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Complexity of Computable Models

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1 Introduction

One of the themes of computable model theory is concerned with the following two related questions. Let T be a first order consistent theory. Does there exist a computable model of T? If T has a computable model then what is the computability-theoretic complexity, e.g. Turing degree, of T? It is well known that if T is decidable then T has a decidable model, that is one for which the satisfaction predicate is decidable. On the other hand, if theory T has a computable model then T is computable in $\mathbf{0}^{\omega}$. For example, the theory of arithmetic $(\omega, S, +\times, \leq, 0)$ is Turing equivalent to $\mathbf{0}^{\omega}$. We also add that there are examples of finitely axiomatizable (and hence computably enumerable) theories which have no computable models. In this paper, for any natural number $n \geq 1$, we present examples of $\aleph_1$–categorical computable models as well as $\aleph_0$–categorical computable models whose theories are Turing equivalent to $\mathbf{0}^n$. Moreover, the languages of these models are finite. We now list some of the related results. In [1] Baldwin and Lachlan showed that all models of any $\aleph_1$-categorical theory T can be listed into the chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots \mathcal{A}_\omega$ of elementary embeddings, where $\mathcal{A}_0$ is the prime model, $\mathcal{A}_\omega$ is the saturated model, and each $\mathcal{A}_{i+1}$ is a minimal proper elementary extension of $\mathcal{A}_i$. Let $SCM(T)$ be the spectrum of computable models of T, that is $SCM(T) = \{i \mid \mathcal{A}_i$ has a computable presentation $\}$. If T is $\aleph_1$–categorical and decidable then, as proved by Harrington

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and Khisamiev in [5] [7], all countable models of T have decidable presentations, that is $SCM(T) = \omega \bigcup \{\omega\}$. In [3] Goncharov showed that there exists an $\aleph_1$-categorical theory T computable in $\mathbf{0}'$ for which $SCM(T) = \{0\}$. Kudeiberganov extended this result by showing that for every $k \geq 0$ there exists an $\aleph_1$-categorical T computable in $\mathbf{0}'$ such that $SCM(T) = \{0, 1 \ldots, k\}$ [10]. In [8] it is shown that there exist $\aleph_1$-categorical theories T_1 and T_2 computable in $\mathbf{0}''$ such that $SCM(T_1) = \omega$ and $SCM(T_2) = \omega \bigcup \{\omega\} \setminus \{0\}$. Thus, all the known $\aleph_1$–categorical theories that have computable models are computable in $\mathbf{0}''$. In [4] the authors, for any given natural number $n \geq 1$, construct examples of $\aleph_1$–categorical computable models whose theories are Turing equivalent to $\mathbf{0}^n$. However, those constructed models have infinite langauges. Lerman and Schmerl in [11] give some sufficient conditions for countably categorical *arithmetic* theories to have a constructive model. More precisely, they show that if T is a countably categorical arithmetical theory such that the set of all sentences beginning with an existential quantifier and having $n+1$ alternations of quantifiers is Σ^0_{n+1} for each n, then T has a constructive model. Knight improves this result in [9] by allowing certain uniformity and omitting the requirement that T is arithmetical. However, all the known examples of $\aleph_0$-categorical computable models have been known to have low complexity, and it has not even been known whether or not there are examples that satisfy the conditions stated in Lerman and Schmerl theorems for sufficiently large n. In this paper we provide such examples.

We now give basic definitions. We fix a computable language L. A structure $\mathcal{A}$ of this language is **computable** if the domain, functions, and predicates of the structure are uniformly computable. This is equivalent to saying that the atomic diagram of $\mathcal{A}$ is computable. A structure $\mathcal{B}$ is **computably presentable** if it is isomorphic to a computable structure. In this case any isomorphism from $\mathcal{B}$ into $\mathcal{A}$ is called a **computable presentation** of $\mathcal{B}$. A complete theory T is $\aleph_1$–**categorical** if all models of T of power $\aleph_1$ are isomorphic. Similarly, a complete theory T is $\aleph_0$–**categorical** if all countable models of T are isomorphic. A model $\mathcal{M}$ is $\aleph_1$–**categorical** ($\aleph_0$-**categorical**) if the theory $Th(\mathcal{M})$ of the model is $\aleph_1$–categorical ($\aleph_0$-**categorical**). Typical examples of $\aleph_1$–categorical theories are the theory of algebraically closed fields of fixed characteristic, the theory of vector spaces over a fixed countable field, the theory of the successor structure (ω, S). Rational numbers with the natural ordering or random structures (see for example [6]) are typical examples of $\aleph_0$-categorical structures.

