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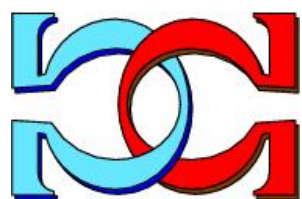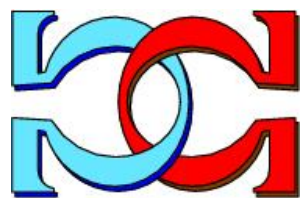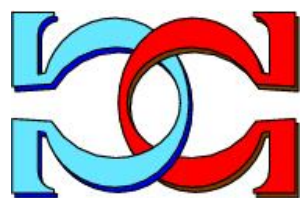**Research**
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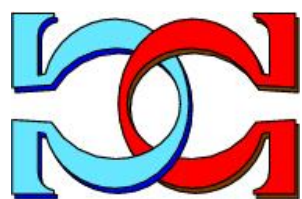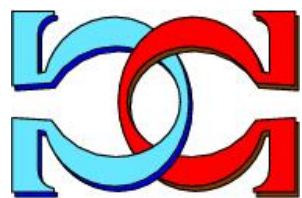
Large Scale Geometries of Infinite Strings

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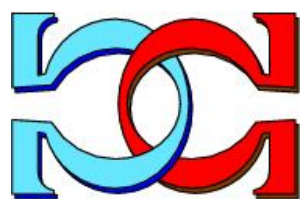
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Large scale geometries of infinite strings

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Abstract: We introduce geometric consideration into the theory of formal languages. We aim to shed light on our understanding of global patterns that occur on infinite strings. We utilise methods of geometric group theory. Our emphasis is on large scale geometries. Two infinite strings have the same large scale geometry if there are colour preserving bi-Lipschitz maps with distortions between the strings. Call these maps quasi-isometries. Introduction of large scale geometries poses several questions. The first question asks to study the partial order induced by quasi-isometries. This partial order compares large scale geometries; as such it presents an algebraic tool for classification of global patterns. We prove there is a greatest large scale geometry and infinitely many minimal large scale geometries. The second question is related to understanding the quasi-isometric maps on various classes of strings. The third question investigates the sets of large scale geometries of strings accepted by computational models, e.g. Büchi automata. We provide an algorithm that describes large scale geometries of strings accepted by Büchi automata. This links large scale geometries with automata theory. The fourth question studies the complexity of the quasi-isometry problem. We show the problem is Σ_3^0-complete thus providing a bridge with computability theory. Finally, the fifth question asks to build algebraic structures that are invariants of large scale geometries. We invoke asymptotic cones, a key concept in geometric group theory, defined via model-theoretic notion of ultra-product. Partly, we study asymptotic cones of algorithmically random strings thus connecting the topic with algorithmic randomness.

Key words: Quasi-isometry, Büchi automata, ultra-filter, asymptotic cones, Σ_3^0-completeness, Martin-Löf randomness.

1   Introduction

Our goal is to introduce geometric considerations into the theory of formal languages. We emphasise the study of large scale geometries of infinite strings. Our hope is to shed light on our understanding of global large scale patterns that occur on infinite strings. Our motivation comes from geometric group theory, a cutting edge research area in group theory linked with geometry, topology, automata, logic, probability, complexity, etc.

In geometric group theory, an important concept is that of quasi-isometry between metric spaces. Let $\mathcal{M}_1 = (M_1, d_1)$ and $\mathcal{M}_2 = (M_2, d_2)$ be metric spaces.

Definition 1.1. A function $f : M_1 \rightarrow M_2$ is called an $(A, B)-$**quasi-isometry** from $\mathcal{M}_1$ to $\mathcal{M}_2$, where $A \geqslant 1$ and $B \geqslant 0$, if for all $x, y \in M_1$ we have

$$(1/A) \cdot d_1(x, y) - B \leqslant d_2(f(x), f(y)) \leqslant A \cdot d_1(x, y) + B,$$

and for all $y \in M_2$ there exists an element $x \in M_1$ such that $d_2(y, f(x)) \leqslant A$.

Note that when $B = 0$, the mapping becomes bi-Lipschitz (and hence continuous). Thus, a quasi-isometry $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ behaves like a bi-Lipschitz map with distortion B between the metric spaces. For instance, the metric spaces $\mathbb{Z}$ (integers) and $\mathbb{R}$ (reals) with their natural metric are quasi-isometric Informally, two metric spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are quasi-isometric if these spaces (such as $\mathbb{Z}$ and $\mathbb{R}$) look the same from far away. The quasi-isometry relation forms an equivalence relation on the class of all metric spaces.

Studying quasi-isometry invariants of groups turned out to be crucial in solving many problems in group theory [6] [7] [8]. Therefore, finding quasi-isometry invariants has become an important theme in geometric group theory. Examples of quasi-isometry invariants are: being virtually nilpotent, virtually free, hyperbolic, having polynomial growth rate, being finitely presentable, having decidable word problem, asymptotic cones [4] [5] [8].

In formal language theory and logic, one of the main objects are infinite strings α over finite alphabets Σ.

These objects are somewhat boring from a geometry point of view. The strings α possess the natural metric inherited from the set of natural numbers ω. The quasi-isometry type of ω is the metric space $\mathbb{R}_{\geqslant 0}$ of all positive reals. So, from a quasi-isometry view point, ω viewed as a metric space is also somewhat uneventful. However, one crucial difference from the setting in geometric group theory is that the metric spaces α are coloured. Namely, for every position $i \in \omega$ in α, the colour of the position i is σ when $\alpha(i) = \sigma$. These observations suggest that the notions and methods of geometric group theory (e.g. quasi-isometry) can be applied to coloured metric spaces. Here we investigate quasi-isometries of coloured metric spaces with strings α as our primary objects, thus initiating the study of large scale patterns on strings.

A **coloured metric space** $\mathcal{M}$ is a tuple $(M; d, C)$, where $(M; d)$ is the **underlying metric space** with metric d, C is a colour function $C : M \rightarrow 2^{\Sigma}$, and Σ is a finite set of colours that we call an alphabet. If $\sigma \in C(m)$ then we say that m *has colour* σ. As mentioned above, infinite strings α are coloured metric spaces of the form $(\omega; d, \alpha)$, where d is the natural metric (so, $d(i, j) = |i - j|$) and $\alpha : \omega \rightarrow \Sigma$ is the colour function. Every element in α has a unique colour. We always assume that the cardinality $|\Sigma|$ of Σ is at least 2.

We denote the set of all infinite strings over Σ, considered as coloured metric spaces, by Σ^{ω}. We now adapt Definition 1.1 for coloured metric spaces.

Definition 1.2. Assume we are given coloured metric spaces $\mathcal{M}_1 = (M_1; d_1, C_1)$ and $\mathcal{M}_2 = (M_2; d_2, C_2)$. A *colour preserving* quasi-isometry from $(M_1; d_1)$ into $(M_2; d_2)$ is called a **quasi-isometry** from $\mathcal{M}_1$ into $\mathcal{M}_2$.

We write $\mathcal{M}_1 \leqslant_{QI} \mathcal{M}_2$ if there is a quasi-isometry from $\mathcal{M}_1$ into $\mathcal{M}_2$. For metric spaces the relation $\leqslant_{QI}$ is an equivalence relation, and hence it is symmetric. In contrast, for coloured metric spaces the relation $\leqslant_{QI}$ is not symmetric. For instance, the mapping $f : \omega \rightarrow \omega$ defined as $f(i) = 2i$ is a quasi-isometry from the string $0^{\omega} = 00000\ldots$ into the string $(01)^{\omega} = 010101\ldots$. There is no quasi-isometric mapping in the opposite direction.

If $\mathcal{M}_1 \leqslant_{QI} \mathcal{M}_2$ and $\mathcal{M}_2 \leqslant_{QI} \mathcal{M}_1$ then we write this by $\mathcal{M}_1 \sim_{QI} \mathcal{M}_2$. The relation $\sim_{QI}$ is an equivalence relation in the class of all coloured metric spaces, in particular on the set Σ^{ω}. Formally:

Definition 1.3. The equivalence classes of $\sim_{QI}$ are called **the quasi-isometry types** or, equivalently, **the large scale geometries**. For the set Σ^{ω}, we define $\Sigma^{\omega}_{QI} = \Sigma^{\omega} / \sim_{QI}$. Denote the large scale geometry of string α by $[\alpha]$. Thus, $\Sigma^{\omega}_{QI} = \{[\alpha] \mid \alpha \in \Sigma^{\omega}\}$.

Non-symmetry of $\leqslant_{QI}$ on coloured metric spaces allows us to compare large scale geometries of these spaces, and consider the partial order $\leqslant_{QI}$ on the quasi-isometry types. Importantly, the partial order $\leqslant_{QI}$ can be restricted to the large scale geometries of coloured metric spaces over a fixed underlying metric space (e.g. ω with its natural metric). In this sense the quasi-isometry on coloured metric spaces is a much refined version of the usual quasi-isometry relation on unclouded metric space. This is because the quasi-isometry type of every uncoloured metric space trivialises to a singleton.

Introduction of large scale geometries and the quasi-isometry relation $\leqslant_{QI}$ poses a vast amount of natural questions. The contribution of this paper consists of investigating the following questions:

The first question is concerned with understanding the partial order $\leqslant_{QI}$ on the set Σ^{ω}_{QI} of all large scale geometries. The order presents an algebraic tool aimed at classification of global patterns that occur on strings. In Section 2 we investigate properties of this partial order. Among several results, we prove that the order has the greatest large scale geometry, infinite chains and infinite anti-chains. We show that there are infinitely many minimal large scale geometries.

The second question is related to understanding the quasi-isometry relation $\sim_{QI}$. This can be done by either restricting the relation $\sim_{QI}$ on various subclasses of infinite strings or by simplifying the definition of quasi-isometry. In Section 3.1 we restrict the relation $\sim_{QI}$ to eventually periodic words, and give a full description of quasi-isometry types in this class. Section 3.2 gives a intuitively more easier yet equivalent definition of the relation $\leqslant_{QI}$ that we call component-wise reducibility. We also give more refined version of $\leqslant_{QI}$, colour-equivalence, that implies quasi-isometry. It is shown that colour-equivalence is strictly stronger than quasi-isometry. It is natural to ask if quasi-isometric maps between strings can be replaced with order preserving quasi-isometric embeddings. We give a negative answer to this question; however, we prove that the answer is positive modulo $\sim_{QI}$ equivalence relation.

The third question is related to describing sets of large scale geometries. We call such sets atlases. Let L be a language of infinite strings. One considers the atlas $[L]$ of all large scale geometries of strings in L. So, the question is related to understanding the atlas $[L]$ given a description of L. In particular, a natural question is if one can decide that $[L_1] = [L_2]$ given descriptions of languages L_1 and L_2. In Section 4, we use Büchi automata as a description language and provide a full characterisation of

the atlases defined by Büchi recognisable languages. As a consequence, we show that for Büchi automata recognisable languages L_1 and L_2 there is a linear time algorithm for deciding the equality of the atlases $[L_1]$ and $[L_2]$. This contrasts with the PSPACE completeness of the equality problem for Büchi languages. This part of the work links large scale geometries with automata theory and complexity theory.

The fourth question addresses the complexity of quasi-isometry relation on infinite strings in terms of arithmetical hierarchy, thus connecting the topic with computability theory. Recall that isometry is colour preserving and distance preserving map. So, quasi-isometry relation is weaker than isometry relation. Hence, one might expect that quasi-isometry is easier to detect than the isometry. We prove that the quasi-isometry relation on computable infinite strings is Σ_3^0-complete. This is in contrast to Π_1^0-completeness of the isometry relation on computable strings. These are explained in Section 5.

