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A SIMPLE CONSTRUCTION OF ABSOLUTELY DISJUNCTIVE LIOUVILLE NUMBERS

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ABSTRACT

A disjunctive sequence is an infinite sequence in which every finite string appears as a substring. An absolutely disjunctive number (or lexicon) is a real whose expansion with respect to every base is disjunctive.

In this note we give a simple construction of absolutely disjunctive Liouville numbers (reals which can be "quite closely" approximated by sequences of rationals).

Keywords: Disjunctive sequences, Liouville numbers, computability

1. Introduction

Disjunctivity is a qualitative form of (Borel) normality: normal sequences are disjunctive, but the converse is false. Like normality [7, 15], disjunctivity is not base-invariant (for more details see [9]).

Jürgensen and Thierrin [11] gave a construction of Liouville numbers disjunctive in base b. Highly incomputable Liouville numbers disjunctive to every base have been presented in [19, Theorem 15].

The recent construction of a computable absolutely normal Liouville number in [1] yields also computable, absolutely disjunctive Liouville numbers. This construction, however, is based on rather complicated measure-theoretic arguments from [2]. The aim of this note is to present a simple algorithm producing weaker examples, that is, computable Liouville numbers disjunctive to every base.

1.1. Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Its elements will be usually denoted by letters $i, \ldots, n$. The set $A_b = \{0, 1, \ldots, b-1\}$, where $b \geq 2$ is a positive integer, is called the *b–base*; the elements of A_b are called b-digits. By A_b^* we denote the set of all finite strings (words) with ε denoting the empty string; A_b^ω is the set of all infinite sequences (ω-words) over A_b; ω-words are usually denoted by $\mathbf{x}, \mathbf{y}$. The length of a finite or infinite string η over A_b is denoted by $|\eta|$.

For $w \in A_b^*$ and $\eta \in A_b^* \cup A_b^\omega$, $w \cdot \eta$ is their *concatenation*. This concatenation product extends in an obvious way to subsets $L \subseteq A_b^*$ and $B \subseteq A_b^* \cup A_b^\omega$. If $w \in A_b^*$ and $i \geq 0$ is an integer, then w^i is the concatenation $ww \cdots w$ (i times) and w^ω is the infinite concatenation $ww \cdots w \cdots$. The $\cdot$ operator can be omitted when the meaning is clear, as in $w\eta$.

By $w \sqsubseteq u$ and $w \sqsubset \mathbf{y}$ we denote that w is a prefix of u and $\mathbf{y}$, respectively. Further, let $\mathbf{pref}(\mathbf{y}) = \{w : w \sqsubset \mathbf{y}\}$ and $\mathbf{infix}(\mathbf{y}) = \{w : \exists v(v \cdot w \sqsubset \mathbf{y})\}$ be the set of prefixes and infixes of $\mathbf{y}$, respectively.

1.2. Preliminary definitions and results

In this section we define the classes of real numbers studied in the paper.

A real number α is called a *Liouville number* if it is irrational and for every positive integer k, there exist integers p_k and q_k with $q_k > 1$ such that

$$\left|\alpha - \frac{p_k}{q_k}\right| < \frac{1}{q_k^k}.$$

A real $\alpha \in [0, 1]$ is called *computable* if for some $b \geq 2$ it has a b-ary computable expansion $\alpha = 0.x_1x_2\ldots$, that is, there is a computable function f_α such that $f_\alpha(n) = x_n$, for all $n \geq 1$. This condition is equivalent to the requirement that there is a computable sequence of rationals $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ such that

$$|\alpha - \frac{p_n}{q_n}| \leq \frac{1}{2^n},$$

for all $n \in \mathbb{N}$. This shows that if α is computable, then its expansions in any base b are computable.

Originally, ω-words $\mathbf{x}$ were called disjunctive because the syntactic monoid of the set $\{\mathbf{x}\}$ is disjunctive, that is, its syntactic congruence is the identity (see [10]). Equivalently, disjunctive ω-words are those which have every finite word as subword.[1] In fact, in a disjunctive ω-word every word appears infinitely many times.

Disjunctivity is also related to randomness: disjunctive ω-words are exactly the ω-words not contained in any null-set definable by finite automata [16, 17]. For more properties of disjunctive sequences see [4].

A real number $\alpha \in [0, 1]$ is *disjunctive* (or *rich*) *in base b* if its b-ary expansion is disjunctive. For example, Champernowne's number $0.0123456789101112\ldots$ is computable and disjunctive in base 10 [8]. No rational number is disjunctive in any base.

[1] In view of this latter property they are also called *rich ω-words*.

An *absolutely disjunctive* number (or *lexicon*) is a real which is disjunctive in every base. Every Martin-Löf random real is a lexicon, but the converse is false [3].

