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G. Kapoulas
Athens Univversity of Economics and
Bussines
Greece

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Infinitesimals Via the Cofinite Filter*

G. Kapoulas
Athens University of Economics and Bussines
Greece
gkapou@math.ntua.gr

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Abstract

We observe that the field of complex and p–adic numbers, the ring of $n \times n$ matrices, the Euclidean spaces $\mathbb{R}^n$ $(n > 1)$, and topological manifolds are models of the notion of infinitesimal. These models do not satisfy the axioms of total order and/or contain zero divisors. As a first consequence the notion of infinitesimal is logically independent of the notions of zero divisor and total order. As second consequence the notion of ultrafilter is not required for the definition of infinitesimal. We define the notion of infinitesimal using the cofinite filter as in C. Schmieden and D. Laugwitz [7]. We prove a translation theorem between expressions using the $\epsilon - \delta$ formalism and expressions using infinitesimals. The language employed is many sorted. The language contains in addition to the basic carrier set (sort), a function symbol intended to interpret the notion of size, absolute value, norm or distance from 0, a unary predicate symbol to be interpreted the range of this distance function, a binary symbol to be interpreted as addition in the range of the metric, a constant symbol to be interpreted as zero as element of the range of metric only, a symbol to denote the total order of the range of the metric function, does not contain symbols to be interpreted as addition or multiplication or their inverses.

Notation: The set of natural numbers ordered by the usual order (to be used as index set for sequences) will be denoted by ω. The field of real numbers will be denoted by $\mathbb{R}$. The field of complex numbers will be denoted by $\mathbb{C}$. The field of p–adic numbers will be denoted by $\mathbb{Q}_p$. The set of $n \times n$ matrices with real numbers as elements will be denoted by $\mathbb{R}^{n\times n}$. The set of Euclidean spaces of dimension n over the real numbers will be denoted by $\mathbb{R}^n$. The set of nonnegative real numbers, intended to be the range of the absolute value functions, or norm, of distance of an element from zero will be denoted by $\mathbb{R}_0^+$. The end of proofs will be denoted by $\Box$.

Keywords: infinitesimal, cofinite filter.

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1 Introduction

We start with the simplest possible definition of infinitesimal and give an argument about the impossibility of certain constructions.

Definition 1 *A positive real infinitesimal is less than any positive real number, nonnegative and different than zero.*

The standard argument about the inconsistency of the above definition is that an infinitesimal cannot exist since any entity (entity meaning real number) less than any positive real number and simultaneously non negative has to be zero. The implicit assumption in the above argument is that an infinitesimal has the same properties as a real number (satisfies the same axioms). The notion of infinitesimal can be realized if we require that an infinitesimal is an entity of different nature than a real number as C. Schmieden and D. Laugwitz [7] and A. Robinson [6] have shown. Since an infinitesimal is an entity different than a real number an infinitesimal cannot satisfy the same properties as a real number. We remark that if the infinitesimals satisfy the same properties as the real numbers they are the real numbers in the monadic second order language of ordered fields.

We review briefly the main points of interpretation of infinitesimals given by A. Robinson. The realization of a (real) infinitesimal in the spirit of the work of A. Robinson and W.A.J. Luxemburg employs the notion of an ultrafilter. The reasons for using an ultrafilter are the following:

1. Elimination of zero divisors.
2. Establishment of total (field) order.
3. Use of classical two valued logic and Compactness Theorem.
4. Establishment of a transfer principle.

The reasons for the requirement that we have a total field order and nontrivial zero divisors comes from the requirement that the infinitesimals should satisfy the same laws as the real numbers. Historically either there were no other models realizing the notion of infinitesimal at the time that the above statement was made or the complex numbers were known. These reasons do no seem very well founded if we consider $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ or topological manifolds where there exists a notion of infinitesimal or elements being infinitesimally close to each other. The argument following definition 1 suggests a flaw in the statement that the enlarged system should satisfy the same laws as the real number system.

We note that Leibniz although mentioned that the laws satisfied the enlarged system of numbers should be the same as the laws satisfied by the real numbers, he mentioned that the introduction of infinitely small and infinitely big entities was not satisfying the Archimedean axiom. In [6] A. Robinson p. 266–267 mentions the possibility that some form of the Archimedean axiom may be true or may fail in a system containing infinitesimals depending on the exact formulation of the Archimedean axiom. In A. Robinson p. 265 *"J' appele grandeurs incomparables dont l'une multipliée par quelqe nombre fini que ce soit, ne saurait excéder l'autre....".* It is also known that the Archimedean axiom has to be modified to be valid in the extended system which contains infinitesimals.

