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STRUCTURAL AND VIBRATION ANALYSIS OF POLYMERIC FUNCTIONALLY GRADED PLATES

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Supervised by:
Prof. Simon Bickerton, Dr. Raj Das, Prof. Brian Mace and Dr. Emilio Calius

Centre for Advanced Composite materials
The University of Auckland

This dissertation is submitted for the degree of

Doctor of Philosophy in Mechanical Engineering

September 2017
“Two things are infinite: the universe and human stupidity; and I am not sure about the universe.”

Albert Einstein
Abstract

Functionally graded materials (FGMs) are advanced engineered materials whereby material composition and properties vary spatially in the macroscopic length scale. Accurate analytical and numerical models for prediction of mechanical behaviour and damage of the FG plate are required as their applications in industry grow. During manufacturing process of FG plates, the reliability requirements for the product should be considered to meet desired or application-specific performance criteria. One approach to produce polymeric FG plates is use of additive manufacturing like 3D printing, which can control local composition and microstructure. In addition, the material characterisation of 3D printed FG plates is a critical factor for prediction of mechanical behaviour and damage. Furthermore, this research is tended to be a first step towards the analytical and numerical solution of structural behaviour of FG plate with in-plane variation of stiffness. This project also develops 3D DIC experimental approach for deflection measurement of FG plates.

Analytical solution for static stress analysis of thin and thick FG plates with variation of stiffness through the length of the plate carries out. Then the numerical simulation achieves using graded elements which implements by user material subroutines (UMAT and USDFLD). After that, an analytical formulation for damage of FGMs with considering coupled damage-plasticity framework presents. The implementation of analytical formulation performs using UMAT subroutine in the ABAQUS software using a robust three-step numerical algorithm. In order to validate the graded elements, a physical model of FG plates designs and manufactures by means of 3D printing to carry out a new 3D-DIC experimental test. Another aim of experimental tests is characterisation and identification of damage parameter and material properties to validate with numerical simulation.

Generally, the results of analytical and computational modelling with physical experiments provide valuable contribution for studying structural prediction of complex FG structure.

The research will be beneficial for real life application of functionally graded plates in engineering structures by localised stiffness improvement in order to reduce localized deformation for light weight structural design in aerospace such as wing component.
Acknowledgements

The work on this Ph.D research has been an inspiring, often exciting, sometimes challenging, and nevertheless always interesting experience. I would like to thank all the people who contributed in some way to work described in this thesis.

Foremost, I would like to express my sincere gratitude to my supervisor Prof. Simon Bickerton for the continuous support of my Ph.D study and research. You have provided the instrumental support and guidance this project which enabled me to pursue the research with success. Thank you for the honest feedback, great advice and engaging me in new idea.

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I thank Distinguished Professor Debes Bhattacharyya for his unconditional commitment to CACM and inspiring me.

Special thanks must also go to Dr. Tom Allen for his patience and helping me to find a solution. Being able to have a chat with you about experimental tests has significantly helped me and I am grateful for all your great feedback.

My greatest thanks go to Dr. Mark Battley. Thank you for all your feedback and idea on how to perform deflection test using DIC even better.

I must grateful the CACM lab technicians who make every practical task possible. Thank you for all your assistance, patience and amazing ability to find a solution for unexpected problems during experiments. Additionally, I would like to thank Sheeja for all her help with the CACM and university tasks.

I am so grateful for all my colleagues and friends at CACM for creating a wonderful and supportive environment during my Ph.D life.

Finally, my greatest thanks go to my family. I owe thanks to a very special person, my husband, Yasser for your continued and unfailing love and support even during your unbelievable medical condition. You constantly encouraged me when the tasks seemed difficult and insurmountable. My heartfelt appreciation goes to my son, Arvin. I could not have completed this path without your smile and support that always inspired me. I will be forever in my parents and siblings’ debt for their love, caring and faith in me to be as ambitious as I wanted.
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Stress analysis of functionally graded plate under different gradient direction*, 6th International Conference on Computational Methods (ICCM2015), Auckland, New Zealand

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An elasto-plastic damage model for functionally graded plates with in-plane material properties variation. Material model and numerical implementation”. Composite Structures, 162, 337-341.

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## Nomenclature

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<thead>
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<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$a$</td>
<td>Plate length</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Eigen-frequency parameter</td>
</tr>
<tr>
<td>$D(x)$</td>
<td>Damage variable</td>
</tr>
<tr>
<td>$\varepsilon_e$</td>
<td>Elastic strain</td>
</tr>
<tr>
<td>$\varepsilon_p$</td>
<td>Plastic strain</td>
</tr>
<tr>
<td>$\varepsilon_{\text{threshold}}$</td>
<td>Initial strain damage threshold</td>
</tr>
<tr>
<td>$E_i$</td>
<td>Inclusion Young’s modulus</td>
</tr>
<tr>
<td>$E_m$</td>
<td>Matrix Young’s modulus</td>
</tr>
<tr>
<td>$f_d$</td>
<td>Damage criteria</td>
</tr>
<tr>
<td>$F_d$</td>
<td>Damage dissipation potentials</td>
</tr>
<tr>
<td>$f_p$</td>
<td>Plasticity criteria</td>
</tr>
<tr>
<td>$F_p$</td>
<td>Plasticity dissipation potentials</td>
</tr>
<tr>
<td>$\bar{G}$</td>
<td>Effective shear modulus</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>Bulk modulus</td>
</tr>
<tr>
<td>$n$</td>
<td>Power law index</td>
</tr>
<tr>
<td>$P_i$</td>
<td>Inclusion phase properties</td>
</tr>
<tr>
<td>$P_m$</td>
<td>Matrix phase properties</td>
</tr>
<tr>
<td>$q$</td>
<td>Transverse load</td>
</tr>
<tr>
<td>$R(x)$</td>
<td>Isotropic hardening</td>
</tr>
<tr>
<td>$r$</td>
<td>Plasticity variable</td>
</tr>
<tr>
<td>$u$</td>
<td>Displacement in x direction</td>
</tr>
<tr>
<td>$v$</td>
<td>Displacement in y direction</td>
</tr>
<tr>
<td>$V_i(x)$</td>
<td>volume fraction of constituents</td>
</tr>
</tbody>
</table>
\( w \) Transverse deflection
\( Y \) Thermodynamic force associated with damage-damage energy release rate
\( Y_{\text{threshold}} \) Damage threshold
\( \xi \) Sinusoidal functions for shear deformation
\( \dot{\lambda}_0 \) Damage multiplier
\( \dot{\lambda}_p \) Plasticity multiplier
\( \bar{\sigma} \) Effective Cauchy stress
\( \sigma \) Cauchy stress
\( \sigma_y(x) \) Initial yield stress
\( \bar{\sigma}_{\text{dev}}^{n+1} \) Effective deviatoric component of Cauchy stress
\( \bar{\sigma}_{\text{hyd}}^{n+1} \) Effective hydrostatic component of Cauchy stress
\( \psi \) Helmholtz free energy
\( \omega \) Natural frequency
\( \Psi_e \) Elastic part of Helmholtz free energy
\( \Psi_p \) Plastic part of Helmholtz free energy

Acronyms / Abbreviations

- CCCC: Four edges completely clamped
- CDM: Continuum Damage Modelling
- CPT: Classical Plate Theory
- DIC: Digital Image Correlation
- DTA: Defence Technology Agency
- FGM: Functionally Graded Materials
- HSDT: Higher Shear Deformation Theory
- LCC: Local Composite Control
- MODEM: MATLAB Optical Displacement and Strain Measurement
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Definition</th>
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<tbody>
<tr>
<td>PDV</td>
<td>Portable Digital Vibrometer</td>
</tr>
<tr>
<td>SLURM</td>
<td>Simple Linux Utility for Resource Management</td>
</tr>
<tr>
<td>SSSS</td>
<td>Four edges simply supported</td>
</tr>
<tr>
<td>VM</td>
<td>Virtual Machine</td>
</tr>
<tr>
<td>VW</td>
<td>Vero White</td>
</tr>
</tbody>
</table>
1 Chapter 1: Introduction

1.1 Background

In examining some natural structures such as the stems of plants, animal bones, mollusk shells, and other biological hard tissues, it can be seen that their geometry or structure changes to accommodate to their physical environment. This implies that they are highly adapted to all boundary and loading conditions defined by their environment. One of the most important consequences in fully understanding the mechanical behaviour of natural materials, e.g. bones, is the inspiration of a new class of advanced materials. The bone sample shown in Figure 1.1 has a graded structure from the surface cortical bone towards the inner cancellous bone. So, the interior structure of a bone has an optimized shape with respect to the direction of principal stress and the magnitude of the shear stress [1].

![Figure 1.1: The functionally graded structure of animal bone](image)

Functionally Graded Materials (FGMs) are advanced engineered materials whereby material composition and properties vary spatially on macroscopic length scales, which are created by specialized manufacturing processes. The introduction of FGMs was made by Japanese scientists in 1984 for a space plane project, to create ultra-high temperature resistant materials [3]. The main advantage of FGMs is the elimination of stress concentration and discontinuity at interfaces, due to monotonous variation of particle
volume fraction, which may also reduce delamination-related problems. The other merits of this type of material are improved residual stress distribution, enhanced thermal properties and high fracture toughness [4]. One example application of FGMs is orthopaedic and dental implants (see Figure 1.2). FGM implants may be used as artificial bone for medical use, and as artificial teeth for dental use. The specified properties are slightly different depending on their use. Orthopaedic implants are used mostly as structurally enforced artificial bone which is inserted inside the corpus [5]. The main advantages of using FGM dental implants are reduction of the stress shielding effect on the surrounding bones, and improvement of biocompatibility with bone tissues [6, 7].

![Figure 1.2: Application of FGMs in orthopaedic and dental implants [8]](image)

Figure 1.3 illustrates another example application of FGMs to a space shuttle structure. The heat source is created by air friction due to high velocity movement. If such structures are made from FGMs, the hot air flow is blocked by the outside surface of ceramics and transfers slightly into the bottom surface. Consequently, the temperature at the lower surface is much reduced, which therefore prevents or minimises structural damage due to thermal stresses and thermal shock.
Figure 1.3: An example of FGM application for aerospace engineering [8]

Figure 1.4: Potentially applicable fields for FGMs [9]

Figure 1.4 presents a range of other applications of FGMs. As shown, these materials can be used for a variety of potential applications in transport systems, energy conversion systems, cutting tools, machine parts, semiconductors, optics, bio-systems, etc.[9]. Moreover, another attractive potential application of FGMs is in the smart structures area. Particularly, it is interesting to explore piezoelectric FGM structures, such as piezoelectric actuators, where the piezoelectric effect could be used to suppress the
dynamic or static response, while grading may result in reduced stresses and deformations [10].

1.1.1 Gradient distribution

Although FGMs are highly heterogeneous, it will be very useful to idealize them as continua, with their mechanical properties changing smoothly with respect to the spatial coordinates. The material properties of an FGM are generally assumed to follow gradation in one direction in a continuous manner. Particular functional forms used to prescribe such non-homogeneity commonly include exponential and power-law variation of material properties, due to their differentiable and simple behaviour. Some scientist have used an exponential function for defining materials’ gradient distribution definition. Guo et al. [11] and Wang et al. [12] formulated an exponential function for variation of material properties, in order to study the thermal shock crack problem for an FG plate and cylindrical shell, respectively. Moreover, other researchers applied this function for variation of volume fractions to investigate crack propagation, fracture mechanics, bending, buckling and vibration ([13],[14],[15]). A common definition of an exponential function of material properties is given by:

\[ f(x) = f_0 e^{\beta x} \]  \hspace{1cm} (1.1)

where \( f(x) \) can represent Young’s modulus \( E \), shear modulus \( G \) and mass density \( \rho \). \( f_0 \) takes the value of the material property at location \( x = 0 \). It is noted that \( \beta \) is a constant parameter defining material property variation through the length, while an isotropic homogeneous material can be represented by setting \( \beta = 0 \). A power law material distribution model has been widely used in a number of studies, exhibiting good agreement with experimental results ([16], [17], [18]). Moreover, Gunes et al. [19] used a power law distribution of volume fraction for an FGM circular plate to investigate elasto-plastic impact response. Also, Hein et al. [20] studied thermo-mechanical analysis for FGMs, applying a power law distribution of volume fraction. For the static and vibration analysis of an FGM piezoelectric beam, Nazargah [21] applied this function for particles distribution. Recently, Natarjan et al. [22] and Farahani et al. [23] used a power law distribution of volume fraction to survey bending and buckling of FGM plates and cylinders analytically. Power law distribution of particles volume fraction have often been represented by the following:

\[ v_p(x) = (1 - \frac{x}{b})^n \]  \hspace{1cm} (1.2)

where the positive number \( n \ (0 \leq n \geq \infty) \) is the power law index, \( x \) is a distance parameter along the variation direction, and \( b \) is the total length of the plate. The power
law index $n$ essentially dictates the amount and distribution of particles in the matrix. As an example application of Equation 1.2, Figure 1.5 shows the change in volume fraction through the thickness for several values of $n$ ($n=0.05, n=0.5, n=1.0, n=2.0, n=10$ (fully matrix)). Note that lower values of $n$ push the curve toward the top, or toward a fully inclusion surface. In fact, at $n\sim 0$, the curve will approach a vertical line corresponding to a volume fraction of inclusion equal to 1. Additionally, higher values of $n$ push the curve toward the bottom, or toward a fully matrix material plate.

![Figure 1.5: Application of Equation 1.2, demonstrating changes in volume fraction through a plate thickness, for different values of $n$](image)

### 1.1.2 Fabrication methods of FGMs

Numerous manufacturing techniques have been proposed to produce components from FGMs, dependent on the intended applications, including a) porosity gradients, b) chemical composition gradients of single-phase materials, and c) volume fraction gradients of constitutive material phases [24]. Table 1.1 depicts the characteristics of conventional processing methods.
In additive manufacturing technology—commonly called “3D printing”—objects are built from selective addition of material rather than by moulding or by traditional methods of subtractive machining, where material is removed by cutting and grinding. In recent years three dimensional printing (3DP) has moved to the foreground as a very competitive process in terms of cost and speed, and sales of related equipment have increased significantly compared to other rapid prototyping machines [25].

The ability of 3DP to make functionally graded material (FGM) parts by using local composition control (LCC) creates novel opportunities for designers to optimise their designs. Jackson [26] gives an example of designing a pulley with more carbide near the hub and rim (to make it harder and therefore more wear resistant). This heralds a completely new way in which parts can be designed. Other interesting examples of 3D printing are described in the work of Oxman and their colleagues [27]. These authors give an example prosthetic, socket and optically transparent glass that was manufactured by 3D printing. Local composition control refers to a capability of the 3D printing process to tailor the material composition of a part locally using precise control of a multiple-material print head [28]. In recent years companies have worked to address some of these challenges, and fabricate devices with no assembly required. Table 1.2 provides some information on a selection of 3D printers used throughout the world.

### Table 1.1: FGM processing methods [24]

<table>
<thead>
<tr>
<th>Process</th>
<th>Type of FGM</th>
<th>Layer Thickness</th>
<th>Versatility in component geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Powder stacking</td>
<td>Bulk</td>
<td>M, L</td>
<td>Moderate</td>
</tr>
<tr>
<td>Sheet lamination</td>
<td>Bulk</td>
<td>T,M</td>
<td>Moderate</td>
</tr>
<tr>
<td>Wet powder spraying</td>
<td>Bulk</td>
<td>UT,T</td>
<td>Moderate</td>
</tr>
<tr>
<td>Slurry dipping and slip casting</td>
<td>Coating</td>
<td>UT,T</td>
<td>Good</td>
</tr>
<tr>
<td>Solid freeform (3DP- Laser Sintering)</td>
<td>Bulk</td>
<td>M,L</td>
<td>Very good</td>
</tr>
<tr>
<td>Sedimentation</td>
<td>Bulk</td>
<td>C</td>
<td>Poor</td>
</tr>
<tr>
<td>Laser Cladding</td>
<td>Bulk Coating</td>
<td>M</td>
<td>Very good</td>
</tr>
<tr>
<td>Thermal spraying</td>
<td>Bulk Coating</td>
<td>T</td>
<td>Good</td>
</tr>
<tr>
<td>Diffusion</td>
<td>Joint, coating</td>
<td>C</td>
<td>Good</td>
</tr>
<tr>
<td>Directed solidification</td>
<td>Bulk</td>
<td>C</td>
<td>Poor</td>
</tr>
<tr>
<td>Electrochemical gradation</td>
<td>Bulk</td>
<td>C</td>
<td>Good</td>
</tr>
<tr>
<td>Foaming of polymers</td>
<td>Bulk</td>
<td>C</td>
<td>Good</td>
</tr>
<tr>
<td>Vapour deposition</td>
<td>Coating</td>
<td>C</td>
<td>Moderate</td>
</tr>
<tr>
<td>Gradient materials by foam compaction (GMFC) process</td>
<td>Bulk</td>
<td>M,L,C</td>
<td>Good</td>
</tr>
</tbody>
</table>

L: large (>1 mm), M: medium (100–1000 micrometre), T: thin (10–100 micrometre), UT: very thin (<10 micrometre), C: continuous.
Table 1.2: Different 3D printers and their applications

<table>
<thead>
<tr>
<th>Name of 3D printer</th>
<th>Type of used material</th>
<th>Application</th>
<th>Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Voxel8</td>
<td>Thermoplastic/Resin/ Conductive Filament</td>
<td>Electronic device</td>
<td>Access to Autodesk Wire 3D Electronics Software The first commercial 3d printer</td>
</tr>
<tr>
<td>OWL MC-2</td>
<td>Resin</td>
<td>Biomedical device</td>
<td>100nm Resolution (The finest resolution)</td>
</tr>
<tr>
<td>XFAB</td>
<td>Acrylate/ABS/PP/Ceramic</td>
<td>-</td>
<td>Cheap</td>
</tr>
<tr>
<td>MultiFAB</td>
<td>-</td>
<td>-</td>
<td>Up to 10 different materials at once Scanning tech.</td>
</tr>
<tr>
<td>CLIP</td>
<td>Polymer</td>
<td>-</td>
<td>100x faster Chemical process</td>
</tr>
<tr>
<td>Object Connex</td>
<td>Rubber/PP/ABS</td>
<td>-</td>
<td>Combine as many as 14 material properties in a single model</td>
</tr>
</tbody>
</table>

Current methods for producing functionally graded materials are limited, and an understanding of their effective properties and subsequent behaviour is scarce in the literature. Current research at MIT’s Media Lab is focussed on creating FGMs based on mixing polymers together uses additive manufacturing concepts [29].

### 1.2 Motivation

As will be discussed later in Chapters 3 and 4, in all the applications reported in the literature the variation of material properties is applied through the thickness direction only. However, no attention has been paid to distribution of volume fraction in length directions, resulting in the variation of material properties along a component. For design and analysis of thin and thick FG plates, with in-plane variation of stiffness, simple closed form and analytical solutions to determine deformations and stresses are useful. Therefore, the development of theoretical and numerical methodologies for accurate structural predictions for FG plates with in-plane property variation is one of the main purposes of this research. Since it is very important to anticipate the progressive damage and failure of a structure, the analytical and numerical solutions were to be developed through this project. Moreover, there is no current coupled elasto plastic-damage formulation for FG plates with in-plane stiffness variation. As will be mentioned, the current analytical models are not able to capture the out of plane deflection for in–plane stiffness variation accurately, especially for very thick plates. Although numerical simulation can be used to determine structural behaviour, current methods cannot predict accurately for FG plates with high stiffness ratio due to the use of homogenous elements with step variations in material properties that approximate an average value at the centre.
of the elements. The research work presented here focuses on the development of a new analytical solution for static analysis of FG plates with in-plane stiffness variation, and a proposes a novel graded solid elements for numerical FE simulation. Moreover, a new experimental tool is developed that can be used to satisfy and compare the proposed analytical solution and numerical simulation outputs.

1.3 Research objectives

The objective of this doctoral research is to develop, implement and validate a methodology for predicting the static behaviour and damage modelling of polymeric functionally graded plates with in-plane stiffness variation. Moreover, this study can be applied to find the optimum material distribution for a controlled-stiffness FG plate corresponding to prescribed characteristics. The overall objectives are mentioned as follows:

1. Development of analytical solutions for static analysis of an FG plate with in-plane property variations, which could be used to predict deflections stresses and strains in closed form.

2. Development of numerical solution for static analysis of an FG plate by means of graded finite elements, which allow gradient properties within an element.

3. Development of an analytical formulation for damage within an FG plate, based on a continuum damage model that can predict failure of the FG plate.

4. FE modelling of continuum damage modelling of an FG plate.

5. Fabrication of 3D-printed samples to carry out experimental tests in order to characterize and identify material parameters to validate analytical solutions and numerical simulations.

6. Experimental studies for static analysis of designed and manufactured FG plates to verify the numerical simulations.

1.4 Potential applications

This research will be beneficial for real life application of FG plates with in-plane variation of material properties, in engineering structures such as wing components, prosthesis and scaffold for biomedical applications and gradient alloys for carbon fibre composite for use in low-temperature spacecraft panels.
For example, aircrafts interact with dynamic fluid flows and incorporate various mechanisms to adapt to changes in the nature of the flow that typically occur with manoeuvres, such as morphing aircraft under development by NASA. Therefore, FG plates with in-plane stiffness variation could be applied for wings for better adaption and efficiency of the aircraft, by localizing stiffness improvements to reduce localized deformation fields. Another aerospace application of FG plates is in the use of gradient alloys for spacecraft panels. Carbon fibre/aluminium foam composites are widely used in spacecraft structures due to their high strength and stiffness, coupled with low density. To attach the panels together or onto other structures, however, metal inserts must be adhered into holes drilled in the panels. The inserts then provide a threaded hole through which other components can be bolted to the panel. Unfortunately, these panels are often used in low temperatures or experience extreme temperature variation. The coefficient of thermal expansion (CTE) mismatch between the low-CTE carbon fiber and the high-CTE metal inserts stresses the epoxy and a phenomenon known as “pullout” occurs.

Another potential application in the biomedical field could be hand prostheses for which gradients in materials could be required, as different parts of a finger need different bending stiffness. Moreover, FG plate with in-plane stiffness variation can be used for implants to minimise the stress field. Traditionally, solid metals such as Ti or Co-Cr alloys are used as hip stems. The problem is the stress shielding caused by the mismatch in stiffness between the implants and the bone. One way to minimise stress shielding is to prepare a functionally graded coating on a conventional solid stem. Moreover, this research can be applied for designing and manufacturing of scaffolds for tissue engineering. In order to improve the acceptance of artificial implants by living tissue, a functionally graded interlayer of biopolymers should be a favourable approach to create optimized interactions with cells living in the surrounding tissue.

This study can be extended and combined with topology optimization to determine the optimum material distribution to produce a controlled-stiffness FG plate corresponding to prescribed structural characteristics. The combination of the derived analytical and numerical results will enable us to understand the behaviour of new materials with controlled macro properties through a specific direction with particular potential relevance to biomedical and aerospace sectors.

1.5 Thesis outline

This thesis has been divided into chapters, addressing the main topic of static behaviour of polymeric FG plates with in-plane stiffness variation. A literature review is presented at the beginning of each chapter with findings relevant to that chapter’s content:
Chapter 2. Characterisation of 3D-printed polymer-polymer composite materials. All polymeric digital materials used in a Stratasys Object 500 3D printer are tested to select the appropriate combination for FG plates to be used in this thesis. This is followed by static and cyclic tensile tests in order to characterize material parameters which are then used to implement the numerical FE modelling. Additionally, the FG plates with desired material properties variation are designed and manufactured using a 3D printer.

Chapter 3. Static analysis of FG plates: analytical and FE solutions. A novel analytical solution based on HSDT theory is proposed to model deflection and free vibration of the designed FG plates. Additionally, graded solid elements are implemented for FE numerical simulations in order to validate the proposed analytical solution.

Chapter 4. Damage modelling of FG plates: material model and numerical implementation. In this chapter, an elasto plastic damage model is proposed to predict damage within an FG plate. Two independent plastic and damage multipliers are introduced to eliminate strong limitation in modelling capacity. Additionally, a robust three-step numerical algorithm is implemented in order to simulate damage numerically.

Chapter 5. Static analysis of FG plates: experimental validation. This chapter presents the experimental facilities and techniques used to determine the out of plane deflection and first natural frequencies of vibration. Complete results are provided which are then compared with the compatible FE models.

The thesis is then concluded with principal achievements, original contributions and recommendation for future work.
2 Chapter 2: Characterisation of 3D-printed polymer-polymer composite materials

2.1 Introduction

The main goal of this chapter is to design and identify the material properties of 3D-printed polymeric functionally graded plates. This includes simple tensile test results on composite materials with different volume ratio of constituents. Then, material properties selected for two distinct FG compositions which show low stiffness (VeroWhite as an inclusion and DM8530 as a matrix) and high stiffness ratio (DM8515 as an inclusion and DM9895 as a matrix) with stiffness ratios of 2 and 18, respectively. After that, verification of material property gradients using simple and five-cycle tensile tests were carried out. It is demonstrated that material properties like Young’s modulus and damage parameters for FG plates follow specific functions with respect to the length of plate (x).

Subsequently, distribution patterns utilising two materials (hard and soft polymers) were designed using the linkage between the GRASSHOPPER, RHINO and MATLAB software packages, in order to manufacture desired FG plates using a 3D-printer.

2.2 Experimentation to characterise digital materials properties

One of the main purposes of this research is to validate the proposed analytical solutions and numerical simulations for two distinct FG plates with varying stiffness ratios. So, the proper selection of the constituent polymeric digital materials that satisfy low and high stiffness ratio is critical.

The properties given in the Stratasys material specifications datasheet are expressed in terms of a range of values and therefore possess uncertainties when trying to design a stiffness gradient [30]. Stratasys, Ltd. is a manufacturer of 3D printers and 3D production systems for office-based rapid prototyping and direct digital manufacturing solutions.
Stratasys is recognised as a leading manufacturer of multimaterial 3D printers. Therefore, in order to determine the material properties of the materials applied to this study, a series of digital material samples were 3D printed using a Stratasys multi-material 3D printer. Their Objet500 Connex printer can produce resolvable inclusion sizes as small as 0.2mm in diameter. Printable materials vary from rigid (VEROX family) to rubber like (TANGOX family). Mixed printing is possible, for which any two materials can be combined to create so called digital materials to deliver any mechanical properties between the limits of the constituents. Table 2.1 shows the list of available materials that can be employed in the Stratasys Objet500 3D printer.

<table>
<thead>
<tr>
<th>Primary Material: VeroWhite</th>
<th>Secondary Material: Tangoblack+</th>
</tr>
</thead>
<tbody>
<tr>
<td>VeroWhite (the hardest)</td>
<td></td>
</tr>
<tr>
<td>DM 8505 Grey20</td>
<td></td>
</tr>
<tr>
<td>DM 8510 Grey25</td>
<td></td>
</tr>
<tr>
<td>DM 8515 Grey35</td>
<td></td>
</tr>
<tr>
<td>DM 8520 Grey40</td>
<td></td>
</tr>
<tr>
<td>DM 8525 Grey50</td>
<td></td>
</tr>
<tr>
<td>DM 8530 Grey60</td>
<td></td>
</tr>
<tr>
<td>DM 9895 Shore 95</td>
<td></td>
</tr>
<tr>
<td>DM 9885 Shore 85</td>
<td></td>
</tr>
<tr>
<td>DM 9870 Shore 70</td>
<td></td>
</tr>
<tr>
<td>DM 9860 Shore 60</td>
<td></td>
</tr>
<tr>
<td>DM 9850 Shore 50</td>
<td></td>
</tr>
<tr>
<td>DM 9840 Shore 40</td>
<td></td>
</tr>
<tr>
<td>TangoBlck+ (the softest)</td>
<td></td>
</tr>
</tbody>
</table>

For the determination of the digital polymeric material properties, according to ASTM D638 [31] for rigid plastic and ASTM D412 [32] for rubbers, three dog-bone test
samples (Type V) for each material were 3D printed. Figure 2.1 presents a schematic of type V samples, as defined by ASTM D638.

All required tests were carried out using an Instron 5564 with a 30KN axial load capacity. To ensure a quasi-static test loading condition, a loading rate of 1 mm/min was applied. For the measurement of strain, a visual extensometer has been used (refer Figure 2.2). The applied extensometer is suitable for measurement of soft materials such as plastics having low elastic modulus compared to metals.
For hard polymeric digital materials, mechanical wedge grips were used, whilst pneumatic grips were utilised for soft polymeric digital materials. It should be noted that the standard procedure for the calculation of Young’s modulus for rubbers (Tangoblack+) has not been mentioned in standard tensile test ASTM D412. Therefore, the chord measurement of 0.05% and 0.25% of strain has been used to calculate the Young’s modulus, based upon ASTM D638. A set of three samples for each digital material mentioned in Table 2.1 were loaded at a rate of 1 mm/min to simulate static tensile tests. The full programme of experiments completed on an Instron 5564 universal testing machine is provided in Appendix A.

The obtained ultimate tensile strengths and Young’s moduli for soft and hard polymeric digital materials are shown in Figure 2.3, Figure 2.4 and Figure 2.5, respectively. During the tests, some hard polymeric materials fractured, failing outside of the gauge length. This was possibly due to the shape of samples, and their relative weakness under small amounts of bending.
Figure 2.3: Experimental values of different digital polymeric materials (a) ultimate tensile strength and (b) ultimate tensile strain.

