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Weighted Finite Automata and Metrics in Cantor Space

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Weighted Finite Automata and Metrics in Cantor Space*

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Abstract

We show how weighted finite automata define topologies on the set of all ω-words over a finite alphabet X. Moreover, we give a characterization of these topologies in terms of topologies on X^ω induces by languages $U \subseteq X^*$.

Keywords: *Weighted Finite Automata, ω-words, topologies*
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Contents

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Weighted finite automata are used to describe fractal images (cf. [CD93, CK93]). In particular, they play some rôle in the computation of the contraction coefficients for so-called Multiple Recursive Function Systems (MRFS are a combinations of finite automata with iterated function systems). This is explained in more detail in [FS01].
Here in a finite automaton $\mathscr{A} = (X, (0,\infty), Z, z_0, f, g)$ with output monoid $((0,\infty),\cdot)$ to each transition the contraction coefficient of the mapping φ_x corresponding to the input letter $x \in X$ is assigned. The resulting output $g(s,w)$ is an upper bound to the contraction coefficient of the mapping $\varphi_w := \varphi_{x_1} \circ \cdots \circ \varphi_{x_n}$ $(w = x_1 \cdots x_2)$.
In case the contraction coefficients of φ_w for $w \to \xi \in X^\omega$ converge to zero the corresponding MRFS " draws a point for $\xi \in X^\omega$ ". The set of all such ξ can be described topologically by a suitable topology (depending on the automaton $\mathfrak{A}$ (cf. [FS01]).
Another kind of topology on X^ω are the U-δ-topologies introduced in [St87] (cf. also [DN92]). Here the distance between two ω-words $\xi, \eta \in X^\omega$ depends of the number of their common prefixes in the given language $U \subseteq X^\omega$.
In the present pager we give a relationship between both topologies. It turns out that every automaton-definable topology is a U-δ-topology for a suitable $U \subseteq X^*$.
An construction for $U \subseteq X^*$ from a given automaton is described. Conversely, we derive a property of automaton-definable topologies which proves that not every U-δ-topology can be defined by a weighted automata. A last result shows that for every regular language $U \subseteq X^*$ the corresponding U-δ-topology is definable by a WFA.

1 Notation and Preliminaries

By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers. Let X be our alphabet of cardinality $|X| = r$, $r \in \mathbb{N}, r \geq 2$.
By X^* we denote the set of finite strings (words) on X, including the *empty* word e. We consider the space X^ω of infinite sequences (ω-words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.
We will refer to subsets of X^* and X^ω as languages or ω-languages, respectively.
By "$\sqsubseteq$" we denote the prefix relation, that is, $w \sqsubseteq \eta$ if and only if there is an η' such that $w \cdot \eta' = \eta$, and $\mathbf{A}(\eta) := \{w : w \in X^* \wedge w \sqsubseteq \eta\}$ and $\mathbf{A}(B) := \bigcup_{\eta \in B} \mathbf{A}(B)$ are the languages of finite prefixes of η and B, respectively.
In the study of ω-languages it is useful to consider X^ω as a metric space (Cantor space) with the following metric (cf. [Th90, St97])

$$\rho(\eta,\xi) := \inf\{r^{-|w|} : w \sqsubset \eta \wedge w \sqsubset \xi\} = r^{1-|\mathbf{A}(\eta)\cap\mathbf{A}(\xi)|} . \quad (1)$$

Then (X^ω, ρ) is a compact metric space. The open balls $\mathbb{B}_\varepsilon(\xi)$ of radius $\varepsilon \in (0,1]$ with center ξ in (X^ω,ρ) can be described as $\mathbb{B}_\varepsilon(\xi) = \{\eta : \rho(\xi,\eta) < \varepsilon\} = w_{\xi,\varepsilon} \cdot X^\omega$ where $w_{\xi,\varepsilon} \in \mathbf{A}(\xi)$ and $|w_{\xi,\varepsilon}| = \lfloor -\log_r \varepsilon \rfloor + 1$. Thus open sets in Cantor space (X^ω,ρ) are sets of the form $W \cdot X^\omega = \bigcup_{w \in W} w \cdot X^\omega$. As usually, closed sets are complements of open sets.
Countable intersections of open sets are known as $\mathbf{G}_\delta$-sets. In Cantor space, we have the following characterization of $\mathbf{G}_\delta$-sets (cf. [Th90, St87, St97]).
We define for a language $U \subseteq X^*$ its δ*-limit*, U^δ, as the set consisting of all infinite words in X^ω having infinitely many prefixes in U,

