an oriented multi-cut. The full-dimensional cone in $R^{n(n-1)}$, generated by all non-zero oriented multi-cuts on V_n, is denoted by $OMCUT_n$.

Given a partition $(S_1, \dots, S_q)$ of V_n, the semi-metric $\delta(S_1, \dots, S_q)$ is called a *multi-cut*, if $\delta(S_1, \dots, S_q)_{ij} = 1$ for $i \in S_\alpha$, $j \in S_\beta$, $\alpha \neq \beta$ and $\delta(S_1, \dots, S_q)_{ij} = 0$, otherwise. Given a subset S of V_n, the semi-metric $\delta(S)$ is called a *cut*, if $\delta(S)_{ij} = 1$ if $|S \cap \{i,j\}| = 1$ and $\delta(S)_{ij} = 1 = 0$, otherwise. Clearly, cuts are multi-cuts with $q = 2$. The $n(n-1)/2$-dimensional cone in $R^{n(n-1)}$, generated by all non-zero cuts on V_n, is denoted by CUT_n.

Note, that in the semi-metric case, the multi-cuts with $q > 2$ are not extreme rays (they are interior points) of CUT_n; it relies on the formula:

$$\delta(S_1, \dots, S_q) = (\sum_{i=1}^{q} \delta(S_i))/2$$

On the other hand,

$$\delta(S_1, \dots, S_q) = \delta'(S_1, \dots, S_q) + \delta'(S_q, \dots, S_1)$$

and so,

$$CUT_n = \{a + a^T, \text{ with } a \in OMCUT_n\}\ .$$

The number of all oriented cuts on n points is 2^n and the number $p'(n)$ of all oriented multi-cuts on V_n is the number of all ordered partitions of n. In fact, $p'(n) = \frac{1}{2}\sum_{\pi \in Sym(n)} x^{1+d(\pi)}$ with $d(\pi) := |\{i \leq n | a_i > a_{i+1}\}|$ for the permutation $\pi = (a_1, \dots, a_n) \in Sym(n)$. So, $p'(3) = 13$, $p'(4) = 75$, $p'(5) = 541$, $p'(6) = 4683$, $p'(7) = 47293$, $p'(8) = 545835$. The number of extreme rays of $OMCUT_n$ is $p'(n) - 1$, since we excluded 0; the number of orbits of those rays is $2, 5, 9, 19, 35, 71$ for $n = 3, 4, 5, 6, 7, 8$.

Let C be a cone in $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$, the inequality $\sum_{i=1}^{n} v_i x_i \geq 0$ is said to be *valid* for C, if it holds for all $x \in C$. Then the set $\{x \in C | \sum_{i=1}^{n} v_i x_i = 0\}$ is called the *face* of C, induced by the valid inequality $\sum_{i=1}^{n} v_i x_i \geq 0$. A face of dimension $\dim(C) - 1$ is called a *facet* of C; a face of dimension 1 is called an *extreme ray* of C.

Two extreme rays of C are said to be *adjacent* on C, if they span a two-dimensional face of C. Two facets of C are said to be *adjacent*, if their intersection has dimension $\dim(C) - 2$ (or codimension 2).

The incidence number of a facet (or of an extreme ray) is the number of extreme rays lying on this facet (or, respectively, of facets containing this extreme ray).

The *skeleton* graph of the cone C is the graph G_C whose node-set is the set of extreme rays of C and with an edge between two nodes if they are adjacent on C. The *ridge* graph of C is the graph G^*_C whose node-set is the set of facets of C and with an edge between two nodes if they are adjacent on C. So, the ridge graph of a cone is the skeleton of its dual.

The cones $QMET_n$ and $OMCUT_n$ have the group $Sym(n)$ of all permutations as a symmetry group. But another symmetry, called *reversal*, exists: associate to each ray d the ray d^r defined by $d^r_{ij} = d_{ji}$, i.e., in matrix terms, the reversal corresponds to the transposition of matrices. This yields that the group $Z_2 \times Sym(n)$ is a symmetry group of the cones $QMET_n$ and $OMCUT_n$. We conjecture that this group is the full symmetry group of those cones; it has been checked by computer for $n = 3, 4, 5$. All orbits of faces considered below, are under action of this group.

The *representation matrix* of skeleton (or ridge) graph is the square matrix where on the place i, j we put the number of members of orbit O_j of extreme rays (or facets, respectively), which are adjacent to a fixed representative of orbit O_i

Comparing with the cones of semi-metrics, the amount of computation and memory is much bigger in the oriented case, because the dimension of the cones $OMCUT_n$ and $QMET_n$ is twice those of CUT_n and MET_n, and because oriented multi-cuts with $q > 2$ are not interior points of the cone generated by oriented cuts. So-called *combinatorial explosion* starts from $n = 5$ (see Table 1) while for corresponding semi-metric cones it starts from $n = 7$. All computations were done using the programs *cdd* (see [Fu95]) and an adaptation, by the second author, of adjacency decomposition method from [ChRe96].

We consider the cones $OMCUT_n$ and $QMET_n$ for $n = 3, 4, 5$; we give the number of facets and of extreme rays for them, their orbits and Tables of their adjacencies and incidences. We study (having the semi-metric case in mind, see [DeDe94], [DeDe95], [DDFu96], and [DeLa97]) the skeleton graphs and the ridge graphs of these cones, i.e. their diameters, adjacency conditions.

In Table 1 there is a synthetic summary of the most important informations concerning the considered cones; the diameters there are those of the skeleton and ridge graph, respectively.

In Appendix 1 we represent all 229 orbits of extreme rays for $QMET_5$ with adjacency, incidence of their representatives and their size. In Appendix 2 we give the same information for facets of $OMCUT_5$. For representation matrices detailing orbit-wise adjacencies and additional information on those cones see *http://www.geomappl.ens.fr/~dutour/*.

This paper is a follow-up to [DePa00], which initiates this subject.

2 General results about $QMET_n$

We got by computations the following new facts about small cones MET_n:

(i) the full symmetry group of MET_n is $Sym(n)$ for $n = 3, 5, ..., 14$, and for $n = 4$ it is $Sym(4) \times Sym(3)$,

(ii) the diameter of G_{MET_7} is three,

(iii) [DFPS] obtained a list of 1550825600 rays of MET_8; we computed that this list consists of 3918 orbits of extreme rays under the symmetry group $Sym(8)$ of this cone,

cone	dim.	# of ext. Rays (orbits)	# of facets (orbits)	diameters
CUT_3=MET_3	3	3(1)	3(1)	1; 1
CUT_4=MET_4	6	7(2)	12(1)	1; 2
CUT_5	10	15(2)	40(2)	1; 2
MET_5	10	25(3)	30(1)	2; 2
CUT_6	15	31(3)	210(4)	1; 3
MET_6	15	296(7)	60(1)	2; 2
CUT_7	21	63(3)	38780(36)	1; 3
MET_7	21	55226(46)	105(1)	3; 2
CUT_8	28	127(4)	$\geq$ 49604520($\geq$ 2169)	1; 3 or 4?
MET_8	28	$\geq$ 119269588, ($\geq$ 3918)	168(1)	3?; 2
$OMCUT_3$=$QMET_3$	6	12(2)	12(2)	2; 2
$OMCUT_4$	12	74(5)	72(4)	2; 2
$QMET_4$	12	164(10)	36(2)	3; 2
$OMCUT_5$	20	540(9)	35320(194)	2; 3
$QMET_5$	20	43590(229)	80(2)	3; 2
$QHYP_5$	20	78810(386)	90(3)	4; 2
$OMCUT_6$	30	4682(19)	$\geq$ 217847040($\geq$ 160728)	2; ?
$QMET_6$	30	$\geq$ 182403032($\geq$ 127779)	150(2)	?; 2

Table 1: Small cones

(iv) the cone MET_7 has 46 orbits of extreme rays and not 41 as, by a technical mistake, was given in [Gr92] and [DeLa97].

Proposition 1 *Any quasi-semi-metric d' on n points is embeddable in $l_\infty^{n,or}$.*

Proof. Let the orinted norm be: $||x-y||_\infty^{or} = max(x_k - y_k, 0)$.

Let $v_1, \dots, v_n$ in R^n defined as $v_i = (d'(i,1), d'(i,2), ..., d'(i,n))$. Then $||v_i - v_j||_\infty^{or} = max(d'(i,k) - d'(j,k), 0)$ (it is $\leq d'(i,j)$ from oriented triangle inequality) and $d'(i,j) - d'(j,j) = d'(i,j)$ so, $||v_i - v_j||_\infty^{or} = d'(i,j)$.

Theorem 1 *Every extreme ray of $QMET_n$ has at least $n-1$ coordinates equal to zero. This lower bound is met for $n = 4, 5, 6$.*

Proof. The rank of the system $(d_{ij} = d_{ik} + d_{kj})_{1 \leq i,j,k \leq n}$ is $(n-1)^2$ (see [DePa00]). Let d be an extreme ray of $QMET_n$ and let $NN = (NN_\alpha)_{\alpha \in A}$ be the set of all non-negativity facets, to which d is incident and $OT = (OT_\beta)_{\beta \in B}$ be the set of all oriented triangle facets, to which it is incident. So, the rank of $NN \cup OT$ is $n(n-1) - 1$.

If $Rank\, OT = (n-1)^2$, then the vector d is incident to all oriented triangle inequalities so, $d_{ij} + d_{ji} = 0$, and since d belongs to $QMET_n$, the equalities $d_{ij} = d_{ji} = 0$ hold. So, we have $Rank\, OT \leq (n-1)^2 - 1$ and

$$n(n-1) - 1 = Rank\,(NN \cup OT) \leq Rank\,(OT) + Rank\,(NN) \leq (n-1)^2 - 1 + |A|$$

implying $n - 1 \leq |A|$.

Two vectors are said to be *conflicting* if there exist a component on which they have non-zero values of different sign.

Theorem 2 *For the ridge graph $G^*_{QMET_n}$ holds:*

(i) A triangle facet is non-adjacent to any facet to which it conflicts;

(ii) the non-negativity facets NN_{ij} and $NN_{i'j'}$ are non-adjacent if, either $i' = j$, or $j' = i$.

Proof. (i) If d is a quasi-semi-metric vector, which is incident to both, $OT_{ij,k}$ and NN_{ij}, then we have $0 = d_{ik} + d_{kj}$. But since d is a quasi-semi-metric, we have $d_{ik} \geq 0$ and $d_{kj} \geq 0$; so, $d_{ki} = d_{kj} = 0$. The vector d must lie in a space of dimension $n(n-1) - 3$. So the facets $OT_{ij,k}$ and NN_{ij} are non-adjacent.
If d is a quasi-semi-metric vector incident to $OT_{ij,k}$ and $OT_{ik,l}$ then we will have

$$d_{ij} = d_{ik} + d_{kj} = d_{il} + d_{lk} + d_{kj} = d_{il} + (d_{lk} + d_{kj})$$

Since d is a quasi-semi-metric we have $d_{ij} \leq d_{il} + d_{lj}$ and $d_{lj} \leq d_{lk} + d_{kj}$. These inequalities must be equalities in order to meet above equality. So, the vector d belongs to a space of dimension $n(n-1) - 4$ and the facets are not adjacent.
(ii) is obvious, since a vector d incident to NN_{ij} and NN_{ki}, is also incident to NN_{kj} and give a lower than expected rank. Similarly, if d satisfy $d_{ij} = 0$ and $d_{ji} = 0$ then we have the equality $d_{ik} = d_{jk}$ for all k and we will again a too low rank again.

Conjecture 1 *(i) A triangle facet is adjacent to a facet if they are non-conflicting;*
(ii) the non-negativity facets NN_{ij} and $NN_{i'j'}$ are adjacent if, neither $i' = j$, nor $j' = i$;
(iii) the diameter of $G^*_{QMET_n}$ is 2.
(iv) The ridge graph $G^*_{QMET_n}$ is induced subgraph of $G^*_{OMCUT_n}$.

We checked by computer (i)-(iv) for $n \leq 7$. Easy to see that (iii) is implied by the criterion of adjacency given by (i) and (ii) together with Theorem 2. Also we checked that for $n \leq 7$ there are no symmetric extreme rays of $QMET_n$. Apropos, any two facets of $QMET_5$ will be not adjacent if, instead of all oriented multi-cuts, we restrict ourself only to oriented cuts.

Note, that the set $E_n = \{e + e^T | e \text{ is an extreme ray of } QMET_n\}$ consists, for $n = 3, 4, 5$ of $1, 7, 79$ orbits (amongst of $2, 10, 229$), including $0, 3, 10$ orbits of path-metrics of graphs. More exactly, a path-metric $d(G)$ belongs to E_4 for the graphs $G = K_4, P_2, C_4, P_4$.

Now, $d(G) \in E_5$ for $G = \{K_{2,3}, K_5 - K_3, K_5 - P_2 - P_3, K_5, C_5, \overline{P_2}, \overline{P_3}, \overline{P_4}, \overline{P_5}, \overline{2P_2}\}$, where $d(K_{2,3})$ is an extreme ray of MET_5; $d(K_{2,3})$, $d(K_5 - K_3)$ and $d(K_5 - P_2 - P_3)$ do not belong to CUT_5, and the remaining seven graphs belong to CUT_5. In fact, those seven path metrics $d(G)$ are all of form $e + e^T$, were e is an extreme ray of $QHYP_5$ (see definition of the cone $QHYP_5$ in the end of section five).