Figure 2.4: Experimental values of Young’s modulus of soft digital polymeric materials.
In order to provide good visualization during test, the grey number is considered as another material selection criteria. Therefore, regarding the Figure 2.4, Figure 2.5 and grey number, the combination of VW_8530 as the low stiffness ratio ($E_i/E_m \sim 2$) and 8515_9895 ($E_i/E_m \sim 17$) as high stiffness ratio combination were considered.

2.3 Verification of material property (stiffness) gradients

2.3.1 Static tensile test

For verification of material property variation through the length of plate, a series of samples with stiffness gradients were 3D printed. These samples were composed from combinations of VW_8530 (VW as a hard polymer) and 8515_9895 (8515 as a hard polymer).

A set of four samples, each measuring 130×13×3 mm, were printed for each combination to determine the material properties of various volume compositions of hard polymer via simple and cyclic tensile testing. Each sample had inclusion size 0.4 mm ×0.4 mm ×0.4 mm and the inclusions were placed randomly throughout the length. For each combination, the four samples were: 20%, 40%, 60% and 80% of the hard polymer.
Computer models and photographs of the samples for the VW_8530 and 8515_9895 combinations are shown in Figure 2.6, Figure 2.7 and Figure 2.8, respectively.

![Computer model of the different composite samples](image)

**Figure 2.6**: Computer model of the different composite samples
20% hard polymer

40% hard polymer

60% hard polymer

80% hard polymer

Figure 2.7: 3D printed composite samples- VW_8530 combination
For static tensile testing, the same Instron machine and test procedures were used as described in Section 2.2. Table 2.2 provides the Young’s modulus and ultimate tensile strength for each sample.
Table 2.2: Experimental results for Young’s modulus and ultimate tensile strength of two combination samples

<table>
<thead>
<tr>
<th>Materials</th>
<th>No of specimens</th>
<th>E_Ave (MPa)</th>
<th>E_Stdv (MPa)</th>
<th>σy_Ave (MPa)</th>
<th>σy_Stdv (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VW_8530_20%</td>
<td>3</td>
<td>1294</td>
<td>267</td>
<td>17.1</td>
<td>0.423</td>
</tr>
<tr>
<td>VW_8530_40%</td>
<td>3</td>
<td>1542</td>
<td>125</td>
<td>20.9</td>
<td>0.342</td>
</tr>
<tr>
<td>VW_8530_60%</td>
<td>3</td>
<td>1664</td>
<td>216</td>
<td>22.5</td>
<td>0.49</td>
</tr>
<tr>
<td>VW_8530_80%</td>
<td>3</td>
<td>1715</td>
<td>112</td>
<td>28.4</td>
<td>0.666</td>
</tr>
<tr>
<td>8515_9895_20%</td>
<td>3</td>
<td>182</td>
<td>15</td>
<td>4.41</td>
<td>0.149</td>
</tr>
<tr>
<td>8515_9895_40%</td>
<td>3</td>
<td>387</td>
<td>48</td>
<td>7.71</td>
<td>0.213</td>
</tr>
<tr>
<td>8515_9895_60%</td>
<td>3</td>
<td>708</td>
<td>86</td>
<td>11.4</td>
<td>0.134</td>
</tr>
<tr>
<td>8515_9895_80%</td>
<td>3</td>
<td>1120</td>
<td>136</td>
<td>19.2</td>
<td>0.078</td>
</tr>
</tbody>
</table>

Figure 2.9 and Figure 2.10 show the variation of the experimental results for Young’s modulus and ultimate tensile strength obtained from the static tensile testing for the high stiffness ratio combination (8515_9895), respectively. From empirical curve fitting, the obtained results display a power law relationship between the material properties and the percentage of the hard polymer in the composite.

\[
E = 0.00012V_i^3 + 0.13V_i^2 + 2.1V_i + 93 \quad (2.1)
\]

\[
\sigma_y = 0.0076V_i^2 + 0.088V_i + 3.4 \quad (2.2)
\]
As can be seen from Figure 2.11 and Figure 2.12, the variation of Young’s modulus and ultimate tensile strength for the low stiffness ratio combination (VW_8530) demonstrates a power law function again.

\[
E = 0.003V_t^3 - 0.35V_t^2 + 17V_t + 1210
\]  

(2.3)
\[ \sigma_y = 0.0000087V_i^3 + 0.00077V_i^2 + 0.93V_i + 16 \]  

(2.4)

Figure 2.11: Variation of Young’s modulus versus volume fraction of hard polymer for VW_8530 combination

Figure 2.12: Variation of ultimate tensile strength versus volume fraction of hard polymer for VW_8530 combination

Lyon [38] showed that the measured Young’s moduli from the elongation at the slow rates (0.08mm/s) and fast rate (0.8mm/s) for 3D printed composite palates with different percentage of inclusion (hard polymer) has different values. They concluded the fast rate of loading has a greater range of Young’s moduli than the slow rate, most likely because the softer samples begin to creep at the slower rate. The proposed model in this research
is considering elastic-plastic damage analysis as the loading rate at experimentation is very slow (1 mm/min) to model quasi-static loading.

2.3.2 Cyclic tensile tests

One of the main novel contributions of this work is that the modelling takes into account variation of the damage parameters through the length of FG plate. Moreover, the proposed damage modelling in Chapter 4 considers only coupled elasto-plastic damage modelling, and does not consider visco-elasto-plastic damage simulation. Therefore, the VW_8530 combination which shows elasto-plastic behaviour has been considered to identify damage parameters for an FG plate composed from a hard polymer (VW) and a soft polymer (8530). The damage evolution parameters of an FG plate have been obtained from the 1D stress-strain response of a uniaxial cyclic tensile test. A set of six VW_8530 samples at each volume ratio were loaded with a gradually increasing prescribed strain according to Figure 2.13 (i.e VW_8530_60%). The tests were performed at a constant temperature of 22°C, and a strain rate of 1 mm/min. During the unloading phase, the load was decreased until the load in the samples was approximately zero.

![Cyclic tensile test graph](image)
From the resulting stress strain diagram (shown in Figure 2.14) the damage parameters can be obtained for each load cycle.

The ultimate tensile strength is taken as the maximum stress value reached during a cycle. After subsequent unloading the residual strain at zero stress equals the accumulated irreversible plastic strain $\varepsilon_p$. The elastic strain $\varepsilon_e$ is the reversible strain that is required to obtain the zero stress point during the unloading path. It is calculated by the difference of the total strain $\varepsilon_{tot}$ and the plastic strain $\varepsilon_p$. Finally, the reduced Young's modulus $E_d$ is obtained by the slope of the unloading path, by relating the elastic strain with the corresponding tensile strength. The reduction in the elastic stiffness provides the damage variable $d$, utilising the following expression:
\[ d = 1 - \frac{E_d}{E_0} \]  \hspace{1cm} (2.5)

\[ E_d = \frac{\sigma_y}{\varepsilon_{\text{tot}} - \varepsilon_p} = \frac{\sigma_y}{\varepsilon_e} \]  \hspace{1cm} (2.6)

The reduction in the Young’s modulus results in an increase in the damage variable during each cycle. The obtained strength, strain and damage parameters for VW_8530_60% samples for each load cycle are listed in Table 2.3. For each sample, three tests were performed and the averaged results were used.

Table 2.3: Identified material parameters for three specimens of VW_8530_60% sample

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Cycle</th>
<th>( \sigma_y ) (MPa)</th>
<th>( \varepsilon_{\text{tot}} )</th>
<th>( \varepsilon_p )</th>
<th>( \varepsilon_e )</th>
<th>( E_d ) (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2.5545</td>
<td>0.001913</td>
<td>0.000323</td>
<td>0.00159</td>
<td>1606.604</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5.109</td>
<td>0.004405</td>
<td>0.000796</td>
<td>0.003609</td>
<td>1415.628</td>
</tr>
<tr>
<td>Specimen 1</td>
<td>3</td>
<td>7.6635</td>
<td>0.007188</td>
<td>0.00155</td>
<td>0.005638</td>
<td>1359.259</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10.218</td>
<td>0.01046</td>
<td>0.002626</td>
<td>0.007834</td>
<td>1304.315</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>12.7725</td>
<td>0.014455</td>
<td>0.004202</td>
<td>0.010253</td>
<td>1245.733</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2.5696</td>
<td>0.001748</td>
<td>0.000263</td>
<td>0.001485</td>
<td>1730.37</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5.1392</td>
<td>0.003507</td>
<td>0.00034</td>
<td>0.003167</td>
<td>1622.734</td>
</tr>
<tr>
<td>Specimen 2</td>
<td>3</td>
<td>7.7088</td>
<td>0.006193</td>
<td>0.001074</td>
<td>0.005119</td>
<td>1505.919</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10.2784</td>
<td>0.009415</td>
<td>0.002031</td>
<td>0.007384</td>
<td>1391.983</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>12.848</td>
<td>0.01346</td>
<td>0.003417</td>
<td>0.010043</td>
<td>1279.299</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2.5589</td>
<td>0.001473</td>
<td>0.000001</td>
<td>0.001463</td>
<td>1863.729</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>5.1178</td>
<td>0.003221</td>
<td>0.00042</td>
<td>0.002801</td>
<td>1827.133</td>
</tr>
<tr>
<td>Specimen 3</td>
<td>3</td>
<td>7.6767</td>
<td>0.005899</td>
<td>0.00084</td>
<td>0.005059</td>
<td>1517.434</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10.2356</td>
<td>0.009015</td>
<td>0.001726</td>
<td>0.007289</td>
<td>1404.253</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>12.7945</td>
<td>0.012944</td>
<td>0.003123</td>
<td>0.009821</td>
<td>1302.77</td>
</tr>
</tbody>
</table>
As can be seen from Equation 2.7, the exponential function for damage evolution function $d$ is defined based on Simon and Ju \cite{33} and Ju \cite{34} damage models

$$
d(x) = 1 - \left(1 - \alpha(x)\right) \frac{\varepsilon_{\text{threshold}}}{\varepsilon} - \alpha(x) \left[ \exp\left(-\beta(x) \times (\varepsilon - \varepsilon_{\text{threshold}})\right) \right]
$$

(2.7)

where $\alpha(x)$ and $\beta(x)$ are the characteristic parameters of an FG plate. Identification of material constants associated with any proposed material model is one of the challenging issues for researchers, where the aim is improved representation of the relevant material models \cite{35}. An identification approach for the damage evolution materials constants has been based on Equation 2.7, utilising the experimental results presented in Figure 2.15, and the least squares minimization solver available in MATLAB. Figure 2.15 presents the result of fitting the experimental data from a set of 6 volume ratios, for the damage evolution function $d$. The characteristic parameters $\alpha(x)$ and $\beta(x)$ for different samples have been calculated based on the least squares minimization method, and are presented in Table 2.4.
\[ D = 1 - (1 - \alpha) \left( \frac{\varepsilon_{th}}{\varepsilon} \right) - \alpha (e - \beta (\varepsilon - \varepsilon_{th})) \]
Figure 2.16: Fitting of the experimental data based on least square minimization solver (in MATLAB) for different samples
Table 2.4: Characteristic parameters for different samples obtained from experimental data using least square minimization method

<table>
<thead>
<tr>
<th>Material</th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>8530</td>
<td>0.8087</td>
<td>0.3114</td>
</tr>
<tr>
<td>VW_8530_20%</td>
<td>0.8907</td>
<td>0.2515</td>
</tr>
<tr>
<td>VW_8530_40%</td>
<td>0.9658</td>
<td>0.1976</td>
</tr>
<tr>
<td>VW_8530_60%</td>
<td>1.0451</td>
<td>0.1901</td>
</tr>
<tr>
<td>VW_8530_80%</td>
<td>1.0505</td>
<td>0.1891</td>
</tr>
<tr>
<td>VW</td>
<td>1.0531</td>
<td>0.1236</td>
</tr>
</tbody>
</table>

The variation of damage characteristic parameters \( α \) and \( β \) with respect to the volume fraction of hard polymer (VW) is shown in Figure 2.17 and Figure 2.18, respectively. A curve fitting approach has been applied using the following cubic polynomials

\[
\alpha(x) = a_1 V_i^3 + b_1 V_i^2 + c_1 V_i + d_1 \tag{2.8}
\]

\[
\beta(x) = a_2 V_i^3 + b_2 V_i^2 + c_2 V_i + d_2 \tag{2.9}
\]
Figure 2.18: Variation of parameter \( \beta \) with respect to the volume fraction of hard polymer and its curve fitting residuals

These characteristic parameter functions act as inputs in the numerical integration of damage modelling in Chapter 4. The code for this was written in FORTRAN (refer Appendix F).

Figure 2.19 shows the variation of the damage variable \( (d) \) with respect to the volume fraction of VW (hard polymer) at different strains. It has been found that the damage variable displayed a decreasing monotonic trend along the plate length for all of strain cases, and reached the minimum value for pure VW.

Figure 2.19: Variation of damage variable \( (d) \) with respect to the volume fraction of VW for different strains.
2.4 Generating the constituents distribution pattern of FG plates

In order to validate the proposed analytical solutions and numerical (FE) simulations presented in Chapter 3, FG plates with specified material property variations should be manufactured. Therefore the main goal of this section is the design and generation of distribution patterns of inclusions in the FG plates that obey a power-law distribution of volume fraction. It has been demonstrated experimentally in Section 2.3.1 that the Young’s modulus and ultimate tensile strength of the materials considered obey the power law function. Therefore, in the analytical and numerical modelling in Chapter 3, it is presented that the volume fraction of inclusions varies continuously through the length of the plate, following the power law function

\[ V_i(x) = \left( \frac{x}{a} \right)^n \]  

(2.10)

where \( x \) and \( a \) are position and length of the FG plate and \( n \) is a parameter that governs the material variation profile through the length. For definition of different distribution patterns, a MATLAB code which is linked to the RHINOCEROS and GRASSHOPPER software packages was written (refer to Appendix B) [36-38]. Then the geometry for inclusion and matrix should be generated separately and saved as .stl files, which are then used to manufacture the samples using the Stratasys Object 500 3D printer. The model logic in GRASSHOPPER and example renders of the generated FG plate models are shown in Figure 2.20 and Figure 2.21, respectively. In Figure 2.21, the matrix and inclusion voxels appear green and red in the render, respectively. It is noteworthy that the voxel size was chosen to be 1 mm. As it can be seen from Appendix B, the plate dimensions and power law index \( n \) were defined as variables in the MATLAB code. In this way, the output of the code can be read as an initial parameter in GRASSHOPPER. The manufactured 3D printed plates for both the VW_8530 and 8515_9895 combinations are shown in Figure 2.22
Figure 2.20: Logic model flow for the FG plates in Grasshopper software

(a)

Figure 2.21: Render of a FG plate in Rhino software (a) power law index $n=1$, and (b) power law index $n=3$
Figure 2.22: Manufactured 3D-printed FG plates (a) 8515_9895 combination, and (b) VW_8530 combination

Composite | Linear FG plate ($n=1$) | Nonlinear FG plate ($n=3$)
2.5 Summary

The summary of the main findings from this chapter are presented here:

- The properties of fourteen digital materials used in Stratasys Object 500 3D printer were obtained using tensile tests based on ASTM D412 and D638 for rubbers and rigid plastic, respectively. Then, two different stiffness ratio combinations, i.e. high stiffness ratio \( E_i/E_m \sim 17 \) material combination of 8515_9895 and low stiffness ratio \( E_i/E_m \sim 2 \) material combination of VW_8530 were considered in order to validate analytical and numerical models in Chapter 3.

- For verification of the material property variation through the length of plate, a set of four samples for each combination of 20%, 40%, 60% and 80% of the hard polymer were tested based on tensile tests and cyclic tensile tests.

- Using curve fitting of the experimental results, a power law relationship was obtained for the Young modulus and ultimate tensile strength with the volume fraction of the hard polymer in the composite.

- Damage material characteristic parameters were obtained using the least squares minimization solver available in MATLAB for experimental data from cyclic tensile tests.

- A decreasing monotonic function for damage parameter \( d \) with respect to the volume fraction of VW (hard polymer) was obtained for different strains.
3 Chapter 3: Static analysis of FG plate: analytical and FE solution

Some of the work presented in this chapter has been published in the following:

- **M. Amirpour, S. Bickerton, R. Das, B. Mace, 2016, “Stress and vibration analysis of the FG plate with in-plane material properties variation”, 14th International Symposium functionally graded materials (ISFGM2016), Bayreuth, Germany.**

3.1 Introduction

In this section a new plate theory is developed, based on the Higher order Shear Deformation Theory (HSDT) for thick Functionally Graded (FG) plate with material properties variation through the length (in-plane). For design and analysis of thick FG plates with in-plane stiffness variation, simple closed form and analytical solutions may be very useful for determination of deformations, stresses and natural frequencies. Therefore, the development of theoretical and numerical methodologies for accurate structural predictions for thick FG plates with in-plane property variation, including stretching effects, is very relevant and represents the main objective of the present chapter.

This chapter extends HSDT to thick FG plates with in-plane variation of stiffness (variation of stiffness through the length of the plate), and also incorporates key physical
effects, namely the sinusoidal shear deformation and the stretching of the thickness. The main challenge in developing analytical models for in-plane property variation, as compared to through-the-thickness property variation, is that the variation of Young’s modulus (material stiffness) through the length (x) leads to a mathematically complex set of five simultaneous governing equations – the solutions of which become relatively cumbersome as several parameters vary with the length (x).

3.2 Literature review of plate theory

In recent decades, there have been various types of research on stress analysis of FG plate under different plate theories, i.e. Classical Plate Theory (CPT) and Shear Deformation Plate Theory (SDT). The deformation solutions for FG plates have been mainly based on classical plate theory (CPT). This theory has been implemented by several researchers to study the mechanical behaviour of FG plates. Abrate [39] used CPT to demonstrate that an FG plate with material properties variation through the thickness behaves like a homogenous plate by choosing the reference surface suitably. Therefore, the bending stretching coupling can be deleted so that the FG plate can be considered as a homogenous plate. Abrate considered a power-law distribution in the thickness direction. An exact solution for exponentially graded thin FG plates with simply-supported boundary condition under a surface load was expanded by Pan [40]. Pan considered an FG plate to behave as a composite laminate with different homogenous layers. For large deflection of rectangular metal/ceramic FG plate with power-law variation of volume fraction and simply-supported boundary condition, Ghannadpour and Alinia [41] implemented the Von-Karman theory by minimisation of the total potential energy with the use of CPT to study static behaviour. Further, Chi and Chung [42, 43] developed an analytical formulation for exponential, power-law and sigmoid gradient distribution in the thickness direction to study the deflection and stress fields in rectangular FGM based on CPT [42]. They validated their exact solution with numerical simulation in MARC FE software. They concluded the stiffness of the FG plate with power-law gradient distribution with power index 0.5 is stronger and stiffer than sigmoid, exponential and power-law gradient function with power index 2. Recently, the 3D exact solution for FG plate with thickness direction exponentially distribution function based on CPT was established by Hadi et al. [44], and they concluded that heterogeneities have significant effect on the mechanical behaviour. The same approach for cylindrical FG plate was used by Navazi et al. [17]. They showed that classical plate theory is insufficient to study static analysis even in the small deflection range. CPT used in the above mentioned studies does not consider shear deformation, and hence is valid only for thin plates. This may lead to inaccurate results for thick plates, those thicker than 1/20 of their larger span [42]. In addition, CPT overestimates frequencies and underestimates deflections [45, 46].
In order to eliminate the inaccuracies due to application of CPT for thick plates, the First-order Shear Deformation Theory (FSDT), including the effects of transverse shear deformation, was employed by many researchers. In FSDT a shear correction factor is needed to satisfy the zero transverse shear stress boundary conditions at the top and bottom of the plate [47]. Reddy et al. [48] developed the deflection, forces and bending of a circular FG plate using FSDT. They considered power-law distributions of the volume fraction of the constituents and various boundary conditions. The identification of the shear factor in FSDT was carried out by Nguyen et al. [49] for a square plate and cylindrical bended sandwich plate with simply-supported boundary conditions. They concluded that the shear correction coefficient is a function of elastic moduli and material heterogeneities. Recently, Saidi et al. developed a new analytical solution to decouple constitutive equations for bending and stretching of thick FG plate based on FSDT [50]. Moreover, Nosier et al. used the Von-Karman equation with FSDT for static analysis of axisymmetric and asymmetric circular FG plates with power-law gradient distribution in z direction. They developed the system of five nonlinear coupled equations into three equations [51]. Although the variation of transverse deflection and radial stresses through the thickness were plotted, the variation of in-plane deformation, angular and shear stresses were not presented.

In order to avoid the use of shear correction factors, several HSDT, such as, the third-order shear deformation theory (TSDT) [52-55], the sinusoidal shear deformation theory (SSDT) [22, 56-58] and the hyperbolic shear deformation theory [46, 59, 60] have been proposed. Bourada et Al [61] used hyperbolic shear deformation theory to study static and free vibration of a simply supported FG beam with material properties variation through the thickness. They used third unknown displacement functions. The effects due to transverse shear and normal deformations were both included. Their theory accounted for adequate distribution of the transverse shear stresses through the beam thickness and tangential stress-free boundary conditions on the beam boundary surface.

In addition, all two-dimensional plate theories ignore the thickness stretching effect. Indeed, a constant transverse displacement through the thickness was considered [62-64]. This assumption is not appropriate for thick FG plates. The importance of the thickness stretching effect in a thick FG plate has been identified in [65], and therefore this effect should be taken into consideration. Recently, Mantari and Guedes Soares [66] conducted static analysis of thick FG plates using a novel trigonometric higher-order theory in which the stretching effect was considered. Moreover, Belabed et. al. [46] presented the HSDT considering the stretching effect and hyperbolic variation of all displacements across the thickness for bending and free vibration of a thick FG plate with variation of stiffness through the thickness.
In most of the applications reported in the literature the variation of material properties is through the thickness only. However, a little consideration has been given to the distribution of volume fraction through the length, resulting in the variation of stiffness along it. The main challenge in developing analytical models for in-plane property variation, as compared to the through-the-thickness property variation, is that the variation of Young’s modulus (material stiffness) through the length (x) leads to a mathematically complex set of five simultaneous governing equations – the solutions of which become relatively cumbersome as several parameters vary with the length (x).

### 3.3 Analytical solution

In this section, two types of plate theory, i.e., Classical Plate Theory (CPT) and Higher order Shear Deformation (HSDT) for FG plate with material properties variation through the length have been developed. The validity of the proposed CPT and HSDT depends on the small deflection assumption. It is assumed that the lateral deflection of the plate is small compared to the thickness of the plate.

rectangular FG plate of uniform thickness \( h \), length \( a \) and width \( b \) is shown in Figure 3.1 with origin \( z = 0 \) coinciding with the mid-surface of the plate.

![Figure 3.1: Schematic representation of the geometry (left) and distribution of volume fraction of the constituents along the length (x) direction of the FG plate (right)](image)

The material properties vary continuously through the length of the plate and obey a simple power-law distribution of volume fraction \( V_i \) of the constituents as given by:

\[
V_i(x) = \left( \frac{x}{a} \right)^n
\]  

(3.1)
where $n$ is a parameter that governs the material variation profile through the length which is called power law index. Since the effect of variation of Poisson’s ratio on the response of FG plates is very small \[67, 68\], this material parameter is assumed to be constant. Other material properties are considered to be variable and can be determined by the rule of mixture as \[64\]

$$P(x) = \left( \frac{x}{a} \right)^n (P_i - P_m) + P_m$$

(3.2)

where $P_i$ and $P_m$ are the inclusion and matrix phase properties, respectively.

The relative difference between the inclusion and matrix properties is an important parameter that controls the material properties gradient such as material stiffness, throughout the length direction. For instance, a low value of $E_i/E_m$ implies a gradual in-plane stiffness variation, and a high ratio indicates a steep in-plane stiffness variation.

### 3.3.1 Classical plate theory (CPT)

In CPT, the deformation and the stress are based on the following assumptions:

1. Line elements perpendicular to the middle surface of the plate before deformation remain normal and un-stretched after deformation.

2. The deflections of the FGM plate are small in comparison with its thickness $h$, such that the linear strain displacement relations are valid.

3. A constant coefficient for Fourier series of the displacement fields.

The transverse strain components $\varepsilon_{xz}$, $\gamma_{xz}$ and $\gamma_{yz}$ are negligibly small. Therefore, the displacements at the point $A_0$ in the $x$, $y$ and $z$ directions are

$$u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w(x, y, t)}{\partial x}$$  \hspace{1cm} (3.3a)

$$v(x, y, z, t) = v_0(x, y, t) - z \frac{\partial w(x, y, t)}{\partial y}$$  \hspace{1cm} (3.3b)

$$w(x, y, z, t) = w_0(x, y, t)$$  \hspace{1cm} (3.3c)

Where $u$, $v$ and $w$ are displacements in $x$, $y$ and $z$ directions, respectively. Moreover $u_0(x, y, t)$ and $v_0(x, y, t)$ are the displacement functions of the middle surface of the plate. Under the assumption of small deformation, the strain field of the FGM plate is
\[ \varepsilon_x = \varepsilon_{x0} - z \frac{\partial^2 W}{\partial x^2} \]  
(3.4a)

\[ \varepsilon_y = \varepsilon_{y0} - z \frac{\partial^2 W}{\partial y^2} \]  
(3.4b)

\[ \gamma_{xy} = \gamma_{xy0} - 2z \frac{\partial^2 W}{\partial x \partial y} \]  
(3.4c)

\[ \varepsilon_z = \gamma_{yz} = \gamma_{xy} = 0 \]  
(3.4d)

The stress–strain relation for plane stress for an FGM plate is

\[ \sigma_x = \frac{E(z)}{1-\nu^2} \left[ \varepsilon_{x0} + v \varepsilon_{y0} - z \frac{\partial^2 W}{\partial x^2} - v z \frac{\partial^2 W}{\partial y^2} \right] \]  
(3.5a)

\[ \sigma_y = \frac{E(z)}{1-\nu^2} \left[ \varepsilon_{y0} + v \varepsilon_{x0} - z \frac{\partial^2 W}{\partial y^2} - v z \frac{\partial^2 W}{\partial x^2} \right] \]  
(3.5b)

\[ \tau_{xy} = \frac{E(z)}{1-\nu^2} \frac{1-\nu}{2} \left[ \gamma_{xy0} - 2z \frac{\partial^2 W}{\partial x \partial y} \right] \]  
(3.5c)

In CPT, the transverse shear deformation can be neglected [43].

