CDMTCS

Research

Report

Series

n-ary quantum information defined by state partitions

Karl Svozil

University of Technology, Vienna

CDMTCS-184

May 2002

Centre for Discrete Mathematics and

Theoretical Computer Science

n-ary quantum information defined by state partitions

Karl Svozil*

Institut für Theoretische Physik, University of Technology Vienna,

Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria

Abstract

We define a measure of quantum information which is based on state partitions. Properties of this measure for entangled many-particle states are discussed. k particles specify k "nits" in such a way that k mutually commuting measurements of n-ary observables are necessary to determine the information.

PACS numbers: 03.65.Ta,03.67.-a

Keywords: quantum information theory,quantum measurement theory

As pointed out many times by Landauer and others (e.g., [1, 2]) the formal concept of *information* is tied to physics, at least as far as applicability is a concern. Thus it should come as no surprise that quantum mechanics requires fundamentally new concepts of information as compared to the ones appropriate for classical physics. And indeed, research into quantum information and computation theory has exploded in the last decade, bringing about a wealth of new ideas, potential applications, and formalisms.

Yet, there seems to be one issue, which, despite notable exceptions (e.g., [3, Footnote 6] and [4]), has not yet been acknowledged widely: the principal irreducibility of n-ary quantum information. An n-state particle can be prepared in a single one of n possible states. Then, this particle carries one "nit" of information, namely to "be in a single one from n different states." Subsequent measurements may confirm this statement. The most natural code basis for such a configuration is an n-ary code, and not a binary or decimal one.

Classically, there is no preferred code basis whatsoever. Every classical state is postulated to be determined by a point in phase space. Formally, this amounts to an infinite amount of information in whatever base, since with probability one, all points are random; i.e., algorithmically incompressible. [5, 6]. Operationally, only a finite amount of classical information is accessible. Yet, the particular base in which this finite amount of classical information is represented is purely conventional.

Let us mention some notation and setup first. Consider a particle which can be observed in a single one of a finite number n of possible operationally distinct and comeasurable properties. The system's state is formalized by a self-adjoint, positive state operator of trace class ρ of an n-dimensional Hilbert space. Any pure state, whether "entangled" or not, is characterized by a onedimensional subspace, which in turn corresponds to a onedimensional projection operator $\rho^2 = \rho$.

Every operationally distinct property corresponds to the proposition (we shall use identical symbols to denote logical propositions and operators) E_i, $1 \leq i \leq k$ that the system, when measured, has the associated property. Propositions are formalized by projection operators, which are dichotomic observables, since $E_i^2 = E_i$ is only satisfied for the eigenvalues 0 and 1 [7].

Thus, any complete set of n base vectors has a dual interpretation: either as an orthonormal set of base states whose linear span is the n-dimensional Hilbert space, or as a maximally comeasurable and operationally distinct set of observables corresponding to propositions such as, "*the physical system is in a pure state corresponding to the respective basis vector.*" Any such proposition is operationalized by a measurement, ideally by registering a click in a particle counter.

All bases corresponding to the k-particle case are obtained in two steps: (i) In the first step, a system of pure basis vectors $\{\rho_1, \ldots, \rho_{n^k}\}$ is formed by taking the tensor products $\rho^s_{i_1} \otimes \rho^s_{i_2} \otimes \cdots \otimes \rho^s_{i_k}$, $1 \leq i_1, i_2, \ldots, i_k \leq n$, of all the single-particle projection operators $\rho^s_{i_1}$, $\rho^s_{i_2} \ldots \rho^s_{i_k}$. (ii) In the second step, this system of basis vector undergoes a unitary transformation $U(n^k)$ represented by a $n^k \times n^k$-matrix. The transformed states need no longer decompose into a product of single particle states, a property called "entanglement" by Schrödinger [8]. From this point of view, entangled state bases are unitary equivalent to nonentangled ones. As a consequence, propositions need no longer refer to attributes or properties of single particles alone, but to joint properties of particles [3, 9].

In what follows, let us always consider a complete system of base states $\mathcal{B}$ associated with a unique "context" [10] or "communication frame" $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$, which are defined as a minimal set of co-measurable n-ary observables. For $n = 2$, their explicit form has been enumerated in [9]. In this particular case, the F's can be identified with certain projection operators from the set of all possible mutually orthogonal ones, whose two eigenvalues can be identified with the two states. For three or more particles, this is no longer possible (see below). A well-known theorem of linear algebra (e.g., [11]) states that there exists a single "context operator" such that all the $F_i \in \mathcal{F}$ are just polynomials of it. That is, a single measurement exists which determines all the observables associated with the context at once.