3 General results about $OMCUT_n$

Conjecture 2 *(i) All oriented multi-cuts are extreme rays of $QMET_n$;*
(ii) the oriented cuts form a dominating clique (i.e. a complete subgraph such that any node is adjacent to one of its elements) in G_{OMCUT_n}; so, the diameter of G_{OMCUT_n} is 2 or 3 for any $n \geq 4$.

We checked it by computer for $n = 7$, using only non-negativity facets and oriented triangle facets. The adjacency between oriented cuts and any other oriented multi-cut are preserved even if we consider only oriented triangle and non-ngativity facets, but already for $n = 4$ (when $OMCUT_n$ has other facets) the adjacency is dimished for any oriented multi-cut orbit different from oriented cut.

We have the inclusion $OMCUT_n \subseteq QMET_n$ with equality if and only if $n = 3$.
Let us consider the following inequalities

- the *zero-extension of an inequality* $\sum_{1 \leq i \neq j \leq n-1} f_{ij} d_{ij} \geq 0$, is an inequality

$$\sum_{1 \leq i \neq j \leq n} f'_{ij} d_{ij} \geq 0 \text{ with } f'_{ni} = f'_{in} = 0 \text{ and } f'_{ij} = f_{ij}, \text{ otherwise;}$$

- the inequality $A_n(c_1,\dots,c_{n-2};a,b) := \sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib})+d_{ba} \geq S_{ab}+S_{c_1\dots c_{n-2}}$, where $S_{c_1,\dots c_{n-2}}$ denotes the sum of distances along oriented cycle $c_1,\dots,c_{n-2}$;
- the inequality $B_n(c_1,\dots,c_{n-2};a,b) := \sum_{i=1}^{n-2}(d_{c_ia}+d_{ac_i}+d_{c_ib}) \geq d_{ab}+S_{c_1\dots c_{n-2}}$;
- the *hypermetric* inequality $H(b) := \sum_{1\leq i\neq j\leq n} b_ib_jd_{ij} \leq 0$, where $b=(b_1,\dots,b_n)\in\mathbb{Z}^n$, $\sum_{i=1}^n b_i = 1$.

Theorem 3 *The following inequalities are valid on (i.e. faces) $OMCUT_n$:*
(i) zero-extensions of valid faces of $OMCUT_{n-1}$;
(ii) symmetric faces coming from CUT_n (so including any inequality $H(b)$)
(iii) $A_n(c_1,\dots,c_n;a,b)$ and $B_n(c_1,\dots,c_n;a,b)$.

Proof (ii) Since we have $CUT_n = \{d+d^r$ s.t. $d\in OMCUT_n\}$, the validity of symmetric faces coming from CUT_n is obvious.

(iii) Consider the inequality A_n on an o-multi-cut $\delta'(S_1,\dots,S_t)$. In this case $S_{c_1\dots c_{n-2}}\leq n-3$ and $S_{c_1\dots c_{n-2}} = n-3$ if and only if $c_1\in S_{\alpha_1},\dots,c_{n-2}\in S_{\alpha_{n-2}}$, where $1\leq\alpha_1<\dots<\alpha_{n-2}\leq n-2$.

Let $b\in S_\alpha, a\in S_\beta$, where $\alpha<\beta$. Then $S_{ab}=d_{ba}=1$ and $S_{ab}+S_{c_1\dots c_{n-2}}\leq n-2$. But $\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib}+d_{ba}) \geq \sum_{i=1}^{n-2} d_{ba} = n-2$ and A_n holds.

Let now $a\in S_\alpha, b\in S_\beta$, where $\alpha<\beta$. Then $S_{ab}=d_{ab}=1$ and $S_{ab}+S_{c_1\dots c_{n-2}}\leq n-2$. But in this case $d_{ac_i}+d_{c_ib}\geq 1$, hence $\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib}+d_{ba}) = \sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib})\geq n-2$, and A_n holds.

If $a,b\in S_\alpha$, then $S_{ab}=0$ and $S_{ab}+S_{c_1\dots c_{n-2}}\leq n-3$. But in this case $d_{ac_i}+d_{c_ib}=1$ for $c_i\notin S_\alpha$ and $d_{ac_i}+d_{c_ib}=0$ for $c_i\in S_\alpha$, hence, $\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib}+d_{ba}) = \sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib})\geq S_{c_1\dots c_{n-2}}$ and A_n holds.

Consider the inequality B_n on an o-multi-cut $\delta'(S_1,\dots,S_t)$. In this case $S_{c_1\dots c_{n-2}}\leq n-3$ and $S_{c_1\dots c_{n-2}}\leq n-3$ if and only if $c_1\in S_{\alpha_1},\dots,c_{n-2}\in S_{\alpha_{n-2}}$, where $1\leq\alpha_1<\dots<\alpha_{n-2}\leq t$.

Let $b\in S_\alpha, a\in S_\beta$, where $\alpha<\beta$. Then $d_{ab}=0$ and $d_{ab}+S_{c_1\dots c_{n-2}}\leq n-3$. But in this case $d_{ac_i}+d_{c_ib}\geq 1$, hence $\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ia}+d_{c_ib})\geq\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib})\geq n-2$, and B_n holds.

Let $a\in S_\alpha, b\in S_\beta$, where $\alpha<\beta$. Then $d_{ab}=1$ and $d_{ab}+S_{c_1\dots c_{n-2}}\leq n-2$. In this case $d_{ac_i}+d_{c_ib}\geq 1$, hence $\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ia}+d_{c_ib})\geq\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib})\geq n-2$, and B_n holds.

If $a,b\in S_\alpha$, then $d_{ab}=0$ and $d_{ab}+S_{c_1\dots c_{n-2}}\leq n-3$. But in this case $d_{ac_i}+d_{c_ia}+d_{c_ib}\geq 1$ for $c_i\notin S_\alpha$ and $d_{ac_i}+d_{c_ia}+d_{c_ib}=0$ for $c_i\in S_\alpha$, hence, $\sum_{i=1}^{n-2}(d_{ac_i}+d_{c_ib}+d_{c_ib})\geq S_{c_1\dots c_{n-2}}$ and B_n holds.

Conjecture 3 *The following inequalities correspond to facets of $OMCUT_n$:*
(i) zero-extensions of any facet of $OMCUT_{n-1}$ (so, including any oriented triangle and non-negativity inequalities);
(ii) any hypermetric inequality $H(b)$, except of non-oriented triangle inequalities;
(iii) $A_n(c_1,\dots,c_{n-2};a,b)$ and $B_n(c_1,\dots,c_{n-2};a,b)$.

We checked this conjecture by computer for $n\leq 7$.

Any oriented triangle inequality is a zero extension of A_3 while any non-negativity inequality is a zero-extension of a degenerated B_3 with $b=c_1$.

Call a face of $OMCUT_n$ *symmetric* if, in matrix terms, it is preserved by the transposition. The first symmetric facet, $H(1,1,1,-1,-1)$, is the only symmetric facet of $OMCUT_5$.

Theorem 4 *(i) Any symmetric facet of the cone $OMCUT_n$ correspond to a facet of the cone CUT_n.*

	12	13	21	23	31	32
$\delta'(\{1\})$:	1	1	0	0	0	0
$\delta'(\{1\},\{2\},\{3\})$:	1	1	0	1	0	0
$OT_{12,3}$	-1	1	0	0	0	1
NN_{12}	1	0	0	0	0	0

Table 2: Representative of the orbits of extreme rays and facets in $OMCUT_3$

(ii) All orbits of symmetric facets of $OMCUT_n$, $n \le 6$ are represented by $H(b)$ with $b = (1,1,1,-1,-1)$, $(1,1,1,-1,-1,0)$, $(2,1,1,-1,-1,-1)$, and $(1,1,1,1,-1,-2)$.

(iii) All orbits of symmetric facets of $OMCUT_7$, are all 9 orbits of hypermetric facets of CUT_7 different from the orbit of triangle facets, and 18 out of 26 non-hypermetric ones (namely, all but O_6, O_{13}, O_{22}, O_{18}, O_{20}, O_{24}, O_{25}, and O_{26} in terms of [DD])

Proof: (ii) and (iii) were obtained by computer using (i). In order to prove (i), let us fix a symmetric facet F of $OMCUT_n$. We set $U_F = \{d \in OMCUT_n \text{ s.t. } F(d) = 0\}$. If F is a symmetric facet, then U_F is a set invariant by the reversal (transposition).
Denote $SU(X) := \{d + d^r \text{ s.t. } d \in X\}$ for any $X \subset OMCUT_n$. Then $SU(U_F)$ is a set of semi-metrics, which are incident to F. Moreover, since $CUT_n = SU(OMCUT_n)$, we have $SU(U_F) \subset OMCUT_n$ By hypothesis F is a facet, so U_F has dimension $n(n-1)-1$. The mapping $d \mapsto d + d^r$ decrease dimension by at most $n(n-1)/2$; so, we get that $SU(U_F)$ has dimension $n(n-1)/2 - 1$, i.e. F is a facet of CUT_n.

Given two oriented partitions, A and B, we will write $A < B$, if A is a proper refinement of B. We will write $Q \dot{<} B$ if, moreover, each part of A is a *proper* subset of a part of B.

Conjecture 4 *(i) An oriented multi-cut $\delta'(A)$ is not adjacent to all oriented multi-cuts $\delta'(B)$ such that $B < A^T$.*

(ii) The orbit of extreme rays represented by oriented cut $\delta'(\{1\},\{2,\dots,n\})$ is unique orbit, such that extreme rays in this orbit is not adjacent only to oriented multi-cuts described in (i) above; the total adjacency is $p'(n) - p'(n-1) - 1$ and it is the maximal total adjacency.

(iii) An extreme ray of the orbit, represented by oriented cut $\delta'(\{1,2\},\{3,\dots,n\})$, is not adjacent only to oriented multi-cuts $\delta'(B)$ such that either $B < (\{1,2\},\{3,\dots,n\})^T$, or B is any cyclic shift of C with $C \dot{<} (\{1,2\},\{3,\dots,n\})$.

4 The cases of 3, 4 points

We start with complete description of the cone $QMET_3 = OMCUT_3$.

There are 12 extreme rays in $OMCUT_3$: 12 non-zero oriented multi-cuts on V_3, including 6 oriented cuts. Under the group action we have, in fact, only 2 orbits to consider: the orbit O_1 of oriented cuts and the orbit O_2 of other oriented multi-cuts. The list of representatives of the orbits is given in the Table 2.

Note that all oriented cuts above can be obtained from $\delta'(\{1\})$ by a permutation ($\delta'(\{2\})$ and $\delta'(\{3\})$) or by a reversal and a permutation ($\delta'(\{1,2\})$, $\delta'(\{1,3\})$, and $\delta'(\{2,3\})$); all oriented multi-cuts with $q = 3$ above can be obtained from $\delta'(\{1\},\{2\},\{3\})$ by some permutation.

The only facet-defining inequalities of $OMCUT_3$ are 6 oriented triangle inequalities $OT_{ij,k}$ and 6 non-negativity inequalities NN_{ij} which form two orbits; see their representatives in Table 2.

Adjacencies of facets and extreme rays of $OMCUT_3$ are shown in Table 3. For each orbit a representative and a number of adjacent ones from other orbits are given, as well as the total number of adjacent ones, the number of incident extreme rays (respectively facets) and the size

Orbit	Representative	O_1	O_2	Adj.	Inc.	Orbit size	Orbit	Representative	OT	NN	Adj.	Inc.	Orbit size
O_1	$\delta'(\{3\})$	5	4	9	8	6	OT	$OT_{12,3}$	3	5	8	7	6
O_2	$\delta'(\{3\},\{2\},\{1\})$	4	2	6	6	6	NN	NN_{12}	5	2	7	6	7

Table 3: Representation matrix of G_{OMCUT_3} and $G^*_{OMCUT_3}$

Or.	Representative	12	13	14	21	23	24	31	32	34	41	42	43
O_1	$\delta'(\{4\})$	0	0	0	0	0	0	0	0	0	1	1	1
O_2	$\delta'(\{3,4\},\{1,2\})$	0	0	0	0	0	0	1	1	0	1	1	0
O_3	$\delta'(\{4\},\{3\},\{1,2\})$	0	0	0	0	0	0	1	1	0	1	1	1
O_4	$\delta'(\{4\},\{2,3\},\{1\})$	0	0	0	1	0	0	1	0	0	1	1	1
O_5	$\delta'(\{4\},\{3\},\{2\},\{1\})$	0	0	0	1	0	0	1	1	0	1	1	1
OT	$OT_{12,3}$	-1	1	0	0	0	0	0	1	0	0	0	0
NN	NN_{12}	1	0	0	0	0	0	0	0	0	0	0	0
A_4	$A_4(1,2;3,4)$	-1	0	1	-1	0	1	1	1	-1	0	0	1
B_4	$B_4(1,2;3,4)$	-1	1	1	-1	1	1	1	1	-1	0	0	0

Table 4: The representatives of orbits of extreme rays and facets in $OMCUT_4$

of orbits.