The fifth question asks how one encodes large scale geometries of coloured metric spaces α into "limit" structures. For this, we define structures obtained from "looking at α from far away". Dries and Wilkie [4], using ultra-filters, formalised this intuitive notion through the concept of asymptotic cone. Their work gave a logical context to Gromov's work in [6] [8]. We invoke the concept of asymptotic cone and study relationship between large scale geometries of infinite strings and their asymptotic cones. In Section 6 we prove theorems akin to results on asymptotic cones in geometric group theory, and we show that asymptotic cones of Martin-Löf random strings coincide when scaling factors are computable. These results bridge the topic of this paper with algorithmic randomness and model theory.

2 The partial order $(\Sigma_{QI}^\omega, \leqslant_{QI})$

2.1 Basic Properties

The quasi-isometry relation $\leqslant_{QI}$ naturally induces the partially ordered set $(\Sigma_{QI}^\omega, \leqslant_{QI})$ on the set of all large scale geometries $[\alpha]$. Denote this partial order by Σ_{QI}^ω thus identifying it with its domain. Say that $[\beta]$ *covers* $[\alpha]$ if $[\alpha] \neq [\beta]$, $[\alpha] \leqslant_{QI} [\beta]$, and for all $[\gamma]$ such that $[\alpha] \leqslant_{QI} [\gamma] \leqslant_{QI} [\beta]$ we have $[\gamma] = [\alpha]$ or $[\gamma] = [\beta]$. An element is an *atom* if it covers a minimal element.

Proposition 2.1. *The partial order Σ_{QI}^ω has the following properties: (1) There exists a greatest element; (2) There exist at least $|\Sigma|$ minimal elements; (3) There exist at least $|\Sigma| \cdot (|\Sigma| - 1)/2$ atoms.*

Proof. Assume that $\Sigma = \{\sigma_1, ..., \sigma_k\}$. For part (1), we claim that $\alpha = (\sigma_1...\sigma_k)^\omega$ is the greatest element in Σ_{QI}. Indeed, take any $\beta \in \Sigma^\omega$. We write α as $v_0 v_1 \ldots$ where each v_i is $\sigma_1 \ldots \sigma_k$. Define $f : \beta \to \alpha$ such that f maps any position n (in β) to the position k_n in the portion v_n of the string α such that $\beta(n) = \alpha(k_n)$. For part (2), consider the quasi-isometry types $[\sigma_i^\omega]$. These form minimal elements in Σ_{QI}^ω. Indeed, if $\alpha \leqslant_{QI} \sigma_i^\omega$, then each element of α has colour σ_i. For part (3), consider $\sigma_i(\sigma_j)^\omega$, where $i \neq j$. Clearly, $\sigma_j^\omega \leqslant_{QI} \sigma_i(\sigma_j)^\omega$. Moreover, for any $\beta \neq \sigma_j^\omega$ if $\beta \leqslant_{QI} \sigma_i(\sigma_j)^\omega$ then β finitely many positions of β are coloured with σ_i, and all other positions are coloured with σ_j. Hence, $\beta \sim_{QI} \sigma_i(\sigma_j)^\omega$. Thus, we have found at least $k \cdot (k-1)/2$ of the atoms. $\square$

A quasi-isometry $g : \alpha \to \beta$ can produce *cross-overs*, e.g., pairs n and m with $n < m$ but $g(m) < g(n)$. The definition of quasi-isometry does not obviously prohibit large cross-overs. Nevertheless, the following lemma shows that there is a uniform bound on cross-overs.

Lemma 2.2 (Small Cross Over Lemma). *Consider a quasi-isometry map $g : \alpha \to \beta$. There is a constant $C \leqslant 0$ such that for all $n < m$ we have $g(m) - g(n) \geqslant C$.*

Proof. Let $n < m$ be given and suppose we have $g(m) - g(n) < C$ for some $C \leqslant 0$. The goal is to provide a lower bound for C. Define $q = \min(g^{-1}(p) \cap [m+1, \infty))$ and $p = \min(g([m+1, \infty)) \cap [g(n)+1, \infty))$. Then, we obtain

$$
\begin{aligned}
d(g(n), p) &\geqslant (1/A) \cdot d(n, q) - B \geqslant (1/A)d(n, m) - B \\
&\geqslant (1/A)(1/A)(d(g(n), g(m)) - B) - B \\
&\geqslant -C/A^2 - ((A^2+1)/(A^2))B.
\end{aligned}
$$

We have $g([m, q]) \subset [0, g(n)] \cup [p, \infty)$ by our definition of p and q. Note $g(m) \in [0, g(n)]$ and $g(q) \in [g(p), \infty)$. Hence, there exists $r_C \in [m, q-1]$ such that $g(r_C) \in [0, g(n)]$ and $g(r_C + 1) \in [g(p), \infty)$, which means

$$
d(g(n), p) \leqslant d(g(r_C), g(r_C + 1)) \leqslant A + B.
$$

From these inequalities we have a lower bound on C. $\square$

Lemma 2.3. *Let $f : \alpha \to \beta$ be a quasi-isometric mapping. Then there exists constants A' and B' such that for all positions $x, y \in \beta$ with $x \leqslant y$ we have*

$$
(1/A') \cdot d_2(x, y) - B' \leqslant
$$
$$
d_1(\min f^{-1}([x, y]), \max f^{-1}([x, y])) \leqslant A' \cdot d_2(x, y) + B'.
$$

Proof. See Appendix. $\square$

Corollary 2.4. *Let $f : \alpha \to \beta$ be a quasi-isometry. There is a $C > 0$ so that $|f^{-1}(y)| < C$ for all $y \in \beta$.* $\square$

2.2 Structure theorems

Elementary properties of the partial order Σ_{QI}^ω are in Proposition 2.1. Now we provide several structure theorems describing algebraic properties Σ_{QI}^ω.

Theorem 2.5. *The partial order Σ_{QI}^ω has a chain $(\alpha_n)_{n\in\mathbb{Z}}$ of the type of integers, that is $\forall n \in \mathbb{Z}[\alpha_n <_{QI} \alpha_{n+1}]$. Furthermore, the partial order Σ_{QI}^ω has a countable anti-chain.*

Proof. We prove the first part. Take two distinct element in Σ, say 0 and 1, respectively. Let

$$\alpha_n = \begin{cases} (01)^{2^n}(011)^{2^n}...(01^{2^k})^{2^n}... & (n \geqslant 0) \\ (01)(01^{2^{2^{-n}}})...(01^{(2^k)^{2^{-n}}})... & (n < 0) \end{cases}$$

The idea is the following. The string α_0 is of the form:

$$(01)(011)(01111)\ldots(01^{2^k})\ldots$$

Let us call the substrings (01^{2^k}) blocks of α_0. The above definition tells us that α_1 is obtained from α_0 by doubling each block of α_0; α_{-1} is obtained from α_0 by removing every other block. This is propagated to all α_n's.

We show that $\alpha_n <_{QI} \alpha_{n+1}$ for all $n \geqslant 0$. For negative n, the proof is similar. To see $\alpha_n \leqslant_{QI} \alpha_{n+1}$, consider the mapping which sends an interval $(01^{2^{k+1}})$ in α_n to $(01^{2^k})^2$ in α_{n+1} in an obvious injective way. This mapping is a quasi-isometry from α_n into α_{n+1}.

To see that the converse is not true, assume that g is a quasi-isometry from α_{n+1} into α_n. Then using Lemma 2.3 and Lemma 2.2, one sees that g needs to be strictly monotone on almost every point of α_{n+1} with colour 0; that is, we need to have $M \in \mathbb{N}$ such that for all $m, m' > M$ and $m > m'$ we have

$$\alpha_{n+1}(m) = \alpha_{n+1}(m') = 0 \Rightarrow g(m) > g(m')].$$

But this is not possible; indeed, let $m_{i,n} \in \omega$ be the position of i-th 0 in α_n. If $m_{i,n+1} > M$ and $g(m_{i,n+1}) = m_{j,n}$, then from monotonicity and injectivity we should have $d(g(m_{i,n+1}), g(m_{i+k,n+1})) \geqslant m_{j+k,n} - m_{j,n}$ for each $k \in \mathbb{N}$. For any i and j we can easily verify

$$\lim_{k\to\infty} \frac{m_{j+k,n} - m_{j,n}}{m_{i+k,n+1} - m_{i,n+1}} = +\infty,$$

which means we cannot have any bound A as in Definition 1. This is contradiction, and hence there is no quasi-isometry from α_{n+1} into α_n.

Now we prove the second part. Consider the following sequence of strings β_n, $n \in \omega$:

$$\beta_n = 010^{2^n}1^{2^n}0^{3^n}1^{3^n}...0^{k^n}1^{k^n}...$$

We claim that this sequence forms an anti-chain in Σ_{QI}^ω. Take any $n, m \in \mathbb{N}$ and suppose $\beta_n \leqslant_{QI} \beta_m$ via f. It suffices to show $n = m$.

Let $A_{n',k'}$ be the "k'-th block" of zeros in $\beta_{n'}$, i.e.

$$A_{n',k'} = [2\sum_{i=0}^{k'-1} i^{n'}, 2\sum_{i=0}^{k'-1} i^{n'} + (k')^{n'} - 1]$$

Then by easy argument we can show that there exists $k, l \in \mathbb{N}$ such that (1) $f(A_{n,k}) \subseteq A_{m,l}$, and (2) for every $k' \in \mathbb{N}$, we have $f(A_{n,k'+k}) \subseteq A_{m,k'+l}$. Intuitively, these say that from some point f maps β_n in "block by block" manner without vacancy. For quasi-isometricity of f we should have an upper and positive lower bound of the rate

$$\frac{|A_{n,k'+k}|}{|A_{m,k'+l}|} = \frac{(k'+k)^n}{(k'+l)^m}$$

with respect to k', since otherwise we do not have any bound A as in Definition 1. Clearly $n = m$ is the only case that satisfies this condition. $\square$

The trivial large scale geometries $[0^\omega]$ and $[1^\omega]$, as noted above, are minimal elements of the partial order Σ_{QI}^ω. The next theorem shows that there are non-trivial minimal large scale geometries.

Theorem 2.6. *Let $\{a_n\}_{n\in\mathbb{N}}$ be an unbounded nondecreasing sequence. Then the large scale geometry of the string $\alpha = 0^{a_0}1^{a_1}0^{a_2}1^{a_3}...0^{a_{2k}}1^{a_{2k+1}}...$ is a minimal element in the partial order Σ_{QI}^ω.*

Proof. Let I_n be the interval that corresponds to p^{a_n} ($p \in \{0,1\}$) in α, that is,

$$I_n = [\sum_{k=0}^{n-1} a_k, (\sum_{k=0}^n a_k) - 1].$$

Suppose $\beta \leqslant_{QI} \alpha$ via f, A, B. We claim that β is of the form $vw_0'w_1w_1'w_2w_2'\ldots w_nw_n'\ldots$, where w_n and w_n' are sequences for which we have constants A', B', D, and n_0 such that for all n we have $|w_n'| \leqslant D$, $(1/A')\cdot a_{n_0+n} - B' \leqslant |w_n| \leqslant A'a_{n_0+n} + B'$ and $w_n = 0^{|w_n|}$ if $n_0 + n$ is even or $w_n = 1^{|w_n|}$ if $n_0 + n$ is odd. If the claim is true, then it is easy to see $\beta =_{QI} \alpha$.