In the sequel we denote by $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{D}$ the set of all Liouville numbers, computable numbers and absolutely disjunctive numbers in $[0,1]$, respectively.

1.3. Co-meagre and dense sets

It is useful to consider the unit interval $[0,1]$ and the spaces of infinite sequences A_b^ω as metric spaces. Suitable metrics are the usual distance $|\alpha - \beta|$ in $[0,1]$ and

$$\rho(\mathbf{x}, \mathbf{y}) = b^{-\inf\{i \in \mathbb{N} \mid i \geq 1, x_i \neq y_i\}},$$

for infinite words $\mathbf{x} = x_1 \cdots x_i \cdots, \mathbf{y} = y_1 \cdots y_i \cdots$ with $x_i, y_i \in A_b$. With these metrics $[0,1]$ and A_b^ω become complete metric spaces.

Let $\mathcal{X}$ be a complete metric space. A set $M \subseteq \mathcal{X}$ is *nowhere dense* if its closure (smallest closed set containing M) does not contain a non-empty open subset. A set $M \subseteq \mathcal{X}$ is *meagre* (or of *first Baire category*) if it is a countable union of nowhere dense sets. A complement of a meagre set is called *co-meagre* (or *residual*).

The following closure property of co-meagre sets is well-known (see [12]).

Fact 1. *In a complete metric space the family of co-meagre sets is closed under countable intersection.*

A set $M \subseteq \mathcal{X}$ is *dense* if $M \cap M' \neq \emptyset$ for every non-empty open set $M' \subseteq \mathcal{X}$. Note that in a complete metric space every co-meagre set is dense, but a dense set might be meagre, even countable.

The following relations hold for subsets $F \subseteq A_b^\omega$ and their counterparts in $[0,1]$.

Lemma 2 [18]. *Let $F \subseteq A_b^\omega$ and $M_F = \{0.\mathbf{x} \mid \mathbf{x} \in F\} \subseteq [0,1]$. Then*

(I) *F is nowhere dense if and only if M_F is nowhere dense.*

(II) *F is co-meagre if and only if M_F is co-meagre.*

(III) *F is dense if and only if M_F is dense.*

Fact 3 [14]. (I) *The set of Liouville numbers $\mathcal{L}$ is co-meagre.*

(II) *The set of computable numbers $\mathcal{C}$ is countable, meagre and dense.*

2. Disjunctive ω-words

As mentioned above disjunctive ω-words are infinite words $\mathbf{x} \in A_b^\omega$ having $\mathbf{infix}(\mathbf{x}) = A_b^*$. By

$$D_b = \{\mathbf{x} \mid \mathbf{x} \in A_b^\omega \wedge \mathbf{infix}(\mathbf{x}) = A_b^*\}$$

we denote the set of all disjunctive ω-words in A_b^ω. Then the set of all absolutely disjunctive numbers in $[0,1]$ is

$$\mathcal{D} = \{\alpha \mid \alpha \in [0,1] \wedge \forall b(b \geq 2 \rightarrow \exists \mathbf{x}(\mathbf{x} \in D_b \wedge \alpha = 0.\mathbf{x}))\}.$$

The set $\mathcal{D}$ has the following topological property:

Lemma 4 [6, 18]. *The set $\mathcal{D}$ is co-meagre in* $[0,1]$.

Then from Fact 1 and Lemma 2 it follows that the set of absolutely disjunctive Liouville numbers is "topologically" large:

Corollary 5. *The intersection $\mathcal{L} \cap \mathcal{D}$ is co-meagre in* $[0,1]$.

Corollary 5 gives only an existence proof, not a constructive one. Further more, since the set of computable reals $\mathcal{C}$ is countable, it does not even show that $\mathcal{L} \cap \mathcal{D} \cap \mathcal{C}$ is not empty.

To show the existence of computable absolutely disjunctive Liouville numbers we use a representation of the b-ary counterparts $\{\mathbf{x} \in A_b^\omega \mid 0.\mathbf{x} \in \mathcal{D}\}$ of $\mathcal{D}$ via computable languages. In Section 4 we then show how this description can be transformed into an algorithm computing an absolutely disjunctive Liouville number.

Theorem 6 [18]. *For every base b there effectively exists a computable language $W_b \subseteq A_b^*$ such that the ω-language $\{\mathbf{x} \in A_b^\omega \mid$ the set $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite$\}$ is the set of all b-ary expansions of absolutely disjunctive reals in* $[0,1]$.

More explicitly, Theorem 6 ([18, Theorem 21]) provides, for every base b, an increasing computable function $g : \mathbb{N} \to A_b^*$ such that $g(\mathbb{N}) = W_b$. This function g naturally induces a computable order on W_b.