3.3.1.1 The axial forces and bending moment

The stress resultants per unit length of the middle surface are defined by integrating stresses along the thickness. The in plane axial forces \( N_x, N_y, \) and \( N_{xy} \) are defined as follows

\[ N_x = \int_{-h/2}^{h/2} \sigma_x \, dz \]

\[ N_y = \int_{-h/2}^{h/2} \sigma_y \, dz \]

\[ N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz \]  
(3.6)

and the bending moments \( M_x, M_y, \) and \( M_{xy} \) are defined as

\[ M_x = \int_{-h/2}^{h/2} z \sigma_y \, dz \]

\[ M_y = \int_{-h/2}^{h/2} z \sigma_x \, dz \]

\[ M_{xy} = \int_{-h/2}^{h/2} z \tau_{xy} \, dz \]  
(3.7)

By substitution of Equations (3.5) to Equations (3.6) and (3.7), the axial forces and bending moments in the matrix forms are obtained as follows
\[
\begin{align*}
\begin{pmatrix}
N_x \\
N_y \\
N_{xy}
\end{pmatrix}
&=
\begin{pmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{11} & 0 \\
0 & 0 & A_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{x0} \\
\varepsilon_{y0} \\
\gamma_{xy0}
\end{pmatrix}
+ \begin{pmatrix}
B_{11} & B_{12} & 0 \\
B_{21} & B_{11} & 0 \\
0 & 0 & B_{66}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2} \\
-2 \frac{\partial^2 w}{\partial x \partial y}
\end{pmatrix}
\end{align*}
\]
(3.8)

\[
\begin{align*}
\begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix}
&=
\begin{pmatrix}
B_{11} & B_{12} & 0 \\
B_{21} & B_{11} & 0 \\
0 & 0 & B_{66}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{x0} \\
\varepsilon_{y0} \\
\gamma_{xy0}
\end{pmatrix}
+ \begin{pmatrix}
C_{11} & C_{12} & 0 \\
C_{12} & C_{11} & 0 \\
0 & 0 & C_{66}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2} \\
-2 \frac{\partial^2 w}{\partial x \partial y}
\end{pmatrix}
\end{align*}
\]
(3.9)

The coefficients of the above equations depend on the material properties of the FG plate and are

\[
A_1 = \int_{-h/2}^{h/2} E(x) \frac{x^w}{(1-v^2)} dx = \int_{-h/2}^{h/2} \left( \frac{x^w}{(1-v^2)} \right) E(x) + E_m \frac{h}{1-v^2} E(x)
\]

\[
A_2 = v A_1 = \frac{v h}{1-v^2} E(x)
\]

\[
A_{60} = \frac{1-v}{2} A_1 = \frac{1-v}{2} \frac{h}{1-v^2} E(x)
\]

\[
B_1 = \int_{-h/2}^{h/2} z E(x) \frac{z}{(1-v^2)} dz = \int_{-h/2}^{h/2} z E(x) \frac{h}{1-v^2} dz = 0
\]

\[
B_2 = v B_1 = 0
\]

\[
B_{60} = \frac{1-v}{2} B_1 = 0
\]

\[
C_{11} = \int_{-h/2}^{h/2} \frac{z^2 E(x)}{(1-v^2)} dz = \int_{-h/2}^{h/2} \frac{z^2 E(x)}{(1-v^2)} \frac{h^3}{12(1-v^2)} = \frac{h^3}{12(1-v^2)} E(x)
\]

\[
C_{12} = v C_{11} = \frac{v h^3}{12(1-v^2)} E(x)
\]

\[
C_{60} = \frac{1-v}{2} C_{11} = \frac{h^3}{12(1-v^2)} E(x)
\]

(3.10a)

(3.10b)

(3.10c)

**3.3.1.2 The equilibrium and compatibility equations**

Here it is assumed that the FGM plate is subjected to the transverse load \( q \) along the \( z \) directions. Consider a small solid element with dimensions \( dx, dy \) and \( dz \). All forces acting on the small element are shown in Figure 3.2.
Figure 3.2: The forces in a small element $dxdydz$ of an FGM plate

When the element is in equilibrium, the resultant forces in the $x$ direction must be zero, i.e.

$$\left( N_x + \frac{\partial N_x}{\partial x} \ dx \right) dy + \left( N_y + \frac{\partial N_y}{\partial y} \ dy \right) dx - N_x \ dy - N_y \ dx = 0 \quad (3.11)$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (3.12)$$

Similarly, the equilibrium equations for $y$ and $z$ directions yield

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_z}{\partial z} = 0 \quad (3.13)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q = 0 \quad (3.14)$$

Moreover, the zero resultant moments in the $y$ and $x$ directions give two equilibrium equations

$$V_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} \quad (3.15)$$

$$V_y = \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} \quad (3.16)$$

The substitution of Equations (3.15) and (3.16) to Equations (3.14) gives the equilibrium equations of FG plate as follows
\[
\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} + 2 \frac{\partial^3 M}{\partial x \partial y^2} = -q
\]  
(3.17)

\[
C_{11} \left[ \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] + \frac{\partial C_{11}}{\partial x} \left[ 2 \frac{\partial^3 w}{\partial x \partial y^2} + 2 \frac{\partial^3 w}{\partial x^3} \right] + \frac{\partial^2 C_{11}}{\partial x^2} \left[ \frac{\partial^3 w}{\partial x^2} + \frac{\partial^3 w}{\partial y^2} \right] = -q
\]  
(3.18)

where \( q \) is the transverse load and its expansion by Fourier series is as follows

\[
q(x, y) = \sum \sum q_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}
\]  
(3.19)

where

\[
q_{mn} = \int \int q(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \, dx \, dy
\]  
(3.20)

If \( q(x, y) \) is a uniform distributed load, i.e. \( q(x, y) = q_0 \), then the quantity \( q_{mn} \) calculated by Equation (3.20) is

\[
q_{mn} = \begin{cases} 
0 & m, n = 2, 4, 6, \\
\frac{16q_0}{\pi^2 mn} & m, n = 1, 3, 5, 
\end{cases}
\]  
(3.21)

And if the applied load is a point load, i.e. \( q(x, y) = P_0 \) acting at \( x = u \) and \( y = v \), then

\[
q_{mn} = \begin{cases} 
0 & m, n = 2, 4, 6, \\
\frac{4P_0}{ab} \sin \frac{m \pi u}{a} \sin \frac{n \pi v}{b} & m, n = 1, 3, 5, 
\end{cases}
\]  
(3.22)

In order to satisfy the equilibrium equation and the simply supported boundary conditions, the deflection function \( w \) of the FG plate should be expressed as

\[
w(x, y, t) = \sum \sum w_{mn} e^{i\omega t} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{a}
\]  
(3.23)

By substitution of Equation (3.23) into Equation (3.9), the formulation for bending moments \( M_x \) and \( M_y \) are as follows:
\[ M_x = -C_{11} \left( \frac{\partial^2 w_{mn}}{\partial x^2} \sin \alpha \sin \beta y + 2 \frac{\partial w_{mn}}{\partial x} \alpha \cos \alpha \sin \beta y + w_{mn} \alpha^2 \sin \alpha \sin \beta y \right) - \]
\[ C_{12} \left( \frac{\partial^2 w_{mn}}{\partial y^2} \sin \alpha \sin \beta y + 2 \frac{\partial w_{mn}}{\partial y} \beta \sin \alpha \cos \beta y + w_{mn} \beta^2 \sin \alpha \sin \beta y \right) \]  
(3.24)

\[ M_y = -C_{11} \left( \frac{\partial^2 w_{mn}}{\partial y^2} \sin \alpha \sin \beta y + 2 \frac{\partial w_{mn}}{\partial y} \beta \sin \alpha \cos \beta y + w_{mn} \beta^2 \sin \alpha \sin \beta y \right) - \]
\[ C_{12} \left( \frac{\partial^2 w_{mn}}{\partial x^2} \sin \alpha \sin \beta y + 2 \frac{\partial w_{mn}}{\partial x} \alpha \cos \alpha \sin \beta y + w_{mn} \alpha^2 \sin \alpha \sin \beta y \right) \]  

By applying a simply supported boundary condition into Equation (3.24) to satisfy zero bending moments at \( x=0 \) and \( y=0 \), we have:

\[ M_{x=0} = 0 \Rightarrow -C_{11} \left( 0 + 2 \frac{\partial w_{mn}}{\partial x} \alpha \sin \beta y + 0 \right) - C_{12} \left( 0 + 0 + 0 \right) = 0 \Rightarrow \frac{\partial w_{mn}}{\partial x} = 0 \]  
(3.25)

\[ M_{y=0} = 0 \Rightarrow -C_{11} \left( 0 + 2 \frac{\partial w_{mn}}{\partial y} \beta \sin \alpha x + 0 \right) - C_{12} \left( 0 + 0 + 0 \right) = 0 \Rightarrow \frac{\partial w_{mn}}{\partial y} = 0 \]

Therefore, Equation (3.24) at \( x=0 \) and \( y=0 \) are as follows:

\[ M_x = -C_{11} \left( w_{mn} \alpha^2 \sin \alpha \sin \beta y \right) - C_{12} \left( w_{mn} \beta^2 \sin \alpha \sin \beta y \right) \]

\[ M_y = -C_{11} \left( w_{mn} \beta^2 \sin \alpha \sin \beta y \right) - C_{12} \left( w_{mn} \alpha^2 \sin \alpha \sin \beta y \right) \]

By substituting Equation (3.23) in the equilibrium Equation (3.18), we can find \( w_{mn} \) as follows

\[ C_{11} \left[ \frac{\partial^2 w_{mn}}{\partial x^2} - 6 \alpha^2 \frac{\partial^2 w_{mn}}{\partial x^2} - \alpha^4 w_{mn} + \frac{\partial^2 w_{mn}}{\partial y^2} - 6 \beta^2 \frac{\partial^2 w_{mn}}{\partial y^2} - \beta^4 w_{mn} + 2 \frac{\partial^4 w_{mn}}{\partial x^2 \partial y^2} + 2 \frac{\partial^2 w_{mn}}{\partial x^2 \partial y^2} - 2 \alpha^2 \frac{\partial^2 w_{mn}}{\partial x^2 \partial y^2} \right] + 4 \alpha \beta \frac{\partial^3 w_{mn}}{\partial x^2 \partial y} - 2 \alpha^3 \frac{\partial w_{mn}}{\partial x} \]  

\[ \frac{\partial C_{11}}{\partial x} \left[ 2 \frac{\partial^2 w_{mn}}{\partial x^2} - 2 \beta^2 \frac{\partial w_{mn}}{\partial x} + 2 \frac{\partial^2 w_{mn}}{\partial x^2} - 6 \alpha \frac{\partial w_{mn}}{\partial x} \right] + \frac{\partial^2 C_{11}}{\partial x^2} \left[ \frac{\partial^2 w_{mn}}{\partial x^2} - \alpha^4 w_{mn} + \frac{\partial^4 w_{mn}}{\partial x^2 \partial y^2} - \alpha^2 \frac{\partial^2 w_{mn}}{\partial x^2 \partial y^2} - \beta^4 \frac{\partial w_{mn}}{\partial y} \right] = -q_{mn} \]  
(3.26)

\[ w_{mn} = \frac{q_{mn}}{C_{11} \left[ \alpha^2 + \beta^2 \right] + \frac{\partial C_{11}}{\partial x} \left[ \alpha^2 \beta + \beta^2 \right]} \]  

Therefore, the strain in the length direction can be expressed as

\[ e_x = -\frac{\partial^2 W}{\partial x^2} = \sum \sum \left( \lambda^{(1)} \left( \frac{m\pi}{a} \right)^2 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} + 2 \lambda^{(1)} \left( \frac{m\pi}{a} \right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \right) \]  
(3.27a)
\[ e_r = -2 \varepsilon \sum \sum \frac{(n\pi)^2}{C_{11} \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{a} \right)^2 \right)^2 + \frac{\partial^2 C_{11}}{\partial x^2} \left( \left( \frac{m\pi}{a} \right)^2 + \nu \left( \frac{n\pi}{a} \right)^2 \right)} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \]  

\[ \gamma_{xy} = -2 \varepsilon \sum \sum \frac{\left( \frac{m\pi}{a} \right) \left( \frac{n\pi}{a} \right) F_0}{C_{11} \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{a} \right)^2 \right)^2 + \frac{\partial^2 C_{11}}{\partial x^2} \left( \left( \frac{m\pi}{a} \right)^2 + \nu \left( \frac{n\pi}{a} \right)^2 \right)} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{a} \]  

where

\[ \lambda = \frac{16 q_{ny} x}{\pi^2} \]  

\[ \lambda^{(1)} = -\frac{16 q_{ny} x}{\pi^2} \left( C_{11} \frac{\pi^4}{a^4} \left( m^2 + n^2 \right)^2 + \frac{\partial^2 C_{11}}{\partial x^2} \frac{\pi^2}{a^2} \left( m^2 + n^2 \right) \right) \]  

\[ \lambda^{(2)} = -\frac{16 q_{ny} x}{\pi^2} \left( C_{11} \frac{\pi^4}{a^4} \left( m^2 + n^2 \right)^2 + \frac{\partial^2 C_{11}}{\partial x^2} \frac{\pi^2}{a^2} \left( m^2 + n^2 \right) \right) \]  

With set of Equations (3.27), it is straightforward to calculate the stress components as follows

\[ \sigma_x = \frac{E(\alpha)}{1 - \nu^2} [e_x + \nu \varepsilon_y] \]  

\[ \sigma_y = \frac{E(\alpha)}{1 - \nu^2} [e_y + \nu \varepsilon_x] \]  

\[ \tau_{xy} = \frac{E(\alpha)}{1 - \nu^2} \frac{1}{2} \gamma_{xy} \]
3.3.2 Higher order Shear Deformation Theory (HSDT)

3.3.2.1 Kinematics

The displacement field of the Higher-order Shear Deformation Theory (HSDT) for thick plates is adopted from that recently developed by Blabed et al. [46] based on the following two assumptions: the transverse displacements are composed of bending, shear and stretching components; and the shear components of the in-plane displacements produce sinusoidal variations of shear strains. Based on these assumptions, the following displacement field relationships can be obtained:

\[
\begin{align*}
  u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_b(x, y, t)}{\partial x} - \xi(z) \frac{\partial w_s(x, y, t)}{\partial x} \\
  v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_b(x, y, t)}{\partial y} - \xi(z) \frac{\partial w_s(x, y, t)}{\partial y} \\
  w(x, y, z, t) &= w_b(x, y, t) + w_s(x, y, t) + \zeta(z) \varphi(x, y, t)
\end{align*}
\]

(3.32)

where \( u_0(x, y) \) and \( v_0(x, y) \) are the displacement functions of the middle surface of the plate; \( w_b \) and \( w_s \) are the bending and shear components of the transverse displacement, respectively; and the additional displacement \( \varphi \) accounts for the stretching effect. The sinusoidal functions for shear deformation are described as [56]:

\[
\xi(z) = \frac{h}{\pi} \sin \left( \frac{\pi z}{h} \right)
\]

(3.33)

and

\[
\zeta(z) = 1 - \xi'(z) = 1 - \cos \left( \frac{\pi}{h} z \right)
\]

(3.34)

Displacements and rotations are assumed to be small and obey Hooke’s law. The linear strains associated with the above displacement field are:

\[
\begin{align*}
  \varepsilon_x &= \frac{\partial}{\partial x} u(x, y, z) = d_1 u_0 - zd_1 w_b - \xi(z) d_1 w_s \\
  \varepsilon_y &= \frac{\partial}{\partial y} v(x, y, z) = d_2 v_0 - zd_2 w_b - \xi(z) d_2 w_s \\
  \varepsilon_z &= \frac{\partial}{\partial z} w(x, y, z) = \zeta'(z) \varphi
\end{align*}
\]

(3.35a, 3.35b, 3.35c)
\[ \gamma_{xy} = \frac{\partial}{\partial y} u(x, y, z) + \frac{\partial}{\partial x} v(x, y, z) = d_1 v_0 + d_2 u_0 - 2z d_{12} w_b - 2 \xi(z) d_{12} w_z (3.35d) \]

\[ \gamma_{xz} = \frac{\partial}{\partial z} u(x, y, z) + \frac{\partial}{\partial x} w(x, y, z) = \zeta(z) \left( d_1 w_x + d_1 \varphi \right) (3.35e) \]

\[ \gamma_{yz} = \frac{\partial}{\partial z} v(x, y, z) + \frac{\partial}{\partial y} w(x, y, z) = \zeta(z) \left( d_2 w_y + d_2 \varphi \right) (3.35f) \]

where \(d_1, d_2, d_{11}, d_{22}, d_{12}\), and \(d_{22}\) represent the following differential operators

\[ d_1 = \frac{\partial}{\partial x}, \quad d_2 = \frac{\partial}{\partial y}, \quad d_{11} = \frac{\partial^2}{\partial x^2}, \quad d_{22} = \frac{\partial^2}{\partial y^2}, \quad d_{12} = \frac{\partial^2}{\partial x \partial y} \]

### 3.3.2.2. Constitutive equations

The Young’s Modulus is defined based on Equation (3.2). So, the linear constitutive relations of a FG plate can be written as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
\]

where the three-dimensional elastic constants vary through the length according to equation (3.2). If \(\varepsilon_z \neq 0\) (thickness stretching), then coefficient of elastic modulus and are defined as

\[ C_{11} = C_{22} = C_{33} = \frac{1-\nu}{(1+\nu)(1-2\nu)} E(x), \]

\[ C_{12} = C_{13} = C_{23} = \frac{-\nu}{(1+\nu)(1-2\nu)} E(x), \]

\[ C_{44} = C_{55} = C_{66} = \frac{1}{2(1+\nu)} E(x). \]

The axial forces, shear forces, bending momentum and shear momentum are determined by integrating stresses along the thickness of the FG plate as follows
\begin{align}
N_x &= \int_{-h/2}^{h/2} \sigma_x dz = A_{11} d_1 u_x + A_{12} d_2 v_y - B_{11} d_1 w_x - B_{12} d_2 w_y + L \phi \\
N_y &= \int_{-h/2}^{h/2} \sigma_y dz = A_{21} d_1 u_y + A_{22} d_2 v_x - B_{21} d_1 w_y - B_{22} d_2 w_x + L \phi \\
N_z &= \int_{-h/2}^{h/2} \sigma_z dz = A_{31} d_1 u_z + A_{32} d_2 v_z - B_{31} d_1 w_z - B_{32} d_2 w_z + L \phi \\
N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} dz = A_{66} d_1 v_z + A_{66} d_z u_x - 2B_{66} d_1 w_x - 2B_{66} d_z w_x \\
M_x^b &= \int_{-h/2}^{h/2} \xi(z) \sigma_x dz = B_{11} d_1 u_x + B_{12} d_2 v_y - D_{11} d_1 w_x - D_{12} d_2 w_y - H_{11} d_1 w_x - H_{12} d_2 w_y + L \phi \\
M_y^b &= \int_{-h/2}^{h/2} \xi(z) \sigma_y dz = B_{21} d_1 u_y + B_{22} d_2 v_x - D_{21} d_1 w_y - D_{22} d_2 w_x - H_{21} d_1 w_y - H_{22} d_2 w_x + L \phi \\
M_z^b &= \int_{-h/2}^{h/2} \xi(z) \tau_{xy} dz = B_{66} d_1 v_z + B_{66} d_z u_x - 2D_{66} d_1 w_x - 2D_{66} d_z w_x \\
M_{xy}^b &= \int_{-h/2}^{h/2} \xi(z) \tau_{xy} dz = A_{41} d_1 w_x + A_{42} d_2 w_y \\
S_{xz} &= \int_{-h/2}^{h/2} \tau_{xz} \xi(z) dz = A_{44} d_2 w_x + A_{45} d_1 w_y \\
S_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} \xi(z) dz = A_{44} d_1 w_x + A_{45} d_2 w_y
\end{align}

where the coefficients in the above Equations (3.9) are functions of the material properties and are

\begin{align}
\begin{bmatrix} A_{11} & B_{11} & D_{11} & B_{11}' & D_{11}' & H_{11}' \\ A_{21} & B_{21} & D_{21} & B_{21}' & D_{21}' & H_{21}' \\ A_{36} & B_{36} & D_{36} & B_{36}' & D_{36}' & H_{36}' \end{bmatrix} &= I_{h/2} \left( \frac{vE(x)}{(1+v)(1-2v)} \begin{bmatrix} 1 & 1 & 1-v & v \end{bmatrix} \right) d\xi \\
\begin{bmatrix} \gamma(z) \\ \xi(z) \\ \phi(z) \end{bmatrix} &= I_{-h/2}^{h/2} \left( \frac{E(x)}{(1+v)(1-2v)} \begin{bmatrix} 1 \\ \xi(z) \\ \phi(z) \end{bmatrix} \right) d\xi \\
A_{44} &= A_{55} = \int_{-h/2}^{h/2} \frac{E(x)}{2(1+v)} \xi(z) dz
\end{align}

The coefficient of the above equation are in Appendix C.
3.3.2.3 Governing equations

The governing equations of the present theory are derived from the static condition of the Principles of Virtual Displacements (PVD), which is called the Hamilton’s principle. The internal virtual work is initially formulated as follows:

\[ \int_0^T (\delta U + \delta V - \delta K)dt = 0 \quad (3.43) \]

where \( \delta U \), \( \delta V \) and \( \delta K \) are the variations of the strain energy, the potential energy and the kinetic energy, respectively, and are specified in the following equations:

\[
\delta U = \int_{h/2}^{h/2} \left[ \sigma_x \delta e_x + \sigma_y \delta e_y + \sigma_z \delta e_z + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{zx} \delta \gamma_{zx} \right] dy dx
\]

\[
= \int_{h/2}^{h/2} \left[ \sigma_x \delta e_x + \sigma_y \delta e_y + \sigma_z \delta e_z + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{zx} \delta \gamma_{zx} \right] dy dx \quad (3.44a)
\]

\[
\delta V = -\int_A q \delta (w_x + w_y + w_z) dA
\]

\[
= -\int_0^1 \int_A q \delta (w_x + w_y + w_z) dx dy \quad (3.44b)
\]

\[
\delta K = \int_{h/2}^{h/2} \int_0^{h/2} \left( \dot{u} \ddot{u} + \dot{v} \ddot{v} + \dot{w} \ddot{w} \right) \rho(x) dx dz = \int_{h/2}^{h/2} \int_0^{h/2} \left( \dot{u} \ddot{u} + \dot{v} \ddot{v} + \dot{w} \ddot{w} \right) \rho(x) dx dz
\]

\[
= \int_{-h/2}^{h/2} \int_0^{h/2} \left[ I_0 \left( \dot{u} \ddot{u} + \dot{v} \ddot{v} + \dot{w} \ddot{w} \right) \right] + I_1 \left( \dot{u} \ddot{w} + \dot{v} \ddot{w} + \dot{w} \ddot{w} \right) + I_2 \left( \dot{w} \ddot{w} + \dot{w} \ddot{w} + \dot{w} \ddot{w} \right) + I_3 \left( \dot{w} \ddot{w} + \dot{w} \ddot{w} + \dot{w} \ddot{w} \right) + \ldots \quad (3.44c)
\]

The dot-superscript convention used in Equations (3.44) corresponds to differentiation with respect to time \( (t) \), and \( I_0 \), \( I_1 \), \( I_2 \), \( I_3 \), \( K_0 \), \( K_1 \), \( K_2 \), \( J_0 \), and \( J_1 \) are the mass inertia

\[
I_0 = \int_{-h/2}^{h/2} \int_0^{h/2} \rho(x) dx dz = h \int_0^{h/2} \rho(x) dx, \quad I_1 = \int_{-h/2}^{h/2} \int_0^{h/2} \rho(x) z dx dz = 0, \quad J_0 = \int_{-h/2}^{h/2} \int_0^{h/2} \rho(x) \zeta(z) dx dz = 0
\]

\[
J_1 = \int_{-h/2}^{h/2} \int_0^{h/2} \rho(x) \zeta(z) dx dz = (h - 2h/\pi) \int_0^{h/2} \rho(x) dx, \quad I_2 = \int_{-h/2}^{h/2} \int_0^{h/2} \rho(x) z^2 dx dz = (h^3 / 12) \int_0^{h/2} \rho(x) dx
\]
\[ J_2 = \int_{0}^{a/h^2} \int_{-h/2}^{h/2} \rho(x)z \xi(z) dz dx = (2h^3 / \pi^3) \int_{0}^{a} \rho(x) dx , \]

\[ K_2 = \int_{0}^{a/h^2} \int_{-h/2}^{h/2} \rho(x) \xi(z)^2 dz dx = (h^3 / 2\pi^2) \int_{0}^{a} \rho(x) dx \]

\[ K'_2 = \int_{0}^{a/h^2} \int_{-h/2}^{h/2} \rho(x) \zeta(z)^2 dz dx = (1.5h - 4h / \pi) \int_{0}^{a} \rho(x) dx \]  

(3.45)

Substituting the expressions for stresses and strains from Equations (3.35) and (3.37) into Equations (3.43) and (3.44), integrating by parts, and then collecting the coefficients of \( \delta u_0, \delta v_0, \delta w_b, \delta w_s, \) and \( \delta \phi, \) the following governing equations of the FG plate with the variation of material property through the length are obtained:

\[ \delta u_0 : d_1 N_x + d_1 N_y = I_0 \ddot{u}_0 - I_1 \ddot{w}_b - J_1 \ddot{w}_s \]  

(3.46a)

\[ \delta v_0 : d_2 N_x + d_1 N_y = I_0 \ddot{v}_0 - I_1 \ddot{w}_b - J_1 \ddot{w}_s \]  

(3.46b)

\[ \delta w_b : d_{11} M^b_x + d_{12} M^b_y + 2d_{12} M^b_y + q = \]

\[ I_0 (\ddot{w}_b + \ddot{w}_b) + J_1 (d_1 \ddot{u}_0 + d_2 \ddot{v}_0) - J_2 (d_1 \ddot{w}_b + d_2 \ddot{w}_b) - J_2 (d_1 \ddot{w}_b + d_2 \ddot{w}_b) + J_1 \ddot{\phi} \]  

(3.44c)

\[ \delta w_s : d_{11} M^s_x + d_{12} M^s_y + 2d_{12} M^s_y + q = \]

\[ I_0 (\ddot{w}_b + \ddot{w}_b) + J_1 (d_1 \ddot{u}_0 + d_2 \ddot{v}_0) - J_2 (d_1 \ddot{w}_b + d_2 \ddot{w}_b) - K_2 (d_1 \ddot{w}_b + d_2 \ddot{w}_b) + J_1 \ddot{\phi} \]  

(3.46d)

\[ \delta \phi : d_{11} S^\phi + d_{12} S^{\phi} + q \zeta(z) - N_\zeta = J'_1 (\ddot{w}_b + \ddot{w}_b) + K'_2 \ddot{\phi} \]  

(3.46e)

By introducing Equation (3.39) into Equation (3.46), the equations of equilibrium can be expressed in terms of the displacement field. The resulting governing equations can take the form:

\[ A_1 d_{11} u_0 + A_{06} d_{22} u_0 + (A_{12} + A_{66}) d_{12} v_0 - B_1 d_{11} w_b - (B_{12} + 2B_{66}) d_{12} w_b \]

\[ - (B_{12} + 2B_{66}) d_{12} w_b - B_{11} d_{11} w_b - L \ddot{d}_1 \ddot{\phi} = I_0 \ddot{u}_0 - I_1 \ddot{w}_b - J_1 \ddot{w}_s \]  

(3.47a)

\[ A_1 d_{11} u_0 + A_{22} d_{22} v_0 + A_{66} d_{11} v_0 + A_{66} d_{12} w_0 + B_{22} d_{22} w_b - (B_{12} + 2B_{66}) d_{12} w_b \]

\[ - (B_{12} + 2B_{66}) d_{12} w_b - B_{11} d_{11} w_b - L \ddot{d}_1 \ddot{\phi} = I_0 \ddot{v}_0 - I_1 \ddot{w}_b - J_1 \ddot{w}_s \]  

(3.47b)

\[ B_1 d_{11} u_0 + B_{22} d_{22} v_0 + (B_{12} + 2B_{66}) d_{12} u_0 + (B_{12} + 2B_{66}) d_{12} v_0 - D_{11} d_{11} w_b \]

\[ - 2(D_{12} + 2D_{66}) d_{12} w_b - D_{22} d_{22} w_b - D_1 d_{11} w_b - 2(D_1 + 2D_{66}) d_{12} w_b \]

\[ - D_{22} d_{22} w_b + L d_1 \ddot{d}_1 \ddot{\phi} + L d_2 \ddot{d}_2 \ddot{\phi} + q = I_0 (\ddot{w}_b + \ddot{w}_b) + I_1 (d_1 \ddot{u}_0 + d_2 \ddot{v}_0) \]

\[ - L_2 (d_1 \ddot{w}_b + d_2 \ddot{w}_b) - L_2 (d_1 \ddot{w}_b + d_2 \ddot{w}_b) + J'_1 \ddot{\phi} \]  

(3.47c)
After substitution of material coefficients (3.42) into Equations (3.47) the simple form of the above equations is

\[ d_i A_i d_i u_0 + d_i A_0 d_i v_0 + A_0 d_12 v_0 + A_0 d_22 v_0 = I_0 (\ddot{w}_0 - \dot{d}_i \dot{w}_0 - J_i \ddot{w}_0) \]  

\[ A_0 d_12 v_0 + A_0 d_22 v_0 + A_0 d_12 v_0 + A_0 d_22 v_0 = I_0 (\ddot{w}_0 - \dot{d}_i \dot{w}_0 - J_i \ddot{w}_0) \]  

Considering the case of a rectangular FG plate with fully clamped supports, the boundary conditions are

\[ u_0 = v_0 = w_0 = w_y = \varphi = 0 \quad \text{at } x=0, a \]

\[ \frac{\partial w_0}{\partial x} = \frac{\partial w_0}{\partial y} = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0 \quad \text{at } x=0, a \]

\[ u_0 = v_0 = w_0 = w_y = \varphi = 0 \quad \text{at } y=0, b \]

\[ \frac{\partial w_0}{\partial x} = \frac{\partial w_0}{\partial y} = \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0 \quad \text{at } y=0, b \]  

(3.47d)

(3.47e)

(3.48a)

(3.48b)

(3.48c)

(3.48d)

(3.48e)
while for a simply supported case, the boundary conditions are represented by:

\[ u_0 = v_0 = w_b = w_y = \varphi = 0 \quad \text{at} \ x=0, \ a \]

\[ M_y = 0 \quad \text{at} \ x=0, \ a \]

\[ u_0 = v_0 = w_b = w_y = \varphi = 0 \quad \text{at} \ y=0, \ b \]

\[ M_y = 0 \quad \text{at} \ y=0, \ b \]

The required displacement fields which satisfy the fully clamped and simply supported boundary conditions are assumed as follows

\[
\begin{bmatrix}
  u_0 \\
v_0 \\
w_b \\
w_y \\
\varphi
\end{bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} U_{mn} e^{i \omega t} \sin \lambda_{mn} x \sin \mu_{mn} y \\ V_{mn} e^{i \omega t} \sin \lambda_{mn} x \sin \mu_{mn} y \\ W_{bmn} e^{i \omega t} (1 - \cos 2 \lambda_{mn} x)(1 - \cos 2 \mu_{mn} y) \\ W_{sbn} e^{i \omega t} (1 - \cos 2 \lambda_{mn} x)(1 - \cos 2 \mu_{mn} y) \\ \phi_{mn} e^{i \omega t} (1 - \cos 2 \lambda_{mn} x)(1 - \cos 2 \mu_{mn} y) \end{bmatrix}
\]

Fully clamped B.C \hspace{1cm} (3.51a)

\[
\begin{bmatrix}
  u_0 \\
v_0 \\
w_b \\
w_y \\
\varphi
\end{bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} U_{mn} e^{i \omega t} \cos \lambda_{mn} x \sin \mu_{mn} y \\ V_{mn} e^{i \omega t} \sin \lambda_{mn} x \cos \mu_{mn} y \\ W_{bmn} e^{i \omega t} \sin \lambda_{mn} x \sin \mu_{mn} y \\ W_{sbn} e^{i \omega t} \sin \lambda_{mn} x \sin \mu_{mn} y \\ \phi_{mn} e^{i \omega t} \sin \lambda_{mn} x \sin \mu_{mn} y \end{bmatrix}
\]

Simply supported B.C \hspace{1cm} (3.52b)

where \( U_{mn}, V_{mn}, W_{bmn}, W_{sbn} \) and \( \phi_{mn} \) are considered to be the constant coefficients in the Fourier series and are the unknown parameters to be determined, \( \omega \) is the eigen-frequency associated with \((m,n)\)th eigen-mode, and \( \lambda = \frac{m \pi}{a} \) and \( \mu = \frac{n \pi}{b} \).

