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On Complexity of Computable $\aleph_1$–Categorical Models[a]

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1 Introduction

One of the themes of computable model theory is concerned with finding computable models for first order theories. It is well known that if a consistent theory T is decidable then T has a decidable model, that is one for which the satisfaction predicate is decidable. On the other hand, if a theory T has a computable model then T is computable in $\mathbf{0}^{\omega}$. For example, the theory of arithmetic $(\omega, S, +\times, \leq, 0)$ is Turing equivalent to $\mathbf{0}^{\omega}$. In this paper, for any natural number $n \geq 1$, we present examples of $\aleph_1$–categorical computable models whose theories are equivalent to $\mathbf{0}^{n}$. The following are related results. In [1] Baldwin and Lachlan showed that all models of any $\aleph_1$-categorical theory T can be listed into the chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots \mathcal{A}_{\omega}$ of elementary embeddings, where $\mathcal{A}_0$ is the prime model, $\mathcal{A}_{\omega}$ is the saturated model, and each $\mathcal{A}_{i+1}$ is a minimal proper elementary extension of $\mathcal{A}_i$. Let $SCM(T)$ be the spectrum of computable models of T, that is $SCM(T) = \{i \mid \mathcal{A}_i$ has a computable presentation $\}$. If T is $\aleph_1$–categorical and decidable then, as proved by Harrington and Khisamiev in [4] [5], all countable models of T have decidable presentations, that is $SCM(T) = \omega \bigcup \{\omega\}$. In [3] Goncharov showed that there exists an $\aleph_1$-categorical theory T computable in

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$\mathbf{0}'$ for which $SCM(T) = \{0\}$. Kudeiberganov extended this result by showing that for every $n \geq 0$ there exists an $\aleph_1$-categorical T computable in $\mathbf{0}'$ such that $SCM(T) = \{0, 1 \ldots, n\}$ [7]. In [6] it is shown that there exist $\aleph_1$-categorical theories T_1 and T_2 computable in $\mathbf{0}''$ such that $SCM(T_1) = \omega$ and $SCM(T_2) = \omega \bigcup \{\omega\} \setminus \{0\}$. Thus, all the known $\aleph_1$–categorical theories that have computable models are computable in $\mathbf{0}''$. Our examples show that for each $n \geq 1$ there is an $\aleph_1$-categorical computable model whose theory is Turing equivalent to $\mathbf{0}^n$.

We now give basic definitions. We fix a computable language L. A structure $\mathcal{A}$ of this language is **computable** if the domain, functions, and predicates of the structure are uniformly computable. This is equivalent to saying that the atomic diagram of $\mathcal{A}$ is computable. A structure $\mathcal{B}$ is **computably presentable** if it is isomorphic to a computable structure. In this case any isomorphism from $\mathcal{B}$ into $\mathcal{A}$ is called a **computable presentation** of $\mathcal{B}$. A complete theory T is $\aleph_1$–**categorical** if all models of T of power $\aleph_1$ are isomorphic. A model $\mathcal{M}$ is $\aleph_1$–categorical if the theory $Th(\mathcal{M})$ of the model is $\aleph_1$–categorical. Typical examples of $\aleph_1$–categorical theories are the theory of algebraically closed fields of fixed characteristic, the theory of vector spaces over a fixed countable field, the theory of the successor structure (ω, S). A theory T is **almost strongly minimal** if in every model of T every element is in the algebraic closure of a strongly minimal set.

Now we briefly outline the paper. In the next section, Section 2, we provide a model–theoretic construction of a model whose theory is an $\aleph_1$–categorical theory. The construction of the model follows the ideas of Marker's construction from [8] but is carried out with an eye towards reducing the computability-theoretic complexity of presentations of the model. In Section 3 we prove a representation lemma about Σ_2^0–subsets of natural numbers. Finally, in the last section we prove the following theorem:

Theorem *For any natural number $n \geq 1$ there exists an $\aleph_1$–categorical theory T with a computable model so that T is equivalent to $\mathbf{0}^n$. Moreover, all (countable) models of T have computable presentations and T is almost strongly minimal.*

We assume that the reader is familiar with basics of model theory and computability theory. We use some standard notions and notations, such as $< \cdot, \cdot >$, l, r Cantor's pairing functions, the concept of X-computable sets (e.g. sets computable with an oracle for X), the jump operation X' for subsets $X \subset \omega$. Standard references are [2] [9].