Now we briefly outline the paper. In the next section, Section 2, we define two a model–theoretic extension operators. The definition of these operators follow the ideas of Marker's construction from [12]. Therefore we call the operators Marker's $\exists$-extension and $\forall$-extension operators. In Section 3 we prove a representation lemma about Σ^0_2–subsets of natural numbers. Finally, in the last two sections we prove the following two theorems:

Theorem 1 *For any natural number $n \geq 1$ there exists an $\aleph_1$–categorical theory T with a computable model of a finite language so that T is equivalent to $\mathbf{0}^n$. Moreover, all (countable) models of T have computable presentations and T is strongly minimal.*

Theorem 2 *For any natural number $n \geq 1$ there exists an $\aleph_0$–categorical theory T with a computable model of a finite language so that T is equivalent to $\mathbf{0}^n$.*

We assume that the reader is familiar with basics of model theory and computability theory. We use some standard notions and notations, such as $< \cdot, \cdot >$, l, r Cantor's pairing functions, the concept of X-computable sets (e.g. sets computable with an oracle for X), the jump operation X' for subsets $X \subset \omega$. Standard references are [2] [14]. Finally, we may interchange the words model and structure.

2 Marker's Extensions

In [12] Marker, for any given natural $n \geq 0$, constructed a totally categorical almost strongly minimal non Σ_n-axiomatizable theory. The construction is carried out for building a particular structure. In this section we adapt Marker's construction for general case.

Let L be a finite language containing no functional symbols, and let $\mathcal{A} = (A, P_0^{n_0}, \ldots, P_m^{n_m})$ be a structure of language L. We assume that for each predicate P of this structure the sets $A^t \setminus P$ and P are both infinite, where t is the arity of P. Take any of the predicates P of this structure whose arity is k.

Marker's $\exists$-extension of this predicate, denoted by $P_\exists$, is defined as follows. Let X be an infinite set disjoint with A. Then $P_\exists$ is the predicate of arity $k+1$ defined by the following rules:

1. If $P_\exists(a_1, a_2, \ldots,_k, a_{k+1})$ then $P(a_1, \ldots, a_k)$ and $a_{k+1} \in X$.

2. For every $a_{k+1} \in X$ there is a unique k tuple $(a_1, \ldots, a_k)$ such that $P_\exists(a_1, a_2, \ldots,_k, a_{k+1})$.

3. If $P(a_1, \ldots, a_k)$ then there is a unique a such that $P_\exists(a_1, a_2, \ldots,_k, a)$.

Marker's $\forall$-extension of the predicate P, denoted by $P_\forall$, is defined as follows. Let X be an infinite set disjoint with A. Then $P_\forall$ is the predicate of arity $k+1$ defined by the following rules:

1. If $P_\forall(a_1, a_2, \ldots,_k, a_{k+1})$ then $a_1, \ldots, a_k \in A$ and $a_{k+1} \in X$.

2. For all $(a_1, \ldots, a_k) \in A$ there is at most one $a_{k+1} \in X$ such that $\neg P_\forall(a_1, a_2, \ldots,_k, a_{k+1})$.

3. If then $P_\forall(a_1, a_2, \ldots,_k, a_{k+1})$ for all $a_{k+1} \in X$ then $P(a_1, \ldots, a_k)$.

4. For every $a_{k+1} \in X$ there is a unique k tuple $(a_1, \ldots, a_k)$ such that $\neg P_\exists(a_1, a_2, \ldots, a_k, a_{k+1})$.

We call the set X in any of the $\exists$ and $\forall$-extensions a **fellow of** P. Here is our definition.

Definition 1 *Let $\mathcal{A} = (A, P_0^{n_0}, \ldots, P_m^{n_m})$ be a model.*

1. *The model $\mathcal{A}_\exists$ is $(A, P_0^{n_0+1}, \ldots, P_m^{n_m+1}, X_0, X_1, \ldots, X_m)$, where each $P_i^{n_i+1}$, $i = 0, \ldots, m$, is Marker's $\exists$-extension of $P_i^{n_i}$ so that fellows of distinct predicates are pairwise disjoint sets.*

2. *Similarly, $\mathcal{A}_\forall$ is the model $(A, P_0^{n_0+1}, \ldots, P_m^{n_m+1}, X_0, X_1, \ldots, X_m)$, where each $P_i^{n_i+1}$, $i = 0, \ldots, m$, is Marker's $\forall$-extension of $P_i^{n_i}$ so that fellows of distinct predicates are pairwise disjoint sets.*

The next simple but important result lists the basic properties of Marker's extensions.