Regarding the existence of zero divisors, the mathematical practice in Analysis avoids the problem of the using nontrivial zero divisors (which they exist) by a suitable definition (definition 4). The issue of total order seems to be irrelevant with the development of Analysis. As we will see there exist natural models satisfying the definition of infinitesimal (or having points infinitesimally close to each other) where there is no total order and there exist zero divisors.

The use of Classical logic is not necessary as we can have a constructive interpretation of the notion of infinitesimal. This constructive interpretation of infinitesimals implicit in J. L. Bell [2] and suggested by P. Schuster will be discussed in the future.

The construction given by A. Robinson avoids the problem of assuming that an infinitesimal is a number (an object of the same status and nature as numbers) by constructing infinitesimals as objects which are not real numbers but elements of a more complicated nature. This construction contains the basic insight about the nature of an infinitesimal. This insight presented by C. Schmieden and D. Laugwitz [7] will be used and explained in the sequel.

The use of (non principal) ultrafilters makes possible the definition of a total order and the elimination of zero divisors in the development of infinitesimal over the field of real numbers by A. Robinson as follows: A infinitesimal over the real numbers is represented by a countable sequence of real numbers. Fix a non principal ultrafilter over the set ω. Given two such sequences $\mathbf{x} = \{x_n \mid n \in \omega\}$, $\mathbf{y} = \{n_n \mid n \in \omega\}$ we define $\mathbf{x} = \mathbf{y}$ iff $\{n \in \omega \ \mid x_n = y_n\}$ belongs to the ultrafilter. We define the order and the rational operations modulo an element of the ultrafilter. The above equivalence relation is compatible with the operations and we obtain a field. For the definition of a total order, we partition the index set ω into three sets $A_1 = \{n \in \omega \ \mid x_n < y_n\}$, $A_2 = \{n \in \omega \ \mid x_n = y_n\}$, $A_3 = \{n \in \omega \ \mid x_n > y_n\}$. Exactly one of the sets A_1, A_2, A_3 will belong to the ultrafilter and hence exactly one of the relations $\mathbf{x} = \mathbf{y}$, $\mathbf{x} < \mathbf{y}$, $\mathbf{x} > \mathbf{y}$ will be true. The above order is compatible with the field operations of the real numbers and gives an ordered field. For the non existence on nontrivial zero divisors given a sequence of real numbers $\mathbf{x} = \{x_n \mid n \in \omega\}$, we partition the index set ω into two disjoint sets $B_1 = \{n \in \omega \ \mid x_n = 0\}$, $B_2 = \{n \in \omega \ \mid x_n \neq 0\}$. Exactly one of the sets B_1, B_2 will belong to the ultrafilter and hence either a sequence (infinitesimal) will be equal to zero or all the terms that are equal to zero can be ignored and we can invert a non zero element.

Regarding the existence of models satisfying the definition of infinitesimal which models contain non trivial zero divisors and they lack a total order relation the basic intuition are the Euclidean spaces and the rings of $n \times n$ matrices. In the rings $n \times n$ matrices it is possible to define the notion of exponential of a matrix and other algebraic or transcendental functions using power series by Taylor expansions.

We observe that whenever we have a Taylor series expansion there ought to be some notion of infinitesimal. In particular we note the definition of the exponential of a square matrix and a formula used for the inversion of a matrix under certain constrains

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}, \qquad \frac{1}{1-A} = \sum_{n=0}^{\infty} A^n. \tag{1}$$

From the two above examples we observe that in the ring of $n \times n$ matrices we have a notion of infinitesimal. The notion of infinitesimal is to be taken in the sense that the sequences of the tails of the infinite sums in equation 1 i.e. the sequences

$$x_k = \sum_{n=k}^{\infty} \frac{A^n}{n!} \qquad y_k = \sum_{n=k}^{\infty} A^n$$

are null sequences or are negligible in size no matter what is the order of magnitude. Similar observations are true for geometric notions as infinitesimal angle, infinitesimal length, infinitesimal area..