By substituting Equation (3.51a) into Equation (3.48), multiplying the first and second of the resulting equations (equations 3.49a and 3.49b) by \( \sin(\lambda x) \sin(\mu y) \) and the third, fourth and fifth (equations 3.48c, 3.48c and 3.48e) by \( (1 - \cos 2 \lambda x)(1 - \cos 2 \mu y) \), and integrating with respect to \( x \) and \( y \) over their respective intervals, the governing equations are

\[
\begin{bmatrix}
  J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\
  J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\
  J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\
  J_{41} & J_{42} & J_{43} & J_{44} & J_{45} \\
  J_{51} & J_{52} & J_{53} & J_{54} & J_{55}
\end{bmatrix} \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} & 0 \\ 0 & m_{22} & m_{23} & m_{24} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ 0 & 0 & m_{53} & m_{54} & m_{55} \end{bmatrix} \begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{sbn} \\ \phi_{mn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q \\ q \\ 0 \end{bmatrix}
\] \hspace{1cm} (3.52)
Eigen-frequencies of the FG plate are obtained by solving the eigenvalue problem in Equation (3.52). Moreover, the coefficients of the matrices of Equation (3.52) with respect to the B.Cs are given in Appendix D.

The closed-form solutions for thick FG plates with in-plane stiffness variation for two different boundary condition cases (fully clamped and simply supported) based on HSDT have been derived in this section. From these results, the deflection and frequencies of the FG plate can be easily determined analytically based on the geometric configurations and the material properties.

### 3.4 Numerical modelling

The method used in this study to model and evaluate the local gradation effects focused on the use of ABAQUS finite element software to model material property gradients. The gradation was modelled at the element level using an ABAQUS user subroutine that mapped the variation in material properties, i.e. stiffness, yield stress, density and etc. as a function of spatial coordinates at different Gauss points within each element. This method results in smooth and continuous variation across the element. With each of the studies conducted, convergence of the Finite Element Analysis (FEA) solution was accomplished to ensure that accuracy was optimized. In addition, the FEA model was evaluated against the closed form solution which was obtained in section 3.3.

#### 3.4.1 Modelling of FGM

There are generally two approaches applied to model the gradation of material properties using finite elements. Homogeneous elements can be utilized in such a fashion that the elements are assembled in rectangular rows that are aligned with the gradation direction. Each row of homogeneous elements is then assigned the varied material property for the midline of the row. This produces a step-wise approximation where the stiffness matrix for a specific element is assumed constant and has the property assigned at the centroid of the element [69]

Although using homogeneous elements can provide reasonable results, it does not lend itself to capturing geometry that is not rectangular in shape. Furthermore, due to the high stress gradients that are inherent to stress concentration problems, a more accurate method of capturing the gradients, without using an extremely large number of elements, is preferred. The more accurate method of modelling material property variation is with the iso-parametric element where the spatial variation in the property can be assigned at each Gauss point within the element. Normally the components of the stiffness matrix contain constant material properties for an element. By assigning spatially dependant properties at each Gauss point, the stiffness matrix provides variation across the element,
resulting in a full, smooth transition across each element. The user subroutine is used to map the modulus gradation over the boundary area of the model. Buttlar et al. [70] implemented the graded element with smooth variation using the UMAT capability of the finite element software ABAQUS. So, the use of graded elements is more desirable for nonhomogeneous materials than standard homogeneous elements (See Figure 3.3)

Graded elements can be compared with conventional homogeneous elements such as those used in traditional layered analysis of pavement, as illustrated by Figure 3.3. Notice that the graded element incorporates the material property gradient at the size scale of the element, while the homogeneous element produces a stepwise constant approximation to a continuous material property field such as the one shown in Figure 3.3. Graded elements are implemented by means of direct sampling properties at the Gauss points of the element [70]. The finite element stiffness matrix relations can be written as [71]:

\[ K' u' = F' \]  

(3.54)

with

\[ K' = \int_{\Omega} B'^T D'(x) B' d\Omega \]  

(3.55)
where $u^e = \text{nodal displacement vector}; \ F^e = \text{load vector}; \ B^e = \text{strain-displacement matrix which contains gradients of the interpolating functions}; \ D^e(x) = \text{constitutive matrix};$ and $\Omega_e$ is the domain of element $e$. In this work the elasticity matrix is assumed to be a function of spatial coordinates. The integral in Equation (3.55) is evaluated by Gauss quadrature, and the matrix $D^e(x)$ is specified at each Gaussian integration point.

### 3.4.2 User subroutine

ABAQUS software provides users with an extensive array of user subroutines that allow adaption of ABAQUS to their particular analysis requirements. A user subroutine UMAT is used to define any complex, constitutive models for materials that cannot be modelled with the available ABAQUS material model. In order to define the values of field variables directly at the integration points of elements, the user subroutine USDFLD is applied. The subroutines are written in the FORTRAN language and run in parallel to the ABAQUS solver. This approach allows the user to establish an algorithm to calculate user variables that will be passed into the ABAQUS solver. While ABAQUS performs the standard finite element procedure using standard types of finite elements, the UMAT and USDFLD govern the behaviour of graded materials during different loading stages, i.e., elastic, inelastic. Combining a UMAT and USDFLD subroutine with the standard finite element procedure will manifest the simplicity and applicability of the proposed model into any engineering problem where the average user will not be exposed to the complexities associated with introducing non-standard finite elements.

### 3.4.3 Finite element FG model verification

Linear hexahedral (C3D8R) graded elements were used in the finite element program ABAQUS to implement an effective continuous variation of material stiffness and density. This type of element is implemented by means of direct sampling of properties at the integration (Gauss) points of the element. The user subroutine UMAT and USDFLD function coded in FORTRAN were used for modelling of the FG plate with graded solid elements in the ABAQUS FE software. In the ABAQUS software, the gradient of properties can be specified via the subroutine, since it is called at the integration points. The Young’s modulus (material stiffness) and density of the FG plate as material properties were defined as functions of length for each integration point throughout the length ($x$) of the plate. This produces an elastic FE model so that its material stiffness and density vary continuously through the length of the plate, having a simple power-law distribution of volume fraction of constituents, as given in equation (3.2). For vibration analysis, a user subroutine UMAT cannot be applied due to a lack of density as one of its arguments. Therefore, a user subroutine USDFLD is used to model
density variation. The user subroutine UMAT and USDFLD for graded elements for bending and vibration analysis are presented in Appendix E.

Although the modelled plate has a symmetric geometry, the material property is not symmetric when considering an in-plane material properties variation. Therefore the full FG plate was considered in the FE model. Two different transverse loading cases, i.e. distributed pressure and concentrated loading are applied along the upper surface of the plate. Figure 3.4 shows a schematic of the modelling approach applied within the ABAQUS software. It is noted that the variation of material properties can be seen in Gauss points and hence a finer mesh will produce a better approximation. So, in order to achieve independency of results from mesh size for a functionally graded plate, a convergence study was conducted that yielded a stable and accurate solution while keeping the computational time to a minimum. So, a convergence study with respect to the mesh size for each example is first performed to ascertain the level of finite element refinement necessary to obtain accurate results. To this aim, the non-dimensional parameter $\kappa$, which represents the ratio of mesh size to the plate thickness, has been considered.

![SDV1 (Avg: 75%)](image)

Figure 3.4: Variation of Young’s modulus of FG plate in ABAQUS- Simply supported boundary conditions

### 3.4.4 Executing FE modelling on pan cluster

Due to the use of different user subroutines and very fine meshes, an alternative computational platform was required rather than the PC that was used for all the previous
analyses to minimize simulation time. The FE simulations were completed on the New Zealand eScience Infrastructure High Performance Computing platforms. All required software was installed on the NeSI Pan Cluster existing in the Auckland university laboratory. A computer cluster is a high performance computing system composed of many smaller computers. Computers in the cluster are linked by a network that allows parallel programs to use many computers simultaneously. Running tasks on a cluster is typically organised by a job scheduler.

The scripts used to carry out the different steps had to be adapted for a Windows system and these were split up into different SLURM (Simple Linux Utility for Resource Management) jobs with dependencies. Moreover, a specific SLURM should be written to run ABAQUS 6.10 and its compatible FORTRAN 11.1 version. Therefore, the FE models were created in ABAQUS 6.10 and written as an input file for the NeSI Pan Cluster along with FORTRAN code for a user subroutine. After that, the simulation was executed by typing “Sbatch SLURM.sl” within the Pan Cluster environment. Figure 3.5 shows a typical Pan Cluster environment during execution of the simulation.

![Pan Cluster environment](image)

Figure 3.5: Pan Cluster environment

### 3.5 Results and discussion

The analytical formulation and numerical implementation to an FG plate with in-plane material properties variation has been derived in the previous sections. In this section, the numerical solutions for bending and free vibration evaluated directly from theoretical
formulation and calculated by finite element method using user subroutines and ABAQUS software, are presented.

### 3.5.1 Bending analysis

A rectangular polymeric FG plate made from digital materials characterized in chapter 2, with different thickness ratio \((\tau = h/l)\) and aspect ratio \((\eta = b/a)\), is considered. The material properties of two types of polymeric FG plates are listed in Table 3.1. The main reason for selecting these materials as inclusion and matrix for the FG plate is that two FG plates with low stiffness ratio (FG I) and high stiffness ratio (FG II) were desired. The achieved stiffness ratios for the FG I and FG II plates are 2 and 18, respectively.

<table>
<thead>
<tr>
<th>FG plate</th>
<th>Material trade name</th>
<th>Young’s modulus (MPa)</th>
<th>Density (Kg/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FG I:</td>
<td>Inclusion</td>
<td>VeroWhite</td>
<td>2420</td>
</tr>
<tr>
<td></td>
<td>Matrix</td>
<td>RGD8530-DM</td>
<td>1210</td>
</tr>
<tr>
<td>FG II:</td>
<td>Inclusion</td>
<td>RGD8515-DM</td>
<td>1685</td>
</tr>
<tr>
<td></td>
<td>Matrix</td>
<td>FLX9895-DM</td>
<td>95</td>
</tr>
</tbody>
</table>

### 3.5.1.1 Comparison of the analytical solutions for different types of plate

Figure 3.6 compares the non-dimensional deflection of the homogenous plate and the FG I plate along the length with two different plate theories, i.e. CPT and HSDT. A simply supported plate with applied distributed load \((q_0 = 1\ \text{bar})\) is considered. As it can be seen, for a homogenous plate, the CPT agrees very well with the proposed HSDT, with the maximum error being 0.76%. However, for FG I plate with in-plane stiffness variation, the application of CPT due to the neglecting of shear deflection and the thickness stretching effect may not be fully valid. This shows that the CPT for a thick FG plate with in-plane stiffness variation is not sufficiently accurate, unlike the case with homogenous plates. So, HSDT as an accurate plate theory for thick and medium thick FG plate will be used in the following work.
Figure 3.6: Deflection of the homogenous and FGI plate using CPT and HSDT with thickness ratio $\tau=0.1$, aspect ratio $\eta=1$ and $n=3$.

Figure 3.7 compares the deflection of the plate along the length regarding HSDT with different types of stiffness variations, i.e., homogenous plate without variation (constant stiffness), through-the-thickness variation and in-plane variation, for an FGI plate. The homogeneous plate was made of the matrix material for each combination.

The homogeneous plate undergoes the largest maximum deflection ($w/h=-0.0337$), as it has the lowest overall stiffness. It can be noted that by introducing the higher stiffness
material (VeroWhite as a hard polymer) as inclusions in the soft polymer as matrix, the overall stiffness of the plate increases, and hence the deflection of the plate decreases significantly for both the stiffness variation cases. Moreover, the FG I plate with variation of the material constituents in the thickness direction has the smallest maximum deflection ($w/h = -0.0255$), with the deflected shape being symmetric about the mid-plane ($x/a = 0.5$). The plate with in-plane variation in stiffness has the intermediate maximum deflection ($w/h = -0.0335$), and the maximum deflection occurs at $x/a = 0.45$. It is noteworthy that the maximum deflection does not occur in the middle of the plate in the case of in-plane stiffness variation. While for homogenous and FG plates with thickness-wise variation, the deflected shape of the plate about its mid-plane is symmetric, and the maximum deflection occurs in the middle of the plates [74]. So, it can be observed that the overall stiffness of the FG plate and the location of the maximum deflection are strongly related to the direction of the material stiffness gradient.

### 3.5.1.2 Comparison of the analytical and numerical solutions for FG plate

In order to effectively implement modelling of an FG plate, both solid and shell elements can be used in ABAQUS UMAT subroutine. Here the solid elements were chosen because the solid element implementation in an ABAQUS UMAT sub-routine allows material stiffness gradients to be specified along both the thickness and length directions. In other words, material stiffness variation through the thickness of the plate cannot be implemented in shell elements. It is noted that the variation of material parameters can be seen in the Gauss points and hence a finer mesh will produce a better approximation. Therefore, in order to demonstrate the independency of results from mesh size for FG plate, a mesh convergence study was first conducted that yielded a stable and accurate solution while keeping the computational time to a minimum. The maximum non-dimensional deflection was measured in each simulation, by increasing the number of elements. The results for FG I are provided in Figure 3.8. It should be noted that the non-dimensional parameter mesh size $\kappa$ represents the ratio of elements size to the plate thickness. Therefore, a non-dimensional parameter mesh size of $\kappa = 0.0625$ is considered.
Figure 3.8: Mesh convergence on FG I plate with thickness ratio $\tau=0.1$, aspect ratio $\eta=1$ and $n=3$
(a) Convergence of maximum non-dimensional deflection and (b) meshing and simulation times

Figure 3.9 presents the deflection of the FG I plate determined from both the analytical and numerical solutions for power-law index $n=3$. It is found that the analytically derived mid-surface transverse displacements of the FG plate exhibit close agreement with those obtained from the FEM predicted solutions. It should be noted that the maximum non-dimensional deflection for FE solution is $w/h =-0.03258$ which occurs at $x/a =0.46$. The discrepancy between analytical and FE results at the maximum deflection point is around 3.07%.
Figure 3.9: Deflection of the FG I plate predicted by the analytical and numerical solutions

Figure 3.10 shows the deviation between the analytical solution and the FE results for the FG I plate. It can be seen that the difference between the analytical and the FE solutions is much more pronounced after the maximum deflection point.

This could be caused by the specific power-law index \( n \) and the resultant steep gradient in the stiffness after this location. Figure 3.11 shows the variation of material stiffness along the length with different power law indices, with low, moderate and high values \( (n=0.1, 1, 3, \text{ and } 7) \). It can be observed from Figure 3.11 that a sudden change in the material stiffness occurs at the maximum deflection point for \( n=3 \) (at \( x/a=0.454 \)). The
Young’s modulus of the FG plate varies rapidly near the inclusion phase dominant (right) side for high values of the power-law index \((n=3\) and \(7\)). Therefore, variation in the Young’s modulus of the FG plate would be steep near the inclusion phase dominant (right) side which, can lead to discrepancy between analytical and numerical results, as also reported in \([43]\).

![Figure 3.11: Variation of the material stiffness through the length with different power-law indices for FGI plate.](image)

Next we determine the strains and stresses in the FG plate from the analytical solution, and compare them with the corresponding numerically predicted values. The variation of the strain \(\varepsilon_x, \varepsilon_y, \varepsilon_z\) along the thickness for FGI plate is shown in Figure 3.12. By comparing the analytical and numerical solutions, the accuracy of the present analytical theory based on the sinusoidal higher deformation theory and consideration of the stretching effect for thick FG I plate, is evaluated. It can be noted from Figure 3.12 that the neutral surface of a FG plate with in-plane stiffness variation is located at the mid-surface \((z=0)\). However, Chi and Chung \([43]\) showed that for an FG plate with through-the-thickness variation of stiffness, the neutral surface moves in the thickness direction to the relatively higher stiffness end.
(a) 

(b)
Figure 3.12: Strain $\varepsilon_i$ at the centre of the FG I plate with thickness ratio $\tau=0.1$, aspect ratio $\eta=1$ and $n=3$.
(a) $\varepsilon_x$, (b) $\varepsilon_y$ and (c) $\varepsilon_z$

Figure 3.13 shows the comparison of the stress in the FG I plate obtained from the analytical and FE solutions. As explained above, good agreement between the two solutions can be observed for the variation of the stress at the centre of the FG I along the thickness direction, with a maximum deviation less than 13.72%. The in-plane stress, $\sigma_{xx}$, of the FG plate calculated by the analytical solution is semi-linear, which is consistent with the formulation in Equation (3.37). It can be noted that the effect of the sinusoidal shear functions, $\xi(z)$ and $\zeta'(z)$, in Equations (3.35) on the stress, $\sigma_x$, is not largely pronounced compared to the bending effect which is a linear function of $z$. In other words, the sinusoidal term is much smaller than the linear term, which implies that the bending is the dominant deformation mechanism.
3.5.1.3 The effect of the stiffness ratio ($E_i/E_m$)

The material stiffness of the FG I plate varies continuously and functionally based on volume fraction, given in Equation (3.2). It should be noted that the Young’s modulus of the matrix is 1207 MPa (see Table 3.1), while Young’s modulus of the inclusion, $E_i$, varies with the ratio of $E_i/E_m$. The variation of Young’s modulus in the FG plate versus the ratio of $E_i/E_m$ are plotted in Figure 3.14 for the case of the power law index $n=3$. 

Figure 3.14: Variation of the Young’s modulus of FG plate for different stiffness ratio $E_i/E_m$
Figure 3.15 shows the effect of different $E_i/E_m$ ratios on the deflection of the FG plate. Generally, it is reported that an increase in the stiffness ratio, $E_i/E_m$, reduces the deflection of the FG plate with property variation through the thickness [43]. However, a different deformation behaviour is observed when in-plane variation of the material stiffness is considered. For the FG plate with property variation through the length, the change in deflection depends on the maximum deflection point. As observed in Figure 3.15, the addition of higher stiffness materials (like inclusions) does not decrease the maximum deflection of FG plates before the maximum deflection point. However, after the maximum deflection point, the results are compatible with general trends in the literature, e.g. as reported in [43].

![Figure 3.15: Deflection of the FG plate with variation of the material stiffness through the length for different $E/E_m$ ratios](image)

Figure 3.16 shows the deviation between the analytical solution and the FE results for the FGI plate for different stiffness ratios, $E_i/E_m$. It can be seen that as the stiffness ratio increases the difference between the analytical and the FE solutions, particularly near the inclusion phase side, increases. There are two main contributing factors leading to discrepancies between the analytical formulation and FE simulation. The first and second reasons are associated to steep variation of properties near the inclusion side and assumption of the constant coefficients for the Fourier series, respectively. As it can be seen from Figure 3.14, the variation of Young’s modulus of FG plate is steep near the inclusion phase dominant (right) side. Therefore, graded elements with defined and converged element size are less able to capture steep gradation of stiffness. The second reason is associated to the constant coefficients for the Fourier series in the boundary condition equations (Equations (3.51)). As it can be seen from Figure 3.9, the maximum discrepancy between analytical and numerical results in absent of high stiffness ratio reason (steep variation contributing factor) is less than 5% which confirms a relatively
good accuracy of proposed analytical formulation for a simply supported plate, considering the assumption of constant Fourier series coefficients.

It could be mentioned that the analytical formulation for a high stiffness ratio thick FG plate cannot capture the deflection of the plate accurately. Therefore the overall stiffness of the FG plate is strongly associated with the direction of the material stiffness variation, and the appropriate selection of the modulus ratio $E_i/E_m$.

![Figure 3.16: Deviation between the analytical formulation and the FE results for deflection of FG plate](image)

(a) $E_i/E_m=2$, (b) $E_i/E_m=4$
The modulus ratio also influences the location of maximum deflection as can be seen from Figure 3.15. As the $E_i/E_m$ ratio increases, the location of the maximum deflection moves towards the matrix phase dominated side of the plate (left side for this study). Table 3.2 presents the location of the maximum deflection point of FG plates with different stiffness ratio.

<table>
<thead>
<tr>
<th>Stiffness Ratio</th>
<th>Solution</th>
<th>Maximum non-dimensional deflection $(w/h)_{max}$</th>
<th>Location of maximum non-dimensional deflection $(x/a)_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_i/E_m = 2$</td>
<td>Analytical formulation</td>
<td>-0.0335</td>
<td>0.454</td>
</tr>
<tr>
<td></td>
<td>FEM-Graded solid element</td>
<td>-0.0325</td>
<td>0.461</td>
</tr>
<tr>
<td></td>
<td>Error (%)</td>
<td>3.07</td>
<td>1.52</td>
</tr>
<tr>
<td>$E_i/E_m = 3$</td>
<td>Analytical formulation</td>
<td>-0.0338</td>
<td>0.427</td>
</tr>
<tr>
<td></td>
<td>FEM-Graded solid element</td>
<td>-0.0314</td>
<td>0.438</td>
</tr>
<tr>
<td></td>
<td>Error (%)</td>
<td>7.64</td>
<td>2.51</td>
</tr>
<tr>
<td>$E_i/E_m = 4$</td>
<td>Analytical formulation</td>
<td>0.0344</td>
<td>0.409</td>
</tr>
<tr>
<td></td>
<td>FEM-Graded solid element</td>
<td>-0.0299</td>
<td>0.425</td>
</tr>
<tr>
<td></td>
<td>Error (%)</td>
<td>15.05</td>
<td>3.76</td>
</tr>
</tbody>
</table>

Utilizing these aspects, i.e. the sensitivity of the magnitude and the location of the maximum deflection, the stiffness of the FG plate can be controlled to meet the desired or application-specific performance criteria.

### 3.5.1.4 The effect of power law index

The effect of the power law index $n$ on the deflection of FG plate is presented in Figure 3.17. It can be observed that the linear FG plate ($n=1$) has the lowest deflection, with a maximum non-dimensional deflection which occurs at $x/a=0.42$. As the power law index $n$ increases, the deflection of FG plate increase due to the decrease of material stiffness based on Figure 3.11. Another interesting point to note is that the location of maximum deflection moves towards the inclusion phase dominant side of the plate (right side for this study) as the power law index $n$ increases. Because, according to Equation (3.2), when the power law index $n$ increases, the FG plate shows the inclusion material behaviour. It can be observed from Figure 3.17 that as the gradient index $n$ increases the
deviation between analytical formulation and FE solution increases, particularly after the maximum deflection point.

Figure 3.17: Deflection of the FG I plate with variation of power law index $n$

3.5.1.5 The effect of geometric ratio (aspect ratio $\eta =a/b$ and thickness ratio $\tau =h/b$)

So far, the problem addressed in the previous figures is assumed to be a square FG plate with thickness ratio $h/b=0.1$. However, it is worthwhile to investigate the effect of aspect ratio $a/b$ and thickness ratio $h/b$ on the deflection of an FG plate with in-plane material property variation. Therefore, in this section, all discussions are based on the assumption that the width of plate $b$ is fixed and length $a$ is varying to investigate the effect of aspect ratio. For each aspect ratio the thickness $h$ varies. The non-dimensional deflection of the FG I plate with different aspect ratios under distributed transverse loading, power law index $n=3$ and thickness ratio $h/b =0.1$ is shown in Figure 3.18. It can be seen that the maximum deflection increased upon increasing the aspect ratio $a/b$ and its location moved towards the inclusion phase dominant side as well. Figure 3.19 presents the maximum dimensionless deflection value. It can be observed that the maximum deflection of the rectangular FG plate changes slightly when $a/b$ is greater than 3. In other words, the maximum deflection for $a/b=2$ is more than two times that of the square plate ($a/b =1$). While, when $a/b$ doubles ($a/b=4$) the maximum deflection increases 1.2 times relative to the FG plate with $a/b =2$. 

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The main application of the proposed Higher order Shear Deformation Theory (HSDT) is modelling of medium thick and thick plates. It is worthwhile to investigate whether the proposed plate theory will hold for the FG plates with different $h/b$ ratio under certain aspect ratio $a/b$. To do this, consider an FG I plate subjected to uniform load and simply supported boundary conditions. Figure 3.20 shows the error between the proposed analytical formulation and graded FE solution of maximum dimensionless deflection for an FG I plate. It is found that the bigger $h/b$ ratio (i.e. a thicker plate) has the highest error.
for all aspect ratios for \( a/b = 1 \) and \( 2 \). Moreover, the error decreases as the aspect ratio \( a/b \) increases. Overall, it could be mentioned that the maximum discrepancy between analytical and numerical results occur for the thick square FG plate with thickness ratio \( h/b = 0.2 \), with an error value of 4.3\%. This indicates good accuracy for the proposed analytical formulation for thick FG plates.

Figure 3.20: Deviation between the analytical formulation and the FE results for maximum dimensionless deflection of FG I plate

### 3.5.1.6 The effect of applied load

In order to investigate the influence of an external load on an FG plate, a square FG plate is considered with elastic modulus ratio \( (E_i/E_m) = 2 \) under a point load at the centre of the plate with magnitude \( P_0 = 10,000 \) kg, in which the total load is equal to that of the uniform load of \( q_0 = 1 \) kg/cm\(^2\). The comparison of the deflection of the FG plate under uniform load and point load are depicted in Figure 3.21. It is shown that dimensionless deflection due to the point load is higher than that of the uniform load. Another interesting point attracting one’s attention is that the maximum deflection point for the point load case occurs in the middle of the plate where the point load is applied. In other words, applying a distributed uniform load would be more reliable for determining the optimum location of maximum deflection of FG plate under different condition.
3.5.1.7 The effect of boundary condition

In this section, two different boundary conditions -fully clamped and simply supported- for the square FG I plate with uniform distributed load are considered. The required displacement fields which satisfy the fully clamped and simply supported boundary conditions are in Equation (3.52). As it can be revealed from Figure 3.22, the clamped boundary condition produces less deflection than the simply supported one, with a maximum dimensionless deflection \( w/h = -0.0087 \). Moreover, the error between the analytical and FE solutions for a fully clamped boundary condition is bigger than for the simply supported boundary condition. The main reason could be due to the assumption of constant coefficients in the Fourier series for the clamped boundary condition (Equation (3.49a)). It is mentioned that for increased accuracy of the analytical results, particularly for clamped boundary condition, the governing equations (3.48) should be solved with considering variable coefficient at Fourier series. However, development of analytical solutions with variable coefficients would represent a significant challenge beyond the methods presented here, for a modest increase in accuracy.
3.5.2 Free vibration analysis

3.5.2.1 Comparison of the analytical solutions for homogenous plate

In order to investigate the accuracy of the present analytical solutions, the results are first compared with those obtained for a thick homogenous plate. For convenience, the non-dimensional parameters

$$\omega = \omega_0 a^2 \sqrt{\frac{\rho h}{D}}, \quad D = \frac{Eh^3}{12(1-\nu)}, \quad \hat{\omega} = \frac{\omega}{\omega_0} \frac{b^2}{h} \sqrt{\frac{\rho}{E}}, \quad \beta = \frac{\omega}{\hat{\omega}} \frac{a^2}{h} \sqrt{\frac{\rho_m}{E_m}}$$

are introduced, for which $\beta$ is the eigenfrequency parameter.

The non-dimensional fundamental frequency parameters ($\sigma$) of the simply supported (SSSS) and fully clamped (CCCC) square plates are presented in Table 3.3 and Table 3.4, respectively, for different thickness ratios ($\tau = h/b$). The good agreement of the present solution with the results reported in the literature confirms the high accuracy of the current approach. It is noted that the FSDT and HSDT solutions by Hosseini-Hashemi et al. [53, 75] were obtained using the Levy method. The small differences between the present results and those predicted by FSDT, TSDT and HSDT [53, 63, 75] from Table 3.3 are due to the thickness stretching effect being ignored. Moreover, the Hyperbolic HSDT (HHSDT) solutions of Blabed et al. [46] used a hyperbolic variation of all displacements across the thickness and incorporated the stretching effect as well. It is noteworthy from Table 3.4 that CPT noticeably over-predicts the natural frequencies of thick plates as mentioned in the literatures [45, 46] due to exclusion of shear deformation effects.
Table 3.3: Comparison of non-dimensional fundamental frequency ($\omega$) for simply supported (SSSS) isotropic plates

<table>
<thead>
<tr>
<th>Method</th>
<th>$r=h/b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>Present-</td>
<td></td>
</tr>
<tr>
<td>analytical</td>
<td></td>
</tr>
<tr>
<td>19.1067</td>
<td>17.7631</td>
</tr>
<tr>
<td>FSDT [75]</td>
<td>19.084</td>
</tr>
<tr>
<td>TSDT [63]</td>
<td>19.0653</td>
</tr>
<tr>
<td>HSDT [53]</td>
<td>19.0652</td>
</tr>
<tr>
<td>HHSDT [46]</td>
<td>19.1002</td>
</tr>
</tbody>
</table>

Table 3.4: Comparison of non-dimensional fundamental frequency ($\omega$) for fully clamped (CCCC) isotropic plate

<table>
<thead>
<tr>
<th>Method</th>
<th>$r=h/b$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>Present-</td>
<td></td>
</tr>
<tr>
<td>analytical</td>
<td></td>
</tr>
<tr>
<td>35.5551</td>
<td>32.6808</td>
</tr>
<tr>
<td>Analytical CPT [76]</td>
<td>36.0117</td>
</tr>
</tbody>
</table>

3.5.2.2 Comparison of the analytical and numerical solutions for an FG plate

A convergence study was first conducted to reduce mesh dependency of the graded FEM results and to increase the accuracy of the solutions. The first natural frequency was measured in each simulation, by increasing the number of elements in the mesh. The results for a simply supported thick FG I plate are provided in Figure 3.23. It should be noted that the non-dimensional parameter mesh size $\kappa$ represents the ratio of mesh size to the plate thickness.