Let $M_n = \max f^{-1}(I_n)$ and $m_n = \min f^{-1}(I_n)$. From Lemma 2.3, there exist A'' and B'' such that

$$(1/A'')\cdot a_n - B'' \leqslant M_n - m_n \leqslant A''a_n + B''$$

for each n. Then there are numbers n_0 and $D > 0$ such that for all $n \geqslant n_0$ we have $m_{n+2} > M_n$ and

$-D \leqslant m_{n+1} - M_n \leqslant 1$. Indeed, let $D = B - A(C-1)$ and n_0 satisfies $a_{n_0} \geqslant A''(-2AC + 3\max\{B, B''\}) + 2$, where C is a constant such that $f(y) - f(x) \geqslant C$ for all $x < y$ (Lemma 2.2). Then $m_{n+1} - M_n \leqslant 1$ holds since otherwise there exists $x \in \mathbb{N}$ such that $M_n < x < m_{n+1}$ which should be mapped to a point other than $I_n \cup I_{n+1}$, and we can find $x \in [M_n, m_{n+1} - 1]$ such that $d(f(x), f(x+1)) > D \geqslant A + B$, which contradicts the quasi-isometricity of f. For a lower bound, if $m_{n+1} < M_n$ then $f(M_n) - f(m_{n+1}) \geqslant C$ due to Lemma 2.2, and hence

$$
\begin{aligned}
m_{n+1} - M_n &= -d(m_{n+1}, M_n) \\
&\geqslant -A \cdot d(f(m_{n+1}), f(M_n)) - B \\
&= A \cdot (f(M_n) - f(m_{n+1})) - B \\
&\geqslant AC - B > -D.
\end{aligned}
$$

We also have

$$
m_{n+2} - M_n =
$$
$$
(m_{n+2} - M_{n+1}) + (M_{n+1} - m_{n+1}) + (m_{n+1} - M_n)
$$
$$
\geqslant 2AC - 2B + (1/A'') \cdot a_{n+1} - B'' > 1.
$$

Now for each $n \geqslant n_0$ let $J_n = [M_{n-1} + 1, m_{n+1} - 1]$ and $J_n' = [m_{n+1}, M_n]$ if $m_{n+1} < M_n$, or otherwise $J_n' = \phi$. Let w_n and w_n' be the strings that corresponds to J_{n_0+n} and J_{n_0+n}', respectively. Then β is of the form $v w_0' w_1 w_1' w_2 w_2' \ldots$, and by the inequality above

$$
|w_n| = m_{n_0+n+1} - M_{n_0+n-1} - 1 \geqslant
$$
$$
(1/A'') \cdot a_{n_0+n} + 2AC - 2B - B''
$$

We also have

$$
|w_n| \leqslant M_{n_0+n} - m_{n_0+n} + 1 \leqslant A'' a_{n_0+n} + B'' + 1.
$$

Hence letting $B' = \max\{-2AC + 2B + B'', B'' + 1\}$ and $A' = A''$ and we have a proof. $\quad\square$

Corollary 2.7. *The partially ordered set Σ_{QI}^{ω} possesses uncountably many minimal elements.* $\quad\square$

We now single out four subsets of Σ_{QI}^{ω} that give some insight into lattice-theoretic properties of the partial order $\leqslant_{QI}$. Consider the string

$$
\alpha = 0^{n_0} 1^{m_0} 0^{n_1} 1^{m_1} \ldots
$$

from the set $\{0,1\}^{\omega}$, where $n_i, m_i \geqslant 1$. Call the substrings 0^{n_i} and 1^{m_i} of the string the 0-**blocks and 1-blocks**, respectively. Define:

- $\mathcal{X}(0) = \{[\alpha] \mid \text{in } \alpha \text{ all the lengths of 0-blocks are universally bounded}\}$,

- $\mathcal{X}(1) = \{[\alpha] \mid \text{in } \alpha \text{ the lengths of all 1-blocks are universally bounded}\}$,

- $\mathcal{X}(u) = \{[\alpha] \mid \text{in } \alpha \text{ there is no universal bound on the lengths of both 0-blocks and 1-blocks }\}$,

- $\mathcal{X}(b) = \{[\alpha] \mid \text{in } \alpha \text{ the lengths of both 0-blocks and 1-blocks are universally bounded}\}$.

Upward closed sets are called *filterers*, and downward closed sets *ideals*. We use this terminology in the next theorem. A proof is in the Appendix.

Theorem 2.8. *The sets $\mathcal{X}(0)$, $\mathcal{X}(1)$, $\mathcal{X}(u)$, $\mathcal{X}(b)$ satisfy the following:*

1. *The sets $\mathcal{X}(0)$ and $\mathcal{X}(1)$ are filters.*

2. *The set $\mathcal{X}(u)$ is an ideal.*

3. *The set $\mathcal{X}(b)$ is the singleton $\{[(01)^{\omega}]\}$.* $\quad\square$

Corollary 2.9. *For all $\alpha \in \mathcal{X}(0)$, $\beta \in \mathcal{X}(1)$, $\gamma \in \Sigma^{\omega}$, if $\alpha \leqslant_{QI} \gamma$ and $\beta \leqslant_{QI} \gamma$ then $\gamma \sim (01)^{\omega}$.* $\quad\square$

Consider the join operation $\oplus$ that, given α and β, produces the string $\alpha \oplus \beta = \alpha(0)\beta(0)\alpha(1)\beta(1) \ldots$. The operation is often used (e.g., in computability and complexity theory) to produce the least upper bounds. It turns out this operation is not well-behaved with respect to the large scale geometries. Indeed, consider the string $\alpha = 010011 \ldots (0)^{2^n}(1)^{2^n} \ldots$ and the string $\beta = 101100 \ldots (1)^{2^n}(0)^{2^n} \ldots$ Then $[\alpha \oplus \beta] = [(01)^{\omega}]$. But, $[(01)^{\omega}] \neq [\alpha]$, $[(01)^{\omega})] \neq [\beta]$, and $[\alpha] = [\beta]$.

Even though the operation $\oplus$ is not well-behaved with respect to $\sim_{QI}$-classes, the operation can still be useful in constructing counter-examples as shown below.

Corollary 2.10. *The sets $\mathcal{X}(0)$, $\mathcal{X}(1)$ are not ideals.*

Proof. Consider the strings $\beta = 010011 \ldots 0^n 1^n \ldots$ and $\alpha = \beta \oplus 1^{\omega}$. It is clear that $\alpha \in \mathcal{X}(0)$. It is also easy to see that $\beta \leqslant \alpha$. However, $\beta \notin \mathcal{X}(0)$. $\quad\square$

Corollary 2.11. *Both $\mathcal{X}(0)$ and $\mathcal{X}(1)$ have countable chains and anti-chains. In addition, $\mathcal{X}(u)$ has an infinite anti-chain consisting of minimal elements.*

Proof. The chain constructed in Theorem 2.5 is in $\mathcal{X}(0)$. Hence, both $\mathcal{X}(0)$ and $\mathcal{X}(1)$ have countable chains. Let β_n be the sequence in the proof of Theorem 2.5. Then in the same manner as in the proof of Theorem 2.5 we can show that $\{\beta_n \oplus 1^{\omega}\}_{n \in \mathbb{N}} \subset \mathcal{X}(0)$ is an anti-chain. Similarly $\{\beta_n \oplus 0^{\omega}\}_{n \in \mathbb{N}} \subset \mathcal{X}(1)$ is an anti-chain. The minimal elements constructed in Theorem 2.6 are in $\mathcal{X}(u)$. $\quad\square$

3 Refining quasi-isometry

The relation $\leqslant_{QI}$ could be analysed in several ways. One is to restrict $\leqslant_{QI}$ to a particular class K of infinite strings, and describe the partial order $\leqslant_{QI}$ restricted to large scale geometers of strings from K. In Section 3.1, we fully describe the partial order $\leqslant_{QI}$ restricted to the class of eventually periodic words. The second way is to refine the definition of $\leqslant_{QI}$ and study its implications. Section 3.2 provides an equivalent, more intuitive, characterisation of quasi-isometry called component wise reducibility.

3.1 Eventually periodic spaces

For a string α (that might be finite) consider the set of all colours in α: $Cl(\alpha) = \{\sigma \in \Sigma \mid \exists i(\alpha(i) = \sigma)\}$. Write $u \sqsubseteq v$ if $Cl(u) \subseteq Cl(v)$. We easily get the following:

Lemma 3.1. *If* $f : \alpha \to \beta$ *is a quasi-isometry then* $Cl(\alpha) \subseteq Cl(\beta)$. *So, if* $\alpha \sim_{QI} \beta$ *then* $Cl(\alpha) = Cl(\beta)$. $\quad\square$

A particularly simple strings are eventually periodic:

Definition 3.2. Metric space $\alpha \in \Sigma^\omega$ is **eventually periodic** if there are finite words $x, u \in \Sigma^\star$ such that $\alpha = xuuu\ldots$. Call u the **period** of α.

Let $\alpha = xu^\omega$ be eventually periodic word. We assume that $Cl(u) \subseteq Cl(x)$ as we can change the prefix x to xu. With this assumption, we have the following theorem:

Theorem 3.3. *For eventually periodic words* α, β *we have* $\alpha \leqslant_{QI} \beta$ *iff there are* $x, y, u, v \in \Sigma^\star$ *such that* $\alpha = xu^\omega$, $\beta = yv^\omega$, $Cl(x) \subseteq Cl(y)$, *and* $Cl(u) \subseteq Cl(v)$.

Proof. If $\alpha \leqslant_{QI} \beta$ then it is easy to select finite strings x, y, u, v that satisfy the statement of the theorem.

Now assume that there are $x, y, u, v \in \Sigma^\star$ such that $\alpha = xu^\omega$, $\beta = yv^\omega$, $Cl(x) \subseteq Cl(y)$ and $Cl(u) \subseteq Cl(v)$. We construct a quasi-isometry f from α to β as follows. First, we map the prefix x into prefix y so that colours are preserved. Secondly, we consider the set $X = \{x_1, \ldots, x_k\}$ of all distinct colours that appear in u. Now we map x_i coloured position in the kth copy of u in the string α into the x_i coloured position in the kth copy of v in the string β. We now set $A = \max\{|x|, |y|, |u|, |v|\}$ and $B = A$. It is clear that f preserves colours and the quasi-isometry inequality between the distances $d(x, y)$ and $d(f(x), f(y))$ with constants A, B as required. $\quad\square$

Let $P_1(\Sigma)$ of be the set of all non-empty subsets of Σ. Consider the following partial order on the domain:

$$\mathcal{X} = \{(A, B) \mid A, B \in P_1(\Sigma) \text{ and } A \supseteq B\},$$

where the partial order on $\mathcal{X}$ is the component-wise inclusion. From the theorem above, we obtain the a full description of the partial order $\leqslant_{QI}$ restricted to large scale geometers of eventually periodic strings. For the proof see the Appendix:

Corollary 3.4. *The partial order* $\leqslant_{QI}$ *restricted to the set* $EP = \{[\alpha] \mid \alpha$ *is eventually periodic string over* $\Sigma\}$ *is isomorphic the partial order* $\mathcal{X}$.

Let $f : \alpha \to \beta$ be a quasi-isometric map, and let P be a property. Say that f is *eventually* P if there is an i such that f restricted to the suffix $\alpha_i = \alpha(i)\alpha(i+1)\alpha(i+2)\ldots$ satisfies P. A quasi-isometry f does not need to be eventually order preserving map; neither f needs to be eventual embedding (that is, eventually injective map). Now we show that if α and β are eventually periodic words, then f can be computable eventually order preserving injective map. For the proof see the Appendix:

Theorem 3.5. *For eventually periodic strings* α *and* β *such that* $\alpha \leqslant_{QI} \beta$ *there exists a computable quasi-isometric map* $f_\alpha : \alpha \to \beta$ *which is eventually order preserving and injective.* $\quad\square$

3.2 Componentwise Reducibility

We formulate an equivalent more intuitive condition for quasi-isometry between coloured metric spaces. Some proofs of results in this subsection are provided in the Appendix.

Definition 3.6. Say α is **componentwise reducible** to β, written $\alpha \leqslant_{CR} \beta$, if we can write $\alpha = u_1 u_2 \ldots$ and $\beta = v_1 v_2 \ldots$ such that $u_i \sqsubseteq v_i$ for each i and $|u_i|, |v_i|$ are uniformly bounded by a constant C. Call these presentations of α and β **witnessing partitions**.