For the next case $n = 4$ we have

1. $OMCUT_4$ has 74 extreme rays (all non-zero oriented multi-cuts on V_4). Under the group action we have, in fact, only 5 orbits to consider. They are orbits with the representatives $\delta'(\{4\})$ (orbit O_1), $\delta'(\{4,3\})$ (orbit O_2) – the only two orbits of oriented cuts in $OMCUT_4$, $\delta'(\{4\},\{3\},\{2,1\})$ (orbit O_3), $\delta'(\{4\},\{3,2\},\{1\})$ (orbit O_4) and $\delta'(\{4\},\{3\},\{2\},\{1\})$ (orbit O_5). We show these representatives in the Table 4.

2. $OMCUT_4$ has 72 facets from 4 orbits, which are induced by 24 oriented triangle inequalities $OT_{ij,k}$, 12 non-negativity inequalities NN_{ij}, 12 inequalities $A_4(i_1,i_2;j_1,j_2)$, and 24 inequalities $B_4(i_1,i_2;j_1,j_2)$. See Table 4 for representatives of the orbits and Table 5 for the representation matrix.

3. The cone $QMET_4$ has 36 facets, distributed in two orbits: 24 oriented triangle facets (orbit OT) and 12 non-negativity facets (orbit NN).

4. There are 164 extreme rays in $QMET_4$. Under the group action we have 10 orbits to consider: orbits O_1 – O_5 with the same representatives as in $OMCUT_4$ and 5 other orbits. The list of representatives of the orbits is given in the Table 7.

Denote by $e(i,j)$ the $\{0,1\}$-vector in $\mathbb{R}^{12}$ with 1 only on the place ij. In Table 7 there is the list of 10 orbits of extreme rays and 2 orbits of facets of $QMET_4$.

The adjacencies and incidences of the facets and extreme rays of $QMET_4$ are given in Tables 8 – 9.

Note, that in $QMET_4$ the adjacencies of facets $OT_{ij,k}$ and NN_{ij} are the same as in $OMCUT_4$ (see Tables 5 and 8); hence, $G^*_{QMET_4}$ is an induced subgraph of $G^*_{OMCUT_4}$. But the adjacencies of oriented multi-cuts from orbits O_3, O_4 and O_5 are decreased in the cone $QMET_4$ (see Tables 6 and 9); hence, G_{OMCUT_4} is not an induced subgraph of G_{QMET_4}.

Orbit	Representative	OT	NN	A_4	B_4	Adj.	Inc.	Size
OT	$OT_{12,3}$	17	11	5	8	41	43	24
NN	NN_{12}	22	6	12	8	48	43	12
A_4	$A_4(1,2;3,4)$	10	12	0	2	24	28	12
B_4	$B_4(1,2;3,4)$	8	4	1	3	16	17	24

Table 5: Representation matrix of $G^*_{OMCUT_4}$

Orbit	Representative	O_1	O_2	O_3	O_4	O_5	Adj.	Inc.	$\|O_i\|$
O_1	$\delta'(\{4\})$	7	6	21	9	18	61	42	8
O_2	$\delta'(\{4,3\})$	8	5	20	12	8	53	48	6
O_3	$\delta'(\{4\},\{3\},\{2,1\})$	7	5	15	7	10	44	29	24
O_4	$\delta'(\{4\},\{3,2\},\{4\})$	6	6	14	6	8	40	33	12
O_5	$\delta'(\{4\},\{3\},\{2\},\{1\})$	6	2	10	4	12	34	24	24

Table 6: Representation matrix of G_{OMCUT_4}

Or.	Representative	12	13	14	21	23	24	31	32	34	41	42	43
O_1	$\delta'(\{1\})$	0	0	0	0	0	0	0	0	0	1	1	1
O_2	$\delta'(\{1,2\})$	0	0	0	0	0	0	1	1	0	1	1	0
O_3	$\delta'(\{1\},\{2\},\{3,4\})$	0	0	0	0	0	0	1	1	0	1	1	1
O_4	$\delta'(\{1\},\{2,3\},\{4\})$	0	0	0	1	0	0	1	0	0	1	1	1
O_5	$\delta'(\{1\},\{2\},\{3\},\{4\})$	0	0	0	1	0	0	1	1	0	1	1	1
O_6	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(1,4)$	0	0	0	1	0	0	1	1	0	2	1	1
O_7	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(4,3)$	0	0	0	1	0	0	1	1	1	1	1	1
O_8	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(3,2)$	0	0	0	1	0	1	1	1	1	1	1	0
O_9	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(2,1)+e(4,3)$	0	0	1	1	1	1	1	1	1	1	0	0
O_{10}	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(1,4)+e(2,1)+e(4,3)$	0	0	1	1	1	1	1	1	2	1	0	0
OT	$OT_{12,3}$	-1	1	0	0	0	0	0	1	0	0	0	0
NN	NN_{12}	1	0	0	0	0	0	0	0	0	0	0	0

Table 7: Representatives of orbits of extreme rays and facets in $QMET_4$

Orbit	Representative	OT	NN	Adj.	Inc.	Size
OT	$OT_{12,3}$	17	11	28	78	24
NN	NN_{12}	22	6	28	80	12

Table 8: The adjacencies of facets in $QMET_4$

Or.	Representative	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	Adj.	Inc.	$\|O_i\|$
O_1	$\delta'(\{1\})$	7	6	21	9	18	6	9	6	3	6	91	27	8
O_2	$\delta'(\{1,2\})$	8	5	20	12	8	12	16	4	4	8	97	24	6
O_3	$\delta'(\{1\},\{2\},\{3,4\})$	7	5	7	5	10	4	4	2	0	2	46	21	24
O_4	$\delta'(\{1\},\{2,3\},\{4\})$	6	6	10	2	8	4	4	0	2	4	46	21	12
O_5	$\delta'(\{1\},\{2\},\{3\},\{4\})$	6	2	10	4	3	4	2	1	0	1	33	18	24
O_6	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(1,4)$	2	3	4	2	4	0	2	1	0	0	18	16	24
O_7	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(4,3)$	3	4	4	2	2	2	0	1	1	2	21	15	24
O_8	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(3,2)$	4	2	4	0	2	2	2	0	0	0	16	15	12
O_9	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(2,1)+e(4,3)$	4	4	0	4	0	0	4	0	0	4	20	12	6
O_{10}	$\delta'(\{1\},\{2\},\{3\},\{4\})+e(1,4)+e(2,1)+e(4,3)$	2	2	2	2	1	0	2	0	1	0	12	12	24

Table 9: The adjacencies of extreme rays in $QMET_4$

	12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54	Orbit	OT	NN	Adj.	Inc.	$\|F_i\|$
$OT_{1,2;3}$	-1	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	OT	19	49	68	13590	60
NN_{12}	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	NN	12	57	69	14359	20

Table 10: The list of the orbits of facets in $QMET_5$ and their representation matrix

	12	13	14	15	21	23	24	25	31	32	34	35	41	42	43	45	51	52	53	54
$\delta'(\{1\})$	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\delta'(\{1,5\})$	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
$\delta'(\{1\},\{2\},\{3,4,5\})$	1	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$\delta'(\{1,2\},\{3\},\{4,5\})$	0	1	1	1	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0
$\delta'(\{1\},\{2,3,4\},\{5\})$	1	1	1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0
$\delta'(\{1\},\{2,3\},\{4,5\})$	1	1	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0
$\delta'(\{1\},\{2\},\{3\},\{4,5\})$	1	1	1	1	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0
$\delta'(\{1\},\{2\},\{3,4\},\{5\})$	1	1	1	1	0	1	1	1	0	0	0	1	0	0	0	1	0	0	0	0
$\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})$	1	1	1	1	0	1	1	1	0	0	1	1	0	0	0	1	0	0	0	0

Table 11: The representative of orbits of extreme rays in $OMCUT_5$

5 The case of 5 points

We present here the complete description of $QMET_5$ and of $OMCUT_5$.

The quasi-semi-metric cone $QMET_5$ has 80 facets, distributed in two orbits: 60 triangle facets (orbit F_1) and 20 non-negativity facets (orbit F_2). The list of representatives of the orbits is given in the Table 10.

There are 43590 extreme rays in $QMET_5$. Under the group action we have 229 orbits to consider (see Appendix 1 for the full list of representatives) from them nine orbits of oriented multi-cuts on V_5 (see 11)

Note, that, for example, the representative of orbit O_{18} is similar to the representative $\delta'(\{1\},\{2\},\{3\},\{4\})+e(1,4)$ of the orbit O_6 in the cone $QMET_4$: it is $\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,5)$, and the representative of orbit O_{11} is similar to the representative $\delta'(\{1\},\{2\},\{3\},\{4\})+e(4,3)$ of the orbit O_7 in the cone $QMET_4$: it is $\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(5,4)$.

Some other representatives from non-oriented multi-cut's orbits of the cone $QMET_5$ have similar form. So, if v_i is the representative of orbit O_i, then

$v_{11}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(5,4)$.

$v_{18}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,5)$,

$v_{25}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(5,3)+e(5,4)$,

$v_{26}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,5)+e(5,4)$,

$v_{32}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,5)+e(2,5)$,

$v_{50}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,4)+e(1,5)+e(5,4)$,

$v_{52}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(5,3)+e(5,4)+e(4,3)$,

$v_{53}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,4)+e(5,3)+e(5,4)$,

$v_{84}=\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})+e(1,4)+e(1,5)+e(2,5)$,

The adjacencies of the facets of $QMET_5$ are given in Table 10.

Oriented cuts (orbits O_1 and O_2 together) form a clique, but, distinctly from cases $n=3,4$, not a dominating clique. For example, a representative from orbit O_{108} (and the same from O_{155}, O_{157}, O_{172}, O_{185}, O_{186}, O_{207}, $O_{216}-O_{227}$, O_{229}) is not adjacent with any oriented cut.

The cone $OMCUT_5$ has 540 extreme rays (all non-zero oriented multi-cuts on V_5), which form 9 orbits. See Table 11 for representatives of the orbits and Table 12 for the representation matrix.

There are 35320 facets in $OMCUT_5$, which form 194 orbits. The list of the representatives of these orbits is given in Appendix 2.

Besides A_i, B_i, and their zero-extensions, the simplest, i.e. with ≤ 8 non-zeros facets of

Or.	Representative	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	Adj.	Inc.	Size
O_1	$\delta'(\{1\})$	9	20	36	30	16	54	108	96	96	465	8840	10
O_2	$\delta'(\{1,5\})$	10	19	38	27	20	57	90	96	60	417	10562	20
O_3	$\delta'(\{1\},\{2\},\{3,4,5\})$	9	19	34	24	16	42	84	72	66	366	3106	40
O_4	$\delta'(\{1,2\},\{3\},\{4,5\})$	10	18	32	24	16	54	72	76	40	342	3172	30
O_5	$\delta'(\{1\},\{2,3,4\},\{5\})$	8	20	32	24	12	42	72	72	54	336	4372	20
O_6	$\delta'(\{1\},\{2,3\},\{4,5\})$	9	19	28	27	14	51	64	66	36	314	3576	60
O_7	$\delta'(\{1\},\{2\},\{3\},\{4,5\})$	9	15	28	18	12	32	57	55	36	262	1598	120
O_8	$\delta'(\{1\},\{2\},\{3,4\},\{5\})$	8	16	24	19	12	33	55	49	36	252	1930	120
O_9	$\delta'(\{1\},\{2\},\{3\},\{4\},\{5\})$	8	10	22	10	9	18	36	36	38	187	1123	120

Table 12: Representation matrix of G_{OMCUT_5}

$OMCUT_5$ are

$$(d_{15} - d_{25}) + d_{54} - d_{14} + d_{34} + d_{31} + d_{23} + d_{12} \geq 0 \text{ and}$$
$$(-d_{51} + d_{52}) + d_{54} - d_{14} + d_{34} + d_{31} + d_{23} + d_{12} \geq 0 \ .$$

Denote by $QHYP_n$ the cone of all quasi-semi-metrics satisfying to all hypermetric inequalities $H(b)$. In fact, $QHYP_n$ is polyhedral (see [DGL93]) and the triangle inequality is redundant. The smallest case when $QHYP_n$ is a proper sub-cone of $QMET_n$ is $n = 5$; see some information on $QHYP_5$ in Table 1 and *http://www.geomappl.ens.fr/~dutour/*

6 Appendix 1: Extreme rays of the cone $QMET_5$

By direct computation 43590 extreme rays of $QMET_5$ were found. Under group action we got 229 orbits.

See below the full list of the representatives of orbits presented as 5×5 matrices. The numbers in parenthesis are, respectively, the orbit number, the adjacency of a member of the orbit, the incidence of a member of the orbit, and orbit-size. Note that adjacency is greater or equal to incidence number with equality only for the last 23 orbits.

