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Twenty Combinatorial Examples of Asymptotics Derived From Multivariate Generating Functions

Robin Pemantle
Department of Mathematics,
University of Pennsylvania,
Philadelphia, USA

Mark C. Wilson
Department of Computer Science,
University of Auckland,
Auckland, New Zealand

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1. Introduction

The purpose of this paper is to review recent developments in the asymptotics of multivariate generating functions, and to give an exposition of these that is accessible and centered around applications. We begin, however, with a brief review of univariate generating functions and of previous results on multivariate generating functions.

1.1. **Background: the univariate case.**

Let $\{a_n : n \geq 0\}$ be a sequence of complex numbers and let $f(z) = \sum_n a_n z^n$ be the associated generating function. Generating functions are among the most useful tools in combinatorial enumeration. In the introductory section of his graduate text [Sta97], Richard Stanley deems the generating function $f(z)$ to be "the most useful but most difficult to understand method for evaluating" a_n, comparing it to a recurrence, an asymptotic formula and a complicated explicit formula.

There are two steps involved in using generating functions to evaluate a sequence: first, one must determine f from the combinatorial description of $\{a_n\}$, and secondly one must be able to extract information about a_n from f. The first step is partly science and partly an art form. Certain recurrences for $\{a_n\}$ translate neatly into functional equations for f, but there are numerous twists and variations. There is no way of telling in advance when f will have a sufficiently nice form to be useful, and a good portion of many texts on enumeration is devoted to the battery of techniques available for producing the generating function f.

The second step, namely estimation of a_n once f is known, is reasonably well understood and somewhat mechanized. Starting with Cauchy's integral formula

$$a_n = \frac{1}{2\pi i} \int z^{-n-1} f(z)\, dz\,,$$

one may apply complex analytic methods to obtain good estimates for a_n. There are, again, many cases and variations, but the small amount of necessary complex analytic machinery, such as the Watson-Deutsch lemma, Darboux' method, and more general saddle point methodology, is catalogued in many places, e.g., [Hen91, Chapter 11]. Furthermore, the consequences of these methods have been worked out for a large class of generating functions. In many, perhaps most cases, f has a finite radius of convergence, R, and is amenable to what is called **singularity analysis**. The accuracy of the estimate of a_n will depend on whether f is analytic on the circle of radius R or beyond this.

Since our main subject is the asymptotics of multivariate generating functions, we give here only the barest summary of the vast literature on asymptotics of univariate generating functions. The transfer theorems of [FO90b] are one strikingly simple and powerful tool that works in the case that f is analytic in the so-called **Camembert-shaped region**

$$\{z : |z| < R + \epsilon, z \neq R, |\arg(z - R)| \geq \pi/2 - \epsilon\} .$$

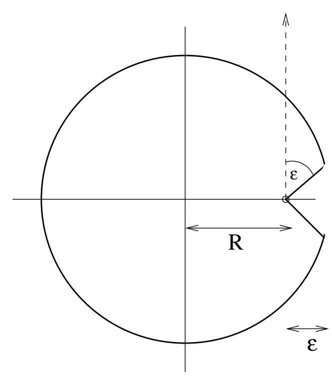

FIGURE 1. the Camembert-shaped region

These theorems allow one to convert information about the asymptotics of f near its minimum modulus singularity into an asymptotic formula for a_n. Pioneering examples of the use of the saddle point method (in the case where f is entire) or of Darboux' method (in the case where f is badly behaved on its circle of convergence) are, respectively, Hayman's estimates for **admissible** entire functions [Hay56] and Brigham's estimates that were developed for partition functions [Bri50]. Among the many reviews of univariate asymptotics, one that predates the work of Flajolet and Odlyzko but is still quite useful is [Ben74, part II]. The 1995 survey article by Odlyzko [Odl95] is somewhat more extensive and up-to-date.

1.2. **Multivariate asymptotics.** At the time of Bender's 1974 review article [Ben74], almost nothing was known about extracting asymptotics from multivariate generating functions. Bender's concluding section urges research in this area:

> Practically nothing is known about asymptotics for recursions in two variables even when a generating function is available. Techniques for obtaining asymptotics from bivariate generating functions would be quite useful.