For a single n-state particle, the nit can be formalized by as a state partition which is fine grained into n elements, one state per element. That is, if the set of states is represented by $\{1, \ldots, n\}$, then the nit is defined by

$$\{\{1\}, \ldots, \{n\}\}. \tag{1}$$

Of course, any labeling would suffice, as long as the structure is preserved. It does not matter whether one calls the states, for instance, "+," "0" and "-", or "1," "2" and "3", resulting in a trit represented by $\{\{+\}, \{0\}, \{-\}\}$ or $\{\{1\}, \{2\}, \{3\}\}$. Thus, nits are defined modulo isomorphisms (i.e., one-to-one translations) of the state labels. To complete the setup of the single particle case, let us recall that any such state set would correspond to an orthonormal basis of n-dimensional Hilbert space.

Before proceeding to the most general case, we shall consider the case of two particles with three states per particle in all details. The methods developed [9] for case of k particles with two states per particle cannot be directly adopted here, since the idempotence ($E_iE_i = E_i$) of the projection operators, which maps the two states onto the two possible eigenvalues $0, 1$ cannot be generalized.

Instead we shall start by considering the optimal partitions of states and construct the appropriate Hilbert space operators from there. Assume that the first and second particle has three orthogonal states labeled by a_1, b_1, c_1 and a_2, b_2, c_2, respectively (the subscript denotes the particle number; a_1a_2 stands for $|a_1\rangle \otimes |a_2\rangle = |a_1a_2\rangle$). Then nine product states can be formed and labeled from 1 to 9 in lexicographic order; i.e.,

$$\begin{aligned} a_1a_2 &\equiv 1, \\ a_1b_2 &\equiv 2, \\ a_1c_2 &\equiv 3, \\ b_1a_2 &\equiv 4, \\ &\vdots \\ c_1c_2 &\equiv 9. \end{aligned} \tag{2}$$

Any maximal set of co-measurable 3-valued observables induces two state partitions of the set of states $S = \{1, 2, \ldots, 9\}$ with three partition elements with the properties that (i) the set theoretic intersection of any two elements of the two partitions is a single state, and (ii) the union of all these nine intersections is just the set of state S. As can be easily checked, an example for such state partitions are

$$\begin{aligned} F_1 &= \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\} \equiv \{\{a_1\},\{b_1\},\{c_1\}\}, \\ F_2 &= \{\{1,4,7\},\{2,5,8\},\{3,6,9\}\} \equiv \{\{a_2\},\{b_2\},\{c_2\}\}. \end{aligned} \tag{3}$$

Operationally, the "trit" F_1 can be obtained by measuring the first particle state: $\{1,2,3\}$ is associated with state a_1, $\{4,5,6\}$ is associated with b_1, and $\{4,5,6\}$ is associated with c_1. The "trit" F_2 can be obtained by measuring the state of the second particle: $\{1,4,7\}$ is associated with state a_2, $\{2,5,8\}$ is associated with b_2, and $\{3,6,9\}$ is associated with c_2. This amounts to the operationalization of the trits (3) as state filters. In the above case, the filters are "local" and can be realized on single particles, one trit per particle. In the more general case of rotated "entangled" states (cf. below), the trits (more generally, nits) become inevitably associated with joint properties of ensembles of particles. Measurement of the propositions, "*the particle is in state* $\{1,2,3\}$" and, "*the particle is in state* $\{3,6,9\}$" can be evaluated by taking the set theoretic intersection of the respective sets; i.e., by the proposition, "*the particle is in state* $\{1,2,3\}\cap\{3,6,9\}=3$." In figure 1, the state partitions are drawn as cells of a twodimensional square spanned by the single cells of the two three-state particles.