(1, 8313, 64, 10)	(2, 6903, 56, 20)	(3, 2534, 48, 30)	(4, 2394, 48, 60)	(5, 1947, 52, 40)	(6, 1932, 52, 20)	(7, 1109, 44, 120)	(8, 1107, 44, 120)	(9, 701, 40, 120)
0 0 0 0 1	0 0 0 1 1	0 0 1 1 1	0 0 1 1 1	0 0 0 1 1	0 1 1 1 1	0 0 1 1 1	0 1 1 1 1	0 1 1 1 1
0 0 0 0 1	0 0 0 1 1	0 0 1 1 1	0 0 1 1 1	0 0 0 1 1	0 0 0 0 1	0 0 1 1 1	0 0 0 1 1	0 0 1 1 1
0 0 0 0 1	0 0 0 1 1	0 0 0 1 1	0 0 0 0 1	0 0 0 1 1	0 0 0 0 1	0 0 0 1 1	0 0 0 1 1	0 0 0 1 1
0 0 0 0 1	0 0 0 0 0	0 0 0 0 0	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1
0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0

(10, 461, 38, 120)	(11, 444, 36, 120)	(12, 413, 40, 60)	(13, 404, 41, 120)	(14, 401, 40, 60)	(15, 398, 40, 120)	(16, 376, 32, 30)	(17, 355, 36, 120)	(18, 345, 37, 120)
0 1 1 1 1	0 1 1 1 1	0 1 1 1 1	0 1 1 1 2	0 0 1 1 1	0 0 1 1 2	0 1 1 1 1	0 1 1 1 1	0 1 1 1 2
0 0 1 1 1	0 0 1 1 1	0 0 0 1 1	0 0 0 1 1	0 0 1 1 1	0 0 1 1 2	0 0 1 1 0	0 0 1 1 1	0 0 1 1 1
0 0 0 0 1	0 0 0 1 1	0 0 0 1 1	0 0 0 1 1	0 0 0 1 1	0 0 0 1 1	0 0 0 1 0	0 0 0 1 1	0 0 0 1 1
0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 0 0 1	0 0 1 0 0	0 0 0 0 0	0 0 0 0 1
0 0 1 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 0 0	0 0 0 1 0	0 0 0 0 0	1 1 1 1 0	0 0 1 1 0	0 0 0 0 0

(19, 342, 40, 60)	(20, 316, 38, 60)	(21, 292, 34, 60)	(22, 287, 33, 120)	(23, 286, 32, 60)	(24, 248, 29, 120)	(25, 186, 33, 240)	(26, 175, 34, 240)	(27, 175, 30, 120)
0 0 1 1 1	0 1 1 1 1	0 0 1 1 1	0 1 1 2 1	0 1 1 1 1	0 1 1 1 1	0 1 1 1 1	0 1 1 1 2	0 1 1 2 1
0 0 1 1 1	0 0 0 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1
0 0 0 1 1	0 0 0 1 1	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 1 1	0 0 0 1 0
0 0 0 0 0	0 0 0 0 0	0 0 1 0 0	0 0 0 0 0	0 0 1 0 0	0 0 1 0 0	0 0 0 0 1	0 0 0 0 1	0 0 1 0 0
0 0 1 1 0	0 1 1 1 0	1 1 1 1 0	0 0 1 1 0	0 1 1 1 0	1 1 1 1 0	0 0 1 1 0	0 0 0 1 0	0 1 1 1 0

(28, 174, 31, 120)	(29, 172, 34, 120)	(30, 172, 34, 120)	(31, 160, 30, 120)	(32, 149, 35, 240)	(33, 143, 28, 240)	(34, 138, 29, 240)	(35, 138, 24, 20)	(36, 133, 26, 60)
0 1 1 2 1	0 0 1 1 1	0 0 1 2 1	0 1 1 2 1	0 1 1 1 2	0 1 1 2 1	0 1 1 1 1	0 1 1 1 1	0 1 2 2 1
0 0 1 2 1	0 0 1 1 1	0 0 1 2 1	0 0 1 1 1	0 0 1 1 2	0 0 1 1 1	0 0 1 1 1	0 0 1 1 0	0 0 1 1 1
0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 1 0 1 0	0 0 0 1 0
0 0 0 0 0	0 0 1 0 0	0 0 1 0 0	0 0 0 0 0	0 0 0 0 1	0 0 1 0 0	0 0 1 0 0	0 1 1 0 0	0 0 1 0 0
0 0 1 1 0	1 1 1 2 0	1 1 1 1 0	0 1 1 1 0	0 0 0 0 0	1 1 1 1 0	1 1 1 2 0	1 1 1 1 0	0 1 1 1 0

(37, 131, 25, 20)	(38, 128, 33, 120)	(39, 122, 26, 120)	(40, 116, 27, 120)	(41, 113, 24, 240)	(42, 113, 25, 120)	(43, 109, 27, 240)	(44, 107, 31, 120)	(45, 102, 25, 240)
0 1 2 1 1	0 1 1 1 1	0 1 2 1 1	0 1 1 1 1	0 1 2 2 1	0 1 2 2 1	0 1 1 2 1	0 1 1 1 1	0 1 2 2 2
0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 1 0	0 0 2 1 1
0 0 0 0 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0
0 1 1 0 1	0 0 0 0 0	0 1 1 0 0	0 1 1 0 0	0 1 1 0 0	0 0 1 0 0	0 1 1 0 0	0 0 1 0 0	0 1 1 0 0
0 1 1 1 0	0 1 1 1 0	1 1 1 1 0	1 1 1 1 0	1 1 1 1 0	1 1 1 1 0	1 1 1 1 0	1 1 1 2 0	1 1 1 1 0

(46, 98, 29, 120)	(47, 95, 31, 240)	(48, 91, 30, 240)	(49, 91, 24, 120)	(50, 91, 30, 120)	(51, 90, 26, 120)	(52, 88, 28, 40)	(53, 87, 31, 240)	(54, 84, 21, 40)
0 1 2 2 1	0 1 1 2 1	0 1 1 2 1	0 1 1 1 1	0 1 1 2 2	0 1 2 1 1	0 1 1 1 1	0 1 1 2 1	0 2 2 2 2
0 0 1 1 0	0 0 1 2 1	0 0 1 2 1	0 0 1 1 0	0 0 1 1 1	0 0 2 1 0	0 0 1 1 1	0 0 1 1 1	0 0 2 1 0
0 1 0 2 1	0 0 0 2 1	0 0 0 1 1	0 1 0 1 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 1	0 0 0 1 1	0 1 0 2 0
0 1 0 0 0	0 0 0 0 1	0 0 0 0 1	0 1 1 0 0	0 0 0 0 1	0 1 1 0 0	0 0 1 0 1	0 0 0 0 1	0 2 1 0 0
1 1 1 1 0	0 0 1 1 0	0 0 1 1 0	1 1 1 2 0	0 0 0 1 0	1 1 2 1 0	0 0 1 1 0	0 0 1 1 0	2 2 2 2 0

(55, 83, 32, 240)	(56, 83, 33, 240)	(57, 81, 31, 240)	(58, 75, 30, 240)	(59, 75, 27, 120)	(60, 72, 27, 120)	(61, 71, 25, 240)	(62, 70, 25, 240)	(63, 70, 22, 240)
0 1 1 2 2	0 1 1 2 2	0 1 1 2 1	0 1 1 2 1	0 1 1 2 1	0 1 2 2 2	0 1 2 2 1	0 1 2 2 1	0 1 2 2 2
0 0 1 1 2	0 0 1 1 2	0 0 1 2 1	0 0 1 1 1	0 0 1 1 1	0 0 1 2 1	0 0 2 1 1	0 0 1 1 1	0 0 2 1 1
0 0 0 1 2	0 0 0 1 2	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0
0 0 0 0 1	0 0 0 0 2	0 0 0 0 0	0 0 1 0 0	0 1 1 0 0	0 0 0 0 0	0 1 1 0 0	0 0 1 0 1	0 1 1 0 0
0 0 0 0 0	0 0 0 0 0	0 0 1 2 0	0 1 1 2 0	1 1 2 1 0	0 0 1 1 0	1 1 1 1 0	0 1 1 1 0	1 1 2 1 0

(64, 69, 26, 240)	(65, 68, 28, 240)	(66, 68, 27, 240)	(67, 65, 30, 240)	(68, 64, 26, 240)	(69, 64, 29, 240)	(70, 64, 26, 240)	(71, 64, 29, 240)	(72, 64, 29, 240)
0 1 2 2 2	0 1 1 1 1	0 1 2 1 1	0 1 1 2 1	0 1 2 2 1	0 1 1 1 1	0 1 2 2 2	0 1 1 2 1	0 1 1 2 1
0 0 1 2 1	0 0 1 1 1	0 0 2 1 1	0 0 1 2 1	0 0 1 1 1	0 0 1 1 1	0 0 1 2 1	0 0 1 1 1	0 0 1 1 1
0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1
0 0 0 0 0	0 0 1 0 1	0 1 1 0 0	0 0 1 0 1	0 0 1 0 0	0 0 1 0 0	0 0 1 0 0	0 0 1 0 1	0 0 0 0 1
0 1 1 1 0	1 1 1 1 0	1 1 1 1 0	0 1 1 1 0	0 1 1 2 0	1 1 1 2 0	0 1 1 1 0	0 1 1 1 0	0 1 1 1 0

(73, 64, 30, 240)	(74, 63, 24, 120)	(75, 63, 24, 120)	(76, 63, 28, 120)	(77, 63, 28, 120)	(78, 62, 28, 240)	(79, 62, 25, 240)	(80, 61, 28, 240)	(81, 60, 32, 240)
0 1 1 2 2	0 1 2 1 1	0 1 1 2 1	0 1 1 1 1	0 1 2 2 1	0 1 1 2 1	0 1 2 2 1	0 1 1 2 1	0 1 1 2 2
0 0 1 1 1	0 0 1 1 1	0 0 1 1 0	0 0 1 1 1	0 0 1 2 1	0 0 1 2 1	0 0 1 2 1	0 0 1 1 1	0 0 1 1 1
0 0 0 1 1	0 0 0 0 1	0 1 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 1
0 0 0 0 0	0 1 1 0 1	0 1 1 0 1	0 1 1 0 0	0 0 0 0 0	0 0 1 0 0	0 0 1 0 0	0 0 1 0 0	0 0 0 0 0
0 1 1 1 0	0 1 1 1 0	1 1 1 1 0	1 1 1 2 0	0 1 1 1 0	1 1 1 1 0	0 1 1 1 0	1 1 2 1 0	0 0 1 1 0

(82, 58, 30, 240)	(83, 58, 23, 240)	(84, 57, 33, 120)	(85, 54, 28, 240)	(86, 54, 25, 120)	(87, 54, 25, 120)	(88, 53, 24, 240)	(89, 53, 27, 240)	(90, 51, 25, 240)
0 1 1 1 1	0 1 1 2 1	0 1 1 2 2	0 1 2 2 2	0 1 2 1 2	0 1 2 2 1	0 1 2 2 1	0 1 2 1 1	0 1 2 2 2
0 0 1 1 1	0 0 1 1 0	0 0 1 1 2	0 0 1 2 1	0 0 1 1 1	0 0 1 1 1	0 0 1 2 1	0 0 2 1 0	0 0 1 2 1
0 0 0 1 0	0 1 0 1 1	0 0 0 1 1	0 0 0 1 1	0 0 0 0 0	0 0 0 0 0	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0
0 0 1 0 1	0 1 1 0 0	0 0 0 0 1	0 0 0 0 0	0 1 1 0 1	0 1 1 0 1	0 0 1 0 0	0 1 1 0 0	0 0 1 0 0
0 1 1 1 0	1 1 1 1 0	0 0 0 0 0	0 0 1 2 0	0 1 1 1 0	0 1 2 1 0	0 1 1 2 0	1 2 2 1 0	0 1 1 2 0

(91, 51, 27, 240)	(92, 51, 25, 240)	(93, 51, 28, 20)	(94, 48, 25, 80)	(95, 47, 26, 240)	(96, 47, 26, 240)	(97, 47, 26, 240)	(98, 47, 27, 240)	(99, 47, 29, 120)
0 1 2 2 2	0 1 2 1 1	0 1 1 1 1	0 2 2 2 2	0 1 2 2 1	0 1 1 2 1	0 1 2 2 2	0 1 2 3 2	0 1 2 2 3
0 0 1 2 2	0 0 2 1 0	0 0 1 1 1	0 0 2 2 2	0 0 1 1 1	0 0 1 1 1	0 0 1 2 1	0 0 2 2 1	0 0 1 2 2
0 0 0 2 1	0 0 0 1 0	0 0 0 0 0	0 0 0 2 1	0 0 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 1 2
0 0 0 0 0	0 1 2 0 0	0 1 1 0 1	0 0 1 0 2	0 0 1 0 0	0 0 1 0 0	0 1 1 0 0	0 0 0 0 0	0 0 0 0 1
0 1 1 1 0	1 2 2 1 0	0 1 1 1 0	0 0 2 1 0	0 1 1 2 0	1 1 1 2 0	0 1 2 1 0	0 1 1 1 0	0 0 0 0 0

(100, 47, 26, 120)	(101, 47, 23, 120)	(102, 47, 31, 120)	(103, 46, 25, 240)	(104, 45, 22, 240)	(105, 45, 22, 120)	(106, 44, 26, 240)	(107, 44, 20, 240)	(108, 44, 20, 120)
0 1 2 2 2	0 1 1 1 1	0 1 1 2 2	0 2 2 2 2	0 1 1 2 1	0 1 2 2 1	0 1 1 2 1	0 2 2 2 2	0 2 3 2 1
0 0 1 1 1	0 0 1 1 0	0 0 1 1 1	0 0 1 1 1	0 0 1 1 0	0 0 1 1 1	0 0 1 1 1	0 0 2 1 0	0 0 1 2 0
0 0 0 1 0	0 1 0 1 0	0 0 0 1 1	0 0 0 1 0	0 1 0 1 1	0 0 0 1 0	0 0 0 1 0	0 1 0 2 0	0 1 0 1 1
0 0 1 0 0	0 1 1 0 1	0 0 0 0 0	0 0 1 0 0	0 1 1 0 0	0 1 1 0 0	0 1 1 0 0	0 2 1 0 0	0 2 1 0 0
0 1 1 1 0	1 1 1 1 0	0 1 1 2 0	0 1 1 2 0	1 1 1 2 0	1 1 2 1 0	1 1 1 2 0	2 2 2 3 0	2 2 2 2 0