It is clear that $\alpha \leqslant_{CR} \beta$ implies $\alpha \leqslant_{QI} \beta$: any colour-preserving map that maps each interval u_i in α to v_i in β is quasi-isometry. Showing the converse is not trivial. The main difficulty is showing that $\leqslant_{CR}$ is transitive.

Definition 3.7. An **atomic crossing map** is a function $f : \alpha \to \beta$ of the following form: we have $\{a_i, b_i\}_{i \in I}$, where I is an at most countable index set such that $\exists C \forall i[(a_i < b_i < a_{i+1}) \wedge (b_i - a_i \leqslant C)]$ and

$$f(a) = \begin{cases} b_i & (a = a_i) \\ a_i & (a = b_i) \\ a & (otherwise) \end{cases}$$

Clearly, every atomic crossing map is bijective. The next result is a step towards transitivity of $\leqslant_{CR}$.

Proposition 3.8. *Any quasi-isometry $f : \alpha \to \beta$ can be decomposed into the following form:*

$$\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma_2 \xrightarrow{f_3} \beta,$$

where f_1 is a monotonic injection, f_2 is a monotonic surjection, f_3 is a bijection, and f_1, f_2, f_3 are all quasi-isometric. Furthermore, f_3 can be decomposed into the following form for some $n \geqslant 1$:

$$\gamma_2 \xrightarrow{g_1} \delta_1 \xrightarrow{g_2} \ldots \xrightarrow{g_n} \beta,$$

where each g_k is an atomic quasi-isometry. □

Corollary 3.9. *If $\alpha \leqslant_{QI} \beta$, then there is $\beta' \sim_{QI} \beta$ so that $\alpha \leqslant_{QI} \beta'$ via strictly monotonic quasi-isometry.* □

Proof. The strings γ_1 is a desired β' as β' is obtained through the composition of quasi-isometric maps f_3^{-1} and $\lambda n. \min f_2^{-1}(n)$ applied to β. □

Informally, the proposition above says that any quasi-isometry can be decomposed into three parts. The first part is the mapping f_1 which can be viewed as a "pollution" of α, the second part is the map f_2 that can be called a "collapsing" map, and the third part is f_3 that can be called "mixing" since it mixes atomic crossing maps. One would notice that the mixing part is the one that makes things complicated: below we show that mixing functions preserve componentwise reducibility. The proof is in the Appendix.

Lemma 3.10. *Suppose $\alpha \leqslant_{CR} \beta$ and $\beta \leqslant_{QI} \gamma$ via an atomic crossing map $f : \beta \to \gamma$. Then $\alpha \leqslant_{CR} \gamma$.* □

Theorem 3.11. *$\alpha \leqslant_{QI} \beta$ implies $\alpha \leqslant_{CR} \beta$.*

Proof. Let $f : \alpha \to \beta$ be a quasi-isometry. Decompose it into $\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma_2 \xrightarrow{f_3} \beta$ according to Proposition 3.8. For any monotonic QI map $g : \alpha' \to \beta'$ we easily have $\alpha' \leqslant_{CR} \beta'$: Indeed, enumerate all elements of an image of g as $\{b_n\}_{n \in \mathbb{N}}$, where $b_n < b_{n+1}$ for all n, let $u_n = g^{-1}(b_n)$ and $v_n = [b_n, b_{n+1} - 1]$. Then $\alpha' = u_1 u_2 \ldots$ and $\beta' = v_1 v_2 \ldots$ are witnessing partitions. Hence we have $\alpha \leqslant_{CR} \gamma_2$, and Lemma 3.10 completes the proof. □

One could attempt to go further, namely say α and β are *colour-equivalent* if they are componentwise reducible to each other via the *same* witnessing partitions. It is clear that if α and β are colour-equivalent, then they are quasi-isometric. Can we show that $\alpha \sim_{QI} \beta$ implies their colour-equivalence? The answer is negative.

Proposition 3.12. *There are colour-equivalent α and β such that no eventual embeddings witness quasi-isometry between α and β.*

Proof. We provide the strings α and β, and refer the reader to the appendix for the proof. Consider the following coloured metric spaces over the alphabet $\{0, 1\}$:

$$\alpha = 001^2 001^4 \ldots 1^{2n} 00 \ldots \text{ and } \beta = 0101^2 \ldots 01^n \ldots.$$

It is not hard to see that α and β are quasi-isometric. □

Proposition 3.13. *There are sequences α and β such that α and β are quasi-isometric via monotonic embeddings but α and β are not colour-equivalent.*

Proof. Let $\Sigma = \{0, 1, *\}$ and define $\alpha, \beta \in \Sigma^\omega$ as follows. We first define β by:

$$010 * \quad 0 * 1 * 0 * * * \quad \ldots \ 0(*)^n 1(*)^n 0(*)^{2n+1} \ \ldots$$

Thus, the nth-part of β is the string $0(*)^n 1(*)^n 0(*)^{2n+1}$. We re-write β with subscripts on each 0 and 1 as pointers for easier readability:

$$0_0 1_0 0_{\frac{1}{2}} * 0_1 * 1_1 * 0_{\frac{3}{2}} * * * \ldots 0_n (*)^n 1_n (*)^n 0_{\frac{2n+1}{2}} (*)^{2n+1} \ldots$$

Let α be obtained from β by omitting the second occurrence of 0 in the nth-substring of β. So,

$$\alpha = 01 * \quad 0 * 1 * * * * \quad \ldots \ 0(*)^n 1(*)^n (*)^{2n+1} \ \ldots$$

So, the nth-substring of α is $0(*)^n 1(*)^n (*)^{2n+1}$. As above we re-write α with with subscripts as pointers:

$$\alpha = 0_0 1_0 * 0_1 * 1_1 * * * * \ldots 0_n (*)^n 1_n (*)^{3n+1} \ldots$$

The Appendix shows that α and β are desired. □

4 Büchi automata and large scale geometries

Let $L \subseteq \Sigma^\omega$ be a language, where we assume that $\Sigma = \{0, 1\}$. The notion of quasi-isometry leads us to consider the quasi-isometry types of strings from L:

Definition 4.1. An **atlas** is a set of quasi-isometry types. In particular, the atlas defined by the language L is the set $[L] = \{[\alpha] \mid \alpha \in L\}$, where $[\alpha]$ is the quasi-isometry type of α.

A question of a general character is concerned with understanding the set $[L]$ given a description of L. In particular, a natural question is if one can decide that $[L_1] = [L_2]$ given descriptions of languages L_1 and L_2. In this section, we study the atlases defined by Büchi automata recognisable languages. We show that for Büchi automata recognisable languages L_1 and L_2 there is an efficient algorithm that decides if $[L_1] = [L_2]$. We recall basic definitions for Büchi automata.

Definition 4.2. A **Büchi automaton** $\mathcal{M}$ is a quadruple (S, ι, Δ, F), where S is a finite set of *states*, $\iota \in S$ is the *initial state*, $\Delta \subset S \times \Sigma \times S$ is the *transition table*, and $F \subseteq S$ is the set of *accepting states*.

A *run* of $\mathcal{M}$ on $\alpha = \sigma_0 \sigma_1 \cdots \in \Sigma^\omega$ is a sequence of states $r = s_0, s_1, \ldots$ such that $s_0 = \iota$ and $(s_i, \sigma_i, s_{i+1}) \in \Delta$ for all $i \in \omega$. The run is *accepting* if the set $Inf(r) = \{s : \exists^\infty i(q_i = s)\}$ has a state from F. The automaton $\mathcal{M}$ *accepts* the string α if it has an accepting run on it. The *language* accepted by $\mathcal{M}$, denoted $L(\mathcal{M})$, is the set of all infinite words accepted by $\mathcal{M}$.

Let $\mathcal{M}$ be a Büchi automaton. Our goal is to describe the atlas defined by the language $L(\mathcal{M})$. We call such $[L]$ *Büchi recognisable*. We need to do some state space analysis of the automation $\mathcal{M}$.

A *loop* (in $\mathcal{M}$) is a path L of states $s_1, \ldots, s_{n+1}$ in the state space S such that $s_1 = s_{n+1}$. We say that a word $\sigma_1 \ldots \sigma_n$ *labels* the loop L if $(s_i, \sigma, s_{i+1}) \in \Delta$ for $1 \leqslant i \leqslant n$. If all symbols σ_i are 0 only (or 1 only), then we say that the loop is *0-loop* (or respectively, *1-loop*). In the loop L we do not require that the states are pairwise distinct. Any state in a loop is a *loop state*.

The state space S can naturally be considered as a directed graph, where the edges between states are transitions of $\mathcal{M}$ with labels removed. Hence, we can write S as a disjoint union of its strongly connected components (sccs). Call a scc $X \subseteq S$ *non-trivial* if $|X| > 1$, where $|X|$ is the cardinality of X. Thus we write S as the union $T \cup S_1 \cup \ldots \cup S_k$, where T is the union of trivial strongly connected components and each S_i is a nontrivial scc. Every successful run of $\mathcal{M}$ on every infinite string α eventually ends up in one of the sccs S_i. Consider Büchi automata $\mathcal{M}_1, \ldots, \mathcal{M}_k$ such that the initial states and the state diagrams of $\mathcal{M}_i$ all coincide with that of $\mathcal{M}$ but $F_i = F \cap S_i$. It is clear that

$$(\star) \qquad L(\mathcal{M}) = L(\mathcal{M}_1) \cup \ldots \cup L(\mathcal{M}_k).$$

Hence describing the atlas $[L(\mathcal{M})]$ boils down to describing the atlases of $[L(\mathcal{M}_i)]$, $i = 1, \ldots, k$. Assume that α is accepted by $\mathcal{M}_i$. Then we can write α as $v\alpha'$ such

that an accepting run of $\mathcal{M}_i$ after reading v stays inside S_i. Therefore, the large scale geometry $[\alpha]$ of α is quasi-isometric to either $[0\alpha']$ or $[1\alpha']$ (or both). Therefore, we can postulate the following assumption till the end of this section unless otherwise stated.

Postulate. The state space S can be written as $\{q_0\} \cup S'$ such that (1) q_0 is the only initial state and $q_0 \notin S'$, (2) S' is the only nontrivial strongly connected component of $\mathcal{M}$, and (3) any transition from q_0 goes into S'.

Lemma 4.3. *If $\mathcal{M}$ has a 0-loop and a 1-loop, then the atlas $[L(\mathcal{M})]$ coincides with the atlas $[\Sigma^\omega] \setminus X$ for some $X \subseteq \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$.*

Proof. Assume that L_0 is a 0-loop and L_1 is a 1-loop. Let α be a string with ω-many 0s and ω-many 1s. Say, $\alpha = 0^{n_0} 1^{m_0} 0^{n_1} 1^{m_1} \ldots$. We can build a β of the form

$$u \quad 0^{n'_1} v 1^{m'_1} w \quad 0^{n'_2} v 1^{m'_2} w \quad 0^{n'_3} v 1^{m'_3} w \quad \ldots \quad 0^{n'_i} v 1^{m'_i} w \quad \ldots$$

such that (1) β is accepted by $\mathcal{M}$, (2) $n_i = n'_i + c_i$ and $m_i = m'_i + c'_i$, where $c_i < |S|$ and $c'_i < |S|$ for all i, (3) v is a string that labels a path from a state s_0 in L_0 to a state s_1 in L_1, and (4) w is a string that labels a path from s_1 to s_0, and (5) v is a string that labels a path from the initial state to s_0. This implies that α and β are colour-equivalent. Hence they are quasi-isometric.