A Hilbert space representation of this setting can be obtained as follows. Define the states in S to be onedimensional linear subspaces of nine-dimensional Hilbert space; e.g.,

$$\begin{aligned} 1 &\equiv (1,0,0,0,0,0,0,0,0), \\ &\vdots \\ 9 &\equiv (0,0,0,0,0,0,0,0,1). \end{aligned} \tag{4}$$

The trit operators are given by (trit operators, observables and the corresponding state partitions will be used synonymously)

$$\begin{aligned} F_1 &= \mathrm{diag}(a,a,a,b,b,b,c,c,c), \\ F_2 &= \mathrm{diag}(a,b,c,a,b,c,a,b,c), \end{aligned} \tag{5}$$

for $a,b,c\in\mathbf{R}$, $a\neq b\neq c\neq a$.

If $F_2=\mathrm{diag}(d,e,f,d,e,f,d,e,f)$ and a,b,c,d,e,f, are six different prime numbers, then, due to the uniqueness of prime decompositions, the two trit operators can be combined to a single "context" operator

$$C=F_1\cdot F_2=F_2\cdot F_1=\mathrm{diag}(ad,ae,af,bd,be,bf,cd,ce,cf) \tag{6}$$

which acts on both particles and has nine different eigenvalues. Just as for the two-particle case [9], there exist $2^3!=9!=362880$ permutations of operators which are all able to separate the nine states. They are obtained by forming a (2×9)-matrix whose rows are the diagonal components of F_1 and F_2 from Eq. (5) and permuting all the columns. The resulting new operators F_1' and F_2' are also trit operators.

A generalization to k particles in n states per particle is straightforward. We obtain (i) k partitions of the product states with (ii) n elements per partition in such a way that (iii) every single product state is obtained by the set theoretic intersection of k elements of all the different partitions.

Every single such partition can be interpreted as a nit. All such sets are generated by permuting the set of states, which amounts to $n^k!$ equivalent sets of state partitions. However, since they are mere one-to-one translations, they represent the same trits. This equivalence, however, does not concern the property of (non)entanglement, since the permutations take entangled states into nonentangled ones. We shall give an example below.

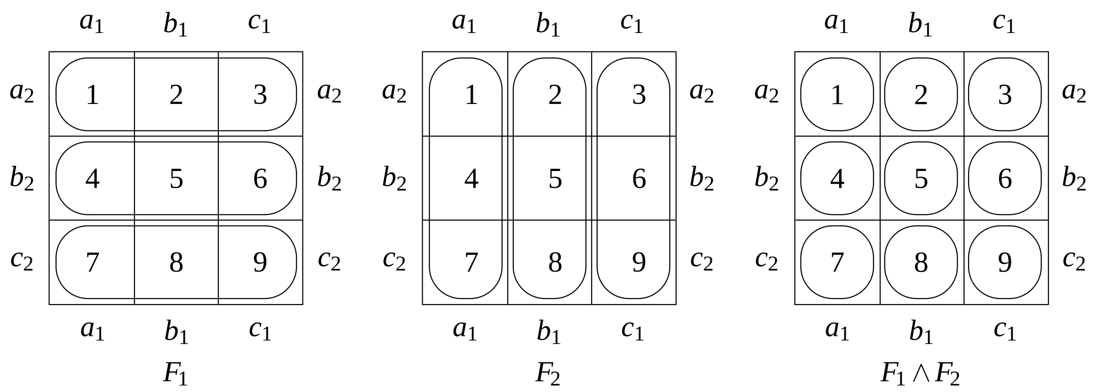

FIG. 1: Representation of state partitions as cells of a twodimensional square spanned by the single cells of the two three-state particles.

Again, the standard orthonormal basis of n^k-dimensional Hilbert space is identified with the set of states $S = \{1, 2, \ldots, n^k\}$; i.e., (superscript "$T$" indicates transposition)

$$
\begin{aligned}
1 &\equiv (1, \ldots, 0)^T \equiv \mid 1, \ldots, 1\rangle = \mid 1\rangle \otimes \cdots \otimes \mid 1\rangle, \\
&\vdots \\
n^k &\equiv (0, \ldots, 1)^T \equiv \mid n, \ldots, n\rangle = \mid n\rangle \otimes \cdots \otimes \mid n\rangle.
\end{aligned}
\tag{7}
$$

The single-particle states are also labeled by 1 through n, and the tensor product states are formed and ordered lexicographically ($0 < 1$).