2 Construction

In [8] Marker provided an example of non Σ_n axiomatizable almost strongly minimal theory for $n \in \omega$. We adapt that construction for our case. The construction uses induction. We first provide the base case and then explain the inductive step.

2.1 Basic 0-Structure

We construct theory T_0. The basic 0-structure will be the model of T_0 denoted by $\mathcal{M}_0$. The language of the structure $\mathcal{M}_0$ is $L_0 =< X, Y, C_0, D_0, P_0, R_0 >$, where X, Y, C_0, D_0 are all unary predicate symbols, P_0 is a binary predicate, and R_0 is a predicate of arity 4. Now we list axioms of the theory T_0.

1. Any model of T_0 is a disjoint union of X, Y, C_0, and D_0.

2. If $P_0(x, y)$ holds then $x \in X$ and $y \in Y$. Thus P_0 defines a bipartite graph between X and Y.

3. Every element of X is connected either to every element of Y or all but one element of Y, and there are infinitely many elements of each type.

4. Every element of Y is connected to all but one element of X.

   Thus, $\neg P_0$ determines a one to one function from Y into x, that is $y \in Y$ is mapped into $x \in X$ if $\neg P_0(x, y)$. Based on P_0 we now define the following two predicates. We say $x \in X$ is **good** if it is connected to every element in Y, and we call x **bad** otherwise. Thus, we have the unary predicates G and B:

   $$G(x) = \forall y(X(x) \& P_0(x, y)) \quad \text{and} \quad B(x) = \exists y(X(x) \& Y(y) \& \neg P_0(x, y)).$$

   What we have defined is not an $\aleph_1$-categorical theory because B and G can realize different cardinalities. Therefore R_0 will be used to remedy this. Using R_0 we define a permutation $\mu : X \to X$ so that $\mu(B) = G$, $\mu(G) = G$ and $\mu(\mu(x)) = x$ for all $x \in X$. Axioms for R_0 are as follows:

5. If $R_0(x, y, c, d)$ then $x \in X$, $y \in X$, $c \in C_0$, and $d \in D_0$.

6. $R_0(x, y, c, d)$ if and only if $R_0(y, x, c, d)$.

   Define the predicate $R^\star(x, y, c)$: $R^\star(x, y, c) = \forall d R_0(x, y, c, d)$. The predicate $R^\star$ will define a bipartite graph between the set $X^{(2)}$ of all unordered pairs of X and the set C. Here are the axioms for $R^\star$:

7. For every unordered pair $\{x, y\}$ in X there is at most one $c \in C_0$ such that $R^\star(x, y, c)$.

8. For every element $c \in C_0$ there is a unique pair $\{x, y\}$ such that $R^\star(x, y, c)$.

9. If $R^\star(x, y, c)$ then one element of the pair $\{x, y\}$ is good and the other is bad.

10. For every $x \in X$ there is unique $y \in X$ so that $R^\star(x, y, c)$ for some $c \in C_0$.

    Thus, $R^\star$ determines a one to one function from C_0 into $X^{(2)}$. We now define a function $\mu : X \to X$ as follows: $\mu(x) = y$ if and only if $\exists c R^\star(x, y, c)$. Clearly, μ is a definable permutation such that $\mu(B) = G$, $\mu(G) = G$ and $\mu(\mu(x)) = x$ for all $x \in X$.

11. For all $x \in X$, $y \in Y$, $c \in C_0$ either $\forall d \in D_0 R_0(x, y, c, d)$ or there is a unique d_0 such that $\neg R_0(x, y, c, d_0)$. Moreover, there are infinitely many elements of each type.

12. For each $d_0 \in D_0$ there are unique pair $\{x, y\}$ and $c_0 \in C$ so that $\neg R_0(x, y, c_0, d_0)$.

    Thus, $\neg R_0$ establishes a one to one mapping from D_0 into $X^{(2)} \times C_0$. This completes the description of T_0.

Here are some properties of the model $\mathcal{M}_0$.

Claim 1 *Every element of $\mathcal{M}$ is in definable closure of B.*

Proof. We denote the closure of B by $cl(B)$. Take any $a \in M$. If $a \in G$ then $\mu(x) = a$ for some $x \in B$. Hence $a \in cl(B)$. If $a \in Y$ then $\neg P_0(x, a)$ for some $x \in B$, and from the axioms for P_0 we see that $a \in cl(B)$. Assume that $a \in C_0$. Then there is a pair $\{x, y\}$ in X such that $R^\star(x, y, c)$. From the axioms for $R^\star$ we derive that $a \in cl(B)$. Similarly, if $a \in D_0$ then from the fact that R_0 is a one-to-one mapping from D_0 into $X^{(2)} \times C_0$, we obtain that $a \in cl(B)$. The claim is proved.