Theorem *Let $\mathcal{A}_\exists$ and $\mathcal{A}_\forall$ be the Marker's extensions of the model $\mathcal{A}$. These extensions satisfy the following properties:*

1. *The model $\mathcal{A}$ is definable in each of the extensions.*

2. *If the theory of $\mathcal{A}$ is $\aleph_0$-categorical then so is the theory of each of the extensions.*

3. *If the theory of $\mathcal{A}$ is $\aleph_1$-categorical then so is the theory of each of the extensions.*

4. *If the theory of $\mathcal{A}$ is (almost) strongly minimal then so is the theory of each of the extensions.*

5. *Any automorphism of $\mathcal{A}$ can be extended to automorphisms of each of the extensions.*

Proof. Part 1). Let $X_0, X_1, \ldots, X_m$ be all the fellows nedded to define the predicates $P_0^{n_0+1}, \ldots, P_m^{n_m+1}$ in any of Marker's extensions. The predicate $\&_{j=0}^m \neg X_i(x)$ defines the original domain of the structure $\mathcal{A}$. Clearly, in the model $\mathcal{A}_\exists$ the predicate $P_i^{n_i}$ is definable by the formula $\exists x P_i^{n_i+1}(x_1, \ldots, x_{n_i}, x)$. Similarly, the formula $\forall x P_i^{n_i+1}(x_1, \ldots, x_{n_i}, x)$ defines the predicate $P_i^{n_i}$ in the model $\mathcal{A}_\forall$. Thus, Part 1) is proved.

Proofs of Parts 2), 3) and 4) follow from the fact that every element in any of the extensions is algebraic over the original domain. Indeed, assume that $a \in X$, and X is the fellow of a predicate P. In the $\exists$-extension there exists a unique tuple $(a_1, \ldots, a_k)$ in the domain of $\mathcal{A}$ so that $P_\exists(a_1, \ldots, a_k, a)$. Similary, in the $\forall$-extension there exists a unique tuple $(a_1, \ldots, a_k)$ in the domain of $\mathcal{A}$ so that $\neg P_\forall(a_1, \ldots, a_k, a)$.

Let $\alpha : \mathcal{A} \to \mathcal{A}$ be an automorphism. We want to extend α to $\mathcal{A}_\exists$. Take an $x \in X$, where X is the fellow of a predicate $P_\exists$. There exists a unique tuple $(a_1, \ldots, a_k)$ in the domain of $\mathcal{A}$ so that $P_\exists(a_1, \ldots, a_k, x)$. Note that $P(\alpha(a_1), \ldots, \alpha(a_k))$. Therefore, by the definition of $P_\exists$ there is a unique $y \in X$ such that $P_\exists(\alpha(a_1), \ldots, \alpha(a_k), y)$. Set $\alpha'(x) = y$. It is not hard to see that α' is an automorphism of $\mathcal{A}_\exists$ that extends α. The automorphism α can be extended to $\mathcal{A}_\forall$ in a similar manner. The theorem is proved.

The Marker's extensions allow us to extend the underlying structures inductively as follows. Let $\mathcal{A}$ be a structure, and w be a word over the alphabet $\{\exists, \forall\}$. Define $\mathcal{A}_w$ by induction as follows. If w is the empty string then $\mathcal{A}_w = \mathcal{A}$. Assume that $w = w'\exists$ or $w = w'\forall$ and $\mathcal{B} = \mathcal{A}_{w'}$. Define $\mathcal{A}_{w'\exists} = \mathcal{B}_\exists$ and $\mathcal{A}_{w'\forall} = \mathcal{B}_\forall$. Therefore we have the following colrollary.

Corollary 1 *Let $\mathcal{A}$ be a structure, and w be a word over the alphabet $\{\exists, \forall\}$. Each of the following is true:*

1. *The model $\mathcal{A}$ is definable in $\mathcal{A}_w$.*

2. *If the theory of $\mathcal{A}$ is $\aleph_0$-categorical ($\aleph_1$-categorical, (almost) strongly minimal) then so is the theory of $\mathcal{A}_w$.*

3. *Any automorphism of $\mathcal{A}$ can be extended to an automorphism of $\mathcal{A}_w$.* $\square$

Our goal in the next sections will be to show that the model $\mathcal{A}_{\exists\forall}$ is less complex than the model $\mathcal{A}$ itself from a computability theoretic point of view.

3 On Presentations of Σ_2^0-Sets

In this section we prove a computability-theoretic lemma needed for the main result of this paper. For the lemma we define the following notion.