Since $\mathcal{D}$ is dense in $[0,1]$, from Lemma 2.iii we deduce that the ω-language $\{\mathbf{x} \in A_b^\omega \mid$ the set $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite$\}$ is dense in A_b^ω. This yields the following.

Corollary 7. *For every $u \in A_b^*$ there is a $v \in W_b$ such that $u \sqsubset v$.*

Proof. As the ω-language $\{\mathbf{x} \in A_b^\omega \mid$ the set $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite$\}$ is dense, every open subset of A_b^ω contains an $\mathbf{x}$ such that $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite.

Consider the open ω-language $u \cdot A_b^\omega$ (see e.g. [18]). Then there is an $\mathbf{x}$ for which $\mathbf{pref}(\mathbf{x}) \cap W_b$ is infinite. Consequently, there is a $v \in \mathbf{pref}(\mathbf{x}) \cap W_b$ such that $u \sqsubset v$. □

3. Expansions of Liouville numbers

For our purposes it is useful to have the following property of b-ary expansions $\mathbf{x}$ of reals which guarantees that $0.\mathbf{x}$ is a Liouville number. A similar criterion was sketched, without proof, by Maillet in [13].

Using finitely or infinitely many strings $w_i \in A_b^*$ and a function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ we construct b-ary expansions of real numbers in the following way.

Define $\Lambda_{j=0}^{\infty} w_j^{f(j)}$ as the concatenation of w_0 ($f(0)$ times), w_1 ($f(1)$ times),w_2 ($f(2)$ times)....

Lemma 8 [5]. *Let* $(w_i)_{i\in\mathbb{N}}$ *be a family of non-empty strings* $w_i \in A_b^*$, $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, *and* $n_i = \sum_{j=0}^{i} f(j) \cdot |w_j|$. *If*

$$\liminf_{i\to\infty} \frac{n_{i-1} + |w_i|}{n_{i-1} + f(i) \cdot |w_i|} = 0\,, \tag{1}$$

then $\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$ *is the b-ary expansion of a rational or a Liouville number.*

4. The Algorithm

The following algorithm computes the b-ary expansion $\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$ of an absolutely disjunctive Liouville number whose b-ary expansion starts with a given word $w_0 \in A_b^*$. It uses the computable injective ordering $g : \mathbb{N} \to W_b$ of the computable language W_b given by Theorem 6.

Algorithm Liouville-disjunctive

0 **initialise** $w_0 = u_0 = v_0, \quad f(0) = 1$

1 **for** $i = 1$ **to** ∞ **do**

2 $v_i =$ first word in $\left(W_b \cap u_{i-1} \cdot A_b^*\right) \setminus \{u_{i-1}\}$

3 **calculate** w_i **from** $v_i = u_{i-1} \cdot w_i$

4 **calculate** $f(i) = \min\left\{k \mid \dfrac{|u_{i-1}| + |w_i|}{|u_{i-1}| + k \cdot |w_i|} < \dfrac{1}{i}\right\}$

5 $u_i = u_{i-1} \cdot w_i^{f(i)}$

6 **endfor**

The algorithm computes three families of words $(u_i)_{i\in\mathbb{N}}$, $(v_i)_{i\in\mathbb{N}}$, and $(w_i)_{i\in\mathbb{N}}$ and a function $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$. Note that at each step the set $(W_b \cap u_{i-1} \cdot A_b^*) \setminus \{u_{i-1}\}$ is effectively ordered according to g.

First, Step 2 implies $v_i \in W_b$ and together with Step 5, by induction, $u_{i-1} \sqsubset v_i \sqsubseteq u_i \sqsubset v_{i+1}$. From the Step 3 and $u_{i-1} \sqsubset v_i$ we have $|w_i| > 0$. Then, again using Step 5, by induction one verifies that

$$u_i = \Lambda_{j=0}^{i} w_j^{f(j)}. \tag{2}$$

It remains to show that the algorithm will produce an infinite computable ω-word, that is, it never stops. To this end it suffices to show that the choice in Step 2 is always possible. From Corollary 7 we know that for every $u \in A_b^*$ there is a $v \in W_b$ such that $u \sqsubset v$. This makes it possible to choose the first element in W_b w.r.t. g which has u as a proper prefix.

Thus the algorithm computes two computable approximations of an ω-word $\mathbf{x} = \Lambda_{j=0}^{\infty} w_j^{f(j)}$ via the families of prefixes $(u_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$. From $v_i \in W_b$ we obtain $0.\mathbf{x} \in \mathcal{D}$ via Theorem 6, and, because of (2), Step 4 shows that the words u_i and w_i satisfy Eq. (1). Thus Lemma 8 verifies that $0.\mathbf{x}$ is also a Liouville number. The computability of $\mathbf{x}$ follows directly from the algorithm.

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