Claim 2 *The set B is strongly minimal.*

Proof. Let $a_1, \ldots, a_n$ be elements of $\mathcal{M}$. We need to prove that there is no infinite and coinfinite subset of B definable with parameters $a_1, \ldots, a_n$. We may assume, by the previous claim, that the parameters $a_1, \ldots, a_n$ are in B. Let b_i and c_i be all elements such that $\mu(b_i) = a_i$ and $R^\star(a_i, b_i, c_i)$ for $i = 1, \ldots, n$. It is not hard to see that for all $x, y \in B$ if $x, y \notin \{a_1, \ldots, a_n\}$ then there is an automorphism α of $\mathcal{M}_0$ for which $\alpha(x) = y$ and $\alpha(a_i) = a_i$, $\alpha(b_i) = b_i$ and $\alpha(c_i) = c_i$ for all $i = 1, \ldots, n$. This proves the claim.

2.2 n-Structure

Suppose that we have constructed the theory T_{n-1} for $n \geq 1$. The language of the theory T_n is $< X, Y, Z_1, Z_2, \ldots, Z_{2n}, C_0, D_0, \ldots, C_{2n}, D_{2n}, P_n, R_n >$, where X, Y, Z_1, Z_2, …, Z_{2n}, C_0, D_0, …, C_{2n}, and D_{2n} are all unary predicate symbols, P_n is a $(2n+2)$–ary predicate symbol, and R_n is $(2n+4)$-ary predicate symbol. The model of T_n will be denoted by $\mathcal{M}_n$. The idea is that we want the previous structure $\mathcal{M}_{n-1}$ to be definable in $\mathcal{M}_n$. The axioms of T_n are the following:

1. Any model of T_n is a disjoint union of the unary predicates X, Y, Z_1, Z_2, …, Z_{2n}, C_0, D_0, …, C_{2n}, and D_{2n}.

   We now describe the predicate P_n.

2. If $P_n(x, y, z_1, \ldots, z_{2n})$ holds then $x \in X$, $y \in Y$, $z_1 \in Z_1$, …, $z_{2n} \in Z_{2n}$.

   We define the following predicate $P^{\star\star}(x, y, z_1, \ldots, z_{2(n-1)})$:

$$P^{\star\star}(x, y, z_1, \ldots, z_{2(n-1)}) = \exists z_{2n-1} \forall z_{2n} P_n(x, y, z_1, \ldots, z_{2n}).$$

3. We postulate that the predicate $P^{\star\star}$ satisfies all the axioms of the predicate P_{n-1}.

   Now consider the following predicate $P^{\star}(x, y, z_1, \ldots, z_{2n-1})$:

$$P^{\star}(x, y, z_1, \ldots, z_{2n-1})) = \forall z_{2n} P_n(x, y, z_1, \ldots, z_{2n-1}, z_{2n}).$$

   Here are the axioms for $P^{\star}$:

4. For all $x \in X$, $y \in Y$, $z_1 \in Z_1$, …, $z_{2(n-1)} \in Z_{2(n-1)}$ there is at most one $z_{2n-1} \in Z_{2n-1}$ such that $P^{\star}(x, y, z_1, \ldots, z_{2n-1})$.

5. For each $z_{2n-1} \in Z_{2n-1}$ there exists a unique tuple $(x, y, z_1, \ldots, z_{2(n-1)})$ such that $P^{\star}(x, y, z_1, \ldots, z_{2(n-1)}, z_{2n-1})$.

   Thus, the predicate $P^{\star}$ determines a one-to-one function from Z_{2n-1} into $X \times Y \times Z_1 \times \ldots \times Z_{2(n-1)}$. The next two axioms finish the description of P_n. These axioms basically tell us that $\neg P_n$ establishes a one-to-one mapping from Z_{2n} into $X \times Y \times Z_1 \times \ldots \times Z_{2n-1}$.