Definition 2 *A Σ_2^0–set A is* **one-to-one representable** *if for some computable predicate $Q \subset \omega^3$ each of the following properties is true:*

1. *For each $n \in \omega$, $\exists a \forall b Q(n,a,b)$ if and only if $n \in A$.*

2. *For each $n \in \omega$, $\exists a \forall b Q(n,a,b)$ if and only if $\exists^{=1} a \forall b Q(n,a,b)$*[1].

3. *For every b there is a unique pair $< n, a >$ such that $\neg Q(n,a,b)$.*

4. *For every pair $< n, a >$ either $\exists b \neg Q(n,a,b)$ or $\forall b Q(n,a,b)$.*

5. *For every a there exists a unique n such that $\forall b Q(n,a,b)$.*

It is not hard to see that every infinite and coinfinite computable set A has a one-to-one representation.

For a Σ_2^0–set A there is a computable H such that $n \in A \leftrightarrow \exists a \forall b H(n,a,b)$. In fact, there is a computable Q for which $\exists a \forall b H(n,a,b) \leftrightarrow \exists^{=1} a \forall b Q(n,a,b)$. To show this we describe the procedure which builds a predicate P_n, $n \in \omega$. To build P_n initially we set the values $a_0 = 0$, $r_0 = 0$, $h_0 = 0$. At stage t the predicate P_n will be defined on all pairs (i,j) so that $j \leq t$, $i \leq r_t$. The intention for a_t is that a_t will be the unique witness for n to belong to A, that is $n \in A$ if and only if $\forall b P_n(a_t, b)$. The intention for h_t is that if $n \in A$ then h_t is the minimal $h \leq t$ for which $(\forall b \leq t) H(n,h,b)$.

Stage $t+1$. Compute $H(n,i,j)$ for all $i, j \leq t+1$. If $(\forall i \leq t+1)(\exists j \leq t+1) \neg H(n,i,j)$ then set $r_{t+1} = r_t + 1$, h_{t+1} and a_{t+1} be undefined, and

[1] $\exists^{=1} x P(x)$ means that there is a unique x satisfying P

make $P_n(i,j)$ false on all (i,j), with $i \le r_{t+1}, j \le t+1$, at which P_n has not been defined. If h_t is undefined and $\forall j \le t+1 H(n,t+1,j)$ is true then set $h_{t+1} = t+1$, $r_{t+1} = r_t+1$, and $a_{t+1} = r_{t+1}$. Make $P_n(a_{t+1}, j)$ to be true for all $j \le t+1$, and make $P_n(i,j)$ false on all (i,j), with $i \le r_{t+1}, j \le t+1$, at which P_n has not been defined. If h_t is defined and $\forall j \le t+1 H(n, h_t, j)$ is true then set $h_{t+1} = h_t$, $a_{t+1} = a_t$, and $r_{t+1} = r_t + 1$, and make $P_n(a_{t+1}, j)$ to be true for all $j \le t+1$, and make $P_n(i,j)$ false on all (i,j), with $i \le r_{t+1}, j \le t+1$, at which P_n has not been defined.

Now define the predicate Q as follows: $(n,a,b) \in Q$ if and only if $P_n(i,j)$. The construction above guarantees that the predicate Q is desired.

Now we prove the following lemma which gives a sufficient condition for Σ^0_2-sets to have one to one representations.

Lemma 1 *Let A be a coinfinite Σ^0_2–set that possesses an infinite computable subset S such that $A \setminus S$ is infinite. Then A has a one-to-one representation.*

Proof. As noted above there is computable set H such that $n \in A$ iff $\exists^{=1} a \forall b H(n,a,b)$. Define the predicate H_1: $H_1(n,a,b)$ if and only if $a = < n, x >$ & $H(n,x,b)$. It is easy to check that the formulas $\exists a \forall b H(n,a,b)$ and $\exists a \forall b H_1(n,a,b)$ are equivalent. Moreover, for every a there exists at most one n such that $\forall b H_1(n,a,b)$. Let H_2 be defined as follows: $\neg H_2(n,a,b)$ if and only if $b = < n,a,x >$ &$\neg H_1(n,a,x)$ & $(\forall z < x) H_1(n,a,z)$. It is not hard to see that the predicate H_2 satisfies the following properties:

1. The formulas $\exists a \forall b H_1(n,a,b)$ and $\exists a \forall b H_2(n,a,b)$ are equivalent.
2. The formulas $\forall b H_1(n,a,b)$ and $\forall b H_2(n,a,b)$ are equivalent.
3. For every pair n, a there exists at most one b such that $\neg H_2(n,a,b)$.
4. For every a there exists at most one n such that $\forall b H_2(n,a,b)$.
5. For every b there exists at most one pair (n,a) such that $\neg H_2(n,a,b)$.