(109, 43, 25, 240)	(110, 43, 25, 240)	(111, 42, 26, 240)	(112, 42, 23, 120)	(113, 41, 27, 240)	(114, 40, 26, 240)	(115, 40, 26, 240)	(116, 40, 26, 240)	(117, 39, 23, 240)
0 1 2 2 1	0 1 2 2 1	0 2 2 2 1	0 1 2 1 1	0 1 1 2 1	0 1 3 2 2	0 1 2 2 1	0 1 2 1 1	0 1 2 2 2
0 0 1 1 0	0 0 1 1 1	0 0 1 2 1	0 0 1 1 1	0 0 1 2 1	0 0 2 1 1	0 0 1 2 1	0 0 2 1 0	0 0 1 2 1
0 1 0 2 1	0 0 0 1 0	0 0 0 2 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 2 1	0 0 0 1 0	0 0 0 1 0
0 1 0 0 0	0 0 1 0 1	0 0 1 0 0	0 1 1 0 1	0 1 1 0 0	0 1 1 0 0	0 0 1 0 0	0 1 1 0 0	0 1 1 0 0
1 1 1 2 0	1 1 1 1 0	1 1 1 3 0	1 1 1 1 0	1 1 2 1 0	1 1 1 1 0	0 1 1 1 0	1 1 3 2 0	1 1 2 1 0

(118, 39, 26, 240)	(119, 39, 23, 240)	(120, 39, 23, 240)	(121, 39, 23, 120)	(122, 38, 24, 240)	(123, 38, 27, 240)	(124, 38, 24, 240)	(125, 38, 27, 240)	(126, 38, 24, 240)
0 1 2 2 2	0 1 3 2 1	0 1 1 2 1	0 1 2 2 2	0 2 2 2 1	0 1 2 2 2	0 1 1 1 1	0 1 2 3 1	0 1 2 2 1
0 0 1 2 1	0 0 2 1 1	0 0 1 1 1	0 0 1 1 1	0 0 1 2 1	0 0 1 1 1	0 0 1 1 0	0 0 1 2 1	0 0 1 2 1
0 0 0 2 0	0 0 0 1 0	0 1 0 2 0	0 0 0 1 0	0 0 0 2 0	0 0 0 1 0	0 1 0 1 1	0 0 0 2 0	0 0 0 1 1
0 0 1 0 0	0 1 1 0 0	0 1 1 0 0	0 1 1 0 0	0 0 1 0 0	0 0 1 0 0	0 1 1 0 0	0 0 0 0 0	0 1 1 0 0
0 1 1 2 0	1 1 2 1 0	1 1 1 2 0	1 1 2 1 0	1 1 1 2 0	0 1 1 2 0	1 1 1 2 0	0 1 2 2 0	1 1 1 1 0

(127, 38, 25, 120)	(128, 37, 24, 240)	(129, 37, 26, 240)	(130, 37, 24, 240)	(131, 36, 25, 240)	(132, 36, 25, 240)	(133, 35, 25, 240)	(134, 35, 25, 240)	(135, 35, 27, 240)
0 1 2 2 2	0 1 2 1 2	0 1 2 2 1	0 1 2 1 1	0 1 1 2 1	0 1 2 1 1	0 2 2 2 2	0 1 2 2 2	0 1 1 2 1
0 0 1 2 1	0 0 2 1 1	0 0 1 2 1	0 0 2 1 0	0 0 1 1 0	0 0 1 1 1	0 0 1 2 2	0 0 1 1 2	0 0 1 2 1
0 0 0 1 1	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0	0 1 0 1 0	0 0 0 1 0	0 0 0 2 1	0 0 0 0 1	0 0 0 1 0
0 0 0 0 0	0 1 1 0 1	0 0 0 0 1	0 1 1 0 0	0 1 1 0 1	0 1 1 0 1	0 0 1 0 0	0 1 1 0 2	0 1 1 0 0
0 1 1 2 0	1 1 1 1 0	0 1 1 2 0	1 1 2 2 0	1 1 2 1 0	0 1 2 1 0	0 1 2 1 0	0 1 1 0 0	1 1 1 1 0

(136, 35, 26, 240)	(137, 34, 24, 240)	(138, 34, 27, 240)	(139, 34, 24, 120)	(140, 32, 26, 240)	(141, 32, 25, 240)	(142, 32, 23, 240)	(143, 32, 25, 240)	(144, 32, 26, 240)
0 1 2 3 3	0 1 1 2 1	0 1 2 3 2	0 1 1 2 1	0 1 2 2 2	0 1 2 2 1	0 1 2 2 1	0 1 2 3 1	0 2 2 3 2
0 0 2 2 2	0 0 1 1 0	0 0 1 2 1	0 0 1 1 0	0 0 1 2 2	0 0 1 2 0	0 0 1 1 1	0 0 1 2 1	0 0 1 1 1
0 0 0 2 1	0 1 0 2 1	0 0 0 1 0	0 1 0 1 0	0 0 0 1 1	0 1 0 2 1	0 0 0 1 0	0 0 0 1 1	0 0 0 1 0
0 0 0 0 0	0 1 1 0 0	0 0 1 0 0	0 1 1 0 0	0 0 0 0 0	0 1 0 0 0	0 1 1 0 1	0 0 1 0 0	0 0 1 0 0
0 1 1 1 0	1 1 1 1 0	0 1 1 1 0	1 1 2 1 0	0 1 1 2 0	1 1 1 3 0	0 1 2 1 0	0 1 1 2 0	0 1 2 1 0

(145, 32, 25, 240)	(146, 32, 26, 240)	(147, 32, 26, 120)	(148, 32, 26, 120)	(149, 31, 23, 240)	(150, 31, 23, 240)	(151, 31, 25, 240)	(152, 31, 25, 240)	(153, 31, 22, 240)
0 1 2 3 2	0 1 3 1 1	0 1 3 3 2	0 1 2 2 1	0 1 2 1 1	0 1 3 2 2	0 1 2 2 1	0 1 2 2 2	0 1 2 2 1
0 0 2 2 2	0 0 2 1 0	0 0 2 3 1	0 0 1 1 1	0 0 2 1 1	0 0 3 2 1	0 0 1 2 1	0 0 1 2 1	0 0 1 1 1
0 0 0 1 0	0 0 0 1 0	0 0 0 1 1	0 0 0 0 0	0 0 0 1 0	0 0 0 2 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0
0 1 1 0 0	0 1 2 0 0	0 0 0 0 0	0 1 1 0 1	0 1 1 0 0	0 1 1 0 0	0 0 1 0 0	0 0 1 0 0	0 1 1 0 1
0 1 2 1 0	1 2 2 1 0	0 1 1 2 0	0 1 2 2 0	1 1 2 1 0	2 2 2 2 0	1 1 1 1 0	1 1 1 1 0	1 1 1 1 0

(154, 31, 22, 120)	(155, 31, 26, 120)	(156, 30, 24, 240)	(157, 30, 24, 240)	(158, 30, 24, 120)	(159, 30, 24, 120)	(160, 29, 24, 240)	(161, 29, 23, 240)	(162, 29, 23, 240)
0 1 1 1 1	0 2 2 3 3	0 1 3 2 1	0 2 2 2 1	0 1 2 2 2	0 1 2 2 2	0 1 2 3 1	0 1 2 2 2	0 1 2 1 1
0 0 1 1 0	0 0 2 2 1	0 0 2 1 1	0 0 2 2 1	0 0 1 1 1	0 0 1 2 1	0 0 1 2 0	0 0 2 2 1	0 0 2 1 0
0 1 0 1 0	0 0 0 2 1	0 0 0 1 0	0 0 0 2 0	0 0 0 1 0	0 0 0 1 0	0 1 0 1 1	0 0 0 1 0	0 0 0 1 0
0 1 1 0 0	0 0 0 0 0	0 1 1 0 0	0 0 1 0 1	0 1 2 0 0	0 1 2 0 0	0 1 1 0 0	0 1 1 0 0	0 1 1 0 0
1 1 2 2 0	0 1 1 3 0	1 2 2 1 0	1 1 3 2 0	1 1 2 1 0	1 2 2 1 0	1 1 1 2 0	0 1 2 1 0	1 2 2 2 0

(163, 29, 23, 240)	(164, 29, 21, 240)	(165, 29, 23, 240)	(166, 29, 24, 240)	(167, 29, 24, 240)	(168, 29, 23, 240)	(169, 29, 21, 120)	(170, 29, 24, 120)	(171, 29, 24, 120)
0 1 2 2 1	0 1 2 2 1	0 1 2 2 1	0 1 2 1 1	0 1 2 2 1	0 1 1 2 1	0 2 2 2 1	0 1 2 1 1	0 1 2 3 2
0 0 1 1 1	0 0 2 1 1	0 0 1 2 0	0 0 1 1 1	0 0 1 1 0	0 0 1 1 1	0 0 2 1 1	0 0 1 1 1	0 0 2 2 1
0 0 0 1 0	0 0 0 1 0	0 1 0 2 1	0 0 0 1 1	0 1 0 2 1	0 1 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 1
0 0 1 0 0	0 1 1 0 0	0 1 0 0 0	0 1 1 0 0	0 1 1 0 0	1 1 1 0 0	0 1 1 0 0	0 1 1 0 1	0 0 0 0 0
1 1 1 2 0	1 1 2 1 0	1 1 1 2 0	1 1 2 1 0	1 1 1 1 0	1 1 2 1 0	1 1 2 1 0	1 1 1 2 0	0 1 1 2 0

(172, 29, 25, 40)	(173, 28, 24, 240)	(174, 28, 24, 240)	(175, 27, 23, 120)	(176, 27, 23, 120)	(177, 26, 23, 240)	(178, 26, 22, 240)	(179, 26, 23, 240)	(180, 26, 24, 240)
0 2 2 2 2	0 1 2 2 2	0 1 2 1 2	0 1 2 2 2	0 1 2 2 2	0 2 2 2 1	0 1 2 2 1	0 1 2 3 2	0 1 2 3 2
0 0 2 2 1	0 0 1 1 1	0 0 1 1 2	0 0 1 2 2	0 0 2 2 1	0 0 1 2 1	0 0 1 1 0	0 0 2 2 1	0 0 1 2 1
0 0 0 0 0	0 0 0 1 0	0 0 0 1 1	0 1 0 2 1	0 0 0 0 0	0 0 0 1 1	0 1 0 1 0	0 0 0 1 1	0 0 0 1 0
0 1 2 0 2	0 1 1 0 0	0 1 1 0 2	0 1 0 0 0	0 1 1 0 1	0 1 2 0 0	0 1 1 0 1	0 0 1 0 0	0 1 1 0 0
0 2 2 1 0	1 1 1 1 0	0 1 1 0 0	1 1 1 1 0	0 1 2 1 0	1 1 2 1 0	1 1 1 1 0	0 1 1 1 0	1 1 2 1 0

(181, 26, 24, 240)	(182, 26, 23, 240)	(183, 26, 23, 240)	(184, 26, 24, 120)	(185, 26, 22, 60)	(186, 26, 22, 40)	(187, 25, 23, 240)	(188, 25, 23, 240)	(189, 25, 23, 240)
0 1 2 2 1	0 1 2 3 2	0 2 2 2 2	0 1 1 2 1	0 2 3 3 2	0 2 4 2 2	0 1 2 2 1	0 1 3 2 2	0 1 2 1 2
0 0 1 2 1	0 0 2 2 1	0 0 1 2 1	0 0 1 1 1	0 0 2 1 2	0 0 2 2 1	0 0 1 2 1	0 0 2 1 2	0 0 1 1 1
0 0 0 1 0	0 0 0 1 1	0 0 0 1 1	0 0 0 1 0	0 0 0 0 0	0 0 0 0 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0
0 1 1 0 1	0 1 1 0 0	0 0 1 0 0	0 1 1 0 0	0 1 3 0 1	0 1 2 0 2	0 1 1 0 1	0 1 1 0 1	0 1 1 0 1
0 1 2 1 0	1 1 1 1 0	0 1 2 1 0	1 1 2 2 0	0 2 2 1 0	0 2 2 1 0	1 1 1 1 0	1 1 1 1 0	1 1 2 1 0

(190, 25, 22, 240)	(191, 25, 23, 240)	(192, 24, 23, 240)	(193, 24, 23, 240)	(194, 24, 23, 240)	(195, 23, 22, 240)	(196, 23, 22, 240)	(197, 23, 22, 240)	(198, 23, 22, 240)
0 1 2 2 1	0 1 1 2 1	0 1 3 2 1	0 1 2 2 2	0 1 2 3 2	0 1 2 2 1	0 1 4 3 2	0 1 3 2 2	0 1 2 2 1
0 0 1 2 0	0 0 1 1 0	0 0 3 1 1	0 0 2 1 1	0 0 2 2 2	0 0 2 1 1	0 0 3 2 2	0 0 2 1 1	0 0 1 2 1
0 1 0 1 1	0 1 0 1 1	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 2 0	0 0 0 1 0	0 0 0 1 0
0 1 1 0 0	0 1 1 0 0	0 1 2 0 0	0 1 2 0 0	0 1 2 0 0	0 1 1 0 0	0 1 1 0 0	0 1 1 0 0	0 1 1 0 0
1 1 1 2 0	1 1 2 1 0	1 2 2 1 0	1 1 3 1 0	1 2 2 1 0	1 2 2 1 0	2 2 2 2 0	1 1 2 1 0	1 1 2 1 0