$$U^\delta = \{\xi \in X^\omega : |\mathbf{A}(\xi) \cap U| = \infty\} .$$

Theorem 1 *In Cantor space, a subset $F \subseteq X^\omega$ is a $\mathbf{G}_\delta$-set if and only if there is a language $U \subseteq X^*$ such that $F = U^\delta$.*

For more background on metric topology see e.g. [Ku66].

2 The U-δ-topology in X^ω

The paper [St87] considered another metric topology on X^ω which turned out to be useful in connection with the study of sequential mappings. In this section we derive some fundamental properties of this topology and relate it to the usual topology in Cantor space.

Definition 1 *For a language $U \subseteq X^*$ and $\xi, \eta \in X^\omega$ we set*

$$\rho_U(\eta,\xi) := \begin{cases} 0 & \text{, if } \eta = \xi \\ r^{1-|\mathbf{A}(\eta)\cap\mathbf{A}(\xi)\cap U|} & \text{, otherwise.} \end{cases}$$

This metric, in some sense, resembles the metric ρ in Cantor space; in fact, $\rho = \rho_{X^*}$. Moreover, since $\rho_U(\xi,\eta) \geq \rho(\xi,\eta)$, the U-δ-topology refines the topology of the Cantor space. In particular, every closed (or open) set in Cantor space is also closed (or open, resp.) in the U-δ-topology of X^ω.
The open balls in (X^ω, ρ_U) are given as follows

$$\mathbb{B}_{\varepsilon,U}(\xi) = \begin{cases} \{\xi\} & \text{, if } \forall\eta(\eta \neq \xi \to \rho_U(\xi,\eta) \geq \varepsilon), \\ X^\omega & \text{, if } \varepsilon > r \text{, and} \\ w_{\xi,\varepsilon}\cdot X^\omega & \text{, otherwise.} \end{cases}$$

Here $w_{\xi,\varepsilon}$ is defined by $w_{\xi,\varepsilon} \in \mathbf{A}(\xi)\cap U$ and $|\mathbf{A}(w_{\xi,\varepsilon})\cap U| = \lfloor -\log_r \varepsilon \rfloor + 2$.
The following topological properties of (X^ω, ρ_U) are useful for our considerations. A point ξ is called an *accumulation point of F* provided $\forall\varepsilon(\varepsilon > 0 \to \exists\eta(\eta \in F \wedge \eta \neq \xi \wedge \rho_U(\xi,\eta) \leq \varepsilon))$.
The following is an easy consequence of Definition 1.

Corollary 2 *A point $\xi \in X^\omega$ is an accumulation point of the whole space (X^ω, ρ_U) if and only if $\xi \in U^\delta$.*

As (X^ω, ρ_U) is a metric space, the smallest closed (with respect to ρ_U) subset of X^ω containing F, $\mathscr{C}_U(F)$, satisfies

$$\mathscr{C}_U(F) = F \cup \{\xi : \xi \in X^\omega \wedge \xi \text{ is an accumulation point of } F \text{ in } (X^\omega, \rho_U)\}. \tag{2}$$

A point $\xi \in \mathscr{C}_U(F)$ which is not an accumulation point of F is called an *isolated point* of F. Thus, ξ is an isolated point of X^ω iff there is an $\varepsilon > 0$ such that $\mathbb{B}_{\varepsilon,U}(\xi) = \{\xi\}$. The *set of isolated points* of (X^ω, ρ_U) is referred to as $\mathbb{I}_U := X^\omega \setminus U^\delta$.
It should be mentioned that an arbitrary set of isolated points of X^ω is open.
In case $U^\delta = \emptyset$, every point of (X^ω, ρ_U) is isolated. Thus, in contrast to the compactness of the Cantor space, in general, we have only the following.