Thus, we may assume that H satisfies the properties $3) - 5)$ above. Now, using the predicate H, we build the desired predicate Q.

At stage t the predicate Q_t will be defined on $[0,t] \times [0, r_2(t)] \times [0, r_3(t)]$, where the functions $r_2(t)$, $r_3(t)$ are given effectively at stage t. The predicate Q_t will satisfy the following properties denoted by P:

P_1: For all $n \leq t$, $a \leq r_2(t)$ either $Q_t(n,a,b)$ holds true for all $b \leq r_3(t)$ or $\exists^{=1} b \leq r_3(t) \neg Q_t(n,a,b)$.

P_2: If $a \leq r_2(t)$ is a (Q,t)-**witness for** $n \leq t$, that is $\forall b \leq r_2(t) Q_t(n,a,b)$ then it is a unique (Q,t)-witness for n.

P_3: No two (Q,t)-witnesses (which may be for distinct n_1 and n_2) coincide.

P_4: For each $b \leq r_3(t)$ there is a unique pair (n,a) such that $\neg Q_t(n,a,b)$.

Let $H_0 \subset H_1 \subset \ldots$ be an approximation of H so that $H = \bigcup_t H_t$, where $H_t = H \bigcap [0,t] \times [0,t] \times [0,b_t]$ and b_t is the minimal $b \geq t$ such that each of the following is true:

1. If $a \leq t$ is a (H,t)-**witness for** $n \leq t$, that is $\forall b \leq t H(n,a,b)$ then it is a unique (H,t)-witness for n.
2. No two (H,t)-witnesses (which may be for distinct n_1 and n_2) coincide.
3. For all $n, a \leq t$ either $(\forall b \leq t) H(n,a,b)$ or $(\exists^{=1} j \leq b) \neg H(n,a,b)$.

Note that b_t is correctly defined. If for an $n \leq t$ there is an (H,t)-witness for n then we denote the witness by $h(n,t)$.

Without loss of generality, we assume that $H(0,0,0)$ is true. In the construction, at Stage t, we use functions $r_2(t)$, $r_3(t)$, $h(n,t)$ and $a(n,t)$. The function $r_2(t)$ and $r_3(t)$ tell us that the second and the third coordinates of Q_t do not exceed $r_2(t)$ and $r_3(t)$, respectively; $h(n,t)$ is the (H,t)-witness for n, and $a(n,t)$ is a (Q,t) witness for n if they exist. The construction guarantees that $h(n,t)$ exists if and only if $a(n,t)$ exists. Initially, we set $r(0) = 0$, $h(0,0) = 0$, and $a(0,0) = 0$. Some of the numbers $a \leq r_2(t)$ will be marked by $\square_s$, where $s \in S$. This will mean that the construction guarantees that a is a Q-witness for s, that is $\forall b Q(s,a,b)$.

We now describe stage t of the construction. We assume that Q_{t-1} has been constructed so that all properties P_1 through P_4 hold. In addition, we assume that each $n \leq r_2(t-1)$ either is a $(Q,t-1)$-witness of the form $a(n,t-1)$ (for some $n \leq t$) or has been marked by a $\square_s$ for some $s \in S$.

Stage t. If $t \in S$ and some $a \leq r_2(t-1)$ is marked with $\square_t$ then make a a (Q,t)-witness for s, set $r_2(t) = r_2(t-1)$, $r_3(t) = r_3(t-1) + t$, extend Q_{t-1} to Q_t in the $[0,t] \times [0, r_2(t)] \times [0, r_3(t)]$ keeping all the $(Q,t-1)$-witnesses as

(Q,t)-witnesses so that Q_t satisfies all properties P_1 through P_4[2]. Otherwise, proceed as follows.

