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A constructive Banach's inverse mapping theorem in F-spaces

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Abstract

Within Bishop's constructive mathematics (BISH) we prove that if T is a one-one sequentially continuous linear mapping of a separable F-space E onto an F-space F, then T^{-1} is sequentially continuous.

1 Introduction

A constructive version of Banach's inverse mapping theorem was proved in [10] for mappings between Banach spaces. In this note we show that the result can be extended to the context of F-spaces. The main difficulty in proving this result constructively is that we cannot apply the open mapping theorem in the form used by the classical argument, because it has no known constructive proof. For details on the constructive versions of this theorem the reader is referred to [7, 4, 5, 8] and Chapter 2 of [6], and for constructive mathematics in general to [1, 2, 12]. The technique the constructive proof is based on is proving that a certain property P holds by showing that there are two alternatives, P and Q, and that Q implies the **limited principle of omniscience**, (LPO)

> If a_n is a binary sequence, then either $a_n = 0$ for all n, or there exists n such that $a_n = 1$.

Then proving the classical form of the open mapping theorem using intuitionistic logic with LPO, we derive a contradiction, so we can rule out alternative Q. Hence P holds.

We introduce some definitions which will be used in the rest of the paper. A metric d on a vector space E is called **invariant** if

$$d(x + z, y + z) = d(x, y)$$

for all x, y, z in E. A topological vector space E is an **F-space** if its topology is induced by an invariant metric d with respect to which E is complete.

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In a metric space (X, d) we denote by $B(x, r)$ the closed ball of radius r centered in x. For a subset S of X we denote by $-S$ the metric complement of S, that is

$$-S = \{x \in X : \exists r > 0 \forall s \in S(d(x, s) > r)\}.$$

The inequality $\neq$ on X is defined by $x \neq y := (d(x, y) > 0)$.

2 Linear maps and F-spaces

The following propositions are generalizations of the results on Banach spaces in [3] to F-spaces.

Proposition 2.1 *Let T be a linear mapping between F-spaces E and F, and let x be a point of E such that $Tx \neq 0$. Then $x \neq y$ for all $y \in \mathrm{Ker}(T)$.*

Proof. Let $y \in \mathrm{Ker}(T)$, and construct an increasing binary sequence $\{\lambda_n\}$ such that

$$\begin{aligned}\lambda_n = 0 \quad &\Rightarrow \quad d(x, y) < \frac{1}{(n+1)^2} \\ \lambda_n = 1 - \lambda_{n-1} \quad &\Rightarrow \quad d(x, y) > 0.\end{aligned}$$

Then we construct a sequence $\{z_n\}$ in E and sequence $\{t_n\}$ in $\mathbf{R}$ as follows. If $\lambda_n = 0$, set $z_n := 0$ and $t_n := 0$; if $\lambda_n = 1 - \lambda_{n-1}$, then set $z_k := n(x - y)$ and $t_k := 1/n$ for all $k \geq n$. We will show that if $m \geq n$, then $d(z_m, z_n) < 1/n$. For $m \geq n$, either $\lambda_m = 0$, or $\lambda_n = 1$, or $\lambda_m = 1$ and $\lambda_n = 0$. In the first case we have $\lambda_n = 0$ and $d(z_m, z_n) = d(0, 0) = 0$. In the second case $\lambda_m = 1$ and there exists $k \leq n$ such that $d(z_m, z_n) = d(k(x - y), k(x - y)) = 0$. In the third case we have $\lambda_{k-1} = 0$, $\lambda_k = 1$ and $z_m = k(x - y)$ for some k with $n < k \leq m$. Hence

$$\begin{aligned}d(z_m, z_n) &= d(k(x - y), 0) \leq \sum_{j=1}^{k} d(j(x - y), (j-1)(x - y)) \\ &= kd(x - y, 0) = kd(x, y) \leq \frac{k}{k^2} = \frac{1}{k} < \frac{1}{n}.\end{aligned}$$