6. For all $x \in X$, $y \in Y$, $z_1 \in Z_1$, …, $z_{2n-1} \in Z_{2n-1}$ either there is a unique $z_{2n} \in Z_{2n}$ for which $\neg P_n(x, y, z_1, \ldots, z_{2n-1}, z_{2n})$ is true or $P_n(x, y, z_1, \ldots, z_{2n-1}, z_{2n}))$ is true for all $z_{2n} \in Z_{2n}$.

7. For each $z_{2n} \in Z_{2n}$ there is a unique tuple $(x, y, z_1, \ldots, z_{2n-1})$ such that $\neg P_n(x, y, z_1, \ldots, z_{2n-1}, z_{2n})$.

   Now we describe the predicate R_n. Here are the lists of axioms for R_n.

8. If $R_n(x, y, c_0, d_0 \ldots, c_n, d_n)$ then $x \in X$, $y \in X$, $c_i \in C_i$, and $d_i \in D_i$ for all $i \leq n$. Moreover, $R_n(x, y, c_0, d_0 \ldots, c_n, d_n)$ if and only if $R_n(y, x, c_0, d_0 \ldots, c_n, d_n)$.

   Define the predicate $R^{\star\star}(x, y, c_0, d_0 \ldots, c_{n-1}, d_{n-1})$ as follows:

$$R^{\star\star}(x, y, \ldots, c_{n-1}, d_{n-1}) = \exists c_n \forall d_n R_n(x, y, \ldots, c_{n-1}, d_{n-1}, c_n, d_n).$$

   Define the predicate $R^{\star}(x, y, c_0, d_0, \ldots, c_{n-1}, d_{n-1}, c_n)$ as follows:

$$R^{\star}(x, y, \ldots, c_{n-1}, d_{n-1}, c_n) = \forall d_n R_n(x, y, \ldots, c_{n-1}, d_{n-1}, c_n, d_n).$$

   Here are the axioms for $R^{\star\star}$ and $R^{\star}$:

9. We postulate that the predicate $R^{\star\star}$ satisfies all the axioms of the predicate R_{n-1}. For all $\{x,y\} \in X^{(2)}$, $c_0 \in C_0$, …, $d_{n-1} \in D_{n-1}$ there is at most one $c_n \in C_n$ such that $R^{\star}(x, y, c_0, d_0 \ldots, c_{n-1}, d_{n-1}, c_n)$.

10. For each $c_n \in C_n$ there exist a unique tuple $(c_0, \ldots, c_{n-1}, d_{n-1})$ and a pair $\{x, y\}$ such that $R^{\star}(x, y, c_0, d_0 \ldots, c_{n-1}, d_{n-1}, c_n)$.

11. For all $\{x, y\} \in X^{(2)}$, $c_0 \in C_0$, $d_0 \in D_0$, …, $d_{n-1} \in D_{n-1}$, $c_n \in C_n$ either $R_n(x, y, c_0, d_0 \ldots, c_{n-1}, d_{n-1}, c_n, d_n)$ for all $d_n \in D_n$ or there is a unique $d \in D_n$ for which $\neg R_n(x, y, c_0, d_0 \ldots, c_{n-1}, d_{n-1}, c_n, d)$.

12. For each $d_n \in D_n$ there is a unique tuple $(c_0, d_0 \ldots, c_{n-1}, d_{n-1}, c_n)$ and a pair $\{x, y\}$ such that $\neg R_n(x, y, c_0, d_0 \ldots, c_{n-1}, d_{n-1}, c_n, d_n)$.

Now by induction on n one can prove the following lemma.

Lemma 1 *For the theory T_n the following are true:*

1. *The unary predicate $B(x)$ is definable by a Σ_n formula in the language of T_n.*
2. *The theory T_n is $\aleph_1$-categorical.*
3. *The predicate $B(x)$ is strongly minimal.*
4. *The theory T_n is almost strongly minimal.* □

3 On Presentations of Σ_2^0-Sets

In this section we prove a computability-theoretic lemma needed for the main result of this paper. For the lemma we define the following notion.

Definition 1 *A Σ_2^0–set A is* **one-to-one representable** *if for some computable predicate $Q \subset \omega^3$ each of the following properties is true:*

1. *For each $n \in \omega$, $\exists a \forall b Q(n, a, b)$ if and only if $n \in A$.*
2. *For each $n \in \omega$, $\exists a \forall b Q(n, a, b)$ if and only if $\exists^{=1} a \forall b Q(n, a, b)$[1].*

[1] $\exists^{=1} x P(x)$ means that there is a unique x satisfying P

3. *For every b there is a unique pair $< n, a >$ such that $\neg Q(n, a, b)$.*

4. *For every pair $< n, a >$ either $\exists b \neg Q(n, a, b)$ or $\forall b Q(n, a, b)$.*

5. *For every a there exists a unique n such that $\forall b Q(n, a, b)$.*

It is not hard to see that every infinite and coinfinite computable set A has a one-to-one representation.