Compute H_t. Let $i_1,\ldots,i_k \leq t$ be in increasing order such that $h(i_j,t)$ is defined and $h(i_j,t) \neq h(i_j,t-1)$, $j=1,\ldots,k$. Note that $h(i_j,t-1)$ could be undefined. Also note that $k \leq 2$. Take the least unused numbers s_1 and $s_2 \in S$, mark each $a(i_j,t-1)$ with $\square_{s_j}$, make sure that $a(i_j,t-1)$ is a (Q,t')-witness for s_j at all stages $t' \geq s_j$, $j=1,\ldots,k$. Further, take numbers $n_1 = r_2(t-1)+1, \ldots, n_k = r_2(t-1)+k$, set $a(i_j,t) = n_j$ for $j=1,\ldots,k$, $r_2(t) = n_k$, $r_3(t) = r_3(t-1)+(k+1)t$, and extend Q_{t-1} to Q_t in the $[0,t]\times[0,r_2(t)]\times[0,r_3(t)]$ making each $a(i_j,t)$ a (Q,t)-witness for i_j, keeping all the other $(Q,t-1)$-witnesses as (Q,t)-witnesses so that Q_t satisfies all properties P_1 through P_4. Note that P_4 can be satisfied as seen from the definition of $r_3(t)$.

Suppose that the sequence $i_1,\ldots,i_k \leq t$ stipulated above does not exist. Take the first unused $s \in S$ and mark t with $\square_s$. Make sure that t is a (Q,t')-witness for s at all stages $t' \geq s$. Set $r_2(t) = r_2(t-1)+1$, and $r_3(t) = r_3(t-1)+2t+1$, and extend Q_{t-1} to Q_t in the $[0,t]\times[0,r_2(t)]\times[0,r_3(t)]$ keeping all the $(Q,t-1)$-witnesses as (Q,t)-witnesses so that Q_t satisfies all properties P_1 through P_4. This ends Stage t.

Set $Q = \bigcup_t Q_t$. Now it is not hard to see that Q is a one to one representation of A. Indeed, note that at every stage t, each $a \leq r_2(t)$ is either marked by $\square_s$ or of the form $a(n,t)$. If a is marked with $\square_s$ then $\forall b Q(s,a,b)$ because a is a (Q,t')-witness for s at each stage $t' \geq s$. Assume that a is not marked with $\square_s$, $s \in S$. Consider stage a. There is an n such that $a = a(n,a)$. Then for all $t \geq a$ we have $a(n,t) = a(n,a)$. Therefore $\forall b Q(n,a,b)$. Thus, each $a \in \omega$ is a Q-witness for some $n \in A$. All the other desired properties of Q follow from the fact that Q_t satisfies properties P_1 through P_4 at each stage t. The lemma is proved.

Clearly the definition of one to one presentations of Σ_2^0-sets can be relativised with respect to any oracle X. The relativised version of the lemma above is the following corollary which will be used in the next section. We state it as a relativised representation lemma

Lemma 2 *Let A be a coinfinite $\Sigma_2^{0,X}$–set that possesses an infinite X-computable subset S such that $A \setminus S$ is infinite. Then there exists an X-computable set $Q \subset \omega^3$ such that Q is a one-to-one representation of A.*$\square$

[2]Note that property P_4 can be satisfied which is seen from the definition of $r_3(t)$.

4 $\aleph_1$-categorical computable models

The main result in this section is the following theorem.

Theorem 1 *For any natural number $n \geq 1$ there exists an $\aleph_1$–categorical theory T with a computable model of a finite language so that T is equivalent to $\mathbf{0}^n$. Moreover, all (countable) models of T have computable presentations and T is strongly minimal.*

Proof. Let X be a Σ_{n+1}-set containing neither 0 nor 1. Consider the structure $\mathcal{M} = (M, P)$, where P is a binary predicate symbol, for which the following properties hold true:

1. The predicate P is antireflexive, that is $\neg P(x, x)$ for all x.
2. $P(x, y)$ if and only if $P(y, x)$.
3. For each n there exists a P-cylce of length n if and only if $n \in X$.
4. For each $n \in X$ there exists exactly one P-cycle of length n.
5. Each element $x \in M$ belongs to a P-cycle.

It is not hard to check that the following properties of the structure $\mathcal{M} = (M, P)$ hold true:

1. The theory T of the structure is $\aleph_1$-categorical.
2. The structure $\mathcal{M}$ has a presentation $\mathcal{A} = (\omega, P)$ such that in the presentation P is computable in $\mathbf{0}^n$.

Using Lemma 2 and Corollary 1 one can construct the sequence $\{\mathcal{A}_i\}_{i \leq n}$ of models so that:

1. $\mathcal{A}_0$ is $\mathcal{A}$
2. The structure $\mathcal{A}_i$, where $1 \leq i \leq n$, is obtained by first applying Marker's $\forall$-extension, followed by Marker's $\exists$-extension to the structure $\mathcal{A}_{i-1}$.
3. The model $\mathcal{A}_i$ is $\mathbf{0}^{n-i}$-computable.