It follows that $\{z_n\}$ is a Cauchy sequence in E, hence it converges to an element z of E. Clearly, $\{t_n\}$ is a Cauchy sequence in $\mathbf{R}$ and converges to a limit t. Since $d(Tx, 0) > 0$, either $d(tTz, 0) < d(Tx, 0)$ or $d(tTz, 0) > 0$. In the first case, if $\lambda_n = 1 - \lambda_{n-1}$, then $z = n(x - y)$ and $t_n = 1/n$; hence $d(tTz, 0) = d(Tx, 0) < d(Tx, 0)$ – a contradiction. It follows that $d(tTz, 0) > 0$. Since the scalar multiplication is continuous, we have $t > 0$. Therefore there exists n with $\lambda_n = 1$, and so $x \neq y$. $\square$

A mapping f between metric spaces is said to be **strongly extensional** if $f(x) \neq f(y)$ implies $x \neq y$.

The following is an immediate corollary of the previous proposition.

Corollary 2.2 *A linear mapping between F-spaces is strongly extensional.*

Proposition 2.3 *A linear mapping T of an F-space E onto an F-space F is well-behaved, that is $Tx \neq 0$ whenever $x \neq y$ for all $y \in \mathrm{Ker}(T)$.*

Proof. Let x be in E such that $x \neq y$ for all y in $\mathrm{Ker}(T)$, and construct an increasing binary sequence $\{\lambda_n\}$ such that

$$\begin{aligned}\lambda_n = 0 \quad &\Rightarrow \quad d(Tx, 0) < \frac{1}{(n+1)^2} \\ \lambda_n = 1 - \lambda_{n-1} \quad &\Rightarrow \quad d(Tx, 0) > 0.\end{aligned}$$

Then we construct inductively a sequence $\{z_n\}$ in F and a sequence $\{t_n\}$ in $\mathbf{R}$ as follows. If $\lambda_n = 0$, set $z_n := 0$ and $t_n := 0$; if $\lambda_n = 1 - \lambda_{n-1}$, then set $z_k := nTx$ and $t_k := 1/n$ for all $k \geq n$. Then $\{z_n\}$ and $\{t_n\}$ are Cauchy sequences in F and $\mathbf{R}$, hence converge to z and t, respectively. Let $y \in E$ be such that $z = Ty$. We will show that $x - ty \in \mathrm{Ker}(T)$. To this end assume that $T(x - ty) \neq 0$. If there exists n such that $\lambda_n = 1 - \lambda_{n-1}$, then $Ty = z = nTx$ and $t = 1/n$, so $T(x - ty) = 0$ – a contradiction. Therefore $\lambda_n = 0$ for all n and so $t = 0, z = 0$ and $Tx = T(x - ty) = 0$. This new contradiction entails that $x - ty \in \mathrm{Ker}(T)$. Hence $x \neq x - ty$ or $ty \neq 0$, which means that $t \neq 0$, and so there exists n with $\lambda_n = 1$. $\square$

The notion of a located set plays an important role in constructive mathematics. If classically it is trivial to compute the distance from any point of the space to a subset, in BISH the infimum in question may not be always computable. A subset S of a metric space X is said to be **located** if for any $x \in X$ we can compute the distance

$$d(x, S) = \inf\{d(x, y) : y \in S\}.$$

Proposition 2.4 *Let M be a closed located subspace of an F-space E. Then E/M is an F-space with the invariant metric*

$$d_{E/M}(x, y) := d(x - y, M).$$

Proof. It is straightforward to see that $d_{E/M}$ is an invariant metric with the equality $=_{E/M}$ on E/M defined by