(199, 23, 22, 240)	(200, 23, 22, 240)	(201, 23, 22, 240)	(202, 23, 22, 240)	(203, 23, 22, 120)	(204, 22, 21, 240)	(205, 22, 21, 240)	(206, 22, 21, 240)	(207, 22, 22, 240)
0 2 2 3 2	0 1 1 1 2	0 1 2 2 2	0 1 2 2 2	0 1 2 2 2	0 1 2 2 1	0 2 2 2 2	0 1 2 2 2	0 2 3 2 1
0 0 2 1 1	0 0 1 0 1	0 0 1 2 1	0 0 1 2 1	0 0 1 2 1	0 0 1 1 0	0 0 2 1 0	0 0 2 1 1	0 0 1 1 1
0 0 0 1 0	0 1 0 1 1	0 0 0 2 0	0 1 0 1 0	0 0 0 1 0	0 1 0 1 1	0 1 0 2 0	0 0 0 1 0	0 1 0 2 0
0 1 1 0 0	1 1 1 0 2	0 1 1 0 0	1 1 1 0 0	0 1 2 0 0	0 1 1 0 0	0 2 1 0 0	0 1 1 0 1	1 1 1 0 0
1 1 2 1 0	1 1 1 0 0	2 2 3 2 0	1 1 2 1 0	1 2 2 3 0	1 1 1 2 0	2 2 4 3 0	1 1 1 1 0	1 1 2 2 0

(208, 21, 21, 240)	(209, 21, 21, 240)	(210, 21, 21, 240)	(211, 21, 21, 240)	(212, 21, 21, 120)	(213, 20, 20, 240)	(214, 20, 20, 240)	(215, 20, 20, 240)	(216, 19, 19, 240)
0 1 2 2 1	0 1 2 2 2	0 1 3 2 1	0 1 2 2 2	0 2 3 2 1	0 1 2 2 1	0 2 2 2 2	0 1 2 2 1	0 2 2 2 2
0 0 2 1 1	0 0 2 2 1	0 0 2 1 1	0 0 2 1 1	0 0 2 1 1	0 0 1 1 1	0 0 2 1 1	0 0 1 2 1	0 0 2 1 0
0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 0 0 1 0	0 1 0 1 0	0 0 0 1 0	0 1 0 1 1	0 1 0 2 0
0 1 2 0 0	0 1 2 0 0	0 1 2 0 0	0 1 2 0 0	0 1 1 0 0	1 1 1 0 0	0 1 1 0 0	0 1 1 0 0	0 2 1 0 1
1 2 2 1 0	1 2 2 1 0	1 2 2 1 0	1 1 2 1 0	1 1 2 1 0	1 1 2 1 0	1 1 2 1 0	1 1 1 1 0	2 2 2 2 0

(217, 19, 19, 240)	(218, 19, 19, 240)	(219, 19, 19, 240)	(220, 19, 19, 240)	(221, 19, 19, 240)	(222, 19, 19, 240)	(223, 19, 19, 240)	(224, 19, 19, 240)	(225, 19, 19, 240)
0 2 3 2 1	0 2 3 3 1	0 2 3 2 1	0 2 3 3 1	0 2 3 2 2	0 2 3 2 1	0 2 3 3 1	0 2 2 2 2	0 2 2 2 2
0 0 1 2 0	0 0 1 2 0	0 0 1 2 0	0 0 1 2 0	0 0 2 1 0	0 0 2 2 0	0 0 1 2 0	0 0 2 1 0	0 0 2 1 0
0 1 0 2 1	0 1 0 1 1	0 1 0 1 1	0 1 0 1 1	0 1 0 2 1	0 1 0 2 1	0 2 0 1 1	0 1 0 2 0	0 1 0 2 1
0 2 1 0 0	0 2 1 0 0	0 2 2 0 0	0 2 2 0 0	0 2 1 0 0	0 2 1 0 0	0 2 2 0 0	0 2 1 0 0	0 2 1 0 0
2 2 2 2 0	2 2 2 2 0	2 2 2 2 0	2 2 2 2 0	2 2 2 3 0	2 2 2 2 0	2 2 2 2 0	1 2 2 3 0	2 2 2 3 0

(226, 19, 19, 240)	(227, 19, 19, 240)	(228, 19, 19, 240)	(229, 19, 19, 240)
0 2 2 3 2	0 2 2 3 2	0 2 2 2 2	0 2 2 3 1
0 0 2 1 0	0 0 2 1 0	0 0 2 1 0	0 0 2 1 0
0 1 0 2 0	0 1 0 2 0	0 1 0 2 0	0 1 0 1 1
0 2 1 0 1	0 2 1 0 0	0 2 1 0 0	1 1 1 0 0
2 2 2 2 0	2 2 3 2 0	1 2 3 3 0	1 3 2 2 0

7 Appendix 2: Facets of $OMCUT_5$

Using a modification by the second author of adjacency decomposition method from [ChRe96], all facets of $OMCUT_5$ were found: 194 orbits consisting all together 35320 facets.

See below the full list of the representatives of orbits presented as 5×5 matrix. The numbers in parenthesis are, respectively, the orbit number, the adjacency of a member of the orbit, the incidence of a member of the orbit, and orbit-size.

(1, 10695, 307, 20)	(2, 6451, 307, 60)	(3, 1590, 198, 60)	(4, 940, 111, 120)	(5, 444, 70, 60)	(6, 352, 111, 120)	(7, 269, 111, 60)	(8, 166, 56, 30)
0 0 0 0 0	0 0 0 0 0	0 −1 0 1 0	0 −1 0 1 1	0 −1 0 1 1	0 −1 0 0 1	0 0 0 0 0	0 −1 1 1 0
0 0 0 0 0	−1 0 0 0 1	−1 0 0 1 0	−1 0 0 1 1	−1 0 0 1 1	−1 0 0 0 1	−1 0 0 1 1	−1 0 1 1 0
0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	−1 −1 0 1 1	−1 −1 0 1 1	−1 −1 0 1 1	0 0 0 −1 1
0 0 0 0 1	0 0 0 0 0	0 0 0 0 1	0 0 0 0 0	1 1 0 0 −1	1 1 0 0 −1	1 1 0 0 −1	0 0 −1 0 1
0 0 0 0 0	1 0 0 0 0	1 1 0 −1 0	1 1 0 −1 0	1 1 0 −1 0	1 1 0 0 0	1 1 0 −1 0	1 1 0 0 0

(9, 162, 125, 120)	(10, 160, 112, 240)	(11, 154, 39, 40)	(12, 129, 61, 240)	(13, 110, 63, 240)	(14, 107, 39, 20)	(15, 99, 81, 120)	(16, 98, 88, 60)
0 −1 0 1 0	0 −1 1 0 0	0 −1 −1 1 2	0 0 0 1 0	0 −1 0 1 1	0 −1 −1 2 2	0 −2 0 1 1	0 −1 1 0 0
−1 0 1 0 0	−1 0 0 1 0	−1 0 −1 1 2	−1 0 0 1 1	−1 0 0 1 0	−1 0 −1 2 2	−1 0 1 0 0	−1 0 1 0 0
0 0 0 1 0	−1 0 0 1 1	−1 −1 0 1 2	−1 0 0 1 0	−1 0 0 1 1	−1 −1 0 2 2	−1 −2 0 2 2	−1 −1 0 2 2
0 1 −1 0 1	1 0 0 0 0	1 1 1 0 −2	1 0 0 0 0	1 0 0 0 0	1 1 1 0 −2	1 2 0 0 −1	1 1 0 0 −1
1 0 1 −1 0	1 1 0 −1 0	1 1 1 −1 0	1 0 1 −1 0	1 1 0 −1 0	1 1 1 −2 0	1 2 0 −1 0	1 1 0 −1 0

(17, 94, 55, 240)	(18, 92, 61, 240)	(19, 92, 59, 240)	(20, 90, 43, 120)	(21, 89, 85, 240)	(22, 88, 73, 120)	(23, 86, 43, 80)	(24, 83, 74, 240)
0 −1 0 1 0	0 0 −1 1 1	0 −1 1 1 0	0 −2 0 2 2	0 −1 0 1 0	0 1 −2 0 1	0 0 −1 1 1	0 0 1 0 0
−1 0 1 1 0	−1 0 1 0 1	−1 0 1 0 1	−1 0 0 1 1	−1 0 1 2 −1	−1 0 0 1 1	−1 0 0 1 1	−1 0 1 1 1
0 0 0 0 0	0 0 0 1 0	0 0 0 0 0	−1 0 0 1 1	−1 0 0 0 1	−2 1 0 −1 2	0 −1 0 1 1	−1 −1 0 1 2
0 1 −1 0 1	0 0 0 0 0	0 1 −1 0 1	1 1 0 0 −1	1 1 −1 0 1	1 0 1 0 −1	0 0 0 0 0	1 1 0 0 −2
1 0 1 0 0	1 0 1 −1 0	1 0 0 0 0	1 1 0 −1 0	1 1 0 −1 0	2 −1 1 1 0	1 1 1 −1 0	2 2 −1 −2 0

(25, 81, 41, 240)	(26, 80, 51, 120)	(27, 79, 56, 120)	(28, 78, 51, 240)	(29, 77, 51, 240)	(30, 69, 42, 240)	(31, 67, 55, 240)	(32, 67, 38, 120)
0 0 0 1 1	0 −1 1 0 1	0 1 −1 0 0	0 0 0 1 0	0 0 1 1 0	0 0 1 0 0	0 −1 0 1 1	0 −1 0 0 1
−1 0 0 1 2	−1 0 1 0 1	−1 0 0 1 1	−1 0 1 0 0	−1 0 0 1 1	−1 0 0 0 1	−1 0 1 0 0	−1 0 0 0 1
−1 −1 0 1 2	−1 −1 0 2 1	−2 0 0 1 1	0 0 0 1 1	−1 0 0 1 1	−1 −1 0 2 1	0 0 0 1 1	−1 −1 0 1 2
1 1 0 0 −2	1 1 0 0 −1	2 0 1 0 −1	0 0 0 0 0	1 0 0 0 −1	1 1 0 0 −1	0 0 0 0 0	1 1 1 0 −2
2 1 0 −2 0	1 1 0 −1 0	2 0 1 −1 0	1 1 0 −1 0	2 0 1 −2 0	1 1 0 0 0	1 1 0 −1 0	2 2 1 −1 0

(33, 66, 54, 10)	(34, 65, 44, 120)	(35, 65, 39, 240)	(36, 64, 51, 240)	(37, 62, 39, 120)	(38, 60, 51, 120)	(39, 60, 40, 120)	(40, 59, 73, 120)
0 −1 −1 1 1	0 −1 1 1 1	0 0 0 0 0	0 −2 0 2 1	0 −1 0 2 2	0 0 −1 1 0	0 −1 0 1 0	0 −1 2 1 0
−1 0 −1 1 1	−1 0 1 1 1	−1 0 1 1 0	−1 0 1 1 0	−1 0 0 2 2	−1 0 1 1 0	−1 0 0 1 0	−1 0 2 1 0
−1 −1 0 1 1	−1 −1 0 1 2	0 −1 0 1 2	0 0 0 0 0	−1 −1 0 1 2	0 0 0 0 0	−1 −1 0 1 2	−1 −1 0 1 3
1 1 1 0 −1	1 1 0 0 −2	1 1 1 0 −2	0 1 −1 0 1	1 1 0 0 −2	0 0 0 0 1	1 1 1 0 −1	1 1 0 0 −2
1 1 1 −1 0	2 2 −1 −2 0	1 2 1 −2 0	1 1 1 −1 0	2 2 0 −3 0	1 0 1 0 0	2 2 1 −2 0	3 3 −2 −3 0

(41, 59, 67, 120)	(42, 58, 47, 240)	(43, 57, 67, 120)	(44, 57, 53, 120)	(45, 57, 43, 240)	(46, 56, 42, 240)	(47, 56, 37, 240)	(48, 53, 63, 40)
0 −1 0 0 1	0 −1 0 1 1	0 −1 1 0 1	0 −1 1 1 0	0 −1 −1 2 1	0 −1 0 0 1	0 0 1 −1 0	0 −1 −1 1 1
−1 0 1 −1 1	−1 0 1 0 1	−1 0 1 0 1	−1 0 1 1 0	−1 0 0 1 1	−1 0 1 −1 2	−1 0 0 1 1	−1 0 −1 1 1
−1 0 0 1 1	−1 −1 0 1 1	−1 −1 0 1 2	0 0 0 0 0	−1 −1 0 2 1	−1 0 0 1 0	0 −1 0 1 1	−1 −1 0 1 1
1 0 1 0 −1	0 1 0 0 0	1 1 0 0 −2	0 0 0 0 1	0 1 0 0 0	1 0 1 0 0	0 1 1 0 −1	0 0 0 0 1
1 1 −1 1 0	2 1 0 −1 0	2 2 −1 −1 0	1 1 −1 0 0	2 1 1 −2 0	1 1 −1 1 0	1 0 0 1 0	2 2 2 −2 0