We observe that in geometric constructions e.g. in calculation of areas, volumes, length of a curve there is no operation of addition or of multiplication of the geometric entities. The difference between a circle and a sequence of inscribed or circumscribed polygons that approximate the area of the circle is a geometric representation of an infinitesimal. In such a geometric representation there is no natural notion of addition or multiplication of these geometric concepts or as total operations. The operator of addition or multiplication is defined among numbers representing these entities (sum of areas, lengths, measure etc). The multiplication is a scalar multiplication. Moreover this geometric association and intuition existed in the form of the method of exhaustion since Archimedes.

We give an alternative definition of infinitesimal which uses a different language than the language used in definition 1 and takes into account expressions similar with the expressions in equation 1.

Definition 2 *An infinitesimal is a non zero entity which has size less than the size of any other entity.*

For the case that there is no zero element in the structure as in topological manifolds we have the following variant.

Definition 3 *Two elements of a structure are infinitesimally close if their distance is positive and less than any real number.*

Henceforth we will use definition 2 or for topological manifolds we will use the variant definition 3. The language used in definition 1, in definition 2 and in definition 3 of an infinitesimal does not include any symbol for addition, multiplication, subtraction, additive or multiplicative inverse, and there is no statement about a total order in any of the definitions.

We note that any of the above definitions does not restrict an infinitesimal to be something which will be only in an extension of the field of real numbers. The order needed in definition 2 is not the field order of the reals or any order of the structure, but the order of the range of the absolute value function or metric. Definition 2 refers to absolute values, norms or distance from 0 to define the notion of infinitesimal. The language required is a language containing a symbol of arity one with intended interpretation size of, norm, distance from zero. Alternatively we can use a two place function symbol to denote the distance of two elements with range the set $\mathbb{R}_+^0$. Definition 2 requires a many sorted structure because of the notion of size. One sort will be the set of entities that are the base set (one instance being the ring of square matrices), and one more sort will be the positive real numbers as the range of the function which determines the size, a constant for 0, a symbol to be interpreted as addition in the range of metric and a symbol for the relation less than in the range of the function size (norm, absolute value). Definition 1 is a definition in the spirit of the of the Dedekind order completion of the rational numbers as an ordered field, which gives one possible definition of the real number field. Definition 2 is in the spirit of the Cauchy completion of the metric space of the rational numbers. The Cauchy completion allows a generalization since it applies to to a variety of topological spaces. It allows also the interpretation of smaller, bigger by the distance of elements form zero admitting as particular models the field of the complex and p–adic numbers, the Euclidean spaces and the ring of $n \times n$ matrices. The definitions of infinitesimals do not characterize the model in the sense of stipulating a total order. The comparison in the definition is among an infinitesimal and a object of the basic set and in a specified order. As we see form definition 2 we need order to compare the sizes of entities. We need order in the range of the metric, norm or distance function. We need infinitesimals to express that some variable quantity (a process) approaches some entity (a limit). The notion of infinitesimal is a topological notion and hence not related to the notion of order or the real number system exclusively. The notion of approximation is closely related with the notion of limit, is a topological notion and as a topological notion no connection with order in a algebraic setting. The notion of infinitesimal expresses the concept of being negligible at any order of magnitude and this is the intuitive idea of a null sequence.

2 A genetic definition of infinitesimals

The construction presented in the sequel uses the notion of the cofinite filter only in contradistinction with the construction given by A. Robinson where ultrafilters are used. The notion of ultrafilter is used in order to preserve the first order properties of the set of real numbers in an appropriate language and transfer them into the product space. This ensures that extending the real numbers with infinitesimals makes possible the development of Analysis using ultrafilters. As a substitute of the transfer principle we prove a translation theorem between expression using the ϵ–δ formalism and expressions using the term infinitesimal (theorem 3). This translation theorem makes possible to develop Analysis using the term infinitesimal in the language.

The basic observation is that there exists a notion of infinitesimal in the sense of definition 2 in the fields of the real, complex, p–adic numbers, $\mathbb{R}^n$ and the set of $n \times n$ matrices and other structures. The argument is presented based on a heuristic approach for the nature of an infinitesimal over some basic structure. The argument takes account definition 2 using the notions of absolute value, size, norm or any other notion of size or metric.