Note that none of the strings 0^ω, 1^ω, 10^ω, 01^ω is quasi-isometric to a string with both ω-many 0s and 1s. Also, these four strings are pairwise not quasi-isometric. Hence, the choice of X is dependent on whether or not $\mathcal{M}$ accepts some (or all) of these four strings. For instance, $0^\omega \in X$ iff 0^ω is not accepted by $\mathcal{M}$. $\qquad \square$

Lemma 4.4. *If $\mathcal{M}$ has no 0-loop and has no 1-loop, then the atlas $[L(\mathcal{M})]$ equals the singleton atlas $[\{(01)^\omega\}]$.*

Proof. Let $\alpha = \sigma_0 \sigma_1 \ldots$ be a string accepted by $\mathcal{M}$. Let $\rho = s_1, s_2, \ldots$ be an accepting run of $\mathcal{M}$ on α. We can write α as $w_1 w_2 \ldots$ so that (1) the length of each w_i is bounded by $|S|$ and (2) the run ρ along each w_i has a loop; that loop within w_i contains both 0 and 1. This implies that $(01)^\omega$ and α are colour-equivalent. Hence they are quasi-isometric. $\qquad \square$

Lemma 4.5. *Suppose that $\mathcal{M}$ contains no 1-loop but has a 0-loop L_0. Then the atlas $[L(\mathcal{M})]$ coincides with one of the following atlases:*

$$[\{0^\omega\}], \ [\{10^\omega\}], \ [\{0^\omega, 10^\omega\}],$$
$$[\{0^{n_1} 10^{n_2} 10^{n_3} \ldots \mid n_i > 0\} \setminus \{0^\omega\}] \ or$$
$$[\{0^{n_1} 10^{n_2} 10^{n_3} \ldots \mid n_i > 0\} \cup \{0^\omega\}].$$

Proof. The assumption of the lemma implies that if α is accepted by $\mathcal{M}$ then (1) α contains infinitely many 0s and (2) the lengths of any sequence of consecutive 1s occurring in α is bounded by $|S|$. If $L(\mathcal{M}) = V0^\omega$ for some regular language $V \subseteq \Sigma^\star$, then the atlas $[L(\mathcal{M})]$ is either $[\{0^\omega\}]$ or $[\{10^\omega\}]$ or $[\{0^\omega, 10^\omega\}]$. So, we assume that $\mathcal{M}$ accepts a string with ω many 1s'. Let α be any string of the form $0^{t_1}10^{t_2}1\ldots$, where all $t_i > 0$ for all i. We can build a string β of the form $u\ 0^{n_1}v\ 0^{n_2}v\ 0^{n_3}v\ \ldots\ 0^{n_i}v\ \ldots$ such that (1) β is accepted by $\mathcal{M}$, (2) $t_i = n_i + c_i$, where $c_i < |S|$ for all i, and (3) v is a string that labels a path from a state s_0 in L_0 back to s_0, and v contains 1. The string β is quasi-isometric to string β' obtained from β by replacing all occurrences of v with 1 and removing the prefix u. In turn β' is colour equivalent to α. Hence, $\alpha \sim_{QI} \beta$.

Now suppose that β is accepted by $\mathcal{M}$ and β has infinitely many 1s. Then β is of the form $u0^{n_1}1^{m_1}0^{n_2}1^{m_2}\ldots$ where we have $n_i, m_i > 0$ for all i. As noted above the numbers m_i are uniformly bounded. The string β is quasi-isometric to string β' obtained from β by replacing all 1^{m_i} with just 1 and removing the prefix u. Clearly β' is of the form desired. Note that $\mathcal{M}$ accepts the string quasi-isometric to 10^ω. Hence, if 0^ω is accepted by $\mathcal{M}$ then the atlas $[L(\mathcal{M})]$ coincides with $[\{0_1^n10^{n_2}10^{n_3}\ldots \mid n_i > 0\} \cup \{0^\omega\}]$. Otherwise, the atlas $[L(\mathcal{M})]$ equals $[\{0_1^n10^{n_2}10^{n_3}\ldots \mid n_i > 0\} \setminus \{0^\omega\}]$. $\qquad\square$

Lemma 4.6. *Suppose that $\mathcal{M}$ contains no 0-loop but has a 1-loop L_0. Then the atlas $[L(\mathcal{M})]$ coincides with one of the following atlases:*

$$[\{1^\omega\}],\ [\{01^\omega\}],\ [\{1^\omega, 01^\omega\}],$$
$$[\{1^{n_1}01^{n_2}01^{n_3}\ldots \mid n_i > 0\} \setminus \{1^\omega\}]\ or$$
$$[\{1^{n_1}01^{n_2}01^{n_3}\ldots \mid n_i > 0\} \cup \{1^\omega\}].\qquad\square$$

We now remove our postulate put on the state space of Büni automata. Using equality $(\star)$ given above, and the lemmas, we obtain the following characterisation result:

Theorem 4.7. *Any Büchi recognisable atlas $[L]$ is a union from the following list of atlases:*

- $[\Sigma^\omega] \setminus X$, *where* $X \subseteq \{[0^\omega], [1^\omega], [10^\omega], [01^\omega]\}$.

- $[\{0^{n_1}10^{n_2}10^{n_3}\ldots \mid n_i > 0\} \setminus \{0^\omega\}]$, $[\{1^\omega\}]$,

- $[\{0^{n_1}10^{n_2}10^{n_3}\ldots \mid n_i > 0\} \cup \{0^\omega\}]$, $[\{0^\omega\}]$,

- $[\{01^\omega\}]$, $[\{10^\omega\}]$, $[\{0^\omega, 10^\omega\}]$, $[\{1^\omega, 01^\omega\}]$,

- $[\{1^{n_1}01^{n_2}01^{n_3}\ldots \mid n_i > 0\} \setminus \{1^\omega\}]$,

- $[\{1^{n_1}01^{n_2}01^{n_3}\ldots \mid n_i > 0\} \cup \{1^\omega\}]$. $\qquad\square$

We obtain the following complexity-theoretic result solving the equality problem for Büchi recognisable atlases. See Appendix for the proof. In comparison, the equality problem for Büchi automata is PSPACE complete.

Corollary 4.8. *There exists an algorithm that, given Büchi automata $\mathcal{A}$ and $\mathcal{B}$, decides if the atlases $[L(\mathcal{A})]$ and $[L(\mathcal{B})]$ coincide. Furthermore, the algorithm runs in linear time on the size of the input automata.* $\qquad\square$

5 The quasi-isometry problem

The quasi-isometry problem QIP is to find if, given strings α and β, there is a quasi-isometry from α to β:

$$QIP = \{(\alpha, \beta) \mid \alpha, \beta \in \Sigma^\omega\ \&\ [\alpha] \leqslant_{QI} [\beta]\}.$$

Let α and β be coloured metric spaces and A, B be constants. Our goal is to construct a rooted tree $T(\alpha, \beta, A, B)$ such that the following properties hold:

1. The tree $T(\alpha, \beta, A, B)$ is finitely branching and computable in α and β. In particular, if α and β are computable then so is $T(\alpha, \beta, A, B)$.

2. For any n one can effectively compute, with an oracle for α and β, the number of nodes in the tree at distance n from the root.

3. The tree $T(\alpha, \beta, A, B)$ is infinite iff there exists an (A, B)-quasi-isometric map from α into β.

4. There is a bijection between (A, B)-quasi-isometric maps from α to β and the set $[T(\alpha, \beta, A, B)]$ of all infinite paths of the tree $T(\alpha, \beta, A, B)$.

Informally, the nodes of $T(\alpha, \beta, A, B)$ are finite partial functions that can potentially be extended to (A, B)-quasi-isometric maps from α to β. Formally:

1. The root of the tree $T(\alpha, \beta, A, B)$ is the empty set $\emptyset$, that is, nowhere defined partial function.

2. Let x be a node of the tree $T(\alpha, \beta, A, B)$ constructed so far, and f_x be the partial function associated with x. We identify x and f_x. We assume that

   (a) x is at distance n from the root, and $f_x = \{(0, i_0), \ldots, (n - 1, i_{n-1})\}$.

   (b) For all k, m from the domain of f_x we have that f_x is colour preserving and the (A, B)-quasi-isometry condition is satisfied:

   $$(1/A) \cdot d(k, m) - B < d(f_x(k), f_x(m))$$
   $$< A \cdot d(k, m) + B.$$

We list all the extension ϕ of f_x such that the domain of ϕ is the set $\{0, 1, \ldots, n\}$ and ϕ satisfies the the (A, B)-quasi-isometry condition. The children of f_x will be all these functions ϕ extending f_x.

In Part 2 of the construction the number of extensions ϕ of f_x is finite since these extensions must satisfy the quasi-isometry condition. Moreover, the number of these extensions is computed with an oracle for α and β. If such extensions ϕ of f_x do not exist, then x is a leaf.

Lemma 5.1. *The tree $T(\alpha, \beta, A, B)$ is infinite iff there is an (A, B)-quasi-isometric map from α into β. There is a bijection between all infinite paths of $T(\alpha, \beta, A, B)$ and (A, B)-quasi-isometric maps from α to β.*

Proof. If $g : \alpha \to \beta$ is an (A, B)-quasi-isometric map from α to β then the sequence $\{g_i\}_{i \in \omega}$, where g_i is the restriction of g on the initial segment $\{0, \ldots, i\}$, is an infinite path through the tree $T(\alpha, \beta, A, B)$. If $x_1, x_2, \ldots$ is an infinite path through the tree $T(\alpha, \beta, A, B)$ then the function f defined as the limit of the sequence $f_{x_1}, f_{x_2}, \ldots$ is an (A, B)-quasi-isometry from α to β. $\square$

The next theorem that solves the quasi-isometry problem. See the Appendix for the proof.

Theorem 5.2. *The following statements are true:*

1. *Given quasi-isometric strings α and β, there exists a quasi-isometry between α and β computable in the halting set relative to α and β.*

2. *The quasi-isometry problem between computable strings is a complete Σ_3^0-set.* $\square$

The isometry problem for coloured computable metric spaces is a refined version of quasi-isometry. Hence, one might expect that quasi-isometry is easier to detect than the isometry. However, the isometry problem for computable coloured metric spaces is Π_1^0-complete as opposed to Σ_3^0-completeness of the quasi-isometry problem.

6 Asymptotic cones

Let $\mathcal{F}$ be a non-principal ultra-filter on ω. Recall that a non-principal ultra-filter is a non-empty maximal subset of $P(\omega)$ that satisfies the following properties: (1) No finite set belongs to $\mathcal{F}$. So, $\mathcal{F}$ does not contain the empty set; (2) For all $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$; (3) For all $A, B \subseteq \omega$ if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

Every ultrafilter $\mathcal{F}$ has the following two properties: (1) For every set $A \subseteq \omega$, either $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$. (2)

For all pairwise disjoint sets $A, B \subseteq \omega$ if $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Let $\alpha \in \Sigma^\omega$ and $s : \omega \to \omega$ be a strictly increasing monotonic function such that $s(0) = 1$. Call the mapping $s(n)$ a *scaling factor*. We define the following sequence of metric spaces:

$$X_{0,\alpha} = (\alpha, d_0), \ X_{1,\alpha} = (\alpha, d_1), \ldots, X_{n,\alpha} = (\alpha, d_n), \ldots$$

where $d_n(i, j) = |i - j|/s(n)$. Informally, we move α away from us by scaling the metric down. For instance, in the metric space $X_{0,\alpha}$ the distance from 0 to $s(n)$ equals $s(n)$, while in $X_{n,\alpha}$ the distance from 0 to $s(n)$ is 1. We assume that the domains of these metric spaces are disjoint pairwise: $X_{i,\alpha} \cap X_{j,\alpha} = \emptyset$ for all $i \neq j$.