(49, 53, 45, 240)	(50, 52, 40, 240)	(51, 50, 44, 120)	(52, 50, 34, 240)	(53, 50, 27, 120)	(54, 49, 63, 40)	(55, 49, 51, 120)	(56, 49, 47, 120)
0 0 1 0 0	0 −2 0 2 2	0 0 1 0 0	0 −1 0 1 1	0 −2 0 1 1	0 0 −1 1 0	0 −1 0 1 1	0 −1 1 1 1
−1 0 0 0 1	−1 0 1 1 2	−1 0 1 0 1	−1 0 −1 2 1	−1 0 1 0 0	−1 0 0 1 0	−1 0 1 0 0	−1 0 1 0 1
−1 −1 0 2 1	−1 0 0 1 0	0 −1 0 1 1	−1 0 0 2 1	−1 0 0 1 1	0 −1 0 1 0	−1 −1 0 2 2	−1 0 0 1 1
1 1 0 0 −1	1 1 0 0 −1	0 1 0 0 −1	1 0 1 0 −1	2 1 1 0 −1	0 0 0 0 2	1 1 0 0 −1	1 0 0 0 −1
1 1 1 −1 0	1 2 −1 −1 0	1 2 −1 0 0	1 1 1 −2 0	2 1 1 −1 0	1 1 1 −1 0	1 1 0 −1 0	2 1 −1 −1 0

(57, 48, 47, 240)	(58, 48, 43, 120)	(59, 47, 32, 240)	(60, 46, 61, 60)	(61, 46, 30, 240)	(62, 45, 49, 240)	(63, 44, 38, 240)	(64, 43, 43, 120)
0 −2 1 2 0	0 −1 1 0 1	0 −2 1 1 1	0 −1 1 −1 1	0 2 −2 0 1	0 −1 1 0 0	0 −2 1 1 1	0 −2 0 1 1
−1 0 1 1 0	−1 0 1 0 1	−1 0 1 0 0	−1 0 1 −1 1	−1 0 0 1 1	−1 0 0 0 1	−1 0 1 0 0	−1 0 0 1 1
0 1 0 0 0	−1 −1 0 1 1	−1 −1 0 1 2	−1 −1 0 2 1	−2 1 0 0 1	−1 1 0 1 1	0 0 0 1 0	−1 −2 0 2 2
0 1 −1 0 1	1 1 −1 0 0	1 1 0 0 −1	1 1 −1 0 0	1 0 1 0 0	1 1 0 0 −1	0 1 0 0 1	1 2 0 0 −1
1 0 1 −1 0	1 1 0 0 0	1 2 0 0 0	1 1 0 1 0	2 −1 1 0 0	2 0 1 −1 0	1 1 0 −1 0	1 2 0 −1 0

(65, 42, 44, 240)	(66, 42, 38, 240)	(67, 42, 36, 240)	(68, 42, 36, 240)	(69, 41, 41, 240)	(70, 41, 40, 240)	(71, 41, 40, 240)	(72, 41, 39, 240)
0 −2 0 1 2	0 0 1 0 0	0 −1 0 1 1	0 −1 1 0 0	0 −2 1 1 1	0 −1 0 1 1	0 0 1 2 0	0 0 0 0 0
−1 0 1 0 1	−1 0 0 0 1	−1 0 1 0 1	−1 0 0 0 1	−1 0 0 1 1	−1 0 1 2 0	−1 0 0 2 2	−1 0 1 1 1
−2 −2 0 2 2	0 −1 0 1 1	−1 0 0 1 1	−1 0 0 1 2	0 0 0 0 1	−1 0 0 0 1	−1 1 0 1 0	−1 −1 0 1 2
1 2 0 0 −1	0 1 1 0 −1	1 0 1 0 −1	1 1 1 0 −2	0 1 1 0 −1	1 1 −1 0 0	1 0 0 0 −1	1 1 1 0 −2
2 2 0 −1 0	1 1 −1 1 0	1 1 −1 0 0	2 1 1 −1 0	1 1 −1 0 0	1 1 0 −1 0	2 0 2 −3 0	2 2 0 −2 0

(73, 41, 37, 240)	(74, 41, 35, 240)	(75, 40, 44, 240)	(76, 40, 41, 120)	(77, 40, 29, 240)	(78, 39, 38, 240)	(79, 39, 37, 120)	(80, 39, 36, 240)
0 0 1 −1 1	0 −2 0 2 2	0 −2 2 0 1	0 −1 2 1 0	0 1 2 0 0	0 −2 0 2 2	0 −1 1 0 0	0 0 1 1 0
−1 0 0 1 1	−1 0 −1 2 2	−1 0 1 0 0	−1 0 2 1 0	−1 0 1 0 1	−1 0 0 1 0	−1 0 1 0 0	−1 0 1 1 1
−1 −1 0 1 1	−1 −1 0 2 1	0 0 0 1 0	−1 −1 0 1 3	0 −1 0 2 1	−1 1 0 1 2	−1 −1 0 2 3	−1 −1 0 1 2
1 0 0 0 0	1 1 0 0 −1	0 1 0 0 1	1 1 0 0 −2	1 1 0 0 −2	1 0 1 0 −1	1 1 1 0 −2	1 1 0 0 −2
1 1 0 0 0	1 2 1 −2 0	1 1 −1 0 0	2 2 −1 −2 0	1 2 −1 0 0	1 1 0 −1 0	2 2 1 −2 0	2 1 0 −2 0

(81, 39, 36, 240)	(82, 39, 32, 240)	(83, 39, 31, 240)	(84, 38, 38, 120)	(85, 38, 30, 240)	(86, 37, 45, 40)	(87, 35, 39, 240)	(88, 35, 39, 120)
0 0 1 0 0	0 0 1 0 0	0 −1 −1 2 2	0 −2 0 2 2	0 0 2 −1 1	0 −1 −1 2 1	0 −1 0 1 1	0 −1 0 1 2
−1 0 1 0 0	−1 0 1 1 1	−1 0 0 2 2	−1 0 1 0 0	−1 0 1 1 1	−1 0 −1 2 1	−1 0 1 1 0	−1 0 0 1 2
0 −1 0 1 2	−1 −1 0 1 2	−1 −1 0 1 2	−1 0 0 2 2	0 1 0 1 −2	−1 −1 0 2 1	0 0 0 0 1	−2 −2 0 2 3
0 1 0 0 0	1 1 0 0 −2	1 1 1 0 −2	1 1 0 0 −1	1 −1 2 0 1	0 0 0 0 1	0 0 −1 0 1	2 2 0 0 −3
1 1 0 −1 0	2 1 0 −1 0	1 2 0 −2 0	1 1 0 −1 0	1 1 −3 1 0	2 2 2 −3 0	1 2 0 −1 0	2 2 0 −2 0

(89, 35, 36, 240)	(90, 35, 33, 240)	(91, 34, 36, 240)	(92, 34, 33, 240)	(93, 34, 32, 240)	(94, 33, 34, 240)	(95, 32, 36, 240)	(96, 32, 33, 120)
0 −1 0 1 1	0 1 −2 1 1	0 −1 1 1 0	0 0 1 1 0	0 −1 0 2 2	0 −2 1 2 1	0 0 1 1 0	0 0 0 1 0
−1 0 1 1 1	−1 0 0 1 1	−1 0 1 1 0	−1 0 0 1 2	−1 0 1 2 1	−1 0 1 1 1	−1 0 −2 1 3	−1 0 −1 1 1
−2 −2 0 2 2	−2 1 0 0 2	−1 0 0 1 2	−1 −1 0 1 2	−1 −1 0 1 2	0 1 0 0 −1	−1 −2 0 1 2	−1 −1 0 1 1
1 2 0 0 −1	1 0 1 0 −1	2 2 0 0 −2	1 1 0 0 −2	1 1 0 0 −2	0 1 −1 0 1	1 2 1 0 −2	0 1 1 0 0
2 2 0 −2 0	2 −1 1 0 0	3 2 0 −3 0	2 1 0 −2 0	2 2 0 −3 0	1 1 0 −1 0	2 0 0 0 0	2 2 2 −2 0

(97, 32, 31, 240)	(98, 32, 27, 240)	(99, 31, 41, 240)	(100, 31, 41, 240)	(101, 31, 37, 240)	(102, 31, 36, 120)	(103, 31, 35, 240)	(104, 31, 31, 240)
0 −1 1 2 2	0 0 1 1 1	0 −1 −1 2 1	0 0 1 1 0	0 0 0 0 1	0 −1 −1 1 1	0 −1 1 0 1	0 −2 1 0 1
−1 0 1 2 1	−1 0 1 2 1	−1 0 1 0 1	−1 0 1 1 0	−1 0 1 0 1	−1 0 −1 1 1	−1 0 0 1 1	−1 0 1 0 1
−1 0 0 1 1	−1 −1 0 2 2	−1 −1 0 2 1	−1 0 0 1 2	−1 0 0 0 1	−1 −1 0 1 1	−1 0 0 1 1	−1 0 0 1 1
2 1 −1 0 −2	2 1 0 0 −3	0 1 0 0 0	1 0 0 0 −1	1 −1 1 0 0	0 1 1 0 0	1 0 1 0 −1	1 1 0 0 −1
2 1 0 −3 0	2 2 −1 −3 0	2 1 1 −2 0	2 2 −1 −2 0	1 1 −1 1 0	2 2 2 −2 0	1 1 0 −1 0	2 1 −1 1 0

(105, 31, 31, 120)	(106, 31, 30, 240)	(107, 31, 30, 240)	(108, 31, 27, 240)	(109, 30, 41, 120)	(110, 30, 37, 120)	(111, 30, 36, 240)	(112, 30, 33, 120)
0 −1 −2 2 3	0 −1 −2 3 2	0 −2 0 1 1	0 1 1 0 −1	0 −1 0 0 1	0 1 1 −1 0	0 0 2 0 0	0 0 1 −1 1
−1 0 −2 2 3	−1 0 −2 3 2	−1 0 2 −1 2	−1 0 −2 1 2	−1 0 2 −2 2	−1 0 −1 2 1	−1 0 1 1 0	−1 0 0 1 1
−1 −1 0 3 3	−1 −2 0 3 3	−1 0 0 1 0	−1 −2 0 2 1	−2 0 0 2 1	−1 −1 0 2 1	−1 −1 0 1 3	0 −1 0 1 0
1 1 2 0 −3	1 1 2 0 −2	1 0 2 0 1	1 1 1 0 0	2 0 1 0 −1	1 1 1 0 −1	1 1 1 0 −2	0 1 1 0 0
1 1 3 −3 0	1 2 2 −3 0	1 2 −2 1 0	1 1 2 −1 0	2 1 −1 1 0	1 1 1 −2 0	2 2 −1 −2 0	1 1 −1 0 0

(113, 30, 30, 120)	(114, 30, 29, 120)	(115, 30, 28, 240)	(116, 29, 32, 240)	(117, 29, 31, 40)	(118, 29, 30, 240)	(119, 29, 28, 240)	(120, 28, 41, 240)
0 −1 1 1 0	0 −1 −1 2 1	0 0 −2 2 0	0 −2 2 1 1	0 −1 −1 1 2	0 1 1 1 0	0 1 −2 0 1	0 −2 1 2 0
−1 0 1 1 0	−1 0 −1 2 2	−1 0 0 1 1	−1 0 1 0 0	−1 0 −1 1 2	−1 0 1 1 1	−1 0 0 1 1	−1 0 1 0 1
−1 −1 0 1 3	−1 −1 0 2 2	−2 1 0 1 0	−1 0 0 1 1	−1 −1 0 1 2	−1 0 0 2 1	−2 1 0 0 1	0 1 0 1 −1
1 1 1 0 −2	0 1 1 0 −1	2 1 1 0 0	1 1 0 0 0	0 0 0 0 0	2 1 −1 0 −2	2 0 1 0 −1	0 1 −1 0 2
3 3 0 −3 0	2 1 1 −2 0	3 1 1 −2 0	2 1 −1 −1 0	2 2 2 −2 0	2 1 0 −3 0	2 0 1 0 0	1 0 1 −1 0

(121, 28, 36, 240)	(122, 28, 33, 240)	(123, 28, 33, 240)	(124, 28, 30, 240)	(125, 28, 28, 240)	(126, 28, 27, 240)	(127, 27, 37, 120)	(128, 27, 34, 240)
0 −1 1 0 1	0 −1 0 0 1	0 −2 0 2 1	0 0 1 2 0	0 −1 2 2 1	0 −2 0 1 1	0 0 0 1 0	0 −1 −2 2 1
−1 0 −1 2 1	−1 0 1 0 1	−1 0 1 1 0	−1 0 0 1 2	−1 0 2 2 1	−1 0 1 0 1	−1 0 −1 1 1	−1 0 −2 2 2
−1 0 0 2 1	−1 −1 0 1 2	−1 −2 0 2 2	−1 1 0 1 0	−1 0 0 1 1	−1 −1 0 1 1	−1 −1 0 1 1	−1 0 0 1 0
1 0 1 0 −1	1 1 1 0 −2	1 2 0 0 −1	1 −1 0 0 0	2 2 −2 0 −2	1 1 1 0 0	0 0 0 0 1	1 1 2 0 −1
1 1 1 −2 0	2 2 0 −1 0	1 2 1 −2 0	2 0 2 −2 0	3 2 −2 −3 0	1 2 −1 0 0	2 2 2 −2 0	2 2 3 −3 0