Theorem 3 *(X^ω, ρ_U) is a complete metric space.*

The close relationship between U-δ-topology and the topology of the Cantor space is documented in the following case of accumulation points.

Theorem 4 *Let $U \subseteq X^*$. Then $\xi \in U^\delta$ is an accumulation point of F in (X^ω, ρ_U) if and only if ξ is an accumulation point of F in (X^ω, ρ).*

Proof. Let ξ be an accumulation point of F in (X^ω, ρ_U). Then for every $n \in \mathbb{N}$ there is an $\eta_n \in F \setminus \{\xi\}$ such that $\rho_U(\xi, \eta_n) \leq r^{-n}$. Since $\rho(\xi, \eta_n) \leq \rho_U(\xi, \eta_n)$, ξ is also an accumulation point of F in Cantor space.
In order to prove the converse consider $\xi \in U^\delta$. Then the function $\psi_\xi : \mathbf{A}(\xi) \to \mathbb{N}$ defined by $\psi_\xi(w) := |\mathbf{A}(w) \cap U|$ is monotone and surjective and satisfies $\rho_U(\xi, \eta) \leq r^{1-\psi_\xi(w)}$ whenever $w \in \mathbf{A}(\xi) \cap \mathbf{A}(\eta)$.
Let ξ be an accumulation point of F in Cantor space, that is, for every $n \in \mathbb{N}$ there is an $\eta_n \in F \setminus \{\xi\}$ such that $\rho(\xi, \eta_n) \leq r^{-n}$. Then there is a word $w_n \in \mathbf{A}(\xi) \cap \mathbf{A}(\eta_n)$ such that $|w_n| \geq n$. By construction, $\rho_U(\xi, \eta_n) \leq r^{1-\psi_\xi(w_n)}$.
Since the function ψ_ξ is monotone and surjective, $\lim_{n\to\infty} \rho_U(\xi, \eta_n) = 0$, and ξ is also an accumulation point of F in U-δ-topology. ❑

From Eq. (2) we obtain immediately the following relation between closed sets in Cantor space and in U-δ-topology.

Corollary 5 *Let $\mathscr{C}(F) := \mathscr{C}_{X^*}(F)$ be the smallest closed set containing F in Cantor space. Then $\mathscr{C}_U(F) = F \cup \left(\mathscr{C}(F) \cap U^\delta\right) = \mathscr{C}(F) \cap (F \cup U^\delta)$.*

As a consequence we obtain Corollary 5.2 of [St01].

Corollary 6 *Every set $F \supseteq U^\delta$ is closed in (X^ω, ρ_U).*

It was mentioned above, every set $J \subseteq \mathrm{I\!I}_U$ of isolated points is an open set in (X^ω, ρ_U), and every set of the form $W \cdot X^\omega$ is open in Cantor space. Then Corollary 5 yields

Corollary 7 *$E \subseteq X^\omega$ is open in (X^ω, ρ_U) iff $E = W \cdot X^\omega \cup J$ for some $W \subseteq X^*$ and $J \subseteq \mathrm{I\!I}_U$.*

The following theorem provides a simple condition when two languages U, V induce the same topology on X^ω.

Theorem 8 *If $U^\delta = V^\delta$ then the U-δ-topology and the V-δ-topology of X^ω coincide, that is,*

$$\{E : E \text{ is open in } (X^\omega, \rho_U)\} = \{E : E \text{ is open in } (X^\omega, \rho_V)\}.$$

If $U^\delta \neq V^\delta$ then the U-δ-topology and the V-δ-topology of X^ω do not coincide.