Let $\mathbf{a} = (a_n)_{n \geqslant 0}$ be a sequence, where each $a_n \in X_{n,\alpha}$ for all n. Call the sequence *bounded* if there is a constant L such that the set $\{n \mid d_n(0, a_n) < L\} \in \mathcal{F}$. Let $\mathbf{B}(\mathcal{F}, s)$ be the set of all bounded sequences. Say that two bounded sequences $\mathbf{a}$ and $\mathbf{b}$ are $\mathcal{F}$-equivalent, written $\mathbf{a} \sim_{\mathcal{F}} \mathbf{c}$, if $\{n \mid d_n(a_n, b_n) \leqslant \epsilon\} \in \mathcal{F}$ for every ϵ.

Definition 6.1. Define the **asymptotic cone of** α, written $Cone(\alpha, \mathcal{F}, s)$, with respect to the scaling function $s(n)$ and the ultra-filter $\mathcal{F}$ to be the factor set

$$\mathbf{B}(\mathcal{F}, s)/ \sim_{\mathcal{F}}$$

equipped with the following metric D and colour C:

1. $D(\mathbf{a}, \mathbf{b}) = r$ iff $\{n \mid r - \epsilon \leqslant d_n(a_n, b_n) \leqslant r + \epsilon\} \in \mathcal{F}$ for all $\epsilon > 0$.

2. $C(\mathbf{a}) = \sigma$ iff $\{n \mid a_n \text{ has colour } \sigma\} \in \mathcal{F}$.

This definition implies that some elements $\mathbf{a}$ of the asymptotic cone might have several colours.

Lemma 6.2. *For each $\sigma \in \Sigma$, the set*

$$\{\mathbf{a} \in Cone(\alpha, \mathcal{F}, s) \mid \sigma \in C(\mathbf{a})\}$$

is closed.

Proof. Let $\mathbf{a}$ be a limit point of the set above, $\mathbf{a_n} = (a_{nm})_{m \geqslant 0} \in Cone(\alpha, \mathcal{F}, s)$ be a point of colour σ such that $D(\mathbf{a}, \mathbf{a_n}) \leqslant 2^{-n}$. Also let $r = D(\mathbf{0}, \mathbf{a})$, $r_n = D(\mathbf{0}, \mathbf{a_n})$, and i_m be a number such that $i_m \leqslant m$ and $|r - d_n(0, a_{i_m m})| \leqslant |r - d_n(0, a_{jm})|$ for all $j \leqslant m$. Then $\mathbf{b} = (a_{i_n n})_{n \geqslant 0} \sim \mathbf{a}$ and $\sigma \in C(\mathbf{b})$: indeed, let $A_n = \{m \in \mathbb{N} \mid |r_n - d_m(0, a_{nm})| \leqslant 2^{-n}\}$. Then for given n,

$$\{m \in \mathbb{N} \mid |r - d_m(0, b_m)| \leqslant 2^{-(n+1)}\}$$
$$\supseteq \{m \in \mathbb{N} \mid |r - d_m(0, a_{i_m m})| \leqslant 2^{-(n+1)}\} \cup [n, \infty)$$
$$\supseteq A_{n+2} \cup [n, \infty) \in \mathcal{F}.$$

Also $\{m \in \mathbb{N} \mid |r - d_m(0, a_m)| \leqslant 2^{-(n+1)}\} \in \mathcal{F}$, hence

$$\{m \in \mathbb{N} \mid d_m(a_m, b_m) \leqslant 2^{-n}\} \in \mathcal{F}. \quad \square$$

Theorem 6.3. *For eventually periodic word $\alpha = uv^\omega$ all asymptotic cones $Cone(\alpha, \mathcal{F}, s)$ equal the coloured metric space $(\mathbb{R}_{\geqslant 0}; d, C)$ such that every real $r > 0$ has colours of v and the real 0 has colours of both u and v.*

Proof. Let $\mathcal{F}$ be any filter and $s : \omega \to \omega$ be a monotonic function. It is easy to see that 0 has all colours of both u and v. Let $r \in \mathbb{R}$ and $r > 0$. For any σ present in v there is a sequence $(a_n)_{n \geqslant 0}$ such that $a_n \in X_{n,\alpha}$, the colours of a_n all are σ and for every $\epsilon > 0$ there is an N_ϵ such that for all $n > N_\epsilon$ we have

$$|d_n(a_n, 0) - r| \leqslant \epsilon.$$

Indeed, for each $n \in \mathbb{N}$ such that $|u|/s(N_\epsilon) < r$ let a_n be any element which has the colour σ and satisfies $(|u| + i|v|)/s(n) \leqslant d_n(0, a_n) < (|u| + (i+1)|v|)/s(n)$, where i is the unique integer which satisfies $(|u| + i|v|)/s(n) \leqslant r < (|u| + (i+1)|v|)/s(n)$. $|d_n(a_n, 0) - r| \leqslant |v|/s(n)$. This inequality implies the claim. $\quad \square$

We now prove a theorem akin to a known result in geometric group theory stating that there are non-quasi-isometric groups that realise the same cones. For instance, in [2] [3] it is proved that all asymptotic cones of non-elementary hyperbolic group are isometric.

Theorem 6.4. *There are non-quasi-isometric strings $\alpha, \beta \in \{0,1\}^\omega$, a scale factor $s(n)$, and filter $\mathcal{F}$ such that $Cone(\alpha, \mathcal{F}, s) = Cone(\beta, \mathcal{F}, s)$ and $\alpha, \beta \in \mathcal{X}(1)$.*

Proof. Our scale factor is $s(n) = 2^n$. The string α is such that $\alpha(i) = 1$ if and only if i is a power of 2. Let $\mathcal{F}$ be any ultra-filter. Consider the cone $Cone(\alpha, \mathcal{F}, s)$. It is note hard to see that the cone coincides with the coloured metric space $(\mathbb{R}_{\geqslant 0}; d, C)$, where d is the usual metric on reals, all reals have colour 0, and a real r has also colour 1 iff r is an integer power of 2 or $r = 0$.

So, we need to construct β and a filter $\mathcal{F}$ such that α and β are not quasi-isometric but the cone $Cone(\beta, \mathcal{F}, s)$ coincides with the above coloured metric space $(\mathbb{R}_{\geqslant 0}; d, C)$. Let β be a string of the form

$$0^{m_0} 1 0^{m_1} 1 0^{m_2} 1 0^{m_3} \ldots 0^{m_k} 1 0^{m_{k+1}} \ldots$$

such that the following properties hold:

1. The positions where β contains 1 are powers of two, and let us list these positions as the sequence $2^{n_0}, 2^{n_1}, 2^{n_2}, \ldots$.

2. For each $i \in \omega$ we have $2n_i < n_{i+1}$.

3. The strings α and β are not quasi-isometric.

Now we construct our filter $\mathcal{F}$. Let X be the set

$$\{n_0, n_0 + 1, \ldots, 2n_0, \ldots, n_k, n_k + 1, \ldots, 2n_k, \ldots\}.$$

Call the sequences of the form $n_k, n_k + 1, \ldots, 2n_k$ *blocks* of the set X. Note that the lengths of the blocks is unbounded. For each integer i, consider the set $X + i = \{x + i \mid x \in X, \ x + i \geqslant 0\}$. Since the sizes of the blocks is unbounded, the collections of sets is a bases of a filter:

$$X, \ X + 1, \ X - 1, \ X + 2, \ X - 2, \ldots, X + i, \ X - i, \ldots$$

Let $\mathcal{F}$ be the ultrafilter that contains the collection.

Consider the cone $Cone(\beta, \mathcal{F}, s)$. We can view the domain of this cone as the set $\mathbb{R}_{\geqslant 0}$. It is not too hard to note that every real r gets colour 0 in the cone $Cone(\beta, \mathcal{F}, s)$. Now we need to show that r gets colour 1 if and only if r is an integer power of 2. It suffices to show that if r is an integer power of 2 then r has colour 1. But, this is implied by the fact that the sets $X + i$ are in $\mathcal{F}$. Indeed, assume that r is of the form 2^i. Then, since $X + i \in \mathcal{F}$, from the definition of $Cone(\beta, \mathcal{F}, s)$ we see that r has colour 1.

To finish the proof, we just need to show that we can select the sequence $n_0, n_1, \ldots$ such that α and β are not quasi-isometric. To see this, let $n_{k+1} = 2n_k + 2$: then we can show that α and β are not quasi-isometric in the same manner as the proof of Theorem 2.5. $\quad \square$

In contrast to the theorem above, we show that the same coloured metric space can produce two asymptotic cones that are not quasi-isometric. This is similar to the result in geometric group theory where one group can realise two non homeomorphic asymptotic cones [1].

Theorem 6.5. *There is a sequence α, scaling factors s_1 and s_2 so that for all ultrafilters $\mathcal{F}$ cones $Cone(\alpha, \mathcal{F}, s_1)$ and $Cone(\alpha, \mathcal{F}, s_2)$ are not quasi-isometric.*

Proof. We construct α and two scaling factors s_0 and s_1 such that in the asymptotic cone $Cone(\alpha, \mathcal{F}, s_0)$ all reals $r > 1$ have colour 0, and in the asymptotic cone $Cone(\alpha, \mathcal{F}, s_1)$ all reals $r > 1$ have colour 1 only. These two metric spaces are clearly not quasi-isometric.

Define p_n and q_n, inductively, by: $p_1 = q_1 = 1$, and

$$p_{n+1} = n \sum_{i=1}^{n} (p_i + q_i),$$

$$q_{n+1} = n \left(\sum_{i=1}^{n} (p_i + q_i) + p_{n+1} \right)$$

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Now consider the string $\alpha = 0^{p_1}1^{q_1}0^{p_2}1^{q_2}\ldots$, and define the following two scale functions:

- $s_0(n) = \sum_{i=1}^{n}(p_i + q_i)$, and

- $s_1(n) = \left(\sum_{i=1}^{n}(p_i + q_i) + p_{n+1}\right)$.

Also let $d_{j,n}(x,y) = |y - x|/s_j(n)$ $(j = 0, 1)$. Then all points a in $(\alpha, d_{j,n})$ such that $1 < d_{j,n}(0, a) < n$ have the colour j. Hence for any $r > 1$, the colour of r is 0 in $Cone(\alpha, \mathcal{F}, s_0)$ and 1 in $Cone(\alpha, \mathcal{F}, s_1)$. $\qquad\square$

The next theorem shows that the asymptotic cones of all Martin-Löf random strings [11] coincide when the scaling factor is a computable function.

Theorem 6.6. *If α is Martin-Löf random, then for all computable scaling factors s and ultra-filters $\mathcal{F}$, the asymptotic cone $Cone(\alpha, \mathcal{F}, s)$ coincides with the space $(\mathcal{R}_{\geqslant 0}; d, C)$, where every real has all colours from Σ.*

Proof. We assume that $\Sigma = \{0, 1\}$. For given $r \in \mathbb{R}_{\geqslant 0}$, assume the following:

$$\exists A \in \mathcal{F} \forall \epsilon \forall^{\infty} n \in A \exists r' \in \alpha_n[r - \epsilon \leqslant r' \leqslant r + \epsilon \wedge C_n(r') = 0].$$

Then we have a sequence (a_n) such that the subsequence $\{a_n \mid n \in A\}$ converges to r and all elements of the subsequence are coloured by 0, and hence $Cone(\alpha, \mathcal{F}, s)$ has the colour 0 at r.

Taking the contraposition of this fact and letting $A = \mathbb{N}$, we have the following: if $Cone(\alpha, \mathcal{F}, s)$ does not have the colour 0 at r (by virtue of Lemma 6.2 we can assume that $r \in \mathbb{Q}_{\geqslant 0}$), then for some $\epsilon > 0$ we have infinitely many $n \in \mathbb{N}$ such that all points r' of α_n in an interval $[r - \epsilon, r + \epsilon]$ has the colour 1.