By the time of Odlyzko's 1995 survey [Odl95], a single vein of research had appeared, initiated by Bender and carried further by Gao, Richmond and others. Let the number of variables be denoted by d so that $\mathbf{z}$ denotes the d-tuple $(z_1, \ldots, z_d)$. We use the multi-index notation for products: $\mathbf{z}^{\mathbf{r}} := \prod_{j=1}^d z_j^{r_j}$. Suppose we have a multivariate array $\{a_{\mathbf{r}} : \mathbf{r} \in \mathbb{N}^d\}$ for which the generating function

$$F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} z_{\mathbf{r}}$$

is assumed to be known in some reasonable form. We are interested in the asymptotic behavior of $a_{\mathbf{r}}$.

The first paper in this vein of research was [Ben73], already published at the time of Bender's survey, but the seminal paper is [BR83]. In this paper, Bender and Richmond assume that F has a singularity of the form $A/(z_d - g(z_1, \ldots, z_{d-1}))^q$ near the graph of a smooth function g, for some real exponent q. They show, under approporiate further hypotheses on F, that the probability measure μ_n one obtains by renormalizing $\{a_{\mathbf{r}} : r_d = n\}$ to sum to 1 converges to a multivariate normal when appropriately rescaled. Their method, which we call the **GF-sequence method**, is to break the d-dimensional array $\{a_{\mathbf{r}}\}$ into a sequence of $(d-1)$-dimensional slices and consider the sequence of $(d-1)$-variate generating functions

$$f_n(z_1, \ldots, z_{d-1}) = \sum_{\mathbf{r}: r_d = n} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} .$$

They show that, asymptotically as $n \to \infty$,

$$f_n(\mathbf{x}) \sim C_n g(\mathbf{x}) h(\mathbf{x})^n \tag{1.1}$$

and that sequences of generating functions obeying (1.1) satisfy a central limit theorem and a local central limit theorem.

We will review these results in more detail in Section 7, but one crucial feature is that they always produce Gaussian (central limit) behavior. The applicability of the entire method is therefore limited to the single, though important, case where the coefficients $a_{\mathbf{r}}$ are nonnegative and possess a Gaussian limit. The work of [BR83] has been greatly expanded upon, but always in a similar framework. For example, it has been extended to matrix recursions [BRW83] and the applicability has been expanded from algebraic to algebraico-logarithmic singularities of the form $F \sim (z_d - g)^q \log^\alpha(1/(z_d - g))$ [GR92]. The difficult step is always deducing asymptotics from the hypotheses $f_n \sim C_n\, g \cdot h^n$. Thus some papers in this stream refer to such an assumption in their titles [BR99], and the term "quasi-power" has been coined for such a sequence $\{f_n\}$.

The theory has also been pushed forward via its use in applications. The nearly-published text by Flajolet and Sedgewick [FS05, Chapter IX] devotes a chapter of nearly 100 pages to multivariate asymptotics, in which many of the basic results on quasi-powers are reviewed and extended. All the limit theorems in [FS05, Chapter IX], however, are Gaussian (outside of the graph enumeration example in Section 11). The large deviation results of Hwang [Hwa96; Hwa98] are not restricted to the Gaussian case, but give asymptotics on a cruder scale.

1.3. **New multivariate methods.** Odlyzko's survey of asymptotic enumeration methods [Odl95], which is meant to be somewhat encyclopedic, devotes fewer than 6 of its 160 pages to multivariate asymptotics. Odlyzko describes why he believes multivariate coefficient estimation to be difficult. First, the singularities are no longer isolated, but form $(d-1)$-dimensional hypersurfaces. Thus, he points out, "Even rational multivariate functions are not easy to deal with." Secondly, the multivariate analogue of the one-dimensional residue theorem is the conisderably more difficult theory of Leray [Ler50]. This theory was later fleshed out by Aizenberg and Yuzhakov, who spend a few pages [AY83, Section 23] on generating functions and combinatorial sums. Further progress in using multivariate residues to evaluate coefficients of generating functions was made by Bertozzi and McKenna [BM93], though at the time of Odlyzko's survey none of the papers based on multivariate residues such as [Lic91; BM93] had resulted in any kind of systematic application of these methods to enumeration.

The topic of the present review article is a recent vein of research, begun in [PW02] and continued in [PW04; BP04; Lla03] and in several manuscripts in progress. The idea, seen already to some degree in [BM93], is to use complex methods that are genuinely multivariate to evaluate coefficients of multivariate generating functions via the multivariate Cauchy formula. By avoiding symmetry-breaking decompositions such as $F = \sum f_n(z_1, \ldots, z_{d-1}) z_d^n$, one hopes the methods will be more universally applicable and the formulae more canonical. In particular, the results of Bender *et al.* and the results of Bertozzi and McKenna are seen to be two instances of a more general result estimating the Cauchy integral via topological reductions of the cycle of integration. These topological reductions, while not fully automatic, are algorithmically decidable in large classes of cases and are the subject of Section 2. An ultimate goal, stated in [PW02; PW04], is to develop computer software to automate all of the computation.