From the definition 2 we have that the infinitesimal may be represented by an entity of dynamic nature. The characterization of dynamic is to be understood in the sense that an infinitesimal incorporates the information that the size of the infinitesimal is less than (in size, absolute value, norm) any other element and the also the size of the infinitesimal has to be different (greater) than zero.

We can simplify the definition of satisfaction of the definition by noting that for any entity x there exists a natural number n_x such $\frac{1}{n_x} < |x|$ so that suffices to find an element y_x such that $0 \neq |y_x| < \frac{1}{n_x}$ (separability of the range of the metric). The above construction is possible in $\mathbb{Q}_p$ $\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{R}^{n\times n}$ and topological manifolds. We can identify the elements x with the constant sequence $x_n = x$. Collecting the information y_{n_x} in a countable sequence $\mathbf{y} = \{y_{n_x} \mid n \in \omega\}$, we have that $\forall x \quad 0 \neq |y_{n_x}| < |x|$ or $\forall x \quad 0 \neq |\mathbf{y}| < |x|$ since $\forall x \quad 0 \neq |y_{n_x}| < \frac{1}{n_x} < |x|$ where the interpretation of the symbol $<$ in the expression $0 \neq |\mathbf{y}| < |x|$ is taken as *eventually less than.*

From the above analysis we obtain the basic insight for the structure that will be used to interpret the notion of infinitesimal. The carrier set A will be the Cartesian product of the real complex or p–adic numbers, the ring of square matrices, the Euclidean spaces etc over a countable index set (the set of natural numbers). In the set A we can embed the set of real, complex and p–adic numbers, and the set of $n \times n$ matrices, where an element x is represented by the constant sequence which has every term equal to x. In order to define the operations we first define equality using the cofinite filter i.e. for $\mathbf{x} = \{x_n \mid n \in \omega\}$, $\mathbf{y} = \{y_n \mid n \in \omega\}$ define $\mathbf{x} = \mathbf{y}$ iff $\exists m_0 \ n > m_0 \rightarrow x_n = y_n$. The inequality, absolute value and the appropriate operations are defined in a similar manner using the cofinite filter.

According to the above definition the sequences

$$\mathbf{i} = \left\{ \frac{1}{n} \middle| n \in \omega \right\}, \quad \mathbf{i} = \left\{ \frac{i}{n} \middle| n \in \omega \right\}, \quad \mathbf{i} = \{p^n | \, n \in \omega\},$$

$$\mathbf{i} = \left\{ \begin{bmatrix} 1/n & 0 \\ 0 & 1/n \end{bmatrix} \middle| n \in \omega \right\}, \quad \mathbf{i} = \left\{ \left(\frac{1}{n}, \dots, \frac{1}{n} \right) \middle| n \in \omega \right\},$$

are examples of infinitesimals for the real, complex, p–adic numbers, the $n \times n$ matrices and the Euclidean spaces $\mathbb{R}^k$. It is straightforward to show that in each case we have for any element x of the above structures we have $|x| > |\mathbf{i}| \neq 0$ where the relation $>$ is to be interpreted as less than in the range of the metric.

We have the following:

Proposition 1 *There exist structures $\mathcal{A}$ realizing the notion of infinitesimal and there is no total order in $\mathcal{A}$.*

Proof. Any of the following structures satisfies the conditions of the statement: The fields of the complex and p–adic numbers. The ring of $n \times n$ matrices for $n > 1$, the Euclidean spaces $\mathbb{R}^n$, $n > 1$. □

Proposition 2 *There exist structures $\mathcal{A}$ realizing the notion of infinitesimal, multiplication is defined in $\mathcal{A}$ and $\mathcal{A}$ contains zero divisors.*

Proof. The ring of $n \times n$ matrices for $n > 1$, satisfies the conditions of the statement. □

Proposition 3 *There exist structures $\mathcal{A}$ realizing the notion of infinitesimal, and there is no field multiplication defined as an operation among elements of $\mathcal{A}$.*

Proof. The Euclidean spaces $\mathbb{R}^n$, the ring of $n \times n$ matrices for $n > 1$, the unit circle and the unit square as topological manifolds satisfy conditions of the statement. □

Proposition 4 *There exist structures $\mathcal{A}$ realizing the notion of infinitesimal, which they do not have a operation of addition, multiplication and there is no constant to be interpreted as zero in $\mathcal{A}$.*

Proof. The unit circle, the unit square as a topological manifolds satisfies the conditions of the statement. □

We have the following statements which follow from the previous propositions:

Theorem 1 *The notion of infinitesimal is logically independent from the notion of total order, from the nonexistence of nontrivial zero divisors, from the notion of addition, from the notion of multiplication and from the existence of zero as an element of the structure.*

The requirement for the existence of zero is not necessary as an element of the structure. The existence of zero is required as an element of the range of the metric defined in the structure.