Proof. The first assertion is immediate from Corollary 7, and the second one follows from the fact that $\{\xi\}$ is open in (X^ω, ρ_U) and not open in (X^ω, ρ_V) whenever $\xi \in V^\delta \setminus U^\delta$. ❑

According to Theorem 8 one has a great variety of languages inducing the same topology. In [St87] and [St97, Section 1.4] the possibilities, depending on the ω-language U^δ, of defining the U-δ-topology via languages V having special properties, e.g. as $V = V \cdot X^*$ or $V = \mathbf{A}(V)$, are considered.
We conclude this section with showing that the space (X^ω, ρ_U) is not compact unless $U^\delta = X^\omega$. To this end we recall that a complete metric space $(\mathscr{X}, d)$ is *compact* iff every family of open sets $\{E_i : i \in I\}$ which covers $\mathscr{X}$, that is, $\bigcup_{i\in I} E_i = \mathscr{X}$, contains a finite subfamily $\{E_i : i \in I'\}$ which also covers $\mathscr{X}$ (e.g. [Ku66]).

Theorem 9 *The space (X^ω, ρ_U) is compact if and only if $U^\delta = X^\omega$.*

Proof. If $U^{\delta} = X^{\omega}$ then Theorem 8 shows that the topology of $U^{\delta} = X^{\omega}$ coincides with the topology of the Cantor space, thus is (X^{ω}, ρ_U) is compact.
Assume $U^{\delta} \neq X^{\omega}$, that is, there is a $\xi \in \mathrm{I\!I}_U$. Then $\{\xi\}$ is open in (X^{ω}, ρ_U). Consider the language $L_{\xi} := \{wx : w \in \mathbf{A}(\xi) \wedge x \in X \wedge wx \notin \mathbf{A}(\xi)\}$. L_{ξ} is infinite and the family $\mathscr{O}_{\xi} := \{E : E = \{\xi\} \vee E = v \cdot X^{\omega}$ for some $v \in L_{\xi}\}$ is an infinite family of pairwise disjoint non-empty open sets covering the whole space. Thus $\mathscr{O}_{\xi}$ cannot contain a proper subfamily covering X^{ω}. ❏

3 Metrics on X^{ω} Defined by Weighted Finite Automata

Another way to describe non-standard metrics on X^{ω} is to use weighted finite automata assigning to an input word a positive real number. In [CK94, DK94] this behaviour led to the computation of real functions. Following the ideas of [FS01] we use weighted finite automata to generate metrics on X^{ω}.
We consider weighted finite automata of the following kind.

Definition 2 *A* (deterministic) weighted finite automaton (WFA) *is a tuple $\mathscr{A} = (X, (0,\infty), Z, z_0, f, g)$ where X, Z are finite nonempty sets of input letters and states, resp., $z_0 \in Z$ is the initial state, $f : Z \times X \to Z$ is the transition function and $g : Z \times X \to (0,\infty)$ is the output function.*

As usual we extend the state transition and output functions to the domain $Z \times X^*$ via

$$\begin{array}{rclcrcl} f(z,e) & := & e & , & f(z,wx) & := & f(f(z,w),x), \\ g(z,e) & := & 1 & \text{and} & g(z,wx) & := & g(z,w) \cdot g(f(z,w),x), \end{array}$$

where "$g(z,w) \cdot g(f(z,w),x)$" is the usual multiplication of real numbers.
As it was explained in [FS01] for valuations, the output function g yields a metric in X^{ω} defined by $\mathscr{A}$:

Definition 3 *Let $\mathscr{A}$ be a weighted finite automaton. Define*

$$\rho_{\mathscr{A}}(\xi,\eta) := \begin{cases} 0 & \text{, if } \xi = \eta \\ \inf\{g(z_0,w) : w \in \mathbf{A}(\xi) \cap \mathbf{A}(\eta)\} & \text{, if } \xi \neq \eta \, . \end{cases}$$

Lemma 10 *If $\mathscr{A}$ is a deterministic weighted finite automaton then $\rho_{\mathscr{A}}$ is a metric on X^{ω}.*