Aside from providing a summary and explication of this line of research, the present survey is meant to serve several other purposes. First, results from [PW02; PW04; BP04] are presented in streamlined forms, stated so as to avoid the scaffolding one needs to prove them. This is to make the results more comprehensible. Secondly, by focusing on combinatorial applications, we hope to create a sort of user's manual: one that contains worked examples akin to those a potential user will have in mind. Many of the applications are to abstract combinatorial structures but we also include direct applications to computational biology and to formal languages and automata theory. Finally, we present a number of results that "pre-process" the basic, general theorems, providing useful computational reductions of the hypotheses or conclusions in specific cases of interest. We now give the notation for generating functions and their asymptotics that will be used throughout, and then briefly describe a prototypical asymptotic theorem.

Let

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} \tag{1.2}$$

be a generating function in d variables, where G and H are analytic and $H(\mathbf{0}) \neq 0$. The representation of F as a quotient of analytic functions is required only to hold in a certain polydisk, described in Section 2.2, though in the majority of the examples F is meromorphic on all of $\mathbb{C}^d$. We will assume throughout that H vanishes somewhere, since the methods in this paper do not give nontrivial results for entire functions. In the bivariate case ($d = 2$), which is the setting for the majority of our examples, it is sometimes clearer to use (x, y) instead of (z_1, z_2):

$$F(x,y) = \frac{G(x,y)}{H(x,y)} = \sum_{r,s=0}^{\infty} a_{rs} x^r y^s \tag{1.3}$$

We are concerned with asymptotics when $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ remaining in some specified set, bounded away from the coordinate planes. Thus for example, when $d = 2$, the ratio s/r will remain in a compact subset of $(0, \infty)$. It is possible via our methods to address the other case, where $r = o(s)$ or $s = o(r)$, (see, for example, [Lla05]), but our main purpose in this paper is to give examples that require, among all the methods and results cited above, only those from [PW02] together with the simplest methods from [PW04; BP04].

The following result is representative of the basic results we quote from [PW02; PW04].

Theorem 1.1 (combines Corollary 2.8 and Theorem 2.10)**.** *Let F be as in (1.2) and suppose $a_{\mathbf{r}} \geq 0$.*

> *(i) For each $\mathbf{r}$ in the positive orthant, there is a unique $\mathbf{z}(\mathbf{r})$ in the positive orthant satisfying the equations (2.5) and on the boundary of the domain of convergence of F.*
>
> *(ii) with $\mathbf{z}(\mathbf{r})$ defined in this way,*
>
> $$a_{\mathbf{r}} \sim (2\pi)^{-(d-1)/2} \mathcal{H}^{-1/2} \frac{G(\mathbf{z}(\mathbf{r}))}{-z_d \partial H/\partial z_d(\mathbf{z}(\mathbf{r}))} |\mathbf{r}|^{-(d-1)/2} \mathbf{z}(\mathbf{r})^{-\mathbf{r}}$$
>
> *uniformly over compact cones of $\mathbf{r}$ for which $\mathbf{z}(\mathbf{r})$ is a smooth point of $\{H = 0\}$ uniquely solving (2.5) on the boundary of the domain of convergence of F, and for which the matrix $\mathcal{H}$ determined by partial derivatives of H (it is the Hessian of $\{H = 0\}$ written as the graph of a function near $\mathbf{z}(\mathbf{r})$) is nonsingular.*

A prototypical companion result is the following result for implicitly defined generating functions, which arise commonly in recursions on trees.

If $f(z) = z\phi(f(z))$ then the n^{th} coefficient of f is given asymptotically by

$$n^{-3/2} \frac{y_0 \phi'(y_0)^n}{\sqrt{2\pi\phi''(y_0)/\phi(y_0)}} \tag{1.4}$$

where y_0 is the unique positive solution to $y\phi'(y) = \phi(y)$.