$$x =_{E/M} y \Leftrightarrow d_{E/M}(x, y) = 0,$$

and that E/M is a linear space. It remains to show that E/M is complete with respect to $d_{E/M}$. Let $\{x_n\}$ be a Cauchy sequence in E/M. Then taking an increasing sequence $\{k_n\}$ such that $d_{E/M}(x_m, x_{n_k}) < 2^{-k}$ for all $m \geq n_k$ and k, construct inductively a sequence $\{y_k\}$ in M such that $y_1 = 0$, and $d(x_{n_k} - x_{n_{k+1}}, y_{k+1} - y_k) < 2^{-k}$ or $d(x_{n_k} + y_k, x_{n_{k+1}} + y_{k+1}) < 2^{-k}$ for each k. Setting $z_k := x_{n_k} + y_k$, it is easy to see that $\{z_k\}$ is a Cauchy sequnce with respect to the metric d, and hence it converges to a limit z in E. Therefore noting that $d_{E/M}(x_{n_k}, z_k) = 0$ and $d_{E/M}(z_k, z) < 2^{-k+1}$, we have for each $m \geq n_k$,

$$\begin{aligned}d_{E/M}(x_m, z) \quad &\leq \quad d_{E/M}(x_m, x_{n_k}) + d_{E/M}(x_{n_k}, z_k) + d_{E/M}(z_k, z) \\ &< \quad 2^{-k} + 0 + 2^{-k+1} = 3 \cdot 2^{-k},\end{aligned}$$

so $\{x_n\}$ converges to z with respect to the metric $d_{E/M}$. $\square$

3 The main results

Lemma 3.1 *Let S be a separable subset of a metric space (X, d), and let C be a closed located subset of X. If LPO holds, then either $S \subset C$ or $S \cap -C \neq \emptyset$.*

Proof. Let $\{s_n\}$ be a dense sequence in S. By LPO, for each n, either $d(s_n, C) = 0$ or $d(s_n, C) > 0$. Hence another application of LPO shows that either $d(s_n, C) = 0$ for all n, or there exists n such that $d(s_n, C) > 0$. In the former case $S = \overline{\{s_n\}} \subset C$; in the latter case $S \cap -C \neq \emptyset$. □

Using LPO we can prove the following version of Baire's category theorem.

Lemma 3.2 *Assuming* LPO*, if $\{C_n\}$ is a sequence of closed located subsets of a complete separable metric space X with $X = \bigcup_{n=1}^{\infty} C_n$, then C_n has nonvoid interior for some n.*

Proof. Note that any ball B in X is separable, and hence either $B \subseteq C_n$ or $B \cap -C_n \neq \emptyset$ by Lemma 3.1. Let $x_0 \in E$ and $r_0 := 1$. Construct an increasing binary sequence $\{\lambda_n\}$ with $\lambda_0 = 0$, a sequence $\{x_n\}$ in X, and a sequence $\{r_n\}$ of real numbers, such that for each $n \geq 1$,

1. $0 < r_n < r_{n-1}/2$,
2. $\lambda_n = 0 \implies B(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \cap -C_n$,
3. $\lambda_n = 1 \implies x_n = x_{n-1}$ and C_k has nonvoid interior for some $k \leq n$.

We proceed by induction. Assume we have constructed λ_n, x_n, and r_n. If $\lambda_n = 1$, set

(*) $\lambda_{n+1} := 1$, $x_{n+1} := x_n$, and $r_{n+1} := r_n/2$.

Otherwise, $\lambda_n = 0$ and either $B(x_n, r_n) \subseteq C_{n+1}$ or $B(x_n, r_n) \cap -C_{n+1} \neq \emptyset$. In the first case, define λ_{n+1}, x_{n+1}, and r_{n+1} by (*). In the second case, set $\lambda_{n+1} := 0$, and choose x_{n+1} in $-C_{n+1}$ and r_{n+1} with $0 < r_{n+1} < r_n/2$, such that $B(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n) \cap -C_{n+1}$. This completes the induction.

If $m > n \geq 0$, we have

$$d(x_m, x_n) \leq \sum_{i=n+1}^{m} d(x_{i-1}, x_i) \leq \sum_{i=n+1}^{m} r_{i-1} < r_n.$$

Hence $\{x_n\}$ is a Cauchy sequence in X, and so converges to a limit x in X such that $x \in B(x_n, r_n)$ for all n. Choose N such that $x \in C_N$. Then if $\lambda_N = 0$, we have $x \in B(x_N, r_N) \subseteq -C_N$, a contradiction. Hence $\lambda_N = 1$, and therefore C_k has nonvoid interior for some $k \leq N$. □

We will only sketch the proof of the following lemma which in the presence of LPO is just the classical argument [11].