The nit operators are defined via diagonal matrices which contain equal amounts n^{k-1} of mutually n different numbers such as different primes $q_1, \ldots, q_n$; i.e.,

$$
\begin{aligned}
F_1 &= \mathrm{diag}(\underbrace{\underbrace{q_1, \ldots, q_1}_{n^{k-1}\text{ times}}, \ldots, \underbrace{q_n, \ldots, q_n}_{n^{k-1}\text{ times}}}_{n^0\text{ times}}), \\
F_2 &= \mathrm{diag}(\underbrace{\underbrace{q_1, \ldots, q_1}_{n^{k-2}\text{ times}}, \ldots, \underbrace{q_n, \ldots, q_n}_{n^{k-2}\text{ times}}}_{n^1\text{ times}}), \\
&\vdots \\
F_k &= \mathrm{diag}(\underbrace{q_1, \ldots, q_n}_{n^{k-1}\text{ times}}).
\end{aligned}
\tag{8}
$$

The operators implement an n-ary search strategy, filtering the search space into n equal partitions of states, such that a successive applications of all such filters renders a single state.

There exist $n^k!$ sets of nit operators, which are are obtained by forming a $(n^k \times n^k)$-matrix whose rows are the diagonal components of $F_1, \ldots, F_k$ from Eq. (8) and permuting all the columns. The resulting new operators $F_1', \ldots, F_k'$ are also nit operators.

All partitions discussed so far are equally weighted and well balanced, as all elements of them contain an equal number of states. In principle, one could also consider nonbalanced partitions. For example, one could take the partition $\overline{F}_1 = \{\{1\}, \{2,3\}, \{4,5,6,7,8,9\}\}$ instead of F_1 in (3), represented the by trit diagonal operator $\mathrm{diag}(a,b,b,c,c,c,c,c,c)$. Yet any such attempt would result in a deviation from the optimal n-ary search tactics, and in an nonoptimal measurement procedures. Another, more principal, disadvantage would be the fact that such a state separation could not reflect the inevitable n-arity of the quantum choice.

The most important feature of entangled states is that the nits, or propositions corresponding to F_i, are no longer single-particle properties, but in general depend on the joint properties of all the k particles. Formally, this feature of entanglement is reflected in the isometric (unitary) transformation of the standard state basis (7), such that this unitary transformation leads outside the domain of single particle unitary transformations. One could define a measure of entanglement by considering the sum total of all parameter differences from the nearest nonentangled states. This amounts to a nontrivial group theoretic question [12]. In terms of partitions, a different approach is possible. Entanglement occurs for diagonal or antidiagonal arrangements of states which do not add up to completed blocks.

Take, for example, the state partition scheme of Fig. 1, which results in nonentangled states and state measurements. A modified, entangled scheme can be established by just grouping the states into diagonal and counterdiagonal groups as drawn in Fig. 2. The corresponding trits are

$$\begin{aligned} F_1 &= \{\{1,5,9\},\{2,6,7\},\{3,4,8\}\}, \\ F_2 &= \{\{1,6,8\},\{2,4,9\},\{3,5,7\}\}. \end{aligned} \tag{9}$$

We can now introduce new 2×3 basis vectors grouped into the two bases $\{a_1',b_1',c_1'\}$ and $\{a_2',b_2',c_2'\}$ by

$$\begin{aligned} |a_1'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1a_2\rangle + |b_1b_2\rangle + |c_1c_2\rangle), \\ |b_1'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1b_2\rangle + |b_1c_2\rangle + |c_1a_2\rangle), \\ |c_1'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1c_2\rangle + |b_1a_2\rangle + |c_1b_2\rangle), \\ |a_2'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1a_2\rangle + |b_1c_2\rangle + |c_1b_2\rangle), \\ |b_2'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1b_2\rangle + |b_1a_2\rangle + |c_1c_2\rangle), \\ |c_2'\rangle &= \tfrac{1}{\sqrt{3}}(|a_1c_2\rangle + |b_1b_2\rangle + |c_1a_2\rangle). \end{aligned} \tag{10}$$

The new orthonormal basis states are "entangled" with respect to the old bases and *vice versa*. Their tensor products generate a complete set of basis states in a new nine-dimensional Hilbert space. In terms of the new basis states, the trits can be written as $F_1 \equiv \{\{a_1'\},\{b_1'\},\{c_1'\}\}$ and $F_2 \equiv \{\{a_2'\},\{b_2'\},\{c_2'\}\}$. The associated bases will be called *diagonal bases*. Note that the permutation which produces the entangled case (9) the nonentangled (3) one is $1\rightarrow 1$, $2\rightarrow 9$, $3\rightarrow 5$, $4\rightarrow 6$, $5\rightarrow 2$, $6\rightarrow 7$, $7\rightarrow 8$, $8\rightarrow 4$, $9\rightarrow 3$, or $(1)(2,9,3,5)(4,6,7,8)$ in cycle form.