For a Σ^0_2–set A there is a computable H such that $n \in A \leftrightarrow \exists a \forall b H(n, a, b)$. In fact, there is a computable Q for which $\exists a \forall b H(n, a, b) \leftrightarrow \exists^{=1} a \forall b Q(n, a, b)$. To show this we describe the procedure which builds a predicate P_n, $n \in \omega$. To build P_n initially we set the values $a_0 = 0$, $r_0 = 0$, $h_0 = 0$. At stage t the predicate P_n will be defined on all pairs (i, j) so that $j \leq t$, $i \leq r_t$. The intention for a_t is that a_t will be the unique witness for n to belong to A, that is $n \in A$ if and only if $\forall b P_n(a_t, b)$. The intention for h_t is that if $n \in A$ then h_t is the minimal $h \leq t$ for which $(\forall b \leq t) H(n, h, b)$.

Stage $t + 1$. Compute $H(n, i, j)$ for all $i, j \leq t + 1$. If $(\forall i \leq t + 1)(\exists j \leq t + 1) \neg H(n, i, j)$ then set $r_{t+1} = r_t + 1$, h_{t+1} and a_{t+1} be undefined, and make $P_n(i, j)$ false on all (i, j), with $i \leq r_{t+1}, j \leq t + 1$, at which P_n has not been defined. If h_t is undefined and $\forall j \leq t + 1 H(n, t + 1, j)$ is true then set $h_{t+1} = t+1$, $r_{t+1} = r_t+1$, and $a_{t+1} = r_{t+1}$. Make $P_n(a_{t+1}, j)$ to be true for all $j \leq t+1$, and make $P_n(i, j)$ false on all (i, j), with $i \leq r_{t+1}, j \leq t+1$, at which P_n has not been defined. If h_t is defined and $\forall j \leq t+1 H(n, h_t, j)$ is true then set $h_{t+1} = h_t$, $a_{t+1} = a_t$, and $r_{t+1} = r_t + 1$, and make $P_n(a_{t+1}, j)$ to be true for all $j \leq t + 1$, and make $P_n(i, j)$ false on all (i, j), with $i \leq r_{t+1}, j \leq t + 1$, at which P_n has not been defined.

Now define the predicate Q as follows: $(n, a, b) \in Q$ if and only if $P_n(i, j)$. The construction above guarantees that the predicate Q is desired.

Now we prove the following lemma which gives a sufficient condition for Σ^0_2-sets to have one to one representations.

Lemma 2 *Let A be a coinfinite Σ^0_2–set that possesses an infinite computable subset S such that $A \setminus S$ is infinite. Then A has a one-to-one representation.*

Proof. As noted above there is computable set H such that $n \in A$ iff $\exists^{=1} a \forall b H(n, a, b)$. Define the predicate H_1: $H_1(n, a, b)$ if and only if

$a =< n, x >$ & $H(n, x, b)$. It is easy to check that the formulas $\exists a \forall b H(n, a, b)$ and $\exists a \forall b H_1(n, a, b)$ are equivalent. Moreover, for every a there exists at most one n such that $\forall b H_1(n, a, b)$. Let H_2 be defined as follows: $\neg H_2(n, a, b)$ if and only if $b =< n, a, x > \& \neg H_1(n, a, x)$ & $(\forall z < x) H_1(n, a, z)$. It is not hard to see that the predicate H_2 satisfies the following properties:

1. The formulas $\exists a \forall b H_1(n, a, b)$ and $\exists a \forall b H_2(n, a, b)$ are equivalent.
2. The formulas $\forall b H_1(n, a, b)$ and $\forall b H_2(n, a, b)$ are equivalent.
3. For every pair n, a there exists at most one b such that $\neg H_2(n, a, b)$.
4. For every a there exists at most one n such that $\forall b H_2(n, a, b)$.
5. For every b there exists at most one pair (n, a) such that $\neg H_2(n, a, b)$.