Now assume such an r exists and consider the set:

$$G_n = \{X \in 2^{\omega} \mid s(n)(r - \epsilon) \leqslant i \leqslant s(n)(r + \epsilon) \Rightarrow X(i) = 1\}$$

In the Cantor space $\{0, 1\}^{\omega}$, the sequence of open sets (G_n) is uniformly computably enumerable and the sum of the measures of G_n is bounded. Such sequences are called Solovay tests. This Solovay test fails the string α, that is, $\alpha \in G_n$ for infinitely many n (cf. Definition 3.2.18 in [11]). It is known that falling Solovay tests is equivalent failing Martin-Löf tests (cf. Proposition 3.2.19 in [11]). This shows that α is not random. $\qquad\square$

A culmination of asymptotic cones construction is that $[\alpha] = [\beta]$ implies bi-Lipschitz equivalence between $Cone(\alpha, \mathcal{F}, s)$ and $Cone(\beta, \mathcal{F}, s)$. The proof is standard with the difference that our metric spaces are coloured. However, since our base metric space is a linear order ω, the conclusion of our theorem is a little stronger as it implies order preservance. The proof is in the Appendix.

Theorem 6.7. *If strings α and β have the same large scale geometry then there are colour-preserving and order preserving homeomorphisms*

$$H : Cone(\alpha, \mathcal{F}, s) \to Cone(\beta, \mathcal{F}, s) \text{ and}$$
$$G : Cone(\beta, \mathcal{F}, s) \to Cone(\alpha, \mathcal{F}, s),$$

constants C_H, C_G such that for all $\mathbf{a}, \mathbf{b} \in Cone(\alpha, \mathcal{F}, s)$ and $\mathbf{c}, \mathbf{d} \in Cone(\beta, \mathcal{F}, s)$ we have:

$$(1/C_H) \cdot D(\mathbf{a}, \mathbf{b}) \leqslant D(H(\mathbf{a}), H(\mathbf{b})) \leqslant C_H \cdot D(\mathbf{a}, \mathbf{b})$$
$$and$$
$$(1/C_G) \cdot D(\mathbf{c}, \mathbf{d}) \leqslant D(G(\mathbf{c}), G(\mathbf{d})) \leqslant C_G \cdot D(\mathbf{c}, \mathbf{d}). \quad\square$$

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7 Appendix

Proof of Lemma 2.3:
The inequality in the definition of quasi-isometry implies:

$$(1/A) \cdot d_2(f(n), f(m)) - (B/A) \leqslant d_1(n, m)$$
$$\leqslant A \cdot d_2(f(n), f(m)) + AB.$$

For given $x \leqslant y \in \beta$, let $M_{xy} = \max f^{-1}([x,y])$ and $m_{xy} = \min f^{-1}([x,y])$. Then we have $d_1(m_{xy}, M_{xy}) \leqslant A \cdot d_2(f(m_{xy}), f(M_{xy})) + AB \leqslant A \cdot d_2(x,y) + AB$. For the lower bound of $d_1(m_{xy}, M_{xy})$, let $M'_{xy} = \max([x,y] \cap f(\alpha))$ and $m'_{xy} = \min([x,y] \cap f(\alpha))$. Then due to the last condition of quasi-isometricity it holds that $d_2(y, M'_{xy}) \leqslant 2A$ and $d_2(x, m'_{xy}) \leqslant 2A$. Also from Lemma 2.2 we have

$$d_2(f(M_{xy}), M'_{xy}) \leqslant -C$$

and

$$d_2(f(m_{xy}), m'_{xy}) \leqslant -C.$$

Hence we obtain:

$$d_2(f(m_{xy}), f(M_{xy})) \geqslant d_2(x,y) - 4A + 2C.$$

In summary, for any $x \leqslant y \in \beta$ it holds that

$$(1/A) \cdot d_2(x,y) - (4A + B - 2C)/A$$
$$\leqslant d_1(m_{xy}, M_{xy}) \leqslant A \cdot d_2(x,y) + AB,$$

and letting $A' = A$ and $B' = (4A + B - 2C)/A + AB$ we have the claim.

Proof of Theorem 2.8
We prove that $\mathcal{X}(0)$ is a filter. Assume that $\alpha \in \mathcal{X}(0)$ and $\alpha \leqslant_{QI} \beta$ through a quasi-isometry f with constants A and B. We need to show that $\beta \in \mathcal{X}(0)$. Assume that the length of 0-blocks in β is unbounded. Let K be the length of the largest 0-block in α. Now, take a 0-block $\beta(i) \ldots \beta(i + n_0)$ in β, where n_0 is sufficiently large. We give bound on the value of n_0 below. Since f is a quasi-isometry, there exists a $\beta(i_0)$ that belongs to the block such that $\beta(i_0)$ has a pre-image $\alpha(j_0)$ and $\beta(i_0)$ is within distance A from the center of the 0-block $\beta(i) \ldots \beta(i + n_0)$. The length of the block in α that contains $\alpha(j_0)$ is bounded by K, hence the string

$$\alpha(j_0 - K - 1) \ldots \alpha(j_0) \ldots \alpha(j_0 + K + 1)$$

must contain 1. The length of this interval is $2K + 2$. There exists a C such that f-image of all intervals of length $2K + 2$ is contained in the intervals in β of length

at most C. So, it must be the case that $n_0 \leqslant C$. Hence $\beta \in \mathcal{X}(0)$. The same proof shows that $\mathcal{X}(1)$ is a filter.

The proof of Part 2 is similar to the proof above.

For the last part for all the strings α, β if in both the lengths of 0-blocks and 1-blocks are universally bounded then, it is not hard to see hat α and β are quasi-isometric to each other since they are colour-equivalent. They are also quasi-isometric to $(01)^\omega$.

Proof of Corollary 3.4:
Let A, B be subsets of $P_1(\Sigma)$ such that $A \subseteq B$. Say, $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_n\}$. We associate with the pair (A, B) the infinite string string $a_1 \ldots a_k (b_1 \ldots b_n)^\omega$. This establishes the mapping from $\mathcal{X}$ into EP. By Theorem 3.3 this is an isomorphism from $\mathcal{X}$ onto EP.

Proof of Theorem 3.5:
Consider $\alpha = xuuu \ldots$ and $\beta = yvvv \ldots$. We assume that $Cl(x) \subseteq Cl(y)$ and $Cl(u) \subseteq Cl(v)$. Our desired mapping f_α maps x into y by preserving colours. (That is where f_α need not be order preserving embedding). So, it suffices to construct a computable quasi-isometric embedding from $\alpha_1 = u^\omega$ to $\beta_1 = v^\omega$. Let $X = \{x_1, \ldots, x_k\}$ be the set of all colours that appear in α_1, and hence in β_1. Let a_i be the number of times that colour x_i appears in v. Clearly $a_i \geqslant 1$. Consider the new string v_1 obtained by writing v exactly $a_1 + \ldots + a_n$ times, that is, $v_1 = (v)^{a_1 + a_2 + \ldots + a_n}$. There exists an embedding f' from u into v_1 that preserves colours and the order. Just like in the theorem above, we can propagate this mapping f' to a colour preserving embedding f_α from α_1 into β_1. The map $f_\alpha : \alpha \to \beta$ thus built is a desired function.

Proof of Proposition 3.8:
Let $f : \alpha \to \beta$ be a quasi-isometry. First we decompose it to $\alpha \xrightarrow{f'} \gamma_2 \xrightarrow{f_3} \beta$, where f' is monotonic and f_3 is a bijection. By Small Cross Over Lemma we have a bound $D < 0$ such that $f(m) - f(n) \geqslant D$ for all $n < m$. Let $\{(n_i, m_i)\}_{i \in I}$ be the set of all pairs of natural numbers such that $n_i < m_i$ and $f(m_i) - f(n_i) = D$, where I is an at most countable index set that depends on f. We define $\tilde{\beta}$ as the sequence that is same as β except that colours of the position n_i and m_i are swapped for each $i \in I$: that is,

$$\tilde{\beta}(n) = \begin{cases} \beta(m_i) & (n = n_i) \\ \beta(n_i) & (n = m_i) \\ \beta(n) & (otherwise) \end{cases}$$

Notice that n_i, m_i, n_j and m_j are all distinct element for distinct $i, j \in I$, and hence the above definition is well-defined. Also let $\tilde{f}$ be a function that executes this swapping, i.e. $\tilde{f}(m_i) = n_i$, $\tilde{f}(n_i) = m_i$ and $\tilde{f}(n) = n$ otherwise. It is easy to show that $\tilde{f} \circ f$ is a QI map from α to $\tilde{\beta}$ such that $\tilde{f} \circ f(m) - \tilde{f} \circ f(n) \geqslant D + 1$ for all $n < m$. Iterating this procedure $(-D)$ times we get γ_2 and a bijective QI map $g : \beta \to \gamma_2$ such that $g \circ f$ is monotonic. let $f' = g \circ f$ and $f_3 = g^{-1}$. From the construction it is clear that g^{-1} is a finite composition of certain atomic crossing functions.

Now we decompose f' to $\alpha \xrightarrow{f_1} \gamma_1 \xrightarrow{f_2} \gamma_2$, where f_1 is a monotonic injection and f_2 is a monotonic surjection. For $\gamma_2 = c_1 c_2 \ldots$, let $\gamma_1 = (c_1)^{n_1}(c_2)^{n_2} \ldots$, where $n_i = \max\{1, |f_1^{-1}(c_i)|\}$. Let f_1 be a map that sends each interval $f'(c_i)$ in α to $(c_i)^{n_i}$ in γ_1 in an obvious injective, monotonic way. Also let f_2 be a map that sends each $(c_i)^{n_i}$ in γ_1 to c_i in γ_2. This is the desired decomposition.

Proof of Lemma 3.10

Let $\alpha = u_1 u_2 \ldots$ and $\beta = v_1 v_2 \ldots$ be witness partitions. Let $\gamma = w_1 w_2 \ldots$ be a partition such that $|w_n| = |v_n|$ for all n. We construct a strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $v_{n_k} \ldots v_{n_{k+1}-1} \sqsubseteq w_{n_k} \ldots w_{n_{k+1}-1}$ for each k. Through the proof we identify a finite sequence u_n in a partition $\alpha = u_1 u_2 \ldots$ and an interval $[|u_1 \ldots u_{n-1}|, |u_1 \ldots u_n| - 1]$ in α, and write as "$a \in u_n$" for given $a \in \mathbb{N}$.

For simplicity we assume f satisfies the following conditions:

1. f is characterised by $\{a_i, b_i\}_{i \in \mathbb{N}}$ such that $C(a_i) \neq C(b_i)$ for all $i \in \mathbb{N}$.

2. For each i, if $a_i \in w_n$ then $b_i \in w_m$ for some $m < n$.

3. For each i, if $a_i \in w_n$ then $b_{i+1} \in w_n$.

This is the most complicated situation we encounter, and the general case can be proved in a similar way. Further we can assume $m = n - 1$ in the condition 2, as otherwise we can regard $v_m \ldots v_{n-1}$ as a single interval in the partition and do the same argument. Similarly we can also assume $a_1 \in u_1$. Hence our assumption on f is simplified as follows:

1. f is characterised by $\{a_i, b_i\}_{i \in \mathbb{N}}$ such that $C(a_i) \neq C(b_i)$ for all $i \in \mathbb{N}$.

2. For each i, $a_i \in u_i$ and $b_i \in u_{i+1}$.

The following claim is the crucial part of our proof.

Claim. Let $n \in \mathbb{N}$ be given. If $C(b_n) \in C(w_{n+i})$ for some $1 \leqslant i \leqslant |\Sigma|$, then there exists j such that $1 \leqslant j \leqslant |\Sigma|(\Sigma + 1)$ such that $v_{n+1} \ldots v_{n+j} \sqsubseteq w_{n+1} \ldots w_{n+j}$ and $c(b_{n+j}) \in c(w_{n+j+i'})$ for some $1 \leqslant i' \leqslant |\Sigma|$.