In [FS05] the problem is reduced via Lagrange inversion to determination of the y^{n-1} coefficient of $\phi^n(y)$, which they then solve via complex integration methods. By viewing this instead as the (n, n) coefficient of

$$F(x, y) := \frac{y}{1 - x\phi(y)} ,$$

one sees that the complex integration step is already done, and that (1.4) is immediate upon identifying that $\mathbf{z}(n, n) = (1/\phi(y_0), y_0)$; see Proposition 5.1, where the hypotheses for (1.4) are stated more completely.

The organization of the remainder of this paper is as follows. The next section outlines results from various sources that give a brief but nearly complete explanation as to how one goes from 1.2, via a multivariable Cauchy formula, to asymptotic formulae for $a_{\mathbf{r}}$. We then quote the precise theorems we will use from [PW02] and [PW04]. Section 3 works through some examples in detail. The next three sections discuss transfer matrices, Lagrange inversion and the kernel method respectively; applications are given of enumerating various kinds of words, paths, trees and graphs. In Section 7 we discuss open questions and extensions of the material presented here, and compare our results with those of other authors.

2. Background and summary of our results

In [PW02; PW04; BP04], asymptotic formulae for $a_{\mathbf{r}}$ are derived, as $|\mathbf{r}| \to \infty$, that are uniform as $\mathbf{r}/|\mathbf{r}|$ varies over some compact set. It is useful to separate $\mathbf{r}$ into a scale parameter $|\mathbf{r}|$, which is a positive real number, and a direction parameter $\overline{\mathbf{r}}$, which is an element of projective space. Although $\mathbf{r}$ is always an element of the positive orthant of $\mathbb{R}^d$, it will make sense to consider it as an element of $\mathbb{C}^d$. When convenient, we identify $\mathbf{r}$ with its class $\overline{\mathbf{r}}$ in projective space or its projection $\hat{\mathbf{r}}$ in the real $(d-1)$-simplex Δ^{d-1}, the set $\{\mathbf{x} \in (\mathbb{R}^+)d : |\mathbf{x}| := \sum_{j=1}^d x_j = 1\}$. Thus $\mathbf{r}$

may appear anywhere in the following diagram, where $\mathcal{O}$ and $\overline{\mathcal{O}}$ denote the positive orthants of $\mathbb{R}^d$ and $\mathbb{RP}^{d-1}$ respectively.

$$\begin{array}{ccccc}
\mathcal{O}^d & \longrightarrow & \mathbb{R}^d & \longrightarrow & \mathbb{C}^d \\
{\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow & & {\scriptstyle -}\downarrow \\
\overline{\mathcal{O}} & \longrightarrow & \mathbb{RP}^{d-1} & \longrightarrow & \mathbb{CP}^{d-1} \\
{\scriptstyle \frown}\downarrow & & & & \\
\Delta^{d-1} & & & &
\end{array}$$

Quantities that depend on $\mathbf{r}$ only through its direction will be denoted as functions of $\overline{\mathbf{r}}$. The results we will quote in this section may be informally summarized as follows.

(i) Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of the pole variety $\mathcal{V} = \{\mathbf{z} : H(\mathbf{z}) = 0\}$ near a finite set, $\mathsf{crit}_{\overline{\mathbf{r}}}$, of **critical points**. [Definition 1.]

(ii) For the purposes of asymptotic computation, one may reduce this set of critical points further to a set $\mathsf{contrib}_{\overline{\mathbf{r}}} \subseteq \mathsf{crit}_{\overline{\mathbf{r}}}$ of **contributing critical points**, usually a single point. [Formula (2.4) and definition 2.]

(iii) One may determine crit and $\mathsf{contrib}$ by a combination of algebraic and geometric criteria. [Proposition 2.2 and Theorem 2.5 respectively.]

(iv) Critical points may be of three types: smooth, multiple and bad. [Definition 5]

(v) Corresponding to each smooth or multiple critical point, $\mathbf{z}$, is an asymptotic expansion for $a_{\mathbf{r}}$ which is computable in terms of the derivatives of G and H at $\mathbf{z}$. [Sections 2.3 and 2.4 respectively.]

The culmination of the above is the following meta-formula:

$$a_{\mathbf{r}} \sim \sum_{\mathbf{z} \in \mathsf{contrib}} \mathbf{formula}(\mathbf{z}) \tag{2.1}$$

where $\mathbf{formula}(\mathbf{z})$ is one function of the local geometry for smooth points and a different function for multiple points. Specific instances of **formula** are given in equations (2.6) - (2.12); the simplest case is

$$a_{\mathbf{r}} \sim C|\mathbf{r}|^{\frac{1-d}{2}} \mathbf{z}^{-\mathbf{r}}$$

where C and $\mathbf{z}$ are functions of $\bar{\mathbf{r}}$. No general expression for **formula** $(\mathbf{z})$ is yet known when $\mathbf{z}$ is a bad point, hence the name bad point.