Corollary 1 *The notion of infinitesimal is logically independent from the notion of the field order.*

Theorem 2 *The notion of ultrafilter is not required for the interpretation of infinitesimals.*

Propositions 1 and 2 entail that the mathematical reasons for the use of ultrafilters in the construction of infinitesimals, namely total order and elimination of nonzero zero divisors are logically independent from the notion of infinitesimal. Hence we can use the cofinite filter for giving a representation of infinitesimal which is represented by a null sequence. The logical reason for the use of ultrafilters is that we can use two valued logic and the Compactness theorem in order to obtain a structure which has the same properties as the real numbers (in an appropriate language). We prove a translation theorem (theorem 3) which is enough to develop Analysis as usual in the monadic second order language of fields. The development of Analysis incorporates the notion of infintesimal via a defintion. It is possible and desirable as suggested by P. Schuster and will be discussed in the future to use null sequences satisfying the principles of some form of Constructive Mathematics and hence use some form of Logic suitable for Constructive Mathematics obtaining constructive infinitesimals such as Intuitionistic Logic.

3 Infinitesimals over the real numbers

From the above we have a ring in the case of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, $\mathbb{R}^{n\times n}$, and a vector space in the case of $\mathbb{R}^n$ where we can interpret the notion of infinitesimal. In essence the present construction is a generalization of the concepts present in C. Schmieden and D. Laugwitz [7]. At this point in order to define the multiplicative inverses the solution given by A. Robinson in [6] is to use ultrafilters in order to ensure that sequences which have infinitely many terms equal to zero are excluded. In the above construction the definition used is the same as the definition used in Analysis (definition 4) and carefully avoids this problem by requiring that the sequences that will represent the infinitesimals have no term equal to zero (this can be relaxed and require that sequences representing infinitesimals eventually have no term equal to zero). This requirement is an inessential variant of the definition of limit in Analysis is postulating as we will see in definition 4. It is interesting that in most cases the discussion of the development of Calculus in the spirit on A. Robinson avoids the discussion of the concept of limit and starts the development from the notion of continuity.

The interpretation of infinitesimals using the cofinite filter as null sequences having no term equal to zero is in accordance with the practice in Analysis. We quote from Spivak the following definition which is the definition of limit [8] (p. 84).

Definition 4 *The function f approaches the limit l near a means: for every $\epsilon > 0$, there exists some $\delta > 0$ such that for all x, if*

$$0 < |x - a| < \delta, \text{ then } |f(x) - l| < \epsilon.$$

We note the explicit use of the clause $0 < |x - a|$ which ensures that when we translate in the language of sequences (using the Axiom of Choice) no term of the sequence will be equal to the limit a.

Definition 4 takes into consideration that in the process of the evaluation of the limit of some expression $E(x)$ as the variable x approaches a, the value $E(a)$ is irrelevant, even $E(x)$ may be undefined at a. In the case that a does not belong in the domain of E taking $x = a$ is impossible. This statement justifies the premise $0 < |x - a| < \delta$, which implies $x \neq a$, hence in the evaluation of limits there will be non (nonzero) zero divisors. As a conclusion we avoid the use but not the existence of zero divisors.

To further elucidate the above statement we briefly review the evaluation of the limit $\lim\limits_{x \to a} \frac{x^2 - a^2}{x - a}$. The expression $\frac{x^2 - a^2}{x - a}$ has a domain the set $\mathbb{R} \setminus \{a\}$. Hence it is not permissible to substitute the value of a for x in the evaluation of the above limit. This last statement rules out nontrivial zero divisors in practice. Similar observations apply in the evaluation of differentials, derivatives of any order, and in similar situations in Euclidean spaces $\mathbb{R}^n$, $n > 1$.