Proof. Obviously the function $\rho_{\mathscr{A}}$ is nonnegative, symmetric in its arguments and vanishes to zero only if the arguments coincide. Finally, $\rho_{\mathscr{A}}$ satisfies the ultra-metric inequality $\rho_{\mathscr{A}}(\xi,\eta) \leq \max\{\rho_{\mathscr{A}}(\xi,\zeta), \rho_{\mathscr{A}}(\eta,\zeta)\}$, because $\mathbf{A}(\xi) \cap \mathbf{A}(\eta)$ contains at least one of the sets $\mathbf{A}(\xi) \cap \mathbf{A}(\zeta)$ or $\mathbf{A}(\eta) \cap \mathbf{A}(\zeta)$. ❏

Next we are going to show that every topology on X^{ω} defined by a WFA is equivalent to a suitably chosen U-δ-topology.

Theorem 11 *For every WFA $\mathscr{A}$ there are a $k \in \mathbb{N}$ and a language $U \subseteq X^*$ such that*

$$\rho_{\mathscr{A}}(\xi,\eta) \leq \rho_U(\xi,\eta) \leq \sqrt[k]{\rho_{\mathscr{A}}(\xi,\eta)} \, .$$

Proof. Define the language $U \subseteq X^*$ via

$$w \in U :\Leftrightarrow \exists n(n \in \mathbb{N} \wedge g(z_0, w) \leq r^{-n} \wedge \forall v(v \sqsubset w \to g(z_0, v) > r^{-n}))\,.$$

To every $w \in U$ we assign the value $n(w) := \max\{n : g(z_0, w) \leq r^{-n}\}$. Thus $n(e) = 0$ and $u, w \in U$, $u \sqsubset w$, imply $n(u) < n(w)$. Then $|\mathbf{A}(v) \cap U| \leq m+1$ yields $\min\{g(z_0, v') : v' \sqsubseteq v\} \leq \min\{r^{-n(w)} : w \in \mathbf{A}(v) \cap U\} \leq r^{-m}$. Letting $\mathbf{A}(v) = \mathbf{A}(\xi) \cap \mathbf{A}(\eta)$, this proves the first inequality.
Now, choose $k \in \mathbb{N}$ such that $r^k \geq \max\{g(z_0, w)/g(z_0, wx) : w \in X^* \wedge x \in X\}$. We are going to show that for words $u, w \in U$ such that $v \notin U$ for all v, $u \sqsubset v \sqsubset w$, the inequality $n(w) - n(u) \leq k$ holds true.
Let $n(u) = m$ and $n(w) = n$. Since $g(z_0, u) > r^{-(m+1)}$ and $v \notin U$ for v, $u \sqsubset v \sqsubset w$, we have $g(z_0, v') > r^{-(m+1)}$ for all $v' \sqsubset w$.
This holds, in particular, for the word $w' \sqsubset w$ with $|w'| = |w| - 1$. Thus $r^k \geq g(z_0, w')/g(z_0, w) > r^{-(m+1)}/r^{-n}$, whence $k \geq n - m$.
Observe that, by definition, the empty word $e \in U$ and $g(z_0, e) = 1 = r^0$ and assume that $|\mathbf{A}(v) \cap U| = m+1$. Let $\mathbf{A}(v) \cap U = \{e, w_1, \ldots, w_m\}$ where $e \sqsubset w_1 \sqsubset \ldots \sqsubset w_m$. Applying repeatedly $n(w_i) - n(w_{i-1}) \leq k$ ($w_0 := e$) one obtains $n(w_m) \leq k \cdot m$.
As w_m is the longest word in $\mathbf{A}(v) \cap U$, each $v' \sqsubseteq v$ satisfies $r \cdot g(z_0, v') > g(z_0, w_m)$ and, since $n(w_m) \leq k \cdot m$ we have $g(z_0, w_m) \geq r^{-k \cdot m+1}$. Consequently, $\min\{g(z_0, v') : v' \sqsubseteq v\} \geq r^{-k \cdot m}$. Again letting $\mathbf{A}(v) = \mathbf{A}(\xi) \cap \mathbf{A}(\eta)$, this proves the second inequality. ❑

As $\lim_{n\to\infty} \rho_U(\xi_n, \xi) = 0$ iff $\lim_{n\to\infty} \rho_{\mathscr{A}}(\xi_n, \xi) = 0$ our Theorem 11 shows that a sequence $(\xi_n)_{n\in\mathbb{N}}$ converges to a limit ξ with respect to the metric $\rho_{\mathscr{A}}$ if and only if it does so with respect to ρ_U. Thus we obtain the following.