Indeed, Corollary 1 tell us that each of the models in the sequence $\{\mathcal{A}_i\}_{i\leq n}$, in particular the model $\mathcal{A}_n$, is $\aleph_1$-categorical. The relativised representation Lemma 2 guarantees that $\mathcal{A}_i$ is $\mathbf{0}^{n-i}$-computable. In particular, $\mathcal{A}_n$ is computable. Note that the original predicate P is definable in the model $\mathcal{A}_n$. Therefore, the statement that there is a P-cycle of length t can be expressed in the theory T of the model $\mathcal{A}_n$. It is not hard to deduce that T is equivalent to $\mathbf{0}^n$. The theorem is proved.

5 $\aleph_0$-categorical computable models

In this section we prove the following theorem.

Theorem 2 *For any natural number $n \geq 1$ there exists an $\aleph_0$–categorical theory T with a computable model of a finite language so that T is equivalent to $\mathbf{0}^n$.*

Proof. Let Y be an infinite subset of ω. We first show how to code this set into an $\aleph_0$-categorical theory T_Y so that Y and T_Y have the same Turing degree. It will then follow that T_Y has a model which is Y-computable.

We basically repeat Peretyat'kin's construction from [13]. The language of T_Y consists of one binary predicate R. For each $n \in \omega$, consider the **cycle** $\mathcal{C}_n = (\{0, 1, \ldots, n+1\}, R)$ of length $n+3$, where $R(x, y)$ is true if and only if $\{x, y\} = \{i, i+1\}$ or $\{x, y\} = \{0, n+2\}$. Clearly R is an antireflexive and a symmetric relation on $\mathcal{C}_n$.

Now consider the class $\mathcal{K}_Y$ that consists of all finite graphs $\mathcal{G}$ (e.g. finite structures of the language) such that no cycle $\mathcal{C}_n$ with $n \notin Y$ is embedded into $\mathcal{G}$. It is clear that $m \in Y$ if and only if $\mathcal{C}_m \in \mathcal{K}_Y$.

Lemma 3 *The class $\mathcal{K}_Y$ has the amalgamation property. In other words, if $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ are in $\mathcal{K}_Y$ and $e : \mathcal{A} \to \mathcal{B}_1$, $f : \mathcal{A} \to \mathcal{B}_2$ are embeddings then there are $\mathcal{C}$ in $\mathcal{K}$ and embeddings $g : \mathcal{B}_1 \to \mathcal{C}$ and $h : \mathcal{B}_2 \to \mathcal{C}$ such that $ge = hf$.*

Proof. We may assume that $A \subset B_1 \bigcap B_2$ and that $\mathcal{A}$ is a subgraph of both $\mathcal{B}_1$ and $\mathcal{B}_2$. We define $\mathcal{C}$ as follows. The domain of $\mathcal{C}$ is the set $C = B_1 \bigcup B_2$. The relation R on $\mathcal{C}$ is defined by taking the union of the sets R_1, R_2 (that is, the relations in $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively) and $\{(a, b) \mid a \in B_1 \setminus A, b \in B_2 \setminus A\}$. Thus, for example, if $b \in B_2 \setminus A$ and $a \in B_1 \setminus A$ then $R(b, a)$ does not hold. Now if $\mathcal{C}_n$ is embedded into $\mathcal{C}$ then clearly, because R

is antisymmetric on any pair (a, b) with $a \in B_1 \setminus A, b \in B_2 \setminus A$, the cycle $\mathcal{C}_n$ is embedded into either $\mathcal{B}_1$ or $\mathcal{B}_2$. Hence, $n \notin Y$. We conclude that $\mathcal{C} \in \mathcal{K}_Y$. The lemma is proved.

The class $\mathcal{K}_Y$ has the **hereditary property**, that is, if $\mathcal{A} \in \mathcal{K}_Y$ and $\mathcal{B}$ is a substructure of $\mathcal{A}$ then $\mathcal{B} \in \mathcal{K}_Y$. The class $\mathcal{K}_Y$ has the **joint embedding property**, that is, for all $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{K}_Y$ there exists $\mathcal{C} \in \mathcal{K}_Y$ such that $\mathcal{A}$ and $\mathcal{B}$ are embedded into $\mathcal{C}$. Hence, as shown in [6], the class $\mathcal{K}_Y$ has ultra-homogeneous structure $\mathcal{A}_Y$ whose theory T_Y is $\aleph_0$-categorical. The theory of the structure is Turing equivalent to Y. We, however, for completeness of the proof, explicitly write down the theory T_Y and build the structure $\mathcal{A}$ of the theory.