Thus, we may assume that H satisfies the properties $3) - 5)$ above. Now, using the predicate H, we build the desired predicate Q.

At stage t the predicate Q_t will be defined on $[0, t] \times [0, r_2(t)] \times [0, r_3(t)]$, where the functions $r_2(t)$, $r_3(t)$ are given effectively at stage t. The predicate Q_t will satisfy the following properties denoted by P:

P_1: For all $n \leq t$, $a \leq r_2(t)$ either $Q_t(n, a, b)$ holds true for all $b \leq r_3(t)$ or $\exists^{=1} b \leq r_3(t) \neg Q_t(n, a, b)$.

P_2: If $a \leq r_2(t)$ is a (Q, t)**-witness for** $n \leq t$, that is $\forall b \leq r_2(t) Q_t(n, a, b)$ then it is a unique (Q, t)-witness for n.

P_3: No two (Q, t)-witnesses (which may be for distinct n_1 and n_2) coincide.

P_4: For each $b \leq r_3(t)$ there is a unique pair (n, a) such that $\neg Q_t(n, a, b)$.

Let $H_0 \subset H_1 \subset \ldots$ be an approximation of H so that $H = \bigcup_t H_t$, where $H_t = H \bigcap [0, t] \times [0, t] \times [0, b_t]$ and b_t is the minimal $b \geq t$ such that each of the following is true:

1. If $a \leq t$ is a (H, t)**-witness for** $n \leq t$, that is $\forall b \leq t H(n, a, b)$ then it is a unique (H, t)-witness for n.
2. No two (H, t)-witnesses (which may be for distinct n_1 and n_2) coincide.

3. For all $n, a \leq t$ either $(\forall b \leq t)H(n,a,b)$ or $(\exists^{=1} j \leq b)\neg H(n,a,b)$.

Note that b_t is correctly defined. If for an $n \leq t$ there is an (H,t)-witness for n then we denote the witness by $h(n,t)$.

Without loss of generality, we assume that $H(0,0,0)$ is true. In the construction, at Stage t, we use functions $r_2(t)$, $r_3(t)$, $h(n,t)$ and $a(n,t)$. The function $r_2(t)$ and $r_3(t)$ tell us that the second and the third coordinates of Q_t do not exceed $r_2(t)$ and $r_3(t)$, respectively; $h(n,t)$ is the (H,t)-witness for n, and $a(n,t)$ is a (Q,t) witness for n if they exist. The construction guarantees that $h(n,t)$ exists if and only if $a(n,t)$ exists. Initially, we set $r(0)=0$, $h(0,0)=0$, and $a(0,0)=0$. Some of the numbers $a \leq r_2(t)$ will be marked by $\Box_s$, where $s \in S$. This will mean that the construction guarantees that a is a Q-witness for s, that is $\forall b Q(s,a,b)$.

We now describe stage t of the construction. We assume that Q_{t-1} has been constructed so that all properties P_1 through P_4 hold. In addition, we assume that each $n \leq r_2(t-1)$ either is a $(Q,t-1)$-witness of the form $a(n,t-1)$ (for some $n \leq t$) or has been marked by a $\Box_s$ for some $s \in S$.

Stage t. If $t \in S$ and some $a \leq r_2(t-1)$ is marked with $\Box_t$ then make a a (Q,t)-witness for s, set $r_2(t) = r_2(t-1)$, $r_3(t) = r_3(t-1)+t$, extend Q_{t-1} to Q_t in the $[0,t] \times [0, r_2(t)] \times [0, r_3(t)]$ keeping all the $(Q,t-1)$-witnesses as (Q,t)-witnesses so that Q_t satisfies all properties P_1 through P_4[2]. Otherwise, proceed as follows.

Compute H_t. Let $i_1, \ldots, i_k \leq t$ be in increasing order such that $h(i_j,t)$ is defined and $h(i_j,t) \neq h(i_j,t-1)$, $j=1,\ldots,k$. Note that $h(i_j,t-1)$ could be undefined. Also note that $k \leq 2$. Take the least unused numbers s_1 and $s_2 \in S$, mark each $a(i_j,t-1)$ with $\Box_{s_j}$, make sure that $a(i_j,t-1)$ is a (Q,t')-witness for s_j at all stages $t' \geq s_j$, $j=1,\ldots,k$. Further, take numbers $n_1 = r_2(t-1)+1$, $\ldots$, $n_k = r_2(t-1)+k$, set $a(i_j,t)=n_j$ for $j=1,\ldots,k$, $r_2(t)=n_k$, $r_3(t)=r_3(t-1)+(k+1)t$, and extend Q_{t-1} to Q_t in the $[0,t] \times [0,r_2(t)] \times [0,r_3(t)]$ making each $a(i_j,t)$ a (Q,t)-witness for i_j, keeping all the other $(Q,t-1)$-witnesses as (Q,t)-witnesses so that Q_t satisfies all properties P_1 through P_4. Note that P_4 can be satisfied as seen from the definition of $r_3(t)$.