Fundamental to all the derivations is the Cauchy integral representation

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z} \tag{2.2}$$

where T is the product of sufficiently small circles around the origin in each of the coordinates, $\mathbf{1}$ is the d-vector of all ones, and $d\mathbf{z}$ is the holomorphic volume form $dz_1 \wedge \cdots \wedge dz_d$. The gist of the analyses in [PW02; PW04; BP04] is the computation of this integral by replacing the cycle T with other cycles where the integral reduces to an exactly computable multivariate residue followed by a saddle point integral for which a complete asymptotic series may be read off in a straightforward manner by well known methods. The first subsection below explains what critical points and contributing critical points are and gives the "big picture", namely the topological context as outlined in [BP04]. The second subsection gives the specific definitions from [PW02] that we need to find the contributing critical points and sort them into types: smooth, multiple and bad. Sections 2.3 and 2.4 quote results from [PW02] and [PW04] that give asymptotics for smooth points if they are respectively smooth or multiple. Section 2.5 restates some of these asymptotics in terms of probability limit theorems.

2.1. **Topological representation.** As $\mathbf{r} \to \infty$, the integrand in (2.2) becomes large. It is natural to attempt a saddle point analysis. That is, we try to deform the contour, T, so as to minimize the maximum modulus of the integrand. If $\hat{\mathbf{r}}$ remains fixed, then the modulus of the integrand is well approximated by the exponential term $\exp(-|\mathbf{r}|(\hat{\mathbf{r}} \cdot \log|\mathbf{z}|))$, where $\log|\mathbf{z}|$ is shorthand for the vector $(\log|z_1|, \ldots, \log|z_d|)$. This suggests that the real function

$$h(\mathbf{z}) := -\hat{\mathbf{r}} \cdot \log|\mathbf{z}| \tag{2.3}$$

be thought of as a **height function** in the Morse theoretic sense, and that we try to deform T so that its maximum height is as low as possible. Stratified Morse Theory [GM88] solves the problem of accomplishing this optimal deformation. We now give a brief summary of this solution; for details, consult [BP04].

The variety $\mathcal{V}$ may be given a Whitney stratification. If $\mathcal{V}$ is already a manifold (which is the generic case), there is just a single stratum, but in general there may be any finite number of strata, even in two dimensions. The set of smooth points of $\mathcal{V}$ constitutes the top stratum, with the set of singular points decomposing into the remaining strata, each stratum being a smooth manifold of lower complex dimension. For details on Whitney stratification, see [GM88, Section I.1.2].

In each stratum, there is a finite set of critical points, namely points where the gradient of the height function restricted to the stratum vanishes. Since the height function depends on $\mathbf{r}$ via $\hat{\mathbf{r}}$, the set of critical points may be defined as a function of $\hat{\mathbf{r}}$ or equivalently of $\overline{\mathbf{r}}$:

Definition 1 (critical points). *If S is a stratum of $\mathcal{V}$, define the set $\mathsf{crit}_{\overline{\mathbf{r}}}(S)$ of critical points of S for H in direction $\overline{\mathbf{r}}$ to be the set of $\mathbf{z} \in S$ for which $\nabla h|_S(\mathbf{z}) = 0$. The set $\mathsf{crit} = \mathsf{crit}_{\overline{\mathbf{r}}}$ of all critical points of $\mathcal{V}$ is defined as the union of $\mathsf{crit}_{\overline{\mathbf{r}}}(S)$ as S varies over all strata.*

If S has dimension k then the condition $\nabla h|_S = 0$ defines an analytic variety of co-dimension k. Generically, then, $\mathsf{crit}_{\overline{\mathbf{r}}}(S)$ is zero-dimensional, that is, finite[1]. If F is a rational function, then the critical points are a finite union of zero-dimensional varieties, so elimination theory (see, e.g., [CLO98, Chapters 1 and 2]) will provide, in an automatic way, minimal polynomials for the critical points; see Proposition 2.2 below. So as to refer to them later, we enumerate the critical points as $\{\mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(m)}\}$.