Based on the present analytical and graded FEM solutions, the first five natural frequencies have been obtained for the simply supported boundary condition, for two types of polymeric square FG plates with a power law index $n=3$. The results are presented in Figure 3.24. The side length of the square plates was set to 0.11 m. Three different thickness ratios, 0.02, 0.07 and 0.1, corresponding to thin, moderately thick and thick plates [42], respectively, were used.
Figure 3.23: Mesh convergence for first natural frequency of FG I plate with thickness ratio $τ=0.1$, aspect ratio $η=1$ and $n=3$

(a) Convergence of maximum non-dimensional deflection and (b) meshing and simulation times.
Figure 3.24: First five natural frequencies for simply supported FG plate with aspect ratio $\eta=1$ and $n=3$

(a) FG I  (b) FG II
Generally speaking, an excellent agreement is observed between the present HSDT solutions and the graded FE solutions for thin simply supported FG I and FG II plates. It is to be noted that the results of the proposed method are close to the FE analysis results at lower frequencies for both FG I and FG II plates. Moreover, as the frequency number increases, the discrepancies between the two solutions increase. Because in the FE solution, the element shape functions provide a better basis (approximation) of the shape of the low modes, and a relatively poorer basis for the higher modes. Also, it can be seen that natural frequencies increase with an increase in the plate thickness. This originates from the increasing stiffness of the plates. Moreover, it is found that the FG II plate with a high elastic modulus ratio has a lower frequency range than FG I with a low elastic modulus ratio. As can be seen from Table 3.1, the material properties of the FG I combination indicate a stiffer plate than FG II. Therefore, the stiffness of the plate for FG I is higher than that of FG II, resulting in higher natural frequencies.

Figure 3.25 shows the error between the proposed analytical solution and the graded FE solution, for determination of the first natural frequency (Hz) for FG I and FG II plates with different thickness ratios.

![Figure 3.25: Deviation between the analytical formulation and the FE results for the first natural frequency of FG I and FG II plates with different thickness ratio](image)

From Figure 3.25, it is seen that the deviation between the analytical and FE solutions for FG I plates is less than for FG II plates for first natural frequency, due to less stiffness variation between the two constituent materials. The maximum deviation between the analytical and FE solutions for the first natural frequency is found for the thick FG II plate \((h/b=0.1)\), with error of approximately 21%. Since the FG II plate (high elastic
modulus ratio) generates a steep in-plane stiffness variation, the graded elements are less able to accurately capture the rapid variation of Young’s modulus with a power law distribution function.

3.5.2.3 The effect of boundary condition

Table 3.5 shows the first natural frequencies for both simply supported and fully clamped FG I and FG II plates. It can be seen that natural frequency decreases when a less restraining boundary condition is used at the edges of the plates, e.g. a simply supported condition. Since the fully clamped case increases the flexural stiffness of the plate, it leads to a higher frequency response. Moreover, as mentioned previously, the FG I plate has the higher natural frequency due to higher total stiffness.

<table>
<thead>
<tr>
<th>τ=h/a</th>
<th>Solution</th>
<th>FG I Simply supported</th>
<th>Fully clamped</th>
<th>FG II Simply supported</th>
<th>Fully clamped</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>Analytical</td>
<td>196.39</td>
<td>266.81</td>
<td>123.83</td>
<td>121.56</td>
</tr>
<tr>
<td></td>
<td>Graded FE</td>
<td>190.62</td>
<td>267.65</td>
<td>110.83</td>
<td>125.66</td>
</tr>
<tr>
<td>0.07</td>
<td>Analytical</td>
<td>683.68</td>
<td>926.44</td>
<td>431.06</td>
<td>353.67</td>
</tr>
<tr>
<td></td>
<td>Graded FE</td>
<td>643.74</td>
<td>863.91</td>
<td>382.84</td>
<td>402.29</td>
</tr>
<tr>
<td>0.1</td>
<td>Analytical</td>
<td>970.81</td>
<td>1311.86</td>
<td>612.11</td>
<td>459.31</td>
</tr>
<tr>
<td></td>
<td>Graded FE</td>
<td>870.12</td>
<td>1161.4</td>
<td>517.46</td>
<td>535.96</td>
</tr>
</tbody>
</table>

The deviation between the proposed analytical solution and FE simulation for both simply supported thin FG I and FG II plates is shown in Figure 3.26. It can be observed that the FG II plate generates higher discrepancies than FG I plate, and this is due to steep stiffness variation in the FG II plate. Moreover, the maximum discrepancy is generated for the FG II plate with simply supported boundary conditions, with error of approximately 12%.
3.5.2.4 Non-dimensional eigenfrequency parameter \((\beta)\)

The variation of the dimensionless eigenfrequency parameter \((\beta)\) versus the power law index \(n\) for FG square plates with two different boundary conditions and thickness to length ratio \(\tau\) based on analytical results are shown in Figure 3.27 and Figure 3.28. These figures show that as the power index \(n\) changes from 0.01 to 100, the FG plate II (with a high elastic modulus ratio) is clearly more sensitive to the eigenfrequency parameter \((\beta)\), as compared to the FG plate I (with a low elastic modulus ratio). For instance, for a simply supported FG II plate the eigenfrequency parameter \((\beta)\) varies from around 25 and reaches the minimum value of about 2 (around 92% variation) while, the variation in \(\beta\) for FG I is from 8.4 and has a minimum value of 5.85 (about 30% variation). The reason is that, as mentioned in section 2.5.1.3, the material properties of the FG plates are strongly dependent on the relative difference between the inclusion and matrix Young’s moduli \((E_i-E_m)\). Figure 3.27 and Figure 3.28 demonstrate that regardless of the applied boundary conditions, thickness to length ratios \(\tau\) and elastic modulus ratios, the eigenfrequency parameter \((\beta)\) has the same minimum values around \(n=5\) for the both FG I and FG II plates with both clamped and simply supported boundary conditions. Moreover, it is found that when power law index \(n\) approaches zero or infinity, the plate behaves as if it is homogenous, being composed completely of inclusion or matrix, respectively. Therefore, the predicted values of the dimensionless eigenfrequency converge to the relevant homogenous plate. Also, it is seen that with an increase in the thickness to length ratio \(\tau\), \(\beta\) decreases for all boundary conditions and elastic modulus ratios. Moreover, simply supported plates have lower eigenfrequency values than fully clamped plates for all different thickness ratio.
3.5.2.5 The effect of density ratio

The above results for FG plates are based on the material properties provided in Table 3.1, and on the assumption that the densities of the two constituents are very similar. It is assumed that the density at the left hand side (matrix phase), $\rho_m$, is 1170 (Kg/m$^3$),
while that at the right hand side (inclusion phase), \( \rho_i \), varies with the ratio of \( \frac{\rho_i}{\rho_m} \). Figure 3.29 shows the effect of variation of density on the first natural frequency for both simply supported FG plates with low material stiffness ratio. From the figure, it is found that the first dimensionless natural frequency (\( \hat{\omega} \)) increases as the density ratio increases. Also, there is no significant difference between non-dimensional frequencies as the thickness ratio varies.

![Graph showing variation of first dimensionless natural frequency with density ratio](image)

**Figure 3.29:** Variation of first dimensionless natural frequency (\( \hat{\omega} \)) with density ratio for simply supported FG plate with low material stiffness ratio (\( E_i/E_m=2 \))

The comparison between analytical and FE solutions for thick FG plates with variation of density is shown in Figure 3.30. It is observed that the error increases as the density ratio rises, and maximum error is generated for a density ratio, \( \rho_i/\rho_m \), of 4 with around 17%.

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Figure 3.30: Deviation between analytical and FE solutions of non-dimensional natural frequency for thick FG plate with low material stiffness ratio \( (E_i/E_m=2) \)

### 3.6 Summary

To end this Chapter, presented here is a summary, together with the main findings and conclusion:

- Higher order Shear Deformation (HSDT) as an analytical solution and graded FE modelling using user subroutine UMAT and USDFLD to incorporate bending and free vibration of FG plates with in-plane stiffness variation.

- Two different polymeric FG plates (refer Table 3.1) called FG I (low stiffness ratio) and FG II (high stiffness ratio) were considered based on design in chapter 2.

- The maximum deflection does not occur in the middle of the FG plate with in-plane variation of material stiffness, as opposed to the case of a homogeneous plate.

- The stress distribution \( (\sigma_{xx}) \) in the FG plate is semi-linear. It implies that sinusoidal term is much smaller that linear term (in Equations (3.37)) which gives bending dominant term.

- Difference between the analytical and the FE solutions increases as stiffness ratio increases, particularly near the inclusion phase side. It could be mentioned that the analytical formulation for a high stiffness ratio thick FG plate cannot capture the deflection of the plate accurately.
• Maximum deflection point for the point load case occurs in the middle of the plate where the point load is applied. In other words, applying a distributed uniform load would be more reliable for determining the optimum location of maximum deflection of FG plate under different condition.

• For the FG plate with property variation through the length, the change in deflection depends on the maximum deflection point. The addition of higher stiffness materials (like inclusions) does not decrease the maximum deflection of FG plates before the maximum deflection point.

• The location of the maximum deflection does not depend on the FG plate thickness. The maximum deflection increases with an increase in the power-law index ($n$).

• Regardless of the applied boundary conditions, the thickness to length ratios ($\tau$) and elastic modulus ratios, the eigen-frequency parameter ($\beta$) has the same minimum values around $n=5$ for the both FG I and FG II plates with both clamped and simply supported boundary conditions.
4 Chapter 4: Damage modelling within an FG plate: material model and numerical implementation

Some of the work presented in this chapter has been published in the following:


4.1 Introduction

The main goal of this chapter is to develop and implement the irreversible thermodynamics-based approach for the modelling of damage mechanisms in FG plates with in-plane material properties variation. This includes implementation of damage models along with associated characteristic material parameters in conjunction with a set of elasto-plastic constitutive relations.

This chapter extends the Ju damage model [33, 77] which is a modification of Lemaitre’s damage model [78] for which the same multipliers for plasticity and damage are used, representing a strong limitation of the Lemaitre’s damage modelling. In contrast, the Ju model treats the evolution of plastic and damage effects in an independent manner. The novelty of the presented damage model is capable of describing the coupled elasto-plastic damage response for FG plates with variation of a damage variable throughout the plane of the plate. The model utilises a simple power law function to describe an FG plate as continua with smooth variation of material properties, such as Young’s Modulus, yield stress, plastic material constants and damage parameters. By introducing two independent plastic and damage multipliers, the model is applicable to different types of materials. Employing the operator splitting methodology, a three-step predictor/multi-corrector algorithm is developed that includes an elastic predictor, a plastic corrector, and a damage corrector. Following presentation of the analytical model, the damage finite element (FE) solutions are obtained using linear hexahedral solid elements with
spatially graded property distribution (at different Gauss points), and subsequently implemented via a user material subroutine (UMAT) in the ABAQUS FE software.

4.2 Literature review of Continuum Damage Modelling (CDM)

There are two different approaches in order to predict damage in a specific structure namely Cohesive Zone Modelling (CZM) and Continuum Damage Mechanics (CDM). One may notice that one of the major limitations for CZM approach is that the crack propagation path needs to be known a priori to lay specialized cohesive elements within that path to simulate the fracture [79]. In addition, the CZM material parameters suffer from mesh dependency in computational modelling. Continuum Damage Mechanics (CDM) is a field of continuum mechanics which expresses the damage and fracture process from the initiation and propagation of micro voids and micro cracks that produce macro defects and cracks, leading to the final fracture of materials [80].

The theory of CDM is based on continuous displacement fields. In other words, damage is an irreversible progressive process of accumulation of microstructural changes leading to degradation of the material stiffness without disrupting the continuity of the displacement field. Kachanov [81] has pioneered phenomenological approaches using a continuous internal state variable to representing the microstructural defects in a material for creep rupture. He introduced a one-dimensional surface damage variable that expressed a very simple measure of the damage amplitude in a given plane, which is obtained by measuring the area of the intersection of all defects with that plane. As shown in Figure 4.1, \( S_D \) represents the area of defects, while the effective area of the sample subjected to uniaxial tension is given by \( S - S_D \). The following positive damage scalar \( D \) is commonly considered as a damage variable:

\[
D = \frac{S_D}{S} , \quad 0 \leq D \leq 1
\]  (4.1)
The effective stress concept which related to the surface that effectively resists the load was stated by Rabotnov[82]. He extended Kachanov’s work by introducing a rate equation for void growth in the context of creep.

Depending on the mechanical properties of the constituent materials, structures can undergo different damage mechanisms. Several approaches to model elasto-plastic damage in anisotropic materials [83-86] and elastic–brittle damage in directionally oriented materials [87-89] have been proposed in the literature. Kuhl et. al [83] studied the anisotropic damage model including a dependence upon the second gradient of strains. Their model had the capabilities to simulate mode I as well as mixed mode failure up to the complete loss of load carrying capacity. The development of an energy-based anisotropic damage model at finite strain for ductile fracture was presented by Zhu and Cescotti [90]. They considered anisotropic elasticity, anisotropic plasticity and anisotropic damage. Their work was based on the hypothesis of damage energy equivalence and provides a good physical representation of damage evolution.

FGMs can undergo elastic–brittle damage and elasto-plastic damage [91-94]. An FGM microstructure is inherently stochastic and can, in fact, be viewed as a statistically
inhomogeneous random field. Therefore, due to the stochastic nature of FGM microstructures, the growth of crack paths can be unpredictable [95, 96]. Therefore, various continuum based theories have been developed to model damage in different materials, such as isotropic materials [33, 34, 77, 97, 98], elasto-plastic damage in anisotropic materials [35, 99-103] and elastic–brittle damage in oriented materials [87, 88]. Damage in FGMs has been studied by several researchers numerically [104-108] and experimentally [109-112] in recent years. The elasto-plastic damage behaviour of metallic FGMs under thermal loading has been studied by FE analysis by Lee et al. [106]. They considered Lemaitre’s damage model for damage evolution. Egner et al. [107] developed a thermo elasto-plastic damage model based on Bielski’s work [113] to investigate a three-layer TBC/FGM behaviour under thermo-mechanical loading numerically. The material was manufactured by means of Plasma Spray Thermal Barrier Coating (PS-TBC). They considered two independent plastic and damage multipliers and damage dissipation potential is dependant on damage as well as on plastic deformation.

Recently, Zivelonghi et al. simulated the ductile failure based on CDM by means of ABAQUS software to understand the deformation and fracture behaviour of the particulate FGM on the basis of its microstructure [108]. They considered the elasto-plastic response of damaged metal in terms of the effective stress and a scalar damage variable.

With regards crack propagation and fatigue in FGM, several studies were carried out to investigate crack propagation. The summaries of highlighted research are as follows:

Chen et al. studied the transient internal crack problem for a functionally graded orthotropic strip analytically [114]. One of the limitations of their work was considering only internal (embedded) cracks. An FE solution for fracture analysis of orthotropic functionally graded materials with arbitrary cracks configurations was conducted by Kim et al. [115]. They evaluated Stress Intensity Factors (SIFs) for mode I and mixed-mode two-dimensional problems. An experimental investigation of fatigue in an alumina/epoxy FGM was conducted by Tilbrook et al. [109]. They monitored crack propagation under both monotonic and cyclic loading, with crack trajectories and growth rates both being recorded. Finite element predictions of the anticipated crack path generated by ANSYS for linear elastic FGM were found to be quite accurate. Recently, Gunes et al. studied the elasto-plastic response of Al/SiC functionally graded material under low-velocity impact loading numerically [19]. They used The Tamura–Tomota–Ozowa (TTO) model to determine the elasto-plastic response of FG circular plates. Moreover, with increasing the compositional gradient exponent, the increase of maximum contact force and decrease of contact duration were concluded. Furthermore, Guo et al. developed an analytical fracture mechanics model based on stochastic uncertainties in phase volume fractions to predict the stress intensity factors and crack propagation [116]. More recently, Hu et al. investigated the crack propagation and failure
mechanics of a magneto-electro-elastic FGM under anti-plane shear and in-plane electric and magnetic loading by the integral transform method [117]. They showed that the SIFs change with the velocity of the moving crack and are dependent on the material properties and geometric size of the composite structure. Another numerical survey of FGM fracture was carried out by Paneda et al. [118]. They implemented the computational analysis quasi-static crack initiation and propagation solution in FGM and showed that their user subroutine code is more suitable for perpendicular crack configuration than other codes presented in the literature. It should be mentioned that the main goal of this chapter is a continuum damage mechanism focusing on coupled elasto-plastic damage. Therefore, crack propagation, fatigue and failure can be considered as a different scope of research.

In applications reported in the mentioned literature, a damage variable has been considered as a constant matrix, even for FGMs. No consideration has been given to the variation of damage parameters as a function of position. Therefore, the development and implementation of damage models for elasto-plastic constitutive materials as a function of in-plane variation of material parameters (e.g., with plate length \( x \)) is the main contribution of this chapter.

The model presented here is capable of describing the coupled elasto-plastic damage response for FG plates with variation of a damage variable throughout the plate length. Damage and plasticity as two dissipation phenomena are incorporated into the model by two coupled dissipation potentials. Moreover, the influence of plastic strain on a continuous micro crack is considered in the proposed model by introducing an additional plastic part in the damage conjugated force. The computational algorithm for the implementation of the coupled elasto-plastic damage is included in a three-step algorithm: elastic predictor, plastic corrector and a damage corrector [99, 103, 119-121]. The plastic corrector step is in the case of the simple von Mises type plasticity usually solved using the return mapping algorithm. This is followed by an additional damage corrector step, which is usually performed in a single iteration step. In the presented numerical modelling using the finite element program ABAQUS, linear hexahedral solid elements with spatially graded property distribution (at different Gauss points) are implemented in a user material subroutine (UMAT) to model material stiffness (Young’s modulus) and damage variable variation in-plane [74]. The utility of the proposed model is that it can be extended to functionally graded materials with different inclusions and matrices, i.e. with different types of property variations.

4.3 Elasto-plastic damage model for FGMs

It is important to note that the damage in such materials is correlated to the history of both elastic and plastic deformation (i.e. state variables) and varies continuously through
the length of the plate. The stress-strain behaviour is affected by the development of micro and macro cracks in the material body. In the polymeric 3D printed FG plates, the initial microcracks may be caused by applied manufacturing process. Under applied loading, further micro-cracking may occur. These microcracks which are initially small (effectively invisible), will eventually lead to visible cracks that extend as the applied external loads are increased. These cracks contribute to the generally observed nonlinear stress-strain behaviour. Since a phenomenological continuum approach for the FG plates is followed in this chapter, these effects are smeared out (i.e. averaged) throughout the plate where the material is considered as a mechanical continuum with degraded (damaged) properties [122].

A set of effective state variables \((\bar{\sigma}, \bar{\varepsilon})\) that represent an undamaged state of material is introduced. From a phenomenological point of view, the stress field \(\sigma\) is related to the effective stress tensor \(\bar{\sigma}\) via the damage variable \(D\) as follows:

\[
\bar{\sigma} = \frac{\sigma}{1-D(x)}
\]  

\(D\) is a fourth order tensor that characterizes the state of damage. Although damage can be considered as an anisotropic phenomenon, the isotropic damage assumption in which the mechanical behaviour of micro cracks is independent of their orientation is often sufficient to render an acceptably accurate prediction in engineering materials [123, 124]. Therefore the damage variable can be considered as a scalar variable \((D)\) to model elasto-plastic damage of FGMs.

In the model of Ju [34], the constitutive relations are derived from a thermodynamic potential, and the Helmholtz free energy \(\psi\), which is decomposed into an elastic \(\psi_e\) and a plastic \(\psi_p\) part. The plastic part also includes the damage variable.

\[
\psi(e^*, D, r) = \psi_e(e^*, D(x)) + \psi_p(r, D(x)) = \frac{1}{2}(1-D(x))e_c : C^0(x) : e_c + (1-D(x))\psi_p(r)
\]  

\(\psi_p(r)\) in Equation (4.3) is part of the Helmholtz Free Energy related to plastic deformation, and it represents the effect of isotropic plastic hardening. Hence, we assume [107, 125]

\[
\psi_p(r) = R_c(x) \times \left( r + \frac{\exp(-r \times b(x))}{b(x)} \right)
\]
where \( R_\omega (x) \) and \( b(x) \) are plastic material constants that change according to the position \((x)\). The Clausius–Duhamel inequality for the pure mechanical theory has the following form for any admissible process:

\[-\Psi + \sigma : \dot{\varepsilon} \geq 0 \tag{4.5}\]

By substituting Equation (4.3) into the Clausius–Duhamel inequality, one obtains the following thermodynamic constraint:

\[(\sigma - \frac{\partial \Psi}{\partial \varepsilon_e}) : \dot{\varepsilon} + \sigma : \dot{\varepsilon}_p - \rho \frac{\partial \Psi}{\partial D} \dot{D} - \rho \frac{\partial \Psi}{\partial r} \dot{r} \geq 0 \tag{4.6}\]

From Equation (4.5), the thermodynamic force \( Y \) associated with damage, the Cauchy stress and driving force \( R \) are as follows:

\[Y = -\frac{\partial \Psi}{\partial D} = \frac{1}{2} \varepsilon^\varepsilon : C^0(x) : \varepsilon + \Psi_p (r) \tag{4.7}\]

\[\sigma (x) = \frac{\partial \Psi}{\partial \varepsilon_e} = (1 - D(x))C^0(x) : \varepsilon \tag{4.8}\]

\[R (x) = -\frac{\partial \Psi}{\partial r} = -(1 - D(x)) \frac{\partial \Psi_p (r)}{\partial r} \tag{4.9}\]

The evolution laws of the internal variables are derived from two independent dissipation potentials \( F_p \) and \( F_d \) and two independent Lagrange multipliers \( \dot{\lambda}_p \) and \( \dot{\lambda}_d \) for plasticity and damage, respectively.

\[\Phi = \Phi (\sigma, Y, R, \varepsilon_e, D, r) = F_p (\sigma, R, D) + F_d (Y, D) \tag{4.10}\]

The increment of the internal state variables associated with plastic deformation and damage is obtained by utilizing the calculus of functions of several variables with the Lagrange multipliers as follows [126]:

\[\dot{\varepsilon}_p = \dot{\lambda}_p \frac{\partial \Phi}{\partial \sigma} = \dot{\lambda}_p \frac{\partial F_p}{\partial \sigma} \tag{4.11}\]

\[\dot{\varepsilon} = -\dot{\lambda}_p \frac{\partial \Phi}{\partial R} = -\dot{\lambda}_p \frac{\partial F_p}{\partial R} \tag{4.12}\]

\[\dot{D} = \dot{\lambda}_d \frac{\partial \Phi}{\partial Y} = \dot{\lambda}_d \frac{\partial F_d}{\partial Y} \tag{4.13}\]

Using the same multiplier for damage and plasticity, as for the Lemaitre damage model [78], the evolution of both internal variables is strongly coupled with each other. In the Lemaitre damage model, it is assumed that the damage variable is governed by the plastic
strain and can only increase during plastic flow. This fact represents a significant limitation of this model, especially for the modelling of materials which show brittle damage behaviour at low strain states. The evolution of the plastic flow is determined by a von Mises type yield function evaluated in the effective stress space as:

\[ F_p(\sigma, R, D) = f_p(\sigma, R, D(x)) = \bar{\sigma}_{eq} - \sigma_y(x) - \bar{R}(x) \] (4.14)

where \( \sigma_y(x) \) is the initial size of the yield surface that varies continuously through the length of the FG plate. The effective isotropic plastic hardening \( \bar{R}(x) \) can be obtained from Equations (4.4) and (4.9) as:

\[ \bar{R}(x) = R_y(x) \times \left(1 + \exp(-r \times b(x))\right) \] (4.15)

The damage criterion relationship \( f_d \) is characterized by a damage threshold in terms of a current energy barrier that limits the conjugated force \( Y \), which has the following form:

\[ f_d = Y(t) - Y_{threshold} \leq 0 \begin{cases} f_d < 0 & \text{Elastic} \\ f_d = 0 & \text{Damage} \end{cases} \] (4.16)

where \( Y_{threshold} \) is the initial damage threshold. The damage multiplier is assumed to obey the following Kuhn–Tucker conditions [35]:

\[ f_d \leq 0 \quad and \quad \begin{cases} f_d < 0 \quad and \quad \dot{\lambda}_d = 0 \quad \text{Undamaged state} \\ f_d = 0 \quad and \quad \dot{\lambda}_d = 0 \quad \text{Damage initiation} \\ f_d = 0 \quad and \quad \dot{\lambda}_d > 0 \quad \text{Damage growth} \end{cases} \] (4.17)

The consistency condition for damage, \( \dot{f}_d = 0 \), can be written as follows:

\[ \dot{f}_d = \frac{\partial f_d}{\partial Y} \dot{Y} = 0 \] (4.18)

Making use of Equations (4.4), (4.7) and (4.15) along with the chain rule, it can be shown that the consistency Equation (4.19) gives the following equation for \( \dot{\lambda}_p \):

\[ \dot{\lambda}_p = \left( \frac{\partial f_d}{\partial Y} \times \frac{\partial Y}{\partial \varepsilon_e} \dot{\varepsilon} \right) \left( 1 - D \times \frac{\partial f_d}{\partial \varepsilon_e} \times \frac{\partial Y}{\partial \varepsilon_e} \times \frac{\partial f_d}{\partial \sigma} \right) \] (4.19)

On the other hand, \( \bar{\sigma} \) and \( \bar{R} \) are functions of \( D \) and their corresponding nominal parts \( \sigma \) and \( R \). In order to find the damage multiplier \( \dot{\lambda}_d \), the corresponding consistency condition \( \dot{f}_p = 0 \) can be written as follows:
\[
\dot{f}_p = \frac{\partial f_{p\sigma}}{\partial \sigma} \dot{\sigma} + \frac{\partial f_{pR}}{\partial R} \dot{R} + \frac{\partial f_{pD}}{\partial D} \dot{D} = 0
\]  
(4.20)

Using Equations (4.9) and (4.12-4.14) along with the chain rule, it can be shown that the consistency Equation (4.20) gives the following equation for \( \dot{\lambda}_D \):

\[
\dot{\lambda}_D = \frac{\frac{\partial f_{p\sigma}}{\partial \sigma} - \left( \frac{\partial f_{pR}}{\partial R} \times \frac{\partial R}{\partial \lambda} \times \dot{\lambda}_p \right)}{\left( \frac{\partial f_{pR}}{\partial R} \times \frac{\partial F_d}{\partial Y} + \frac{\partial f_{pD}}{\partial D} \times \frac{\partial F_d}{\partial Y} \right)}
\]  
(4.21)

Simo and Ju [33] and Ju [34] proposed to define the damage evolution function \( D \) as an exponential function of strain energy \( Y \) as follows:

\[
D(x) = 1 - (1 - \alpha(x)) \frac{Y_{\text{threshold}}}{Y} - \alpha(x) \left[ \exp \left( -\beta(x) \times (Y - Y_{\text{threshold}}) \right) \right]
\]  
(4.22)

where \( \alpha(x) \) and \( \beta(x) \) are characteristic parameters of FGMs that change according to the experimental material characteristic results following a function of \( x \).

### 4.4 Numerical integration of the constituent model

The development of a computational algorithm that is consistent with the proposed theoretical formulation is given in detail in this section, to facilitate the numerical integration of the constitutive equations in the context of the FE method. The behaviour of elasto-plastic damage should be considered as a strain driven problem where stress history is obtained from the strain history by means of an integration algorithm [127].

The aim of this section is to update the set of variables \( \{ \sigma_n, \varepsilon_n, D_n, r_n \} \) at time \( t_n \) to the corresponding values at time \( t_{n+1} \). According to the operator split concept in [128], Equation (4.8) can be decomposed into elastic, plastic and damage parts, leading to the corresponding numerical algorithm incorporating elastic-predictor, plastic-corrector and damage-corrector steps [103, 129]. In the elastic-predictor step, the problem is solved with the initial conditions while keeping the irreversible variables unaltered. This step is known as a trial stress state, \( \sigma'' = \sigma'' + \Delta \sigma'' \). The second and third steps are defined to restore the generalized plasticity and damage consistency conditions by returning back the trial stress to the plastic surface and the damage surfaces. A fully Implicit (Backward-Euler) scheme is used for the stress computation problem in the effective space, followed by an explicit integration scheme for the updated damage variables and Cauchy stress tensor. The details of a three-part algorithm are explained as follows:
4.4.1 Step I: Elastic predictor

The elastic predictor problem establishes the stress in the trial state assuming an entirely elastic strain increment as follows:

\[
\sigma_{\text{trial}}^{n+1} = (1 - D_n) C^0(x) \left( \epsilon_{p}^{n+1} - \epsilon_p^n \right) = (1 - D_n) C^0(x) \left( \epsilon_p^n + \Delta \epsilon - \epsilon_p^n \right) \\
= (1 - D_n) C^0(x) \left( \epsilon_p^n + \Delta \epsilon \right)
\]

(4.23)

\[
\bar{\sigma}_{\text{trial}}^{n+1} = C^0(x) \left( \epsilon_p^n + \Delta \epsilon \right)
\]

(4.24)

\[
D_{\text{trial}}^{n+1} = D^n ; \quad \bar{r}_{\text{trial}}^{n+1} = \bar{r}^n
\]

(4.25)

In order to verify the correctness of this elastic prediction, the trial stress is applied to the plastic yield function \( f_p \) (Equation (4.15)). If \( f_p(\bar{\sigma}_{\text{trial}}^{n+1}, D^n, \bar{r}^n) < 0 \), the process is elastic and the trial state is admissible. This should be considered as the final state, \( \bar{\sigma}^{n+1} = \bar{\sigma}_{\text{trial}}^{n+1} \) and then the damage corrector step (step III) should be applied. On the other hand, if \( f_p(\bar{\sigma}_{\text{trial}}^{n+1}, D^n, \bar{r}^n) > 0 \), the following plastic corrector step has to be applied.