Suppose that the sequence $i_1,\ldots,i_k \leq t$ stipulated above does not exist. Take the first unused $s \in S$ and mark t with $\Box_s$. Make sure that t is a

[2]Note that property P_4 can be satisfied which is seen from the definition of $r_3(t)$.

(Q,t')-witness for s at all stages $t' \geq s$. Set $r_2(t) = r_2(t-1)+1$, and $r_3(t) = r_3(t-1)+2t+1$, and extend Q_{t-1} to Q_t in the $[0,t]\times[0,r_2(t)]\times[0,r_3(t)]$ keeping all the $(Q,t-1)$-witnesses as (Q,t)-witnesses so that Q_t satisfies all properties P_1 through P_4. This ends Stage t.

Set $Q = \bigcup_t Q_t$. Now it is not hard to see that Q is a one to one representation of A. Indeed, note that at every stage t, each $a \leq r_2(t)$ is either marked by $\Box_s$ or of the form $a(n,t)$. If a is marked with $\Box_s$ then $\forall b Q(s,a,b)$ because a is a (Q,t')-witness for s at each stage $t' \geq s$. Assume that a is not marked with $\Box_s$, $s \in S$. Consider stage a. There is an n such that $a = a(n,a)$. Then for all $t \geq a$ we have $a(n,t) = a(n,a)$. Therefore $\forall b Q(n,a,b)$. Thus, each $a \in \omega$ is a Q-witness for some $n \in A$. All the other desired properties of Q follow from the fact that Q_t satisfies properties P_1 through P_4 at each stage t. The lemma is proved.

Clearly the definition of one to one presentations of Σ^0_2-sets can be relativised with respect to any oracle X. The relativised version of the lemma above is the following corollary which will be used in the next section.

Corollary 1 *Let A be a coinfinite $\Sigma_2^{0,X}$–set that possesses an infinite X-computable subset S such that $A \setminus S$ is infinite. Then there exists an X-computable set $Q \subset \omega^3$ such that Q is a one-to-one representation of A.*$\Box$

4 The Main Result

Consider the basic 0-structure $\mathcal{M}_0$ of the theory T_0. The following lemma shows that $\mathcal{M}_0$ can have presentations of arbitrarily high complexity.

Lemma 3 *For any set $X \subset \omega$ there exists an X-computable presentation of $\mathcal{M}_0$ such that the following properties hold:*

1. *The predicates X, Y, C_0, D_0 are computable.*
2. *The predicate $B(x)$ is T-equivalent to X'.*

Proof. We prove the lemma for the case when $X = \emptyset$. The case when $X \neq \emptyset$ can essentially be repeated. Thus, we need to prove that there exists a computable presentation of $\mathcal{M}_0$ such that the set $B(x)$ is Turing equivalent to the halting set K. We build the model $\mathcal{M}_0$ by stages. At stage t we will have a finite model $\mathcal{M}_0^t$ with finite predicates $X^t, Y^t, C_0^t, D_0^t, P_0^t, R_0^t$.

We may assume that ω is the disjoint union of infinite computable sets X, Y, C_0, D_0, and that $K \subset X$. At stage t the sequence of unordered pairs $\{a_0, b_0\}$, …, $\{a_t, b_t\}$ is called **active** if $X^t = \{a_0, b_0, \ldots, a_t, b_t\}$, $a_0, \ldots, a_t \in K_t$, and $b_0, b_1, \ldots, b_t \notin K_t$, and $a_0 < a_1 < \ldots < a_t$, $b_0 < b_1 < \ldots < b_t$, where $K_0 \subset K_1 \subset K_2 \subset \ldots$ is an approximation of K with $K = \bigcup_t K_t$. A pair $\{a, b\}$ is **active** if $\{a, b\} = \{a_i, b_i\}$ for some $i \leq t$. It is clear that for any unordered pair $\{a, b\}$ in X there exists a stage t such that $\{a, b\}$ is active at stage t if and only if $\{a, b\}$ is active at stages $n \geq t$. Now we describe the construction of $\mathcal{M}_t$ at stage t. At the initial stage, the model $\mathcal{M}_0^0$ is empty.