Corollary 12 *If U is defined as in Theorem 11 then the WFA-topology defined by $\mathscr{A}$ and the U-topology coincide.*

So far we have shown that every topology defined by a WFA is definable as a U-δ-topology for a suitable language $U \subseteq X^*$. Next we are going to show that the converse is not the case. To this end let $Ult := \{w \cdot v^\omega : w, v \in X^*\}$ be the ω-language of all *ultimately periodic ω-words*. As for languages, by $\mathrm{I\!I}_{\mathscr{A}}$ we denote the set of isolated points of the space $(X^\omega, \rho_{\mathscr{A}})$.

Proposition 13 *If $\mathrm{I\!I}_{\mathscr{A}} \supseteq Ult$ for some WFA $\mathscr{A}$ then $\mathrm{I\!I}_{\mathscr{A}} = X^\omega$.*

Proof. Assume $\xi \notin \mathrm{I\!I}_{\mathscr{A}}$. Then $\liminf_{w\to\xi} g(z_0, w) = 0$. Since $\mathscr{A}$ is a finite automaton there is an infinite family $(w_i)_{i\in\mathbb{N}}$ of prefixes of ξ such that $\lim_{i\to\infty} g(z_0, w_i) = 0$ and $f(z_0, w_i) = z$ for some $z \in Z$. Choose w_i and w_j in such a way that $w_i \sqsubset w_j$ and $g(z_0, w_i) > g(z_0, w_j)$. Define $v \in X^*$ by the identity $w_i \cdot v = w_j$. Then $g(z, v) = g(z_0, w_j)/g(z_0, w_i) < 1$.
Hence, $\lim_{l\to\infty} g(z_0, w_i \cdot v^l) = 0$, and the ω-word $w_i \cdot v^\omega \in Ult$ does not belong to $\mathrm{I\!I}_{\mathscr{A}}$. ❑

The next proposition gives the announced example.

Proposition 14 *There is a language $U \subseteq X^*$ such that $\mathrm{I\!I}_U = Ult$ but there is no WFA $\mathscr{A}$ such that $\mathrm{I\!I}_{\mathscr{A}} = Ult$.*

Proof. The ω-language Ult is countable, and every subset $F \subseteq X^\omega$ having a countable complement is a $\mathbf{G}_\delta$-set. According to Theorem 1 there is a language $U \subseteq X^*$ such that $U^\delta = X^\omega \setminus Ult$.
Proposition 13 proves that $\mathrm{I\!I}_{\mathscr{A}} = Ult$ is impossible. ❑

Next, we exhibit a class of languages for which the U-δ-topology is definable by a WFA.

Proposition 15 *If $U \subseteq X^* \setminus \{e\}$ is a language accepted by a finite automaton then there is a WFA $\mathscr{A}$ such that $\rho_U = \rho_{\mathscr{A}}$.*

Proof. Let $\mathscr{B} = (X, Z, z_0, f, Z_{fin})$ be a finite automaton accepting U, that is, $U = \{w : w \in X^* \wedge f(z_0, w) \in Z_{fin})\}$. Set $\mathscr{A} := (X, Z, (0,\infty), z_0, f, g)$ where the output function $g : Z \times X \to (0,\infty)$ is defined as follows

$$g(z,x) := \begin{cases} r^{-1} & \text{, if } f(z,x) \in Z_{fin} \text{ and} \\ 1 & \text{, otherwise.} \end{cases}$$

It is easy to see that $g(z_0, w) = r^{-|\{v : e \sqsubset v \sqsubseteq w \wedge v \in U\}|}$. This proves our assertion. ❑

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