(129, 27, 33, 120)	(130, 27, 32, 120)	(131, 27, 31, 240)	(132, 27, 29, 240)	(133, 27, 29, 240)	(134, 27, 29, 240)	(135, 26, 38, 240)	(136, 26, 36, 120)
0 −2 −2 3 2	0 −1 0 1 1	0 1 1 −1 1	0 −1 0 2 1	0 −2 2 0 1	0 −2 1 0 1	0 1 −1 1 0	0 −1 2 2 0
−1 0 −1 2 1	−1 0 0 1 1	−1 0 1 2 1	−1 0 1 2 1	−1 0 1 0 0	−1 0 2 −1 2	−1 0 0 1 1	−1 0 2 2 0
−1 −1 0 2 1	−1 −1 0 1 1	0 2 0 0 −2	−1 0 0 1 0	−1 0 0 2 1	−1 1 0 1 0	−2 0 0 2 1	−1 −1 0 1 3
0 1 1 0 0	0 0 0 0 1	1 −2 3 0 1	1 1 −1 0 0	1 1 0 0 0	1 −1 2 0 1	1 0 0 0 0	1 1 0 0 −2
2 2 2 −3 0	2 2 1 −2 0	1 1 −3 1 0	1 1 1 −2 0	1 1 0 −1 0	1 2 −2 1 0	2 1 1 −2 0	2 2 0 −3 0

(137, 26, 36, 120)	(138, 26, 35, 240)	(139, 26, 33, 120)	(140, 26, 32, 240)	(141, 26, 32, 240)	(142, 26, 31, 240)	(143, 26, 30, 240)	(144, 26, 29, 240)
0 2 2 0 −2	0 −1 0 1 0	0 −1 1 1 1	0 0 2 1 −1	0 0 −1 1 0	0 0 2 0 0	0 −1 −2 1 2	0 0 2 −2 2
−1 0 −1 1 3	−1 0 1 1 0	−1 0 1 1 1	−1 0 1 2 0	−1 0 0 1 1	−1 0 1 1 0	−1 0 −2 2 2	−1 0 0 1 1
−1 −1 0 1 3	−1 −1 0 1 2	−2 −2 0 2 3	0 0 0 −2 2	0 −1 0 1 1	−1 −1 0 1 3	−1 0 0 0 1	0 −1 0 1 0
1 1 1 0 −3	1 1 1 0 −1	2 2 0 0 −3	1 0 −2 0 1	0 0 0 0 1	1 1 0 0 −1	1 1 2 0 −2	0 0 1 0 1
2 1 1 −2 0	2 2 0 −2 0	2 2 0 −2 0	1 1 0 0 0	1 1 1 −1 0	2 2 −1 −2 0	2 2 3 −2 0	1 1 −1 1 0

(145, 26, 29, 240)	(146, 26, 29, 240)	(147, 26, 29, 120)	(148, 26, 29, 120)	(149, 26, 29, 120)	(150, 26, 28, 240)	(151, 26, 27, 120)	(152, 25, 39, 120)
0 0 1 0 0	0 1 0 −1 2	0 −1 1 2 1	0 −1 0 1 1	0 1 2 −1 −1	0 −3 1 2 0	0 −1 2 1 1	0 −1 1 0 0
−1 0 1 −1 1	−1 0 0 0 1	−1 0 1 2 1	−1 0 0 1 1	−1 0 −3 2 2	−1 0 1 1 −1	−1 0 2 1 1	−1 0 1 −1 2
−1 −1 0 2 1	−2 1 0 1 2	−1 −1 0 1 2	−2 −2 0 2 2	−1 −2 0 2 2	−1 2 0 1 1	−2 −2 0 2 4	−1 0 0 1 1
1 1 −1 0 0	1 1 −1 0 0	1 1 0 0 −2	1 1 0 0 0	1 1 2 0 −1	0 1 −1 0 2	2 2 0 0 −4	1 1 1 0 −1
1 1 0 1 0	2 −1 1 1 0	2 2 0 −3 0	2 2 1 −2 0	1 1 2 −1 0	2 2 0 0 0	3 3 −1 −3 0	2 0 −1 1 0

(153, 25, 34, 240)	(154, 25, 33, 240)	(155, 25, 33, 240)	(156, 25, 31, 120)	(157, 25, 29, 240)	(158, 25, 28, 240)	(159, 25, 28, 120)	(160, 24, 35, 120)
0 −1 1 1 0	0 1 2 0 0	0 1 1 0 0	0 −2 0 1 1	0 −2 0 2 1	0 −2 1 1 1	0 −2 −2 2 3	0 −1 −1 2 0
−1 0 1 0 0	−1 0 1 1 0	−1 0 1 0 1	−1 0 1 0 0	−1 0 1 0 0	−1 0 1 −1 1	−1 0 −1 1 2	−1 0 1 0 1
−1 −2 0 3 2	−1 −2 0 2 4	0 −1 0 2 1	−1 0 0 1 1	−1 −1 0 2 1	−1 0 0 1 1	−1 −1 0 1 2	−1 −1 0 1 1
1 2 0 0 −1	1 2 2 0 −3	1 1 0 0 −2	2 2 0 0 −1	0 2 0 0 1	1 1 0 0 −1	0 1 1 0 −1	0 1 1 0 1
1 2 1 −2 0	2 2 0 −3 0	1 1 1 −2 0	2 2 0 −1 0	2 1 1 −2 0	1 1 0 1 0	2 2 2 −2 0	2 1 0 −1 0

(161, 24, 34, 240)	(162, 24, 31, 240)	(163, 24, 29, 240)	(164, 24, 29, 240)	(165, 24, 29, 120)	(166, 24, 28, 240)	(167, 24, 28, 120)	(168, 24, 25, 240)
0 −1 1 1 0	0 0 −1 1 1	0 1 2 0 0	0 1 1 0 0	0 −1 −1 2 0	0 1 1 2 0	0 1 3 0 0	0 2 −3 1 1
−1 0 2 1 0	−1 0 1 1 2	−1 0 1 1 0	−1 0 0 1 1	−1 0 1 1 0	−1 0 2 1 0	−2 0 1 0 2	−1 0 0 1 1
−1 −1 0 1 3	−2 −2 0 2 3	−1 −2 0 2 4	0 1 0 1 −2	−1 −1 0 1 1	−1 0 0 3 2	0 −2 0 3 1	−3 2 0 0 3
1 1 1 0 −2	1 2 0 0 −2	1 2 0 0 −1	1 −1 1 0 1	0 1 0 0 1	2 1 −1 0 −2	1 2 0 0 −2	1 0 2 0 −1
3 3 −1 −3 0	2 2 0 −2 0	2 2 0 −3 0	1 1 −2 1 0	2 1 1 −1 0	2 2 0 −4 0	2 3 −2 1 0	3 −2 1 1 0

(169, 23, 32, 240)	(170, 23, 29, 240)	(171, 23, 29, 120)	(172, 23, 27, 240)	(173, 23, 26, 240)	(174, 23, 26, 120)	(175, 23, 26, 120)	(176, 23, 25, 240)
0 −1 −3 2 2	0 −2 0 1 1	0 −1 1 1 0	0 1 1 −2 2	0 −2 1 1 0	0 −1 1 1 0	0 −1 2 2 0	0 −2 0 1 1
−1 0 −3 3 3	−1 0 1 0 0	−1 0 1 1 0	−1 0 1 0 1	−1 0 1 1 0	−1 0 1 1 0	−1 0 2 2 0	−1 0 1 0 0
−1 1 0 0 0	−1 −2 0 2 3	−2 −2 0 3 5	0 0 0 2 1	0 0 0 1 1	−1 −1 0 2 3	0 0 0 −2 2	−1 0 0 1 1
1 1 3 0 −2	1 2 1 0 −2	2 2 2 0 −4	0 2 0 0 −1	0 1 −1 0 1	2 2 1 0 −3	1 1 −3 0 1	2 1 1 0 −1
2 2 4 −3 0	2 3 1 −2 0	4 4 2 −5 0	1 0 −1 1 0	2 1 0 −1 0	3 3 1 −4 0	1 1 0 0 0	2 2 0 −1 0

$$(177, 22, 30, 120)\quad \begin{array}{rrrrr} 0 & -1 & 1 & 1 & 2 \\ -1 & 0 & 1 & 1 & 2 \\ -2 & -2 & 0 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 3 & 3 & -1 & -3 & 0 \end{array}$$

$$(178, 22, 29, 120)\quad \begin{array}{rrrrr} 0 & -1 & 2 & 2 & 1 \\ -1 & 0 & 2 & 2 & 1 \\ -2 & -2 & 0 & 3 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 4 & 4 & -2 & -5 & 0 \end{array}$$

$$(179, 22, 28, 120)\quad \begin{array}{rrrrr} 0 & -2 & 3 & 1 & 1 \\ -1 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 2 & 3 & -3 & 0 & -2 \\ 2 & 3 & -3 & -2 & 0 \end{array}$$

$$(180, 22, 28, 120)\quad \begin{array}{rrrrr} 0 & -1 & 3 & 1 & 0 \\ -1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 1 & 1 & -3 & 0 & 1 \\ 2 & 2 & -2 & 0 & 0 \end{array}$$

$$(181, 22, 26, 240)\quad \begin{array}{rrrrr} 0 & -1 & 1 & 1 & 1 \\ -1 & 0 & 2 & -1 & 2 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \end{array}$$

$$(182, 22, 25, 240)\quad \begin{array}{rrrrr} 0 & 1 & -2 & 1 & 0 \\ -1 & 0 & -2 & 2 & 3 \\ -2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 2 & 0 & 2 & -1 & 0 \end{array}$$

$$(183, 22, 25, 240)\quad \begin{array}{rrrrr} 0 & 1 & -2 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 2 & 0 \end{array}$$

$$(184, 22, 25, 80)\quad \begin{array}{rrrrr} 0 & 0 & -1 & 1 & 2 \\ -1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & -3 \\ 1 & 1 & 1 & -2 & 0 \end{array}$$

$$(185, 22, 25, 40)\quad \begin{array}{rrrrr} 0 & -1 & -1 & 3 & 3 \\ -1 & 0 & -1 & 3 & 3 \\ -1 & -1 & 0 & 3 & 3 \\ 1 & 1 & 1 & 0 & -3 \\ 2 & 2 & 2 & -5 & 0 \end{array}$$

$$(186, 22, 23, 120)\quad \begin{array}{rrrrr} 0 & -3 & -1 & 2 & 2 \\ -1 & 0 & 1 & 1 & 0 \\ -3 & -3 & 0 & 3 & 3 \\ 2 & 3 & 0 & 0 & -1 \\ 2 & 3 & 1 & -1 & 0 \end{array}$$

$$(187, 21, 25, 120)\quad \begin{array}{rrrrr} 0 & -4 & 0 & 3 & 3 \\ -2 & 0 & 1 & 1 & 1 \\ -4 & -4 & 0 & 4 & 4 \\ 3 & 4 & 0 & 0 & -2 \\ 3 & 4 & 0 & -2 & 0 \end{array}$$

$$(188, 21, 24, 240)\quad \begin{array}{rrrrr} 0 & 1 & -2 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 & 3 \\ 1 & 0 & 2 & 0 & -1 \\ 2 & -2 & 1 & 1 & 0 \end{array}$$

$$(189, 21, 23, 120)\quad \begin{array}{rrrrr} 0 & 2 & 2 & -2 & -1 \\ -1 & 0 & -1 & 3 & 2 \\ -1 & -1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 0 & -3 \\ 2 & 2 & 2 & -4 & 0 \end{array}$$

$$(190, 21, 23, 120)\quad \begin{array}{rrrrr} 0 & -1 & 1 & 2 & 2 \\ -1 & 0 & 1 & 2 & 2 \\ -1 & -1 & 0 & 2 & 2 \\ 2 & 2 & -1 & 0 & -3 \\ 2 & 2 & 0 & -4 & 0 \end{array}$$

$$(191, 21, 22, 240)\quad \begin{array}{rrrrr} 0 & -1 & -2 & 3 & 3 \\ -1 & 0 & 1 & 2 & 2 \\ -2 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 2 & 1 & 1 & -2 & 0 \end{array}$$

$$(192, 20, 21, 240)\quad \begin{array}{rrrrr} 0 & -3 & 1 & 4 & 3 \\ -1 & 0 & 2 & 4 & 2 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & 3 & -3 & 0 & -1 \\ 2 & 2 & 0 & -2 & 0 \end{array}$$

$$(193, 20, 21, 120)\quad \begin{array}{rrrrr} 0 & -2 & -2 & 2 & 2 \\ -1 & 0 & 0 & 1 & 1 \\ -3 & 1 & 0 & 1 & 1 \\ 4 & 3 & 1 & 0 & -2 \\ 4 & 3 & 1 & -2 & 0 \end{array}$$

$$(194, 19, 19, 40)\quad \begin{array}{rrrrr} 0 & -1 & -1 & 2 & 3 \\ -1 & 0 & -1 & 2 & 3 \\ -1 & -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 0 & -3 \\ 2 & 2 & 2 & -4 & 0 \end{array}$$

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