Stage t. We extend $\mathcal{M}_0^{t-1}$ satisfying the following conditions:

1. If $R_0(a, b, c, d_0)$ is false then we guarantee that $R_0(a, b, c, d)$ is true for all $d \neq d_0$ with $d \in D_0^t$.

2. For every pair $\{a, b\}$ that was not active at the previous stage but which has become active at stage t we take an unused element c, put it into C_0^t, and then guarantee that $R_0(a, b, c, d)$ holds for all $d \in D_0^t$. We call the element c the **t-witness** for the pair $\{a, b\}$.

3. If $(t-1)$-active pair $\{a, b\}$ is still active then we guarantee that the $(t-1)$-witness for $\{a, b\}$ is also a t-witness.

4. For every pair $\{a, b\}$ that is not t-active but which was $(t-1)$-active with the $(t-1)$-witness c, we enumerate an unused element d_0 into D_0^t and make $R_0(a, b, c, d_0)$ false.

5. We guarantee that for any t-active pair $\{a, b\}$, where $b \in K_t$, we have the following:

   (a) $P_0(a, y)$ is true for all $y \in Y^t$.

   (b) There is a unique $y \in Y^t$ such that $P_0(b, y)$ is false.

   (c) For any $y \in Y^t$ there exists a unique t-active pair $\{a, b\}$ such that $P_0(a, b, y)$ is false.

It is clear that the model $\mathcal{M}_0^t$ can be constructed effectively. Let $\mathcal{M}_0 = \bigcup_t \mathcal{M}_0^t$. It is not hard to see that $\mathcal{M}_0$ is the desired model. We also note that for the constructed model the permutation $\mu : X \to X$ is such that the ith element of K is sent to the ith element in $X \setminus K$. Thus, we have proved the lemma.

Now we are ready to prove our main theorem.

Theorem *For any natural number $n \geq 1$ there exists an $\aleph_1$–categorical theory T with a computable model so that T is equivalent to $\mathbf{0}^n$. Moreover, all (countable) models of T have computable presentations and T is almost strongly minimal.*

Proof. By the previous lemma there exists a $\mathbf{0}^n$-computable presentation of $\mathcal{M}_0$ such that the predicate $B(x)$ is equivalent to $\mathbf{0}^{n+1}$. From Corollary 1 we can construct the sequence $\{\mathcal{A}_i\}_{i \leq n}$ of models so that:

1. The model $\mathcal{A}_i$ is isomorphic to the i-structure $\mathcal{M}_i$.
2. The sets $X, Y, Z_1, Z_2, \ldots, Z_{2i}, C_0, D_0, C_1, D_1, \ldots, C_i, D_i$ in each model $\mathcal{A}_i$ are computable.
3. The model $\mathcal{A}_i$ is $\mathbf{0}^{n-i}$-computable.

Thus, each of the models, in particular the model $\mathcal{A}_n$, is $\aleph_1$-categorical. Now expand the model $\mathcal{A}_n$ by adding constant symbols c_x for each $x \in X$. Thus, we have the model $\mathcal{A} = (\mathcal{A}_n, c_x)_{x \in X}$. Let T be the theory of $\mathcal{A}$. The following now can easily be verified:

1. $\mathcal{A}$ is computable.
2. The theory of $\mathcal{A}$ is $\aleph_1$-categorical and is almost strongly minimal.
3. The set $\{B(c_x) \mid \mathcal{A} \models B(c_x)\}$ is a Σ_n^0–set and is c.e. in $\mathbf{0}^n$.
4. All models of T have computable presentations.

The theorem is proved.

5 Future Work

We are currently working on improving or generalizing the main theorem of this paper in the following directions. First of all, we hope to construct an $\aleph_1$-categorical model of a finite language for which the main result of this paper holds true. Secondly, we plan to adapt the construction of this paper to build $\aleph_0$–categorical models whose theories are Turing equivalent to $\mathbf{0}^n$. Finally, we are investigating a possibility of constructing $\aleph_1$–categorical or $\aleph_0$-categorical computable models whose theories have hyperarithmetical degrees.

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