4.4.2 Step II: Plastic corrector

If the current step is not an elastic state, i.e., \( f_p(\bar{\sigma}_{\text{trial}}^{n+1}, D^n, \bar{r}^n) > 0 \), the plastic strain tensor will change. Then consistency is achieved by the return-mapping algorithm, the plastic consistency multiplier \( \lambda^{n+1} \) being obtained via a local Newton Raphson iteration of the following nonlinear equations:

\[
\Delta \lambda_p = \frac{\sqrt{3/2} \left\| \sigma_{\text{dev}}^{n+1} \right\| - \sigma_y(x) - \bar{R}_{\text{trial}}^{n+1}(x)}{3G + \frac{\partial f_p}{\partial r_{\text{trial}}}}
\]

(4.26)

\[
\bar{\sigma}^{n+1} = \bar{\sigma}_{\text{trial}}^{n+1} - \Delta \lambda_p \left( \sqrt{6G} \sigma_{\text{dev}}^{n+1} \right)
\]

(4.27)

\[
\bar{\sigma}_{p}^{n+1} = \bar{\sigma}_{p}^n + \Delta \lambda_p \left( \sqrt{3/2} \left\| \sigma_{\text{dev}}^{n} \right\| \right)
\]

(4.28)

\[
\bar{r}^{n+1} = \bar{r}^n + \Delta \lambda_p
\]

(4.29)
Once the plastic consistency condition is enforced, an effective configuration of the variables \( \{ \sigma_{\text{eff}}^{n+1}, \epsilon_p^{n+1}, D_n, \bar{r}^{n+1} \} \) is obtained. Upon convergence, the updated effective stress \( \sigma^{n+1} \) can be used in the damage-corrector step in order to update the damage variable \( D^{n+1} \) and the Cauchy stress tensor \( \sigma^{n+1} \). This is the reason why local iterations are required to obtain the plastic multiplier \( \Delta \lambda_p \). In each iteration the new evaluated trial stress will be used to update the plastic variables and the hardening functions in order to obtain an updated plastic multiplier. The process goes on until a convergence tolerance for the yield function is satisfied.

### 4.4.3 Step III: Damage corrector

In order to update the damage variable \( D^{n+1} \), the damage energy release rate needs to be compared with the damage threshold. If the former exceeds the later, the damage variable is updated. If \( f_d = Y_{n+1}(t) - Y_{\text{threshold}} < 0 \) the damage-corrector step ends and the Cauchy stress tensor takes the form of the updated effective stress tensor. If the damage criterion is activated, \( f_d \geq 0 \), damage evolution takes place.

\[
D^{n+1} = 1 - \left(1 - \alpha(x)\right) \frac{Y^n_{\text{threshold}}}{Y^{n+1}} - \alpha(x) \left[ \exp \left(-\beta(x) \times (Y^{n+1} - Y^n_{\text{threshold}})\right) \right] \quad (4.30)
\]

\[
Y^{n+1} = \frac{\left(\sigma^{n+1}_{\text{dev}}\right)^2}{6G} + \frac{\left(\sigma^{n+1}_{\text{hyd}}\right)^2}{2K} + R_n(x)(\bar{r}^{n+1} + \frac{1}{b(x)} \exp(-b(x)\bar{r}^{n+1})) \quad (4.31)
\]

\[
Y^n_{\text{threshold}} = \max \left\{ Y^n_{\text{threshold}}, Y^{n+1} \right\} \quad (4.32)
\]

\[
\sigma^{n+1} = (1 - D^{n+1}) \bar{\sigma}^{n+1} \quad (4.33)
\]

This concludes the damage-corrector step. Figure 4.3 shows an entire step of the integration scheme demonstrating the effective elastic-predictor and plastic corrector steps followed by the damage corrector step.
Figure 4.3: Integration scheme flowchart

Elastic Predictor

Check for yielding

Plastic return mapping

Initial guess for $\Delta \lambda^p$

Using Newton method for finding the exact value $\lambda^p = \frac{\Pi^u}{F}$

Update:

Check for damage

Damage corrector

Update $d_{n+1}$

Update

$\sigma_{n+1} = (1-d_{n+1})\bar{\sigma}_{n+1}$

$g(Y_{n+1}, d_n) > 0$

$\epsilon_{v+1}^{trial} = \epsilon_n^r + \Delta \epsilon = \epsilon_n^r + \Delta \epsilon$

$r_{n+1}^{trial} = r_n$

d_{n+1}^{trial} = d_n$

$s_{n+1}^{trial} = 2G \epsilon_{v+1}^{trial}$

$\sigma_{v+1}^{trial} = \frac{3}{2} \frac{\epsilon_{v+1}^{trial}}{\sigma_{v+1}^{trial}}$

$f(\bar{\sigma}_{n+1}^{trial}) > 0$

yes

no

$\sigma_{n+1}^{trial} = \sigma_{v+1}^{trial} - \sigma_{v+1}^p - R_{v+1}^{trial} (1-d_n)$
4.5 Verification of the proposed model

This section presents several numerical examples to demonstrate the capability, applicability and effectiveness of the proposed elastic-plastic-damage model. The numerical algorithm of the proposed model for FGMs was implemented in the non-linear FE code ABAQUS via a user material subroutine (UMAT). It should be noted that linear hexahedral (C3D8R) graded elements were used to implement an effective continuous variation of material properties through the length of a plate. This type of element is implemented by means of direct sampling of properties at the integration (Gauss) points of an element. In the ABAQUS software, the gradient of properties can be specified via a UMAT, since it is called at the integration points. The Young’s modulus, initial yield stress, isotropic hardening and damage variables of the FG plate were defined as a function of length for each integration point throughout the length ($x$) of the specimen [174], and can be defined from experimental tests (refer Chapter 2). The FORTRAN code for the integration scheme of damage modelling is shown in Appendix F.

The first presented example demonstrates the elasto-plastic response of an isotropic material under uniaxial displacement control. The second and the third examples represent the elastic-damage in isotropic and composite materials, such as clay/epoxy nanocomposites. The model predictions are compared with other numerical and experimental results from the literature [35, 130]. A fourth example demonstrates elasto-plastic damage of high strength steel under a force controlled tension test. Then the numerical results of the present model are compared with numerical results of Hesebeck with the same geometry and boundary conditions [125].

4.5.1 Example 1: Elasto-plastic response of a steel plate

A steel plate with a length and width of 200 mm and a thickness of 20 mm, respectively is considered as an example. The right edge is subjected to an axial displacement control of 0.01 m applied through different time increments. Material properties are: initial yield stress $\sigma_y = 414$ MPa, Young’s modulus $E = 200$ GPa, Poisson ratio $\nu = 0.3$ and plastic material constants $R_\infty = 509$ MPa and $b = 0.93$ [35]. Figure 4.4 shows the axial stress against axial strain curve for the plate, with the results from the present model being compared with those of Taqieddin [122] with the same plate geometry, material properties and boundary conditions. Using the chosen material parameters, the strain at yield is equal to $\varepsilon_{\text{yield}}=0.00207$. Good agreement between the two solutions can be observed for the uniaxial stress variation with respect to strain, with a maximum deviation less than 5.47% after the initiation of yielding and plasticity. This deviation could be due to the difference between non-linear and linear plasticity models which are used in the present model and Taqieddin [122], respectively.
4.5.2 Example 2: Elastic damage of Silicon Carbide (SiC) plate

The second example studies the damage behaviour of a Silicon Carbide (SiC) plate under tensile axial loading. The dimensions of the plate are $266.7 \, \text{mm} \times 88.9 \, \text{mm} \times 5 \, \text{mm}$. The material and geometric properties of the plate are: Young’s modulus $E = 400$ GPa, Poisson ratio $\nu = 0.22$ and initial damage threshold $Y_{\text{threshold}} = 9 \, \text{MPa}$ [103]. An axial displacement increment of $3 \, \text{mm}$ (or $1.1\%$ strain) is applied at the edge of the plate [35]. Figure 4.5 shows the axial stress against axial strain curve for the plate and the results from the present model are compared with work from previous research [103, 131] utilising the same plate geometry, material properties and boundary conditions. It can be observed from Figure 4.5 that less damage is obtained for the present model than for that of Voyiadjis et al. [131]. This is because the description of damage in ref. [131] is through the fourth-order operator tensor, while a scalar variable for damage is assumed for the present model. As it can be seen from the Figure 4.5, the proposed model provides quite a close result with ref. [60]. After the initiation of damage at the damage threshold point, the deviation between the two solutions increased, but the maximum error is less than $6\%$. 

Figure 4.4: Uniaxial stress variation with strain for the steel plate and its comparison with the results of Taqieddin [122]
4.5.3 Example 3: Elastic damage of clay/epoxy nanocomposite

In this example, the proposed model is validated by numerical and experimental testing of a dog-bone shaped sample (based on ASTM D638) made of 2% clay/epoxy nanocomposite [130, 132]. The material properties are: Young’s modulus $E = 2.978$ GPa, Poisson’s ratio $\nu = 0.35$, $Y_{\text{threshold}} = 6.8$ MPa and damage characteristic parameters $\alpha = 0.9389$ and $\beta = 7.349$ [132]. The damage versus strain curve for the sample in the gage section is shown in Figure 4.6 and compares with the results of Silani et al. [130]. The numerical simulation presented in ref. [130] was conducted using a displacement control method and due to symmetry, only a quarter of the specimen was simulated. The exponential formulation (Equation (4.22)) for scalar damage variable is used for both the present model and that presented in ref. [130]. As can be observed from Figure 4.6, a fairly good agreement is obtained between two solutions, with the maximum deviation being less than 5.84%. It could be due to the fact that Silani et al. used the effective strain and the initial damage threshold strain in Equation (4.22) instead of strain energy $Y$, as less damage is obtained for Silani et al. [130].
Figure 4.6: Comparison of the present model with that of Silani et al. [130]: Damage versus strain curve for 2% clay/epoxy nanocomposite sample.

Figure 4.7 shows the damage distribution in a dog-bone sample after 1.6% strain for both the present model and the numerical results of Silani et al. [130]. Two contours show very good agreement with the maximum damage value of about 0.16 which occur within the initial curved part of the specimen.

Figure 4.7. Damage distribution for 2% clay/epoxy nanocomposite sample: (a) XFEM simulation [130] (b) Present numerical simulation.

The stress-strain curve of the 2% clay/epoxy nanocomposite dog-bone sample is presented in Figure 4.8 for the proposed model, and the results are compared with an XFEM simulation and the experimental results obtained by Silani et al. [132]. The
Lemaitre damage model is implemented and used for modelling damage using the eXtended Finite Element Method (XFEM), and the experimental results were obtained from a simple tensile test according to ASTM D638 [132].

A fairly good agreement exists between the present model and the experimental and numerical results from Silani et al. [132]. At higher strain the present model matches better with the experimental results. This is due to the fact that Silani et al. used the effective strain concept and Lemaitre damage model in their numerical simulation instead of the strain energy concept, and as can be observed from Figure 4.6 and was mentioned in Section 4.3, this leads to lower damage levels within the specimen. It is also noticeable that the present model predicts slightly higher stresses than the experimental results, probably because of particle-matrix debonding that can occur during experimentation, which is not accounted for in the present model. The maximum stress error is approximately 10.07%, when comparing the present model and the numerical XFEM simulation.

![Figure 4.8: Comparison of the stress–strain curve for 2% clay/epoxy nanocomposite sample obtained using the present model with the experimental and numerical results from [132]](image)

### 4.5.4 Example 4: Elasto-plastic damage of high strength steel plate

The experimental results of Hesebeck [125] for high strength steel (30CrNiMo8) are numerically simulated using the proposed model. In the mechanical experiments, force controlled tension tests with partial unloading were performed at a stress rate of \( \dot{\sigma} = 30 \text{MPa/s} \). The material properties of the plate are: Young’s modulus \( E = 199 \text{ GPa} \),
Poisson’s ratio $\nu = 0.3$, $Y_{\text{threshold}} = 3.8$ MPa, initial yield stress $\sigma_y = 870$ MPa and material hardening parameters $R_\infty = 509$ MPa and $b = 29.8$ [35].

The resulting stress versus plastic strain curve obtained by the proposed model and that by Hesebeck [125] are plotted in Figure 4.9. As can be seen from Figure 4.9, a good agreement is obtained between the proposed numerical solution and the experimental results from ref [125]. The maximum deviation between the two results is approximately 1.56%.

![Stress–plastic strain curve for damaged high strength steel plate](image)

Figure 4.9: Stress–plastic strain curve for damaged high strength steel plate as compared to the experimental data in [125]

**4.5.5 Example 5: Cyclic elastic plastic response of VW_8530 composite plate**

In this example, the proposed model is compared with the experimental results obtained from Section 2.3.2 for a VW_8530 composite plate made of 80% of inclusion (VW). Figure 4.10 shows a 5-cycle increasing load which is imported as a ramp in the FE modelling.
In order to simulate the boundary conditions which were applied by mechanical wedge grips, the ENCASTRE condition was applied to the top edge (around 10 mm of the plate length) in the FE modelling. All displacement and rotation excluding displacement in the loading direction ($x$ direction) are defined zero at the bottom edge. Data presented in Figure 4.10 is imported to be applied as a periodic displacement in the $x$ direction (loading direction) at the bottom edge of the plate. All boundary condition in the FE analysis are shown in Figure 4.11.
Figure 4.12 shows the tensile stress contours at the end of the loading path at the fifth cycle for the VW_8530 composite plate.

It can be seen from Figure 4.12 that there is a stress concentration at the interface of the clamped edge and the plate due to applied boundary condition at 10 mm offset from the edges, as applied to approximate boundary conditions in the experimental tests.

Figure 4.13 shows the axial tensile stress against strain for a five-cycle periodic load for an VW_8530_80% plate, with the results from the present model being compared with the experimental results presented in Section 2.3.2.
It can be observed from Figure 4.13 that lower tensile stresses at the fourth and fifth cycles is obtained for the present model than for that of experimental results. This is likely because of the description of boundary condition and applied loading using mechanical grips, applied in the FE analysis. Moreover, the proposed continuum damage model is not able to consider viscous-plasticity behaviour of 3D printed digital material which occurs during experimentation and leads obtained lower stress at FE analysis.

### 4.5.6 Elasto-plastic damage of 3D-printed polymeric FG plates

The previous examples have verified the accuracy of the proposed model when applied to different materials utilising an isotropic damage parameter. In this numerical example, the elasto-plastic damage of a 3D-printed functionally graded plate consisting of VW (hard polymer) as an inclusion and 8530 (soft polymer) as a matrix with in-plane material properties variation is evaluated. Table 4.1 shows material properties for the VW and 8530 digital polymers. These material properties were obtained from the simple and cyclic tensile tests presented in Chapter 2.

<table>
<thead>
<tr>
<th>Inclusion- VW</th>
<th>Matrix-8530</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Young’s modulus</strong></td>
<td>$E_i= 2420 \text{ GPa}$</td>
</tr>
<tr>
<td><strong>Poisson’s ratio</strong></td>
<td>$\nu = 0.4$</td>
</tr>
<tr>
<td><strong>Initial yield stress</strong></td>
<td>$\sigma_{y0,i}= 34.3 \text{ MPa}$</td>
</tr>
<tr>
<td><strong>Material hardening parameters</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\alpha= 1.0531$</td>
</tr>
<tr>
<td><strong>Material damage parameters</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Plate dimensions (length, wide, height)</strong></td>
<td></td>
</tr>
</tbody>
</table>

As explained in Chapter 2, material properties like Young’s modulus, initial yield stress and damage parameters in the damage evolution function (Equation (4.2.22)) follow the specific function of volume fraction of hard polymer VW in equations (2.3), (2.4), (2.8) and (2.9). Moreover, linear hexahedral (C3D8R) graded finite elements were used in ABAQUS to implement an effective continuous variation of material properties. This
type of element is implemented by means of direct sampling of properties at the integration (Gauss) points of the element. The UMAT function coded for modelling an FG plate with graded solid elements has been linked with another UMAT routine that implements the proposed elasto-plastic damage model (see Appendix E and F). Figure 4.14 shows a schematic representation of the geometry and volume fraction distribution through the component length.

Figure 4.14: Schematic representation of the geometry (left) and distribution of volume fraction along x direction of the FG plate (right)

The emphasis of this work is to better understand the stress-strain behaviour of materials with in-plane material property variations. It should be noted that the damage threshold of the original matrix and inclusion materials corresponds to a value of strain at first cycle in the experimental cyclic tensile test presented in Chapter 2. An axial displacement increment of 0.004 m (or 2% strain) is applied at the right side (at x=a) of the plate. Figure 4.15 shows the boundary condition and axial displacement applied to the FG plate in ABAQUS.

Figure 4.15: Applied boundary condition and axial displacement of the FG plate in ABAQUS
In order to achieve independency of results from mesh size for a functionally graded plate, a convergence study was conducted that yielded a stable and accurate solution while keeping the computational time to a minimum. So, a convergence study with respect to the mesh size is first performed to ascertain the level of finite element refinement necessary to obtain accurate results. The maximum damage parameter (D) was measured in each simulation, by increasing the number of elements. The result is provided in Figure 4.16. It should be noted that the non-dimensional parameter mesh size $\kappa$ represents the ratio of mesh size to the plate thickness.

Figure 4.16: Mesh convergence on FGI plate for damage parameter

Figure 4.17 presents the stress-strain curves at the centre plane of the FG plate for both the VW phase dominant (at x=a) side and 8530 phase dominant (at x=0) side, respectively. Both the inclusion and matrix phase dominated sides (due to their brittle nature) exhibit elastic brittle damage. The failure strain for the inclusion phase dominant (at x=a) side is 0.0134, which is lower than the value of 0.02 for the matrix phase dominant (at x=0) side. In other words, the VW_8530 FG plate under uniaxial tensile loading shows more brittle damage behaviour with lower strain at the VW phase dominant side (at x=a) and brittle damage behaviour with higher strain at the other side (x=0). It can be observed from Figure 4.17 that at the same strain value, the VW dominant side experienced higher stress values due a to lower damage parameter ($D$). The contour of damage parameter variation for FG plate is shown in Figure 4.18.
Figure 4.17: Stress–strain curves for the VW and 8530 phase dominant side of FG plate

Figure 4.18: Contour of damage parameter (D) through the length of FG plate

The variations of the damage variable $D(x)$ along the length direction for the VW_8530 FG plate and homogenous plates (VW and 8530 materials) after applying 2% strain at the right side (at x=a) of the plate are shown in Figure 4.19. The homogenous plates have the same geometry, applied boundary conditions and matrix and inclusion and material properties. As can be observed from Figure 4.19, the damage variable is constant at the final state (after 2% strain) for the homogenous plates. This is compatible with the fact that for the homogenous plate with a clamped boundary condition at one side and application of a strain at the other, the stress and strain are constant along the length of the structure. However, there is a significant gradient in the damage distribution along
the length of the FG plate. It is noteworthy that the variation of damage is a monotonic function for the FG plate with in-plane material properties variation. Moreover, it is found that the trend of the graph is a decreasing monotonic function that is in consistent with Figure 19 in Chapter 2, which was obtained from experimental data.

Figure 4.19: Variation of damage variable for homogeneous and FG plates along the length direction of the plate

Figure 4.20 demonstrates the effect of varying the stiffness index $n$ on the damage variable $D(x)$. Index $n$ is a parameter that governs the material variation profile through the length. It can be seen that the damage within the FG plate is strongly associated with the material variation profile, governed by $n$. It is found that when the gradient index, $n$, approaches zero or infinity, that the plate would be homogenously composed completely of inclusion or matrix, respectively. So, the predicted values of the damage variable converge to the relevant homogenous plates. As the power index $n$ increases, meaning that the FG plate material properties approach the matrix material properties, the damage variable treats the saturation effects the same as for a homogenous plate consisting of 8530. Utilizing these aspects, i.e. the sensitivity of the magnitude damage variable, the material variation profile for FG plates can be controlled to meet the desired or application-specific performance criteria.
Figure 4.20: Damage variation for the VW_8530 FG plate with in-plane material property variation for different indices ($n$)
4.6 Summary

To end this chapter, presented here is a summary, together with the main findings and conclusions:

- A continuum based damage model for elasto-plastic damage in functionally graded plates with in-plane material properties variations has been presented. This model uses the interaction mechanism exhibited by the plasticity and the damage morphologies.

- Two independent plastic and damage multipliers have been introduced to eliminate strong limitations in modelling capacity.

- Material properties, such as Young’s Modulus, yield stress, plastic material constants and damage characteristic parameters, have been assumed to follow simple power law functions to idealize the FG plate as a continua with smooth variation of material properties.

- A robust three-step numerical algorithm was implemented in the user material subroutine UMAT by using the graded solid elements (in ABAQUS) which describe effective continuous variation of material properties.

- A range of different examples for different materials have been presented in order to validate the proposed model i.e., elasto damage for a SiC plate and nano-composite, and elasto plastic damage for a high strength steel plate. Therefore, it has been demonstrated that there is no limitation to model damage in various materials with different material property variations.

- A decreasing monotonic function of the damage variable within an FG plate, which is consistent with Figure 2.19, has been observed.
Chapter 5: Static analysis of FG plate: experimental validation

Some of the work presented in this chapter has been published in the following:


5.1 Introduction

One of the main novelties of this research is the set of experimental tests performed to capture the out of plane deformation of a selection of 3D printed FG plates, in order to validate the proposed graded FE solutions. This chapter details the experimental work carried out to capture both static deflection and natural frequencies of FG plates. For the experimental static deflection tests, 3D printed FG plates with both low stiffness ratio (VW_8530) and high stiffness ratio (8515_9895) material combinations were subjected to a concentrated load at their centre. All four edges of the plate are simply supported. In order to capture out of plane deflection (in z direction) a 3D Digital Image Correlation technique (3D- DIC) was employed. This technique involves tracking the motion of geometric features on the specimen over the course of an experiment, to generate an experimental displacement field. Portable Digital Vibrometer (PDV) tool was used to measure surface vibration without contact utilizing Laser Doppler Vibrometry (LDV) technology. The non-contact laser vibrometer PDV used to measure the natural frequency of the simply supported FG plates. All experimental results are compared with the appropriate FE model to validate the accuracy of the proposed graded FE solutions.
5.2 Deflection measurement

5.2.1 3D- digital image correlation

As an experimental technique, Digital Image correlation (DIC) is becoming increasingly popular, largely due to advances in computing power and digital camera performance, along with decreasing hardware costs. Although the technique is most commonly referred to as digital image correlation, it has also been called electronic speckle photography [133], texture correlation [134] and computer-aided speckle interferometry [135]. The advantages of DIC include: the modest hardware requirements, the large range of measurable strains and a relative insensitivity to the nature of the specimen being analysed. During the experiment images are typically captured using one or more digital cameras and sample preparation is limited to the application of a speckle pattern to the specimen if the surface being analysed does not already have sufficient contrast for tracking. The disadvantages of DIC include the moderate level of processing required to generate strain fields from the measured displacement fields, a relative unsuitability for analysis of cyclic experiments and relatively high sensitivity to noise and vibration in the surrounding environment.

The accuracy with which geometric features on the specimen can be tracked relies largely on the lighting conditions and the visible contrast of the surface being analysed. Surface contrast is often improved through the application of a random speckle pattern. If the experiment is limited to in-plane deformation of a planar surface then 2D DIC is sufficient and only a single camera is required. If the surface is not flat, or if out-of-plane deformation is expected, then 3D DIC is required and at least two cameras must be used. This is because, from the perspective of an observer, the apparent size of a sample changes with distance. If only one camera is used then the distance of a given point from the camera is unknown and is unable to be accounted for. 3D DIC removes this limitation by making use of multiple cameras viewing the surface from different perspectives, allowing for features to be located in 3D space and ensuring that out-of-plane motion does not cause artificial strains to be reported. Images captured over the course of the experiment are processed after the fact, due to the computational expense of this process. The motion of individual points on the specimen, referred to as a 'feature' or 'tracking subset', is determined by considering a small subset of the image which is centred on its location. The tracking solution is the imposed displacement which corresponds to the smallest change in this subset's appearance between the current frame and the reference frame, and is determined using nonlinear optimisation. When performed for a sufficient number of points the DIC tracking process yields a full field description of the surface's displacement. Therefore, in order to validate out of plane deflection obtained using the graded FE model, a 3D DIC technique was employed.
5.2.3.1 MODEM software

The Defence Technology Agency’s (DTA) Applied Vehicle systems group has recently developed an in-house DIC system: MODEM (MATLAB Optical Displacement and Strain Measurement) [136]. This system includes three main components: the DIC software itself, which processes the captured images and allows for the results to be interrogated and visualised; the requisite hardware, such as lighting and cameras; and the image capture software through which the cameras are controlled and camera calibration can be performed. MODEM’s primary software component has been developed in MATLAB and is capable of performing both 2D and 3D DIC. As it has not been compiled to a standalone executable, the software requires MATLAB to be installed on the host machine in order to be run. This limitation could however be overcome if necessary using the ‘MATLAB Compiler’ toolbox. The basic structure of the MODEM software is illustrated in Figure 5.1

Before feature tracking is performed, local regions are identified in the image which contain the greatest contrast and are therefore the most likely to provide an accurate tracking result. Tracking in MODEM is performed using a pyramidal implementation of the inverse compositional Gauss-Newton tracking algorithm [137], and optionally makes use of a rigid, affine, irregular or quadratic shape function (also known as a warp). In the case of a rigid shape function, the appearance of the tracking subset under consideration is assumed to remain unchanged between the current frame and the reference frame. The algorithm may however fail to converge on a tracking solution or produce a poor result if the subset’s appearance has in fact changed appreciably between the two frames.

For each frame, tracking is reattempted if more than 2 percent of the currently active features are not successfully tracked. The tracking results for those features which were successfully tracked are first used to produce estimates of the displacements and warps across the region of interest. These estimates are then used as initial guesses for the tracking solutions of each of the features which were lost. Tracking attempts are performed iteratively in this manner until the recovery success rate of the latest iteration is less than 10 percent.

In MODEM, stereo correlation is achieved by performing an initial correlation between the first frames of each of the two image streams, and then processing each of the image streams as independent jobs. This minimises the number of times that correlation must be performed between the image streams, which is desirable as this correlation is relatively computationally expensive [138].
The initial correlation is performed using the same algorithms that are employed for intra-sequence tracking, but using independently selected tracking parameters. Specifically: tracking must be performed using a quadratic shape function to allow for the perspective distortion that takes place between the two cameras; and a large pyramidal height is typically selected to allow for the large apparent displacement that may occur between the two views. The wide array of variables in MODEM that determine the measurement precision achievable when using DIC makes it difficult to identify a specific metric quantifying this system’s accuracy. However, a conservative estimate of MODEM’s performance is that deformation measurements can be obtained with a precision of 1000
microstrain or better in most cases. Greater precision may be achievable in a number of instances, particularly for cases involving smooth strain fields where significant smoothing can be applied to the measured displacements without detrimentally affecting the results [139]. Interpolation of the greyscale intensities from the images allows this tracking to be carried out with subpixel accuracy. The nature of this approach however, combined with sources of experimental error such as image noise and imperfect camera optics, means that the greatest measurement precision which is currently claimed to be achievable using DIC is no better than 50 με (0.005%) under ideal conditions [139]. The region being considered can range in size from tens of microns [140] to tens of metres [141] and claimed strain measurement ranges encompass strains of 0.005%-2000% [142]. The measurement precision achievable when using MODEM is dependent on a wide range of factors including, but not limited to: image noise, image spatial resolution, lens quality, the quality of the surface’s visual texture, the presence of motion blur, the selected tracking parameter and the amount of smoothing applied.

5.2.3.2 Image capture software
A program called “uEye Sequence Capture” has been developed at Defence Technology Agency (DTA) to facilitate the capturing of images using the AVS group’s iDS uEye cameras for the purpose of DIC. “uEye Sequence Capture” has been written in the Python programming language. The program has been packaged as a portable executable file using Python’s PyInstaller module. The software provides the ability to both capture image sequences during experiments and perform a camera calibration. In both cases this can be done either for a single camera or for a pair of cameras. Camera calibration is performed using the calibration functions provided in OpenCV [143, 144]. OpenCV’s camera calibration functions are capable of solving for a range of optical distortion parameters, including radial distortion coefficients up to the 6th order and tangential distortion coefficients. It is however possible to set any number of the parameters to zero when the functions are called, which provides flexibility regarding the optical distortion model that is used to describe the combined characteristics of the chosen cameras and lenses.