Lemma 3.3 *Assuming* LPO, *if T is a sequentially continuous linear mapping of a separable F-space E onto an F-space F, then T is open.*

Proof. Let d be the invariant metric on E and let V be a neighbourhood of 0 in E. Define

$$V_n := \{x : d(x, 0) < 2^{-n}r\},$$

where $r > 0$ is chosen such that $V_0 \subset V$. We will show that there exists a neighbourhood W of 0 in F such that

$$W \subset \overline{T(V_1)} \subset T(V).$$

Note that V_2 is separable, and by sequential continuity of T, we have $T(V_2)$ separable. Also $T(V_2)$ is located in F by [[10], Lemma 2]. Hence writing $F = \bigcup_{k\geq 1} \overline{kT(V_2)}$ we can use Lemma 3.2 to see that $\overline{T(V_2)}$ has nonempty interior; thus $\overline{T(V_1)}$ has nonempty interior.

Fix now $y_1 \in \overline{T(V_1)}$. For $n \geq 1$ assume y_n has been constructed in $\overline{T(V_n)}$. Using a similar argument as above, we can show that $\overline{T(V_{n+1})}$ has nonempty interior, so

$$\{y_n - z : z \in \overline{T(V_{n+1})}\} \cap T(V_n) \neq \emptyset.$$

Hence there exists $x_n \in V_n$ such that

$$Tx_n \in y_n + \overline{T(V_{n+1})}.$$

Set $y_{n+1} := y_n - Tx_n$; then $y_{n+1} \in \overline{T(V_{n+1})}$. Inductively, we construct a sequence $\{x_n\}$ such that $d(x_n, 0) < 2^{-n}r$ for each $n \geq 1$, and the sums $x_1 + x_2 + \ldots + x_n$ form a Cauchy sequence which converges to some $x \in E$, with $d(x, 0) < r$. Thus $x \in V$, and by continuity of T, we show that $y_1 = Tx \in T(V)$. □

Now we can prove the following version of Banach's inverse mapping theorem.

Theorem 3.4 *Let T be a one-one sequentially continuous linear mapping of a separable F-space E onto an F-space F. Then T^{-1} is sequentially continuous.*

Proof. Let $\{Tx_n\}$ be a sequence in F converging to 0, and note that T^{-1} is strongly extensional by Corollary 2.2. Then by [9, Lemma 2], for each $\epsilon > 0$, either $d(x_n, 0) > \epsilon/2$ for infinitely many n or $d(x_n, 0) < \epsilon$ for all sufficiently large n. In the former case, taking a subsequence, we may assume that $d(x_n, 0) > \epsilon/2$ for all n. Therefore there exists a neighbourhood V of 0 such that $x_n \in\sim V$, and so $Tx_n \in\sim T(V)$ by the strong extensionality of T^{-1}. Moreover LPO holds by [10, Lemma 1]. Hence by Lemma 3.3, there exists a neighbourhood W of 0 with $W \subseteq T(V)$, a contradiction. Thus $d(x_n, 0) < \epsilon$ for all sufficiently large n. Since $\epsilon > 0$ is arbitrary, it follows that T^{-1} is sequentially continuous and this concludes the proof. □

A linear mapping T between F–spaces is **sequentially open** if, whenever $Tx_n \to 0$, there exists a sequence $\{y_n\}$ in $\mathrm{Ker}(T)$ such that $x_n + y_n \to 0$.

We omit the proofs of the following corollaries. For details the reader is referred to [10].

Corollary 3.5 *Let T be a sequentially continuous linear mapping of a separable F-space E onto an F-space F such that* $\mathrm{Ker}(T)$ *is located. Then T is sequentially open.*

Corollary 3.6 *Let T be a linear mapping between F-spaces such that* graph(T) *is closed and separable. Then T is sequentially continuous.*

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