A generalization to diagonal bases associated with an arbitrary number of nits is straightforward. We conjecture that the basis generated by the diagonal bases are "maximally entangled" with respect to group theoretic measures of entanglement.

If Hilbert space is taken for granted as a valid formalization of quantum mechanics, then Gleason's theorem [13] and many later results [14–17], most notably the theorems by Bell [18, 19], Kochen & Specker [20], and Greenberger, Horne & Zeilinger [21] state the impossibility of (noncontextual) value definiteness. That is, there does not seem to exist elements of reality to every conceivable observable but a single (complete) one.

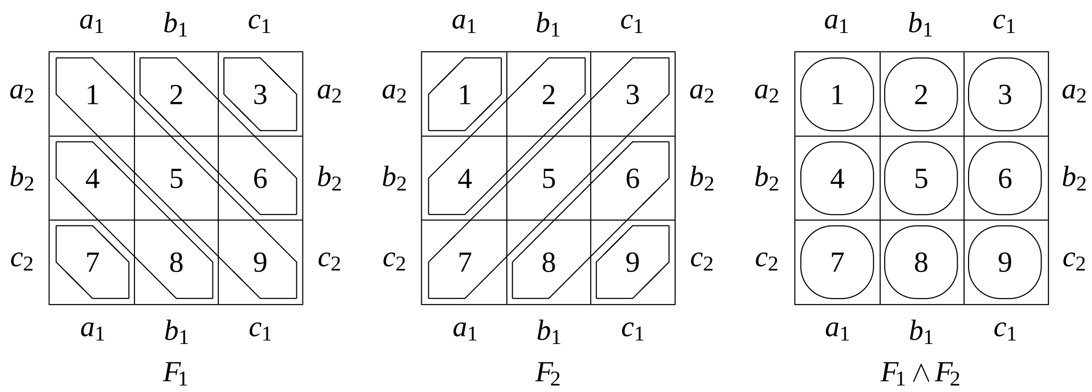

FIG. 2: Entangled schemes through diagonalization and counterdiagonalization of the states.

One possible consequence (among many others, not necessarily compatible ones) of this fact may be the assumption that a quantum state of k n-state particles carries just k nits, and no more information. It should be stressed that, from this point of view, all other conceivable information relating to counterfactual elements of physical reality corresponding to state bases different from the encoded one are nonexistent. The measurement outcomes in such improper communication frames are randomized (maybe because the measurement apparatus serves as an randomizing interface to the proper communication frame, maybe because the information cannot be interpreted correctly). If interrogated "correctly" (i.e., in the proper communication frame), it will pass on this information unambiguously. In such a frame, but only in this particular one, this information is pseudo-classical and therefore can also be copied or "cloned." Such a point of view is suggested, to some extend, by Peres' statement that "unperformed experiments have no results" [22], and is compatible to Zeilinger's foundational principle for quantum mechanics [3] stating that "*the most elementary quantum system represents the truth value of one proposition.*"

The quantum logical description allows for two possible representations. (i) Every single partition F_i generates a subalgebra of the Boolean algebra of states. All the subalgebras corresponding to the k partitions and their can then be pasted together to form a nonboolean lattice. (ii) All partitions can be used to generate the context operator (cf. above) which generates the finest partition. Every element of the latter partition can then be identified with the atoms of a Boolean subalgebra.

Every such system can be modeled by a finite automaton partition logic [23–27] or a generalized urn model [28–30]. All filters combined render a Boolean algebra 2^{n^k} with n^k atoms.

As regards the binary codes and their relation to n-ary ones, by a well known theorem for unitary operators [31], any quantum measurement of an k-ary system can be decomposed into binary measurements. Also, it is possible to group the k possible outcomes into binary filters of ever finer resolution; calling the successive outcomes of these filter process the "binary code." Yet, all these attempts result in codes with undesirable features. Unitary decompositions in general yield noncomeasurable observables and thus to nonoperationalizability. Filters are inefficient, and so may be binary codes [32].