First notice that for given n, there are $n_1, n_2 \in \{n, \ldots, n+|\Sigma|\}$ such that $n_1 < n_2$ and $C(b_{n_1}) = C(b_{n_2})$: this is immediate from the pigeonhole principle. Notice that this implies that $C(b_{n_1}) \in C(w_{n_2})$ and $n_2 - n_1 \leqslant |\Sigma| - 1$. Also for given n, let

$$A = \{m \mid m > n \wedge C(a_m) \notin C(w_{n+1} \ldots w_m)\}.$$

Then $|A| \leqslant |\Sigma| - 1$, as $m \in A$ only if $C(a_m) \neq C(a_n)$, and for any distinct $m, m' \in A$ we have $C(a_m) \neq C(a'_m)$.

Now assume $C(b_n) \in C(w_{n+i})$ holds for some $1 \leqslant i \leqslant |\Sigma|$. From the facts above, we have $1 \leqslant m \leqslant |\Sigma|^2$ such that for all $m' \leqslant |\Sigma|$ we have $C(b_n), C(a_{n+m+m'}) \in C(w_{n+1} \ldots w_{n+m+m'})$. Notice that from this we have $v_{n+1} \ldots v_{n+m+m'} \sqsubseteq w_{n+1} \ldots w_{n+m+m'}$. Finally, we can find m' and i' such that $m' < m' + i' \leqslant |\Sigma|$ and $C(b_{n+m+m'}) = C(b_{n+m+m'+i'})$, hence $C(b_{n+m+m'}) \in C(w_{n+m+m'+i'})$, where $1 \leqslant i' \leqslant |\Sigma|$. This $m + m'$ is the desired j.

Now let find n such that $u_1 \ldots u_n \sqsubseteq w_1 \ldots w_n$ and $C(b_n) \in C(w_{n+i})$ for some $1 \leqslant i \leqslant |\Sigma|$, then iterate finding j in the claim above. This is a procedure that gives the desired $\{n_k\}$.

Proof of Proposition 3.12:

We now prove that no eventual embeddings exist witnessing quasi-isometry between α and β.

Assume that $f : \alpha \to \beta$ and $g : \beta \to \alpha$ are eventual embeddings with constants A_f, B_f and A_g, B_g witnessing quasi-isometry. Let i_n and $i_n + 1$ be the sequence of consecutive positions in α with colour 0. Since f is quasi-isometry we have

$$|f(i_n + 1) - f(i_n)| \leqslant A_f|i_{n+1} - i_n| + B_f = A_f + B_f.$$

There exists a position i_n such that $f(i_n) < f(i_n+1)$ and the number of consecutive 1s immediately on the right side of $f(i_n)$ is greater that $A_f + B_f$. Hence, $f(i_n + 1) - f(i_n) > A_f + B_f$. This is a contradiction with quasi-isometry.

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Proof of Proposition 3.13:
Constructing a quasi-isometry $f : \alpha \to \beta$ is straightforward, since eliminating $0_{\frac{2n+1}{2}}$ for each $n \in \mathbb{N}$ from β we have the sequence α. We define $g : \beta \to \alpha$ as a map that sends certain intervals in β to intervals in α monotonically and as equally as possible, as follows:

- g maps each sequence $0_n(*)^n 1_n$ in β to the sequence $0_{2n}(*)^{2n} 1_{2n}$ in α;

- $1_n(*)^n 0_{\frac{2n+1}{2}}$ in β to $1_{2n}(*)^{6n+2} 0_{2n+1}$ in α; and

- $0_{\frac{2n+1}{2}}(*)^{2n+1} 0_{n+1}$ in the string β to the sequence $0_{2n+1}(*)^{2n+1} 1_{2n+1}(*)^{6n+3} 0_{2(n+1)}$ in α.

Then g is a monotonic embedding and satisfies $d(i,j) \leqslant d(g(i), g(j)) \leqslant 6d(i,j)$, thus it is a witness for quasi-isometric embedding.

For non-colour equivalence, assume the contrary and let $\alpha = u_1 u_2...$ and $\beta = v_1 v_2...$ be a witness. We should have only finitely many u_i and v_i that contain more than one occurrences of 0 and/or 1, since otherwise we do not have a bound of $|u_i|$ or $|v_i|$. Let $I \in \mathbb{N}$ be a number such that for all $i \geqslant I$ this does not happen.

Let $p \geqslant I$ and suppose u_p contains 1_n. Then v_p contains 1_{n+k} for some $k \in \mathbb{Z}$, and for all $1_{n'}$ and $p' \geqslant I$ such that $u_{p'}$ contains $1_{n'}$, $v_{p'}$ contains $1_{n'+k}$ (otherwise some $u_{p'}$ or $v_{p'}$ contains multiple 1s).

Now suppose $p' > p \geqslant I$ and u_p and $u_{p'}$ contain 1_n and 1_{n+1}, respectively. Then from the construction of α there should be exactly one $q \in [p+1, p'-1]$ with the corresponding interval u_q contains 0. On the other hand v_p and $v_{p'}$ contain 1_{n+k} and 1_{n+k+1}, respectively, and from the construction of β we should have two elements in $[p+1, p'-1]$ with the corresponding interval contains 0, which is the contradiction.

Proof of Corollary 4.8:
Let $\mathcal{M}$ be a Büchi automaton. Our goal is to find atlases from the list provided in Theorem 4.7 such that the union of these atlases coincides with the atlas $[L(\mathcal{M})]$. For this, we decompose $\mathcal{M}$ into k automata $\mathcal{M}_1$, ..., $\mathcal{M}_k$ so that the equality $(\star)$ holds and each $\mathcal{M}_i$ satisfies the postulate. This decomposition can be done in linear time on the size of $\mathcal{M}$ (e.g. using Tarjan's algorithm that decomposes a directed graph into strongly connected components). Now for each $\mathcal{M}_i$, we check the assumptions of Lemma 4.3 through Lemma 4.6. This can also be done in linear time.

As an example, the assumptions of Lemma 4.3 can be checked as follows. Assume that the automaton under consideration is $\mathcal{M}_i$. In order to check if $\mathcal{M}_i$ contains a 0-loop, we remove all transitions labeled with 1 from the state diagram of $\mathcal{M}_i$. This gives us a directed graph whose edges are transitions labelled with 0. In this graph we check if there is a loop. If there exists a loop, then $\mathcal{M}_i$ has a 0-loop. Otherwise $\mathcal{M}_i$ does not have a 0-loop. Similarly, to check if $\mathcal{M}_i$ contains a 1-loop, we remove all transitions labeled with 1. This gives us a directed graph. If there is a loop in this directed graph, then $\mathcal{M}_i$ has a 1-loop; Otherwise, not. To find the set X from (the statement of) the Lemma we just need to check which of the strings 0^ω, 1^ω, 10^ω, 01^ω is accepted by $\mathcal{M}_i$. This can also be done in linear time on the size of $\mathcal{M}_i$.

The argument shows that we can find, in linear time on size of $\mathcal{M}$, the atlases from the list provided in Theorem 4.7 such that the union of these atlases is the atlas $[L(\mathcal{M})]$. Thus, given Büchi automata $\mathcal{A}$ and $\mathcal{B}$ we can decide in linear time if $[L(\mathcal{A})] = [L(\mathcal{B})]$.

Proof of Theorem 5.2:
Assume that α and β are (A, B)-quasi-isometric. The tree $T(\alpha, \beta, A, B)$ is infinite by the Lemma above. Such a tree has a path computable in the halting set for the tree [10]. The tree, as we constructed above, is computable in α and β. This proves the first part.

We prove the second part of the theorem. By Lemma 5.1 computable strings α and β are quasi-isometric if and only if there exists are constants A, B such that the computable tree $T(\alpha, \beta, A, B)$ is infinite. This is a Σ_3^0-statement. Hence, the quasi-isometry problem between computable strings α and β is a Σ_3^0-set. To prove that the problem is complete, we reduce a Σ_3^0-complete problem to the quasi-isometry problem. Below is an informal explanation of the reduction.

Let $W_0, W_1, \ldots$ be a standard enumeration of all c.e. sets. It is known that the set $Fin = \{i \mid W_i \text{ is finite}\}$ is a Σ_3^0-complete problem [12]. For each W_i we need to construct α_i and β_i two infinite strings such that W_i is finite if and only if α_i and β_i are quasi-isometric. For this we start enumerating W_i by stages 1, 2, ... At stage 0, we consider $\alpha_{i,0}$ and $\beta_{i,0}$ finite prefixes that can be extended to a $(1,1)$-quasi-isometry and set $A_0 = B_0 = 1$. By stage s we will have a finite prefixes $\alpha_{i,s-1}$ of α_i and $\beta_{i,s-1}$ of β_i built. If at stage s the enumeration of W_i outputs a new element, then we start extending α_s and β_s so that the following holds. If W_i never increases its size from stage s on then α and β are (A_s, B_s)-quasi-isometric but

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not (A_{s-1}, B_{s-1})-quasi-isometric, where $A_s > A_{s-1}$ and $B_s > B_{s-1}$. This can easily be achieved in two steps. In the first step, one ensures that a large part of α contains a long consecutive sequence of elements of one colour, and the same positions of β have another colour. This will guarantee that α_i and β_i are not (A_{s-1}, B_{s-1})-quasi-isometric. In the second step, one ensures that those long sequences in the first step do not conflict with (A_s, B_s)-quasi-isometry and one can continue on extending $\alpha_{i,s}$ and $\beta_{i,s}$ so that they are (A_s, B_s)-quasi-isometric. Thus, if W_i is infinite then there is an infinite sequence s_1, s_2, ... of increasing stages at which the enumeration of W_i increases the size of W_i. Hence, from stage s_i, it is guaranteed that α and β are not (A_{s_i}, B_{s_i})-quasi-isometric. This implies that W_i is infinite then α_i and β_i are not quasi-isometric.

Proof of Theorem 6.7:
Let $h : \alpha \to \beta$ be an (A, B)–quasi-isometry, and $g : \beta \to \alpha$ be a (C, D)–quasi-isometry. They induce quasi-isometries between $X_{n,\alpha}$ and $X_{n,\beta}$. In particular, for the function h and each $X_{n,\alpha}$ and $X_{n,\beta}$ we have:

$$(1/A) \cdot d_n(i, j) - B/s(n) \leqslant$$
$$d_n(h(i), h(j)) \leqslant A \cdot d_n(i, j) + B/s(n),$$

where i, j and $h(i), h(j)$ are elements of the spaces $X_{n,\alpha}$ and $X_{n,\beta}$, respectively. Hence, we have

$$(1/A) \cdot |a_n - b_n|/s(n) - B/s(n) \leqslant d_n(h(a_n), h(b_n))$$
$$\leqslant A \cdot |a_n - b_n|/s(n) + B/s(n).$$

Taking the limits and setting $H(\mathbf{a}) = (h(a_n))_{n \geqslant 0}$, we get

$$(1/A) \cdot D(\mathbf{a}, \mathbf{b}) \leqslant D(H(\mathbf{a}), H(\mathbf{b})) \leqslant A \cdot D(\mathbf{a}, \mathbf{b}).$$

So, the desired C_H is just A. The function G is constructed the same way. Both H and G preserve colours.

Continuity and injectivity of G and H immediately follow from the bi-Lipschitz condition. This implies that both G and H are order preserving mappings as well. Surjectivity of H follows from the condition put on quasi-isometry h: for every $a' \in \beta$ there exists an $a \in \alpha$ such that $|a' - h(a)| \leqslant A$. Hence for any bounded sequence $(a'_n)_{n \geqslant 0}$ there exists a sequence $(a_n)_{n \geqslant 0}$ such that $D(H(a), \mathbf{a}') = 0$, that is, $h(\mathbf{a}) = \mathbf{a}'$.