We also quote some relevant comments following the definition of continuity from Apostol [1] p. 75:

We require that p will be an accumulation point of A to make certain that they will be points sufficiently close to p with $x \neq p$.

The above statement translated in the language of sequences using the Axiom of Choice allows to use the cofinite filter and rules out explicitly the use of zero divisors as all the terms in any null sequences occurring have to be non zero.

From the above discussion we see that we can use the cofinite filter and avoid the use of zero divisors which are not logically related with the notion of infinitesimal. In other words the working Analyst works with sequences (uses the cofinite filter) and avoids the existence of zero divisors by using an appropriate definition.

As further support of the above observation we note that in the expression defining the derivative i.e.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

the fact that we have a null sequence (infinitesimal) present in the denominator suggests that we should be able to invert null sequences (by some method taking as a simple solution to require that no term will be equal to zero) and hence null sequences (infinitesimals) are not zero divisors. As further support of the above remarks we quote from A. Robinson [6] (p. 1) “For any positive number ϵ there exists a positive number δ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - a \right| < \epsilon$$

for all x such that $0 < |x - x_0| < \delta$."

This last clause $0 < |x - x_0| < \delta$, ensures that we avoid the zero divisors whenever they might occur. From the above quotations from Analysis we conclude than in Analysis the problem of the existence of (non zero) zero divisors was addressed and solved during the early stages of the development of Calculus.

We note also that in the case of occurrence of the division we can reformulate the expressions without the division operation.

We give an example of of eliminating division in the definition of the notion of differential of a function $f : \mathbb{R}^m \to \mathbb{R}^n$. The linear transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ is called the differential of f at $a \in \mathbb{R}^m$ if there is a function $K : \mathbb{R}^m \to \mathbb{R}^n$ which satisfies $||f(a+h) - f(a) - Th|| = |K(h)|\,||h||$, with $\lim_{h \to 0} K(h) = 0$ (A. Taylor and W. Man [9]). This formula is an alternative of the expression

$$\lim_{x \to x_0} ||f(x) - f(x_0) - A(x - x_0)|| = 0.$$

As we see when we eliminate division we can substitute $x = a$ which in the presence of division may give rise to a zero divisor, but the formula is trivial.

In Euclidean spaces $\mathbb{R}^n$ the division for the evaluation of derivatives is not division among elements of the space $\mathbb{R}^n$ but among elements of the metric space (quotients of size, norms). In these cases we try to determine the rate of change using the notion of size of an element, not the elements themselves.

The above allows the interpretation of infinitesimals using the cofinite filter and the notion of the cofinite filter is used to formulate the notion of limit, derivative, differential, integral in all Euclidean spaces, the complex and p–adic numbers.

The fact that we can interpreted infinitesimals using only the cofinite filter makes the development of Non Standard Analysis to coincide with the development of Analysis via an easy translation theorem (theorem 3 which uses the Axiom of Choice). One more consequence of the above remarks we can interpret infinitesimals according to C. Schmieden and D. Laugwitz [7].

Regarding the field of real numbers we see from the above that it is possible to define the notion of infinitesimal using the notion of the cofinite filter and obtain a structure which extends the real numbers, it fails to be a field and has no natural field order. In such an extension it is possible to develop Analysis as usual. This formulation is in accordance with the actual practice of Analysis and Numerical Analysis and the content of Non Standard Analysis according to the above interpretation is the same as the content of ordinary Analysis.

We have the following main theorem which is the analogue of the transfer principle in Non Standard Analysis:

Theorem 3 *Every theorem of Analysis which uses the $\epsilon - \delta$ formalism can be translated to a statement where we employ null sequences which have no term equal to zero which represent infinitesimals and conversely.*

Proof. Translate the $\epsilon - \delta$ expression to expressions which use sequences and conversely. □

One consequence is that under the interpretation of infinitesimals presented here the methods and the content of Non Standard analysis and Analysis are the same. The above theorem can be seen as a completeness – correctness characterization since it shows that the two formulations the one that uses infinitesimals and the one that uses sequences (or $\epsilon - \delta$ expressions) are the same.