5.2.3.3 DIC system hardware
In order to acquire high quality images the surface of interest must be well lit. This was achieved through the use of LED panels in combination with linear polarising filters. LED lighting is generally utilised for DIC as the general confinement of their emitted energy to the visible spectrum minimises the introduction of thermal strains. The use of polarising filters with both the lights and the cameras allows for specular reflections to be blocked; these are undesirable as they can cause the local appearance of the surface to change significantly depending on its orientation, leading to the acquisition of poor or invalid tracking results. The cameras used are a pair of iDS UI-2280SE-M-GL USB cameras incorporating 5 megapixel Sony ICX655 monochrome sensors. All FG plates
were covered with mat black paint and random sparkle pattern having a white colour. Figure 5.2 shows the experimental hardware utilised in the 3D DIC system employed in this project.

![Hardware implemented for the MODEM DIC system](image)

**5.2.2 Experimental setup**

The goal for this experimental programme is the study of deflection of simply supported FG plates under concentrated transverse loading using 3D DIC. Therefore, the experimental fixture shown in Figure 5.3 was designed to model simply supported boundary conditions for all plate edges. All technical drawings of this fixture are provided in Appendix G. The experimental fixture has been manufactured from 10 mm thick aluminium plates, and has been fixed within an Instron universal testing machine (INSTRON 5567 at Centre for Advanced Composite Materials CACM), operating under load/displacement control. The full experimental setup is shown in Figure 5.4.
Figure 5.3: Schematic of experimental Aluminium fixture
5.2.3 Results

To carry out 3D image processing and utilise the MODEM code for deflection analysis of FG plates, the installed memory (RAM) on the local PC should be at least 16.0 Gb. As it was more convenient and more computationally powerful, a Virtual Machine (VM) from the Centre of e-research (CeR) has been assigned for this project. There are a number of advantages provided by the use of a VM. In contrast to the Pan cluster, the VMs enable Windows-only software running on large memory machines. They provide a stable environment for very long running jobs. VMs can also be check-pointed at any time.

This section provides the DIC results from deflection experiments on plates with different gradient distributions, under applied concentrated transverse loading. Moreover, the experimental results are compared with the developed FE simulation using graded elements. It should be noted that the dimension of the 3D printed plates are 110 mm × 110 mm × 2 mm.
It should also be noted that for analysis of data obtained from the MODEM code, another MATLAB code was written to select the correct location of nodes in 3D space and their displacement in the z direction.

### 5.2.3.1 Homogenous plate

A homogenous plate made of digital material 8530 was subjected to a 15 N concentrated load. There are two main factors that should be considered for the choice of load magnitude: load should be large enough to generate deflection to measured accurately using current DIC approach and on the other hand avoid large deformation/nonlinear effects. Therefore, a preliminary FE simulation using different applied load carried out to measure the maximum deflection and maximum yield stress to define nonlinearity limit. For the designed square homogenous plate made of 8530 digital material, the FE yield stress below experimental yield stress (15.7 MPA) to simulate linearity and small deflection case, obtained under 17.4 N applied load. Therefore, 15N was conservatively considered to apply in experimental tests. The calibration images for the first and second cameras are shown in Figure 5.5. In order to obtain a good calibration result, the chessboard calibration pattern should be viewed from a wide range of camera angles and positions, that as a collection should cover as much of the field of view (of both cameras) as can be practically achieved. Therefore, it can be seen in Figure 5.5 that there is a good variation in the position and orientation of the calibration pattern.
Figure 5.5: Calibration images for homogenous 8530 plate (a) first camera (b) second camera

The Table 5.1 shows the preliminary data that is used in the DIC calculation and results.

<table>
<thead>
<tr>
<th>Calibration error (pixels)</th>
<th>Stereo correlation factor (%)</th>
<th>Applied force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.646</td>
<td>88</td>
<td>15</td>
</tr>
</tbody>
</table>

It would be ideal to have full stereo correlation (100%). However in reality, due to experimental issues such as sparkling patterns, size of the applied sparkled white points, lighting conditions and software criteria, the correlation factor was reduced. As can be seen from Figure 5.6, the stereo correlation factor achieved by division of tracked and defined features (red points) to non-defined in tracking performance step.
One of the logical limitations in the MODEM code is that it cannot consider infinite nodes as tracked features for image processing. To reduce logical errors generated during DIC processing, only the tracked area around the middle of the plate and applied load was considered. After image processing, two matrices are generated for the definition of node locations, NODES and NODES_CORRECTED. It should be mentioned that the NODES variable contains the 2D coordinates of the nodal points from the perspective of the primary camera, while NODES_CORRECTED contains their coordinates in 3D space (with the primary camera’s optical focal point being the origin). The smoothed description of the displacement measurements is evaluated at a set of points that (from the perspective of the primary camera) forms a uniformly spaced rectangular grid when the surface of interest is in its undeformed state and this happens where the optical axis of the primary camera is not normal to the surface in its undeformed state. The 2D coordinates are utilised to calculate the results, because it is the distance between features and nodes in the 2D image that determines how strongly the tracking result for each feature contributes to the result at each node. So the variable of most interest should be NODES_CORRECTED. Example distributions of NODES and NODES_CORRECTED are shown in Figure 5.7. X and Y coordinates corresponds the greatest contrast and most likely an accurate tracking results. Then using Shi-Tomasi corner detection algorithms [145], a local quality value to each pixel coordinate assigned.
Figure 5.7: The location of (a) nodes and (b) nodes_corrected for tracked features of homogenous 8530 plate.

Figure 5.8 shows the 3D deflection of the homogenous 8530 plate under concentrated transverse applied force, and is compared with predicted contour results obtained directly from data analysis of MODEM software. All coordinates are expressed in terms of calibration square widths, which should be linearly converted into a length dimension using the conversion calibration factor defined as follows:
Conversion factor = \[
\frac{\text{calibration square (\text{mm})}}{\text{calibration square (\text{pixel})}} = \frac{6.68}{252} = 0.02651
\]

Figure 5.8: 3D deflection contour of the homogenous 8530 plate directly from (a) MODEM (b) data analysis

The x, y and z axis used in Figure 5.8 show the dimensionless plate length, width and out of plane displacement, respectively. It can be seen that zero deflection occurred at the point of application of the point load (at the middle of the plate). Moreover, the observed non-smooth surface is due to the noise level (or other random variables) from various different sources during the experimental tests. Noise during experimentation is random and unavoidable. The main sources of noise during experimental tests are: test set up (focal length, lighting, contrast, stereo angle), speckle pattern, calibration, contamination.
and subset size. The sensitivity of the DIC method to noise levels has been studied previously [146].

In order to compare between FE modelling and the DIC experiments, deflection along a band through the middle of the plate (an average around the indenter location) along the length are shown in Figure 5.10, for both the FE modelling and experimental results. The applied boundary conditions in the ABAQUS FE model are based on the experimental set up in which simply supported boundary conditions (four rollers) are implemented at a 5 mm offset from all four edges of the plate. This means that the fixed point, or zero displacement, due to the boundary conditions occurs at \( x/L = 5/110 = 0.045 \). The boundary conditions and force applied within ABAQUS are shown in Figure 5.9. As it can be comprehensively concluded from mesh convergence in section 3.5, the non-dimensional mesh size 0.0625 considered for FE modelling. It should be noted that the experimental DIC results are shifted based on a difference of displacement (from zero point displacement-applied concentrated load location- to maximum displacement, end of edges) which is 3.34 mm, and has been measured from the crosshead movement of the Instron universal testing machine.

As can be seen from Figure 5.10, good agreement is found between the FE modelling and the experimental 3D-DIC displacement measurements. It should be mentioned that the experimental result is an average of 3 samples. The maximum deviations are noted around the location of the applied concentrated load. As it discussed before, one of the main reason for more variation in experimental results is high sensitivity to noise and vibration in the surrounding environment. Moreover, optical axis of the primary camera is not normal to the surface and has an angle which results in non-smooth surface for deflection.

Figure 5.9: Geometric model of the simulated plate, showing boundary conditions applied in the FE model

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5.2.3.2 Composite Plate

In this section, the two composite plates with the designed and manufactured material combinations (VW_8530 and 8515_9895) with a constant 50% volume fraction of inclusion (hard polymer) are considered. Table 5.2 shows the preliminary data that was used for the DIC calculation and results.

<table>
<thead>
<tr>
<th>Composite plate</th>
<th>Calibration error (pixels)</th>
<th>Stereo correlation factor (%)</th>
<th>Applied force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VW_8530</td>
<td>0.631</td>
<td>83</td>
<td>25</td>
</tr>
<tr>
<td>8515_9895</td>
<td>0.776</td>
<td>86</td>
<td>15</td>
</tr>
</tbody>
</table>

It should be noted that as the cameras and experimental set up were shut down between experiments, the calibration procedure should be completed before each trial. So, the different camera parameters such as focal length and calibration error were obtained for each of the different composite plates. Calibration images obtained for the test on the VW_8530 composite plate are in Appendix H.

The tracked and recognised features (red points) that were used in the DIC calculations for VW_8530_composite and 8515_9895_composite are shown in Figure 5.11.
Figure 5.11: Tracked features for the (a) VW_8530 composite plate, and (b) the 8515_9895 composite plate.

The 3D deflection contours obtained from the DIC analysis and the MODEM software for the VW_8530 and 8515_9895 composite plates are shown in Figure 5.12 and Figure 5.13, respectively.
Figure 5.12: 3D deflection contour of the VW_8530 composite plate directly from (a) MODEM, and (b) from the data analysis
The symmetric deflection contour observed in Figure 5.12 and Figure 5.13 were as expected for the composite plate. Another notable point is that the maximum displacement in the z direction for 8515_9895 composite plate (6.85 mm) is higher than VW_8530 composite plate (5.24 mm), which is due to the lower stiffness of the 8515_9895 composite plate.

These figures show deflection at the middle of each plate along its length, for both the FE modelling and experimental DIC results.
Figure 5.14: Deflection of the composite samples along the band through the middle of the plate, for both the FE modelling and 3D-DIC; (a) VW_8530, and (b) 8515_9895

As can be observed from Figure 5.14, the 3D-DIC results for the VW_8530 composite plate exhibit stronger fluctuations, possibly due to greater experimental errors resulting from high noise levels and a lower stereo correlation factor. Moreover, it was not easy to determine the DIC results for the exact middle line of the plate. It should be mentioned that the force applied for the experiment with the VW_8530 composite plate was 25 N, higher than the 15 N applied for the 8515_9895 composite plate. Because the VW_8530
composite plate is stiffer, to ensure that measurable deflections could be captured by two cameras accurately, the applied load was increased.

### 5.2.3.3 Functionally Graded Plate

One of the main novelties of this research is to experimentally capture the bending behaviour of designed and manufactured 3D printed FG plates under out-of-plane concentrated loading using the 3D-DIC method. In this section, FG plates with the VW_8530 and 8515_9895 combinations and with different inclusion distribution patterns \( n = 1 \) “linear” and \( n = 3 \) “non-linear”) are considered. Table 5.3 shows the preliminary data that was used in the 3D-DIC calculation and results.

<table>
<thead>
<tr>
<th>FG plate</th>
<th>Calibration Error (Pixels)</th>
<th>Stereo Correlation Factor (%)</th>
<th>Applied Force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VW_8530_Linear</td>
<td>0.912</td>
<td>88</td>
<td>25</td>
</tr>
<tr>
<td>VW_8530_Nonlinear</td>
<td>0.651</td>
<td>90</td>
<td>25</td>
</tr>
<tr>
<td>8515_9895_Linear</td>
<td>0.806</td>
<td>89</td>
<td>15</td>
</tr>
<tr>
<td>8515_9895_Nonlinear</td>
<td>0.785</td>
<td>86</td>
<td>15</td>
</tr>
</tbody>
</table>

The tracked and recognised features (red points) which have been used in the DIC calculations for the FG plates are shown in Figure 5.15. As mentioned previously, the Stereo Correlation Factor (SCF) is calculated by division of the active and tracked features to the total features in the area of interest.

Figure 5.15 shows the tracked and recognised features (red points) employed within the MODEM software for the different functionally graded plates. It should be mentioned that the gradient stiffness direction is from left side of the plate (soft polymer) to the right side (hard polymer). The 3D deflection contours obtained from MODEM for the linear \( n = 1 \) and nonlinear \( n = 3 \) FG plates with different material combinations are shown in Figure 5.16.
Figure 5.15: Tracked features for the FG plates

Figure 5.16: Measured 3D deflection contours for the FG plates
As can be observed from Figure 5.16, deflection contours for the FG plate are not symmetric, differences being more obvious for the linear high stiffness ratio combination (8515_9895_Linear). Therefore, unlike the homogenous and composite plates, the deflection contours for FG plates are not symmetric, which is compatible Figure 3.9 that shows an unsymmetrical deflection trend for an FG plate.

Figure 5.17 and Figure 5.18 show deflections along the middle bands for the linear and nonlinear FG plates (an average around the indenter location), along the lengths for both the VW_8530 and 8515_9895 combinations, respectively. As can be seen from Figure 3.11, the FG plate with a power law index of $n=3$ has lower stiffness than the linear FG plate with $n=1$. It can be observed that the deviation between the FE and experimental results for the nonlinear plate is higher than that for the linear FG plate, which can be due to the high gradient distribution of inclusions. As mentioned in Chapter 3 (refer to Figure 3.11), the Young’s modulus of the FG plate varies rapidly near the inclusion phase dominant (right) side for high values of the power-law index ($n=3$). So, the variation in the Young’s modulus of the FG plate will be steepest near the inclusion phase dominant (right) side.

Figure 5.17: Deflection of the VW_8530 FG sample along the band through the middle of the plate, for both the FE modelling and 3D- DIC
Figure 5.18: Deflection of the 8515_9895 FG sample along the band through the middle of the plate, for both the FE modelling and 3D-DIC

Another interesting point is an unsymmetrical deflection trend for FG plates. As it can be revealed form the Table 5.4, the deflection of FG plates varies with the same points away from the centre (X/L=0.5) which comprehensively discussed in section 3.5.1.1. The graph with labelled data for deflection is in appendix I.

<table>
<thead>
<tr>
<th>x/a</th>
<th>VW_8530_Linear</th>
<th>VW_8530_Nonlinear</th>
<th>8515_9895_Linear</th>
<th>8515_9895_Nonlinear</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>-0.96</td>
<td>-0.98</td>
<td>-2.62</td>
<td>-4.35</td>
</tr>
<tr>
<td>0.3</td>
<td>-1.28</td>
<td>-1.63</td>
<td>-3.77</td>
<td>-7.04</td>
</tr>
<tr>
<td>0.4</td>
<td>-1.86</td>
<td>-2.14</td>
<td>-5.70</td>
<td>-7.39</td>
</tr>
<tr>
<td>0.5</td>
<td>-2.25</td>
<td>-2.45</td>
<td>-6.83</td>
<td>-8.55</td>
</tr>
<tr>
<td>0.6</td>
<td>-1.97</td>
<td>-2.33</td>
<td>-6.23</td>
<td>-9.29</td>
</tr>
<tr>
<td>0.7</td>
<td>-1.37</td>
<td>-1.87</td>
<td>-4.72</td>
<td>-7.59</td>
</tr>
<tr>
<td>0.8</td>
<td>-1.10</td>
<td>-1.52</td>
<td>-2.44</td>
<td>-7.20</td>
</tr>
</tbody>
</table>
5.3 Experimental validation of natural frequency

5.3.1 Experimental set up and procedure

For measurement of the first natural frequency of designed and manufactured FG plates, a Portable Digital Vibrometer (PDV) used. This instrument is intended for laboratory use and for use in an industrial environment. This instrument uses a laser beam and its light source is a helium neon laser. It is designed to be powered by a low voltage source (nominal 12 V) provided externally, which is accessible for constructional reasons. PDV is used for non-contact measurements of surface vibrational velocities. In the frequency range from 0.05 Hz to 22 kHz the PDV allows achievement of good measurement accuracy. PDV systems use the principal of heterodyne interferometry to acquire the characteristics of mechanical vibrations or transient motion processes. With this type of interferometer, a high frequency carrier signal is generated on the photo detector. To make the vibration measurement, the beam of the helium neon laser is pointed at the vibrating object, and is subsequently scattered back from it. As can be seen from Figure 5.19, the following components were used to measure the natural frequency of FG plates.

- Laser Vibrometer PDV-100
- Plug-in power supply
- Piezoelectric impulse force hammer
- Digital audio cable
- Experimental Aluminium fixture
It should be mentioned that the PDV reaches its optimal properties after a warm-up period of approximately 20 minutes. Moreover, PDV offers three velocity measurement ranges, for which full scale values are shown in mm/s (20 mm/s, 100 mm/s and 500 mm/s) on the software display. When using the digital output, the full scale value is needed as a reference for a numerical value transmitted. Therefore, for achieving the respective scaling factor in mm/s/V, the full scale value should be divided by four, corresponding to the output voltage swing of ±4 V.

For determination of the first natural frequency of the polymeric FG plates, a set of three samples for each plate were tested using the Portable Digital Vibrometer (PDV) tool. Due to rubbery and polymeric nature of all tested plates, the required tests were carried out using a small hammer to avoid noise and unwanted vibration. For measurements of the exact value for natural frequency, a MATLAB code using single degree freedom circle fit method was written by Prof. Brain Mace at the University of Auckland in order to plot the acceleration magnitude, acceleration phase and real-image frequency circle.

5.3.2 Results

In order to measure the exact value for first natural frequency, the frequency versus acceleration phase should be plotted and frequency defined from the value at which the phase changes 180 degree. For example the frequency versus acceleration phase and magnitude curves for an 8515_9895 nonlinear FG plate is shown in Figure 5.20. It can be seen that at approximately 100 Hz to 150 Hz the acceleration phase changed, and the
magnitude hit the maximum, indicating the natural frequency response area. Therefore, to obtain the exact value for natural frequency the real-image frequency circle can be plotted and the first natural frequency 117.53 Hz obtained for the linear 8515_9895 nonlinear FG plate.

Figure 5.20: Frequency versus acceleration phase and magnitude curves for 8515_9895 nonlinear FG plate

Figure 5.21 shows the first natural frequency for the different plates obtained from experimental tests. All acceleration magnitude and phase plots for different FG plates are provided in Appendix B.
As can be seen from Figure 5.22, stiffness of the plate for VW_8530_linear is the highest, which results in a higher frequency response than the other FG plates. Moreover, the natural frequency of the 8515_9895_nonlinear FG plate has the lowest value compared with other FG plates, due to the lower stiffness of this material combination.

In order to validate the proposed graded FE solution, the experimental results for first natural frequency are compared with results from the FEM. Table 5.5 shows the first
natural frequency for different plates captured using both the Portable Digital Vibrometer (PDV) (average value), and by application of FEM.

Table 5.5: First natural frequency for different plates captured using Portable Digital Vibrometer (PDV) (average value) and FEM results

<table>
<thead>
<tr>
<th>Plate</th>
<th>FEM- Graded elements (Hz)</th>
<th>Experimental (Average Value) (Hz)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homogenous -8530</td>
<td>130.1</td>
<td>117.8</td>
<td>10.4</td>
</tr>
<tr>
<td>Composite- VW_8530_50%</td>
<td>158.3</td>
<td>132.9</td>
<td>19.1</td>
</tr>
<tr>
<td>Composite-8515_9895_50%</td>
<td>110.2</td>
<td>97.6</td>
<td>12.9</td>
</tr>
<tr>
<td>FG- VW_8530_Linear</td>
<td>213.4</td>
<td>187.1</td>
<td>14.1</td>
</tr>
<tr>
<td>FG- VW_8530_Nonlinear</td>
<td>180.7</td>
<td>151.4</td>
<td>19.3</td>
</tr>
<tr>
<td>FG- 8515_9895_Linear</td>
<td>136.9</td>
<td>120.7</td>
<td>13.4</td>
</tr>
<tr>
<td>FG- 8515_9895_Nonlinear</td>
<td>123.8</td>
<td>111.1</td>
<td>11.4</td>
</tr>
</tbody>
</table>

The results for nonlinear FG plates utilising two material combinations are obtained from Figure 3.24 for a thin plate (h/b=0.02). These results differ slightly from the experimental results here, due to the approach to model boundary conditions. Hence, as the simply supported boundary condition provided by the four rods in the experimental setup are 5 mm offset from each edge, boundary conditions that more accurately represented this situation were applied in the FE modelling. It should be noted that applied simply supported boundary condition in FE modelling at Figure 3.24 are at the middle of the each edge.

It can be observed from Table 5.5 that the first numerically derived natural frequency exhibits good agreement with those obtained from the experimental results, with a maximum deviation of 19.3 % for the VW_8530_Nonlineara FG plate. Figure 5.23 compares the deviations between linear and nonlinear FG plates, considering both the predicted FEM solutions and experimental results. As can be seen, the error between linear and non-linear distributions for the VW_8530 combination for FE solution is 15% while the experimental result is 19%. Moreover, the deviation between the linear and
non-linear distributions for the 8515_9895 combination for FE solution and experimental results are 10% and 8%, respectively. This implies good reliability of the proposed graded FE simulation for modelling of FG plates.

Figure 5.23: Deviation between linear and non-linear FG plates for FE and experimental results
5.4 Summary

To end this chapter, presented here is a summary, together with the main findings and conclusion:

- A 3D- DIC method was utilised to capture out of plane deflection for designed and manufactured FG plates, using the MODEM DIC system.
- An experimental fixture was manufactured to simulate bending of a simply supported plate under concentrated loading.
- A set of three repeat samples for each plate (homogenous, composite, linear and nonlinear FG plates) were tested to obtain out of plane deflection.
- Good agreement between the deflection contours from the MODEM software and results extracted from data analysis, was obtained.
- The deviation between FE and experimental out of plane deflection for nonlinear FG plates is higher than for linear FG plates, which is likely to be due to the high gradient distribution of inclusion.
- A Portable Digital Vibrometer (PDV) was used for non-contact measurements of surface vibrational velocities, allowing measurement of the first natural frequency of the plates.
- The VW_8530_linear FG plate exhibited the highest first natural frequency, due to the highest average stiffness of the material combination.
6 Chapter 6: Conclusions and outlook

6.1 Conclusions

The objective of this doctoral research was to develop, implement and validate a methodology for predicting the static behaviour and damage mechanics of polymeric functionally graded plates using analytical, numerical and experimental techniques.

A particular emphasis was placed on the development of a novel analytical modelling approach based on Higher order Shear Deformation Theory (HSDT), which is able to predict the deflection and free vibration of thick FG plates with in-plane material property variation. In order to validate the analytical predictions of FG plate behaviour, graded finite elements, as a novel approach to allow gradients of properties within an element, has been introduced.

The comprehensive static and damage prediction tool that has been developed using HSDT theory and graded numerical modelling can be used to model an FG structure with a wide range of thickness and material distribution direction.

Despite the limitations of the HSDT theory for very thick plates and high stiffness ratios, an automated and easy approach is provided to use graded solid elements. The new numerical approach provides an attractive alternative to carrying out costly experiments. Moreover, the predicted values for deflection under transverse loading were validated through the results obtained experimentally using 3D-DIC method.

The major contributions of the different chapters in this thesis are as follows:

- Chapter 2. Material properties for different polymeric digital materials, printed using a Stratasys Object500 3D printer with different combinations, have been obtained using standard tensile and cyclic tensile tests. From these results, it was decided that the best combination to obtain different stiffness ratios for FG plates is VW_8530 and 8515_9895, with stiffness ratios of 2 and 18, respectively. Using an empirical curve fitting approach, a power law relationship between material properties and percentage of hard polymer (inclusion) has been obtained. The obtained material properties have been used as input data for experimental validation of deflection prediction of FG plates. A decreasing monotonic function
for a damage parameter, with respect to the percentage of hard polymer (inclusion), was obtained from cyclic tensile tests.

- Chapter 3. Higher order Shear Deformation Theory (HSDT) and graded FEM using UMAT and USDFLD user subroutines were developed to incorporate bending and free vibration of FG plates with in-plane stiffness variation. Differences between the analytical and FEM solutions increase as stiffness ratio increases, particularly near the inclusion phase side, because the variation in Young’s modulus are steepest near the inclusion phase dominant side. It could be interpreted that the analytical formulation for a high stiffness ratio thick FG plate cannot capture the deflection of the plate accurately, and the graded elements are less able to accurately capture the rapid variation of Young’s modulus with a power law distribution function. This approach can be used to determine the optimum material distribution to produce a controlled-stiffness FG plate corresponding to prescribed structural characteristics.

- Chapter 4. An efficient continuum based damage model for elasto-plastic damage in FG plates, with two independent plastic and damage multipliers, has been developed. After that, a robust three - step numerical algorithm has been implemented in the user material subroutine UMAT by using the graded solid elements. After validation for different materials, it has been demonstrated that there is no limitation to use the proposed damage tool for prediction of damage for various materials with different material property variations.

- Chapter 5. A 3D Digital Image Correlation method using the MODEM DIC tool was applied to measure the out of plane deflection of manufactured polymeric FG plates. A fair agreement has been obtained between experimental results and FE predicted solutions, with stereo correlation factors between 85% to 90%. The non-smooth surface for the observed deflection contours were due to noise levels from various different sources during the experimental tests. A non-symmetric profile for deflection of FG plates was verified under point applied load in the middle of the plates.

Moreover, a Portable Digital Vibrometer (PDV) tool was used as a non-contact frequency measurement of first natural frequencies of the FG plates. A comparison was made between graded FEM and the obtained experimental results for natural frequency.
6.2 Novel Contributions

The novelties of this research are as follows:

1. Characterization and identification of a damage parameter for 3D-printed functionally graded plates, which is correlated with plate length.

2. Solution of advanced analytical formulations for deflection and free vibration of graded plates with in-plane stiffness variation.

3. Numerical implementation of material gradation through Gauss points of a single solid finite element.

4. Capturing experimentally the non-uniform shape for deflection of functionally graded plates, which is associated with in-plane stiffness variation, using a 3D-DIC method.

6.3 Recommendations for future work

The research objectives presented in Chapter 1 were achieved in full. There are however, a number of potential avenues for future work based on the methodology developed and results presented. These extensions will increase the accuracy of predictions, for robust use of complex FG structures in industrial applications.

First, higher order Shear Deformation theory (HSDT) can be extended to be able to predict the deformation and free vibration of FG plates with high stiffness ratios and with high power law index profiles. In the presented work, the HSDT based on sinusoidal shear deformation were developed, however there are other shear functions which can be used for complex materials. To enable deflection predictions for high stiffness ratio FG plates, the HSDT will need to be altered.

Second, material gradation through three directions for different coordinate systems can be developed, and implemented using user subroutines within ABAQUS.

Third, being able to experimentally validate the damage modelling proposed in Chapter 4 would be beneficial. In order to achieve this, further experimental tensile tests on 3D-printed FG plates should be performed to model numerical simulation for damage under axial displacement. In addition, the effect of viscoelasticity (due to polymeric characteristic of the FG plates) will have to be incorporated by numerical implementation. To support this, an additional step including the rate of material
properties variation with respect to time will have to be added into the integration scheme.

Fourth, further effort can be expended to improve the 3D- DIC measurement, to prevent unwanted noise sources during experimental tests. It could be done by capturing measured deflection for the first images which shows potential noises for the experimental tests. This will lead to smoother and more reliable contour deflection plots.

Lastly, in order to comprehensively compare deflection results obtained from graded FE simulation and experimental results, a more comprehensive modelling of the applied boundary conditions and loading will need to be completed. To achieve this, the simply supported boundary condition at four rods, and a concentrated point load from the indenter, will have to be modelled in the FE software.
Appendix A: Full programme of experiments completed on an Instron 5564

Column 2: Width
Column 3: Thickness
Column 4: Axial Gauge Length (Strain Source)
Column 5: Modulus (Chord 0.05 % - 0.25 %)
Column 6: Yield Strength
Column 7: Ultimate Tensile Strength
Column 8: Tensile strain at Yield (Zero Slope)
Column 9: Tensile strain at Break (Standard)
Column 10: Maximum Load
Column 11: Extension at Break
Column 12: Grip Pressure (bar)
Column 13: Comment

Notes:
- Measure the width and thickness of flat specimens at the centre of the specimen and within 5 mm of each end of the gauge length.
- Injection moulded specimen dimensions may be determined by actual measurement of only one specimen from each sample when it has previously been demonstrated that the specimen-to-specimen variation in width and thickness is less than 1%.
- Enter 'Distance between the grips' in mm in the Length box.

Notes:
- Measure the width and thickness of flat specimens at the centre of the specimen and within 5 mm of each end of the gauge length.
- Injection moulded specimen dimensions may be determined by actual measurement of only one specimen from each sample when it has previously been demonstrated that the specimen-to-specimen variation in width and thickness is less than 1%.

Make sure the specimen is properly installed in the test machine.
Keep clear of the machine.
Press the Start button to start the test.