Let $\mathcal{M}$ denote the domain of holomorphy of the integrand in (2.2), that is,

$$\mathcal{M} := \mathbb{C}^d \setminus \left\{ \mathbf{z} : H \cdot \prod_{j=1}^{d} z_j = 0 \right\} .$$

Given a real number c, let $\mathcal{M}^c$ denote the set of points of $\mathcal{M}$ with height less than c. Let $\mathcal{M}^+$ denote the $\mathcal{M}^c$ for some c greater than the greatest height of any critical point and let $\mathcal{M}^-$ denote $\mathcal{M}^c$ for some c less than the least height of any critical point. If the interval $[c, c']$ does not contain a critical value of the height function, then $\mathcal{M}^c$ is a strong deformation retract of $\mathcal{M}^{c'}$, so the particular choices of c above do not matter.

[1] It does happen sometimes that for a small set of values of $\overline{\mathbf{r}}$ the critical set is not finite; for example it happens when $\mathcal{V}$ is a binomial variety, in which case $\nabla h|_S$ vanishes everywhere on S for a particular $\overline{\mathbf{r}}$ and nowhere on S for other values of $\overline{\mathbf{r}}$. We address such a case in Section 3.10; aside from that, such a case will not arise among our examples.

Let X denote the topological pair $(\mathcal{M}^+, \mathcal{M}^-)$. The homology group $H_d(X)$ is generated by the homology groups $H_d(\mathcal{M}^{c'}, \mathcal{M}^{c''})$ as $[c', c'']$ ranges over a finite set of intervals whose union contain all critical values. These groups in turn are generated by quasi-local cycles. At any critical point $\mathbf{z}^{(i)}$ in the top stratum, the quasi-local cycle is the Cartesian product of a patch $\mathcal{P}_i$ diffeomorphic to $\mathbb{R}^{d-1}$ inside $\mathcal{V}$ whose maximum height is achieved at $\mathbf{z}^{(i)}$ with an arbitrarily small circle γ_i transverse to $\mathcal{V}$.

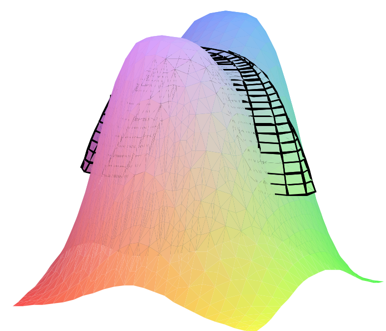

FIGURE 2. a piece of a quasi-local cycle at top-dimensional critical point

Thus for example when $d = 2$, the quasi-local cycles near smooth points look like pieces of macaroni: a product of a circle with a rainbow shaped arc whose peak is at $\mathbf{z}^{(i)}$. We have not described the quasi-local cycles for non-maximal strata, but for a description of the quasi-local cycles in general, one may look in [BP04] or [GM88][2].

Since the quasi-local cycles generate $H_d(\mathcal{M}^+, \mathcal{M}^-)$, it follows that the integral (2.2) is equal to an integer linear combination

$$\sum n_i \int\int_{\mathcal{C}_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, d\mathbf{z} \tag{2.4}$$

[2]For generic $\overline{\mathbf{r}}$, the function h is Morse. Each attachment map is a d-dimensional complex, so the attachments induce injections on H_d. It may happen for some $\overline{\mathbf{r}}$ that h is not Morse, but in this case one may understand the local topology via a small generic perturbation. Two or more critical points may merge, but the fact that the attachments induce injections on H_d implies that such a merger produces a direct sum in H_d; in particular, it will be useful later to know that a cycle in the merger is nonzero if and only if at least one component was nonzero.

where $\mathcal{C}_i$ is a quasi-local cycle near $\mathbf{z}$ and $T = \sum n_i \mathcal{C}_i$ in $H_d(\mathcal{M}^+, \mathcal{M}^-)$. When $\mathbf{z}$ is in the top stratum, i.e., $\mathbf{z}$ is a smooth point of $\mathcal{V}$, then $\mathcal{C}_i = \mathcal{P}_i \times \gamma_i$, the product of a d-patch with a transverse circle, so the integral may be written as $\int_{\mathcal{P}_i} \int_{\gamma_i} \exp[-|\mathbf{r}| h(\mathbf{z})] F(\mathbf{z})\, d\mathbf{z}$. The inner integral is a simple residue and the outer integral is a standard saddle point integral. The asymptotic evaluation of $a_{\mathbf{r}}$ is therefore solved if we can compute the integers n_i in the decomposition of T into $\sum n_i \mathcal{C}_i$. Observe that the contribution from $\mathbf{z}^{(i)}