The above theorem can be seen as a weak translation theorem. The construction does not imply that we have any kind of elementarily equivalent structure with the field of real numbers in some language, the Compactness Theorem is not used. The representation of infinitesimals by null sequences which uses the cofinite filter is an extension of Analysis by a definition. We see the requirement of elementarily equivalence as strong requirement. Furthermore the Compactness Theorem is applied in a language that is not the usual language for the development of Analysis. The usefulness of the above interpretation of infinitesimals over the real numbers system is that we can express the notion of approximation of a real number by some entity which is familiar to the working mathematician. The language used to define an ultrafilter is considerably more complex than the language used to define the cofinite filter.

The cofinite filter can be defined by Π_2 sentences which makes possible a constructive interpretation. The above quantifier complexity of the definition of an infinitesimal is the same complexity as most of the notions of Analysis. The real number that is approximated using th notion of infinitesimal can represent the integral, the derivative, length of curve, work of a force etc. These notions (where applicable) can be defined in a straightforward manner using expressions where converging sequences occur in the appropriate structures (fields of the complex and p–adic numbers etc).

We can see the ϵ, δ development of Calculus as a quantitative version of the development which uses infinitesimals. The development of Calculus using infinitesimals can be regarded as a qualitative version of Calculus. These two version quantitative and qualitative are equivalent as we see from theorem 3.

We proceed to define some notions of Analysis in the spirit of the above exposition.

Definition 5 *Let $x \in \mathbb{R}$, $x = \lim_{n\to\infty} x_n$ if and only if the sequence $x - x_n$ is an infinitesimal.*

As a consequence of the above definition of the limit it is possible to express all the definitions of Analysis that involve the notion of limit (i.e. derivative, differential, integral, Taylor and Mac Laurent expansion, length of curves etc) in a manner that uses the notion of infinitesimal as defined above. The standard definitions that use the notion of convergence (or null sequence) can be interpreted as definitions that use infinitesimals and vice versa. The logical complexity of the expressions that are used is not increasing.

Definition 6 *Let $f : [a, b] \to \mathbb{R}$. The real number a is the derivative of f at $x_0 \in [a, b]$ iff for any infinitesimal i the quantity $\frac{f(x_0+i)-f(x_0)}{i} - a$ is infinitesimal.*

The definition of the integral requires some technical elaboration. The integral of a function should be defined as the limit of the sums of products. Each product represents the size of a rectangle such that the bases of the rectangles tend to zero (i.e. the sequence of the lengths is infinitesimal). As the notion of limit can be expressed using the notion of infinitesimal we have the following definition.

Definition 7 *Let $f : [a, b] \to \mathbb{R}$ be a function and $[c, d] \subseteq [a, b]$. Let $\mathcal{P}_n = \{x_0 = c < x_1 < \ldots < x_n = d\}$ be a (finite) partition of the interval $[c, d]$. For such a partition let $d(\mathcal{P}_n) = \max\{|x_{i+1} - x_i|\}$. The number A is the integral of f over* *the interval* $[c,d]$ *iff for any infinitesimal* I *for any sequence of partitions* $\{\mathcal{P}_n\}$ *of* $[c,d]$ *such that* $\{d(\mathcal{P}_n) | n \in \omega\} \leq I$*, and for any* $\xi_i \in [x_i, x_{i+1}]$ *the quantity* $\sum_0^{n-1} f(\xi_i)(x_{i+1} - x_i) - A$ *is infinitesimal.*

It would be interesting to see applications of the above method in the following sense. "One might express the hope that some branches of modern Theoretical Physics, in particular those afflicted with divergence problems, night be treated with profit by Non–standard Analysis." (A. Robinson [6], p. 5).

We also give the following theorem as an application which is inspired by a theorem of A. Kock [4].

Theorem 4 *A function* $f : [a,b] \to \mathbb{R}$ *is continuous at a point* x_0 *then* $f(x_0)$ *equals the average of the values of the function over any infinitesimal around* x_0*.*

Proof. For any sequence $\{x_n \mid n \in \omega\}$ that converges to x we have that $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = x$ (Cesaro limit). Since the function f is continuous for any infinitesimal (null sequence) $I = \{x_0 - x_n \mid n \in \omega\}$ around x_0 that $\lim_{n\to\infty} f(x_n) = f(x_0)$. $\square$

Geometrically in the case of the real numbers we can represent the infinitesimals as a cone in a plane that contains the real line, with apex at zero and the cone does not contain the line that corresponds to the real numbers. In the case of the complex numbers similarly we can represent the infinitesimals as a cone with apex at zero and not containing the complex plane.

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