Here are the axioms of T_Y. First of all, we postulate antireflexiveness of R. Secondly, for each finite $\mathcal{B} \notin \mathcal{K}_Y$, we postulate that $\mathcal{B}$ can not be embedded into models of T_Y. This corresponds to a listing an infinitely many universally quantifired sentences. Finally, the next list of axioms gurantees the following. For each $\mathcal{A}, \mathcal{B} \in \mathcal{K}_Y$ and an embedding $f : \mathcal{A} \to \mathcal{B}$ there are an extension $\mathcal{A}'$ of $\mathcal{A}$ and an isomorphic mapping $f' : \mathcal{A}' \to \mathcal{B}$ that extends f. Thus, the list of the axioms is, in fact, a list of $\forall\exists$–formulas.

For any $\mathcal{A} \in \mathcal{K}_Y$, by applying the lemma above sufficiently many times, we can find $\mathcal{A}^\star$ that satisfies the following property. For all $\mathcal{B}, \mathcal{C} \in \mathcal{K}_Y$ and f such that f is an embedding of $\mathcal{B}$ into $\mathcal{A}$, $\mathcal{B}$ is a substructure of $\mathcal{C}$, and $card(\mathcal{C}) = card(\mathcal{B}) + 1$ there is an embedding of $g : \mathcal{C} \to \mathcal{A}^\star$ that extends f. Note that given $\mathcal{A}$ the model $\mathcal{A}^\star$ can be constructed effectively with an oracle for Y.

Now we construct a model $\mathcal{A}$ of the theory T_Y as follows. Let $\mathcal{A}_0$ be a model from $\mathcal{K}_Y$. Consider the chain

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \ldots$$

of models from $\mathcal{K}_A$ such that $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ by applying the procedure above, that is $\mathcal{A}_{n+1} = \mathcal{A}_n^\star$. Let $\mathcal{A}$ be the union of this chain. It is not hard to see that $\mathcal{A}$ is a model of T_Y.

Let now $\mathcal{A}$ and $\mathcal{B}$ be two countable models of T_Y. Consider the set of all finite partial isomorphisms between these models $\mathcal{A}$ and $\mathcal{B}$. Using the axioms of T_Y and Ehrenfeucht-Fraïsse back-and-forth games, one can now show that these two models are in fact isomorphic. Therefore the theory T_Y is $\aleph_0$-categorical[3].

To show that Y and T_Y have the same Turing degree it suffices to note that for all $n \in \omega$, $n \in Y$ if and only if the cycle $\mathcal{C}_n$ is embeddable into a model of T_Y, that is the sentence saying that there is a cycle of length n belongs to T_Y.

Now to finish the proof of the theorem, assume that $Y = \mathbf{0}^n$. Then the theory T_Y has a model $\mathcal{A} = (\omega, R)$ computable in $\mathbf{0}^n$. From Lemma 2 and Corollary 1 we can construct the sequence $\{\mathcal{A}_i\}_{i \leq n}$ of models so that:

1. $\mathcal{A}_0$ is $\mathcal{A}$
2. The structure $\mathcal{A}_i$, where $1 \leq i \leq n$, is obtained by first applying Marker's$\exists$-extension and then Marker's $\forall$-extension to the structure $\mathcal{A}_{i-1}$.
3. The structure $\mathcal{A}_i$ is $\mathbf{0}^{n-i}$-computable.

Indeed, Corollary 1 tells us that each of the structures in the sequence $\{\mathcal{A}_i\}_{i \leq n}$, in particular the structure $\mathcal{A}_n$, is $\aleph_0$-categorical. The relativized representation Lemma 2 guarntees that $\mathcal{A}_i$ is $\mathbf{0}^{n-i}$-computable. In particular, $\mathcal{A}_n$ is computable. Note that the original predicate R is definable in the model $\mathcal{A}_n$. Therefore, the statement that there is a R-cycle of length n can be expressed in the theory T of the model $\mathcal{A}_n$. Therefore, it can be checked that T is equivalent to $\mathbf{0}^n$. The theorem is proved.

6 Future Work

The following questions still need to be answered. Does there exist a computable $\aleph_1$-categorical model whose theory is equivalent to $\mathbf{0}^\omega$? Does there exist a computable $\aleph_0$-categorical model whose theory is equivalent to $\mathbf{0}^\omega$? We suspect that the techniques developed in this paper could be used to answer both of these questions positively.

[3]It is also not hard to see that T_Y admits quantifier elimination

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