Notes:
- For any required properties other than tensile modulus, please do not Balance the load using Key1, instead use the Fine Position turnwheel to balance the load which will not affect determination of e.g. yield stress, max load, etc.
- Strain1 and Tensile Strain will be automatically balanced upon pressing the Start button.
Appendix B: MATLAB code linked to the Rhinoceros and Grasshopper software packages

% Create Matrix and Inclusion csv files that is an input for Grasshopper
clear all; clc;
warning ('OFF');

M2GHfileName = 'H:\Matrix3.csv';
inclusionFile = 'H:\Inclusions3.csv';

% The dimension of the plate
x = 110; % length mm
y = 110; % wide mm
z = 2; % thickness mm
nn = 0.5; % voxel size

[inclusions, matrix] = CalcInclusions_v3(x, y, z, nn);

parametres = [x/2 y/2 z/2];
csvwrite(inclusionFile, inclusions);
csvwrite(M2GHfileName, matrix);

function [ result_inc, result_mat ] = CalcInclusions_v3( x, y, z, s, pos)

l = linspace(-x/2, x/2, x/s+1);
w = linspace(-y/2, y/2, y/s+1);
h = linspace(-z/2, z/2, z/s+1);

% Calculate gradient distribution along beam
temp = linspace(0, 1, length(l)-1);
grad = temp.^3; % power law distribution function for n=3
ele = (length(w)-1)*(length(h)-1);
result_mat = [];
result_inc = [];
for i = 1: length(l)-1
    counter = 1;
    matrix = [];
    for j = 1:length(w)-1
        for k = 1:length(h)-1
            inclusions(counter,:) = [l(i),w(j),h(k)];
            counter = counter+1;
        end
    end
    for n = 1:round(grad(i)*ele)
        [o,~] = size(inclusions);
        ran = randi([1 o], 1, 1);
        matrix(n,:) = inclusions(ran,:);
        inclusions(ran,:) = [];
    end
    result_mat = vertcat(result_mat, matrix);
end
result_inc = vertcat(result_inc, inclusions);
Appendix C: Coefficients of Equation (3.38)

\[ A_1 = A_{22} = \int_{-h/2}^{h/2} C_{11}dz = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \int_{-h/2}^{h/2} E(x)dz = \frac{(1 - \nu)h}{(1 + \nu)(1 - 2\nu)} E(x) \]

\[ A_2 = \int_{-h/2}^{h/2} C_{12}dz = \frac{v}{(1 + \nu)(1 - 2\nu)} \int_{-h/2}^{h/2} E(x)dz = \frac{vh}{(1 + \nu)(1 - 2\nu)} E(x) \]

\[ A_{66} = \int_{-h/2}^{h/2} C_{66}dz = \frac{1}{2(1 + \nu)} \int_{-h/2}^{h/2} E(x)dz = \frac{h}{2(1 + \nu)} E(x) \]

\[ B_1 = B_{22} = \int_{-h/2}^{h/2} C_{11}zdz = \frac{(1 - \nu)E(x)}{(1 + \nu)(1 - 2\nu)} \int_{-h/2}^{h/2} zdz = 0 \]

\[ B_2 = \int_{-h/2}^{h/2} C_{12}zdz = \frac{vE(x)}{(1 + \nu)(1 - 2\nu)} \int_{-h/2}^{h/2} zdz = 0 \]

\[ B_{66} = \int_{-h/2}^{h/2} C_{66}zdz = \frac{E(x)}{2(1 + \nu)} \int_{-h/2}^{h/2} zdz = 0 \]

\[ B'_1 = B'_{22} = \int_{-h/2}^{h/2} \frac{h}{\pi} \sin \left( \frac{\pi z}{h} \right)dz = \frac{(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \int_{-h/2}^{h/2} \frac{h}{\pi} \sin \left( \frac{\pi z}{h} \right)dz = 0 \]

\[ B'_2 = \int_{-h/2}^{h/2} \frac{h E(x)}{\pi} \sin \left( \frac{\pi z}{h} \right)dz = 0 \]

\[ B'_{66} = \int_{-h/2}^{h/2} \frac{h}{2(1 + \nu)} \sin \left( \frac{\pi z}{h} \right)dz = 0 \]

\[ D_1 = D_{22} = \int_{-h/2}^{h/2} C_{11}z^2dz = \frac{(1 - \nu)h^3}{12(1 + \nu)(1 - 2\nu)} E(x) \]

\[ D_2 = \int_{-h/2}^{h/2} C_{12}z^2dz = \frac{vh^3}{(1 + \nu)(1 - 2\nu)} E(x) \]

\[ D_{66} = \int_{-h/2}^{h/2} C_{66}z^2dz = \frac{h^3E(x)}{24(1 + \nu)} \]

\[ H'_{11} = H'_{22} = \int_{-h/2}^{h/2} C_{11}g^2(z)dz = \int_{-h/2}^{h/2} C_{11} \left( \frac{h}{\pi} \right)^2 \sin^2 \left( \frac{\pi z}{h} \right)dz = \frac{(1 - \nu)h^3}{2\pi^2(1 + \nu)(1 - 2\nu)} E(x) \]

\[ H_{12} = \int_{-h/2}^{h/2} C_{12}g^2(z)dz = \frac{vh^3}{2\pi^2(1 + \nu)(1 - 2\nu)} E(x) \]

\[ H_{66} = \int_{-h/2}^{h/2} C_{66}g^2(z)dz = \frac{h^3}{4\pi^2(1 + \nu)} E(x) \]

\[ L' = \int_{-h/2}^{h/2} C_{13}g'(z)dz = \frac{v}{(1 + \nu)(1 - 2\nu)} E(x) \int_{-h/2}^{h/2} \frac{\pi}{h} \sin \left( \frac{\pi z}{h} \right)dz = 0 \]

\[ L' = \int_{-h/2}^{h/2} C_{13}zg'(z)dz = \frac{v}{(1 + \nu)(1 - 2\nu)} E(x) \int_{-h/2}^{h/2} \frac{\pi}{h} \sin \left( \frac{\pi z}{h} \right)dz = \frac{2hv}{\pi(1 + \nu)(1 - 2\nu)} E(x) \]

\[ L' = \int_{-h/2}^{h/2} C_{33}g'(z)dz = \frac{v}{(1 + \nu)(1 - 2\nu)} E(x) \int_{-h/2}^{h/2} \sin^2 \left( \frac{\pi z}{h} \right)dz = \frac{hv}{2(1 + \nu)(1 - 2\nu)} E(x) \]

\[ L' = \int_{-h/2}^{h/2} C_{33}zg'(z)dz = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} E(x) \int_{-h/2}^{h/2} \left( \frac{\pi}{h} \right)^2 \sin^2 \left( \frac{\pi z}{h} \right)dz = \frac{\pi^2(1 - \nu)}{2h(1 + \nu)(1 - 2\nu)} E(x) \]

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\[ A'_{44} = A'_{55} = \int_{-h/2}^{h/2} C_{44} \xi^2(z) \, dz = \frac{1}{2(1 + \nu)} E(x) \left[ \frac{3h}{2} - \frac{4h}{\pi} \right] \frac{E(x)}{2(1 + \nu)} \]
Appendix D: Coefficients of Equation (3.50)

Fully clamped boundary condition:

\[ J_{11} = \left( \frac{ab}{4} \right) \left( -\left( \lambda \right)^2 A_{11} - \left( \mu \right)^2 A_{10} \right), \quad J_{12} = J_{13} = J_{14} = J_{15} = 0 \]

\[ J_{22} = \left( \frac{ab}{4} \right) \left( -\left( \lambda \right)^2 A_{22} - \left( \mu \right)^2 A_{10} \right), \quad J_{23} = J_{24} = J_{25} = 0 \]

\[ J_{33} = \left( \frac{3ab}{4} \right) \left( 2(\lambda)^2 d_1 D_{11} + (2\mu)^2 d_1 D_{10} - (2\lambda)^2 D_{11} - (2\mu)^2 D_{10} - \frac{2}{3}(2\lambda)^2 (2\mu)^2 D_{12} - \frac{4}{3}(2\lambda)^2 (2\mu)^2 D_{01} \right) \]

\[ J_{44} = \left( \frac{3ab}{4} \right) \left( 2(\lambda)^2 d_1 D_{11} + (2\mu)^2 d_1 D_{10} - (2\lambda)^2 D_{11} - (2\mu)^2 D_{10} - \frac{2}{3}(2\lambda)^2 (2\mu)^2 D_{12} - \frac{4}{3}(2\lambda)^2 (2\mu)^2 D_{01} \right) \]

\[ J_{55} = \left( \frac{3ab}{4} \right) \left( 3d_1 L^2 - L' (2\lambda)^2 - L' (2\mu)^2 \right) \]

\[ J_{13} = J_{23} = 0 \]

\[ J_{43} = \left( \frac{3ab}{4} \right) \left( 2(\lambda)^2 d_1 D_{11} + (2\mu)^2 d_1 D_{10} - (2\lambda)^2 D_{11} - (2\mu)^2 D_{10} - \frac{2}{3}(2\lambda)^2 (2\mu)^2 D_{12} - \frac{4}{3}(2\lambda)^2 (2\mu)^2 D_{01} \right) \]

\[ J_{44} = \left( \frac{3ab}{4} \right) \left( 2(\lambda)^2 d_1 H_{11} + (2\mu)^2 d_1 H_{10} - (2\lambda)^2 H_{11} - (2\mu)^2 H_{10} - \frac{2}{3}(2\lambda)^2 (2\mu)^2 H_{12} - \frac{4}{3}(2\lambda)^2 (2\mu)^2 H_{01} \right) \]

\[ J_{54} = \left( \frac{3ab}{4} \right) \left( 3d_1 L^2 - L' (2\lambda)^2 - L' (2\mu)^2 \right) \]

\[ J_{55} = \left( \frac{3ab}{4} \right) \left( -\left( 2\lambda \right)^2 A_{03} - \left( 2\mu \right)^2 A_{04} - 3L' \right) \]

\[ J_{31} = J_{32} = 0 \]

\[ m_{31} = m_{32} = -\left( \frac{3ab}{4} \right) I_{0}, m_{33} = 0, m_{34} = 0, m_{35} = 0 \]

\[ m_{33} = 0, m_{34} = 0 \]

\[ m_{31} = m_{32} = m_{33} = -\left( \frac{3ab}{4} \right) \left( 3I_{0} + J_{1} \left( 2\lambda \right)^2 + 2(2\mu)^2 \right), m_{34} = -\left( \frac{3ab}{4} \right) \left( 3I_{0} + J_{1} \left( 2\lambda \right)^2 + 2(2\mu)^2 \right) \]

\[ m_{35} = -\left( \frac{9ab}{4} \right) J_{1}, m_{41} = m_{42} = m_{43} = m_{44} = m_{45} = \left( \frac{3ab}{4} \right) \left( 3I_{0} + J_{1} \left( 2\lambda \right)^2 + 2(2\mu)^2 \right), m_{45} = m_{35} \]

\[ m_{33} = m_{34} = m_{43} = m_{44} = -\left( \frac{9ab}{4} \right) K_{1} \]

Simply supported boundary conditions:
\[ J_{11} = -\left( \frac{m\pi}{a} \right)^2 A_{11} - \left( \frac{n\pi}{b} \right)^2 A_{66}, \quad J_{12} = -\left( \frac{m\pi}{a} \right) \left( \frac{n\pi}{b} \right) A_{12} - \left( \frac{m\pi}{a} \right) A_{66}, \quad J_{13} = J_{14} = J_{15} = 0 \]

\[ J_{21} = -\left( \frac{m\pi}{a} \right) \left( \frac{n\pi}{b} \right) A_{21} - \left( \frac{m\pi}{a} \right) A_{66}, \quad J_{22} = -\left( \frac{n\pi}{b} \right)^2 A_{22} - \left( \frac{m\pi}{a} \right)^2 A_{66}, \quad J_{23} = J_{24} = J_{25} = 0 \]

\[ J_{33} = \left( \frac{m\pi}{a} \right)^2 d_1 D_1 + \left( \frac{n\pi}{b} \right)^2 d_1 D_1 - \left( \frac{m\pi}{a} \right)^4 D_1 - \left( \frac{n\pi}{b} \right)^4 D_1 - 2 \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 D_1 - 4 \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 D_6 - \left( \frac{m\pi}{a} \right)^2 A_5 - \left( \frac{n\pi}{b} \right)^2 A_4 \]

\[ J_{44} = \left( \frac{m\pi}{a} \right)^2 d_1 H_1 + \left( \frac{n\pi}{b} \right)^2 d_1 H_1 - \left( \frac{m\pi}{a} \right)^4 H_1 - \left( \frac{n\pi}{b} \right)^4 H_1 - 2 \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 H_1 - 4 \left( \frac{m\pi}{a} \right)^2 \left( \frac{n\pi}{b} \right)^2 H_6 \]

\[ -\left( \frac{m\pi}{a} \right)^2 A_5 - \left( \frac{n\pi}{b} \right)^2 A_4 \]

\[ J_{45} = d_1 L^2 - L^2 \left( \frac{m\pi}{a} \right)^2 - L^2 \left( \frac{n\pi}{b} \right)^2 \]

\[ J_{41} = J_{42} = 0 \]

\[ J_{32} = -L^2 \left( \frac{m\pi}{a} \right)^2 - L^2 \left( \frac{n\pi}{b} \right)^2 \]

\[ J_{55} = -\left( \frac{m\pi}{a} \right)^2 A_5 - \left( \frac{n\pi}{b} \right)^2 A_4 - L^2 \]

\[ J_{51} = J_{52} = 0 \]

\[ m_{11} = m_{22} = -I_0, m_{13} = \left( \frac{m\pi}{a} \right) I_1, m_{24} = \left( \frac{m\pi}{a} \right) J_1 \]

\[ m_{33} = \left( \frac{n\pi}{b} \right) I_1, m_{34} = \left( \frac{n\pi}{b} \right) J_1 \]

\[ m_{31} = m_{32} = m_{33}, m_{35} = - \left( I_0 + I_2 \left( \frac{n\pi}{b} \right)^2 + \left( \frac{m\pi}{a} \right)^2 \right), m_{44} = - \left( I_0 + J_2 \left( \frac{n\pi}{b} \right)^2 + \left( \frac{m\pi}{a} \right)^2 \right) \]

\[ m_{55} = -J_1, m_{41} = m_{42} = m_{43} = m_{44} = - \left( I_0 + K_2 \left( \frac{n\pi}{b} \right)^2 + \left( \frac{m\pi}{a} \right)^2 \right), m_{45} = m_{55} \]

\[ m_{51} = m_{52} = m_{53} = m_{55} = -K_1 \]
Appendix E: User subroutine for graded elements

User subroutine UMAT for graded elements for bending solution:

```fortran
SUBROUTINE UMAT(STRESS, STATEV, DDSDDE, SSE, SPD, SCD, RPL,
                  DDSDDT, DRPLDE, DRPLDT, STRAN, DSTRAN, TIME, DTIME, TEMP, DTEMP,
                  PREDEF, DPRED, CMNAME, NDI, NSHR, NTENS, NSTATV, PROPS, NPROPS,
                  COORDS, DROT, PNEWDT, CELENT, DFGRD0, DFGRD1, NOEL, NPT, LAYER,
                  KSPT, KSTEP, KINC)
!
! INCLUDE 'ABA_PARAM.INC'
!
CHARACTER*8 CMNAME
DIMENSION STRESS(NTENS), STATEV(NSTATV), DDSDDE(NTENS, NTENS),
               DDSDDT(NTENS), DRPLDE(NTENS), STRAN(NTENS), DSTRAN(NTENS),
               PREDEF(1), DPRED(1), PROPS(NPROPS), COORDS(3), DROT(3, 3),
               DFGRD0(3, 3), DFGRD1(3, 3)
DIMENSION EELAS(6), EPLAS(6), FLOW(6)
!
PARAMETER(ZERO=0.D0, ONE=1.D0, TWO=2.D0, THREE=3.D0,
           SIX=6.D0, EIGHT=9.D0, NEWTON=10, TOLER=1.0D-6)
!
----------------------------------------------------------------
! UMAT FOR ISOTROPIC ELASTICITY AND ISOTROPIC MISES PLASTICITY
! CANNOT BE USED FOR PLANE STRESS
! PROPS(1) - E(i)
! PROPS(2) - NU
! PROPS(3) - E(m)
! PROPS(4) - L - LENGHT OF PLATE
! COORDS(1) IS X-COORDINATE OF GAUSS POINTS.
! COORDS(2) IS Y-COORDINATE OF GAUSS POINTS.
! COORDS(3) IS Z-COORDINATE OF GAUSS POINTS.
!----------------------------------------------------------------
!
IF (NDI.NE.3) THEN
  WRITE (7, *) 'THIS UMAT MAY ONLY BE USED FOR ELEMENTS
WITH THREE DIRECT STRESS COMPONENTS'
  CALL XIT
ENDIF
!
! ELASTIC PROPERTIES

EMOD=(PROPS(1)+PROPS(3))/2
STATEV(1)=EMOD
ENU=PROPS(2)
!
EBULK3=EMOD/(ONE-TWO*ENU)
EG2=EMOD/(ONE+ENU)
EG=EG2/TWO
EG3=THREE*EG
ELAM=(EBULK3-EG2)/THREE
!
! ELASTIC STIFFNESS
!
DO K1=1, NDI
  DO K2=1, NDI
    DDSDDDE(K1, K2)=ELAM
  END DO
END DO
DDSDDDE(K1, K1)=EG2+ELAM
```

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User subroutine USDFLD for graded elements for vibration solution:

```
c------------------
        USDFLD subroutine
```

```
---

**subroutine** usdfld(field, statev, pnewdt, direct, t, celent, 
1 time, dtime, cmname, orname, nfield, nstatv, noel, npt, layer, 
2 kspt, kstep, kinc, ndi, nshr, coord, jmac, jmatyp, matlayo, 
3 laccfla)

**include** 'aba_param.inc'

**character** *80 cmname, orname
**character** *3 flgray(15)
**dimension** field(nfield), statev(nstatv), direct(3,3), 
1 t(3,3), time(2)
**dimension** array(15), jarray(15), jmac(*), jmatyp(*), 
1 coord(*)

---

**first inc. of first step ?**

**if** (kstep .eq. 1 .and. kinc .eq. 1) **then**

**read** x-coordinate

x = coord(1)

**calculate Young module**


ro = 1170.d0 + ((1180.d0-1170.d0) * (x / 1.d0)**3.d0)

**define depend state variable (E(y))**

field(1) = ro

field(2) = E

**save E value for state dependend variable**

statev(1) = ro

statev(2) = E

**do it for all others inc. and steps**

**else**

**assign initial values calculated in first inc.**

field(1) = statev(1)

field(2) = statev(2)

**end if**

**return**

**end**
Appendix F: The FORTRAN code for the integration scheme of damage modelling for graded elements

SUBROUTINE UMAT(STRESS, STATEV, DDSDE, SSE, SPD, SCD, RPL, DDSDT, DRPLDE, DRPLDT, STRAN, DSTRAN, TIME, DTIME, TEMP, DTEMP, PREDEF, DPRED, CMNAME, NDI, NSHR, NTENS, NSTATV, PROPS, NPROPS, COORDS, DROT, PNEWDT, CELENT, DFGRD0, DFGRD1, NOEL, NPT, LAYER, KSPT, KSTEP, KINC)

INCLUDE 'ABA_PARAM.INC'

CHARACTER*8 CMNAME

DIMENSION STRESS(NTENS), STATEV(NSTATV), DDSDE(NTENS, NTENS), DDSDDT(NTENS), DRPLDE(NTENS), STRAN(NTENS), DSTRAN(NTENS), PREDEF(1), DPRED(1), PROPS(NPROPS), COORDS(3), DROT(3, 3), DFGRD0(3, 3), DFGRD1(3, 3)

DIMENSION EELAS(6), EPLAS(6), FLOW(6), ELAS(6), STRIAL(6), ESTRESS(6)

PARAMETER(ZERO=0.D0, ONE=1.D0, TWO=2.D0, THREE=3.D0, SIX=6.D0, NEWTON=10, TOLER=1.0D-6)

!---------------------------------
! UMAT FOR ISOTROPIC PALSTIC DAMAGE OF FGM WITH VARIATION OF MATERIAL THROUGH
! THE LENGTH COUPLED WITH PLASTICITY
! CANNOT BE USED FOR PLANE STRESS
! PROPS(1) - E(i)
! PROPS(2) - NU
! PROPS(3) - E(m)
! PROPS(4) - YIELD STRESS FOR INCLUSION
! PROPS(5) - YIELD STRESS FOR MATRIX
! PROPS(6) - RINF AT HARDENING LAW FOR INCLUSION
! PROPS(7) - b AT HARDENING LAW FOR INCLUSION
! PROPS(8) - INITIAL DAMAGE TRESHOLD (STRAIN) Y0 FOR INCLUSION
! PROPS(9) - A in damage evolution law FOR INCLUSION
! PROPS(10) - B in damage evolution law FOR INCLUSION
! PROPS(11) - RINF AT HARDENING LAW FOR MATRIX
! PROPS(12) - b AT HARDENING LAW FOR MATRIX
! PROPS(13) - INITIAL DAMAGE TRESHOLD (STRAIN) Y0 FOR MATRIX
! PROPS(14) - A in damage evolution law FOR MATRIX
! PROPS(15) - B in damage evolution law FOR MATRIX
! PROPS(16) - LENGHT OF THE PLATE
! EELAS - ELASTIC STRAINS
! EPLAS - PLASTIC STRAINS
!COORDS(1) IS X-COORDINATE OF GAUSS POINTS.
!COORDS(2) IS Y-COORDINATE OF GAUSS POINTS.
!COORDS(3) IS Z-COORDINATE OF GAUSS POINTS.
!---------------------------------

IF (NDI.NE.3) THEN
WRITE (7, *) 'THIS UMAT MAY ONLY BE USED FOR ELEMENTS'
WITH THREE DIRECT STRESS COMPONENTS

CALL XIT
ENDIF

! GET THE STATE VARIABLE FROM THE PREVIOUS INCREMENT
! CALCULATE YIELD STRESS BASED ON RULE OF MIX.

EELAS=STATEV(1)
EPLAS=STATEV(NTENS+1)
HARD=STATEV(1+2*NTENS)
D=STATEV(2+2*NTENS)
STRESS=STATEV(3+3*NTENS)

! --------------------------------------------------------
SIGMA=PROPS(4)-PROPS(5)
YIELD=PROPS(5)+SIGMA*((COORDS(1)/PROPS(16))**3)

SIGMA2=PROPS(11)-PROPS(6)
RINF=PROPS(11)+SIGMA2*((COORDS(1)/PROPS(16))**3)

SIGMA3=PROPS(12)-PROPS(7)
BX=PROPS(12)+SIGMA3*((COORDS(1)/PROPS(16))**3)

SIGMA4=PROPS(8)-PROPS(13)
Y0=PROPS(13)+SIGMA4*((COORDS(1)/PROPS(16))**3)

SIGMA5=PROPS(9)-PROPS(14)
A=PROPS(14)+SIGMA5*((COORDS(1)/PROPS(16))**3)

SIGMA6=PROPS(15)-PROPS(10)
B=PROPS(15)+SIGMA6*((COORDS(1)/PROPS(16))**3)

! --------------------------------------------------------
! CALCULATE ISOTROPIC HARDENING
R=RINF*(1-exp(-BX*HARD))
YH=YIELD+R

! ELASTIC PROPERTIES WITHOUT DAMAGE
BETTA=PROPS(1)-PROPS(3)
EMOD=PROPS(3)+BETTA*((COORDS(1)/PROPS(16))**100)
ENU=PROPS(2)
EBULK3=EMOD/(ONE-TWO*ENU)
BULK=EBULK3/THREE
EG2=EMOD/(ONE+ENU)
EG=EG2/TWO
EG3=THREE*EG
ELAM=(EBULK3-EG2)/THREE

! COMPUTE ELASTIC STIFFNESS WITHOUT DAMAGE
!
DO K1=1, NDI
  DO K2=1, NDI
    DDSDDE(K2, K1)=ELAM
  END DO
  DDSDDE(K1, K1)=EG2+ELAM
END DO
DO K1=NDI+1, NTENS
  DDSDDE(K1, K1)=EG
END DO

! TRIAL STRESS FOR STRESS PREDICTOR
DO K1=1, NTENS
  DO K2=1, NTENS
    ESTRESS(K2)=ESTRESS(K2)+DDSDDE(K2, K1)*DSTRAN(K1)
  END DO
END DO
EELAS(K1)=EELAS(K1)+DSTRAN(K1)

! CALCULATE VON MISSES STRESS
ESMISES=(ESTRESS(1)-ESTRESS(2))**2+(ESTRESS(2)-ESTRESS(3))**2
+ (ESTRESS(3)-ESTRESS(1))**2
DO K1=NDI+1, NTENS
  ESMISES=ESMISES+SIX*ESTRESS(K1)**2
END DO
ESMISES=SQRT(ESMISES/TWO)

!!!!!! CHECK YIELDING
IF (ESMISES.GT.YH) THEN
  ! ACTIVELY YIELDING
  ! SEPARATE THE HYDROSTATIC FORM DEVIATORIC STRESS AND FLOW
  ESHYDRO=(ESTRESS(1)+ESTRESS(2)+ESTRESS(3))/THREE
  DO K1=1, NDI
    FLOW(K1)=(ESTRESS(K1)-ESHYDRO)/ESMISES
  END DO
  DO K1=NDI+1, NTENS
    FLOW(K1)=ESTRESS(K1)/ESMISES
  END DO
  ! SOLVE FOR TRUE PLASTIC MULTIPLIER
  ! INITIAL GUESS FOR PLASTIC MULTIPLIER
  PLMULT=ZERO
  HARD0=HARD
  R0=R
  ! INITIAL GUESS FOR HARDENING (r HARD=STATEV(3))
  ! NEWTON-RAPHSON ITERATION FOR FINDING THE PLASTIC MULTIPLIER
  DO KEWTON=1, NEWTON
    ! CURRENT YIELD STRESS
    RHS=ESMISES-EG3*PLMULT-YH
    DEFH=RINF*BX*(exp(-BX*HARD))
    HARD=HARD0+PLMULT
    R=R0+RINF*(1-exp(-BX*HARD))
    YH=YIELD+R
    IF (ABS(RHS).LT.TOLER) GOTO 10
  END DO
  ! WRITE WARNING MESSAGE TO THE.MSG FILE
  WRITE(7,2) NEWTON
  2 FORMAT//(//,30X,'***WARNING-PLASTICITY ALGH. DID NOT',
     !'CONVERGE')
  10 CONTINUE

! HAVING TRUE PLASTIC MULTIPLIER AND UPDATE EFFECTIVE YIELD STRESS
STATEV(1+2*NTENS)=HARD
R=RINF*(1-exp(-BX*HARD))
!YH=YIELD+R
! UPDATE STRESS, EELASTIC AND EPLASTIC
DO K1=1, NDI
  ESTRESS(K1)=FLOW(K1)*YH+ESHYDRO
  EPLAS(K1)=EPLAS(K1)+(THREE/TWO)*FLOW(K1)*PLMULT
  EELAS(K1)=EELAS(K1)-(THREE/TWO)*FLOW(K1)*PLMULT
END DO
DO K1=NDI+1, NTENS
  ESTRESS(K1)=FLOW(K1)*YH
  EPLAS(K1)=EPLAS(K1)+THREE*FLOW(K1)*PLMULT
  EELAS(K1)=EELAS(K1)-THREE*FLOW(K1)*PLMULT
END DO

! FORMULATE THE JACOBIAN MATRIX FOR PLASTICITY
EFFG=EG*YH/ESMISES
EFFG2=THREE*EFFG
EFFLAM=(EBULK3-EFFG2)/THREE
DEFH2=RINF*BX*(exp(-BX*HARD))
EFFHRD=EG3*DEFH2/(EG3+DEFH2)-EFFG3
DO K1=1, NDI
  DO K2=1, NDI
    DDSDDE(K2,K1)=EFFLAM
  END DO
  DDSDDE(K1,K1)=EFFG2+EFFLAM
END DO
DO K1=NDI+1, NTENS
  DDSDDE(K1,K1)=EFFG
END DO
ENDIF

! CALCULATE EFFECTIVE STRAIN
DO K1=1, NTENS
  ELAS(K1)=EELAS(K1)+EPLAS(K2)
END DO
Eeq=SQRT(ELAS(1)*ELAS(1)+ELAS(2)*ELAS(2)+ELAS(3)*ELAS(3))

!!!!!! CHECK DAMAGE
IF (Eeq.LT.Y0) THEN
  ! DAMAGE AT NEXT INCREMENT
  D=ZERO
ELSE
  D=(1-Y0*(1-A)/Eeq-A/exp(B*(Eeq-Y0)))
END IF

! REFORMULATE THE JACOBIAN MATRIX FOR DAMAGE
DO K1=1, NTENS
  DO K2=1, NTENS
    DDSDDE(K2,K1)=(ONE-D)*DDSDDE(K1,K2)
  END DO
END DO
DO K1=1, NTENS
  STRESS(K1)=(ONE-D)*ESTRESS(K2)
END DO
write(6,*) kinc, NOEL, Eeq, Y0, D

Y0 = MAX(Eeq, Y0)

STATEV(2+2*NTENS) = D

! STORE EFFECTIVE STRESS & STORE ELASTIC AND PLASTIC AND HARD IN
STATE VARIABLE ARRAY
DO K1=1, NTENS
   STATEV(K1) = EELAS(K1)
   STATEV(K1+NTENS) = EPLAS(K1)
   STATEV(2+2*NTENS) = ELAS(K1)
   STATEV(2+3*NTENS) = STRESS(K1)
END DO

write(6,*) kinc, NOEL, ESMISES, YH, ESTRESS(1), Eeq, D, STRESS(1), PLMULT
RETURN
END
Appendix G: Technical drawings of the experimental fixture
Appendix H: Calibration images for VW_8530 composite plate

First camera

Second camera
Appendix I: Labeled deflection for FG plates

VW_8530

8515_9895
Appendix J: Acceleration magnitude and phase plots

Homogenous plate (8530)
Composite plate (VW_8530_50%)
Composite plate (8515_9895_50%)
FG plate (VW_8530_Linear)

FG plate (VW_8530_Nonlinear)
FG plate (8515_9895_Linear)
FG plate (8515_9895_Nonlinear)
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