So far, the main emphasis in the area of quantum computation has been in the area of binary decision problems. It is suggested that these investigations should be extended to k-ary decision problems (e.g., [33, pp. 332-340]), for which quantum information theory seems to be extraordinarily well equipped.

* Electronic address: svozil@tuwien.ac.at; URL: http://tph.tuwien.ac.at/~svozil

[1] R. Landauer, Physics Today **44**, 23 (1991).

[2] R. P. Feynman, *The Feynman lectures on computation* (Addison-Wesley Publishing Company, Reading, MA, 1996), edited by A.J.G. Hey and R. W. Allen.

[3] A. Zeilinger, Foundations of Physics **29**, 631 (1999).

[4] A. Muthukrishnan and J. C. R. Strod, Physical Review **A62**, 052309 (2000).

[5] G. J. Chaitin, *Information, Randomness and Incompleteness* (World Scientific, Singapore, 1990), 2nd ed., this is a collection of G. Chaitin's publications.

[6] C. Calude, *Information and Randomness—An Algorithmic Perspective* (Springer, Berlin, 1994).

[7] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), english translation: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955.

[8] E. Schrödinger, Naturwissenschaften **23**, 807 (1935), english translation in [34] and [35, pp. 152-167]; a copy can be found at www.emr.hibu.no/lars/eng/cat/.

[9] N. Donath and K. Svozil, Physical Review A **65**, 044302 (2002), arXiv:quant-ph/0105046, also published in [36].

[10] K. Svozil (2001), quant-ph/0012066.

[11] P. R. Halmos, *Finite-Dimensional Vector spaces* (Springer, New York, Heidelberg, Berlin, 1974).

[12] R. Dirl and K. Svozil (2002), in preparation.

[13] A. M. Gleason, Journal of Mathematics and Mechanics **6**, 885 (1957).

[14] F. Kamber, Nachr. Akad. Wiss. Göttingen **10**, 103 (1964).

[15] F. Kamber, Mathematische Annalen **158**, 158 (1965).

[16] N. Zierler and M. Schlessinger, Duke Mathematical Journal **32**, 251 (1965).

[17] C. Calude, P. Hertling, and K. Svozil, Foundations of Physics **29**, 349 (1999).

[18] J. S. Bell, Physics **1**, 195 (1964), reprinted in [35, pp. 403-408] and in [37, pp. 14-21].

[19] N. D. Mermin, Reviews of Modern Physics **65**, 803 (1993).

[20] S. Kochen and E. P. Specker, Journal of Mathematics and Mechanics **17**, 59 (1967), reprinted in [38, pp. 235–263].

[21] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, American Journal of Physics **58**, 1131 (1990).

[22] A. Peres, American Journal of Physics **46**, 745 (1978).

[23] K. Svozil, *Randomness & Undecidability in Physics* (World Scientific, Singapore, 1993).

[24] M. Schaller and K. Svozil, International Journal of Theoretical Physics **35**, 911 (1996).

[25] A. Dvurečenskij, S. Pulmannová, and K. Svozil, Helvetica Physica Acta **68**, 407 (1995).

[26] C. Calude, E. Calude, K. Svozil, and S. Yu, International Journal of Theoretical Physics **36**, 1495 (1997).

[27] K. Svozil, *Quantum Logic* (Springer, Singapore, 1998).

[28] R. Wright, in *Mathematical Foundations of Quantum Theory*, edited by A. R. Marlow (Academic Press, New York, 1978), pp. 255–274.

[29] R. Wright, Foundations of Physics **20**, 881 (1990).

[30] K. Svozil (2002), e-print `arXiv:quant-ph/in preparation`.

[31] F. D. Murnaghan, *The Unitary and Rotation Groups* (Spartan Books, Washington, 1962).

[32] C. Calude and C. Câmpeanu, Complexity **5**, 47 (1996).

[33] S. C. Kleene (North-Holland, Amsterdam, 1952).

[34] J. D. Trimmer, Proc. Am. Phil. Soc. **124**, 323 (1980), reprinted in [35, pp. 152-167].

[35] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1983).

[36] N. Donath and K. Svozil, Virtual Journal of Quantum Information **2** (2002).

[37] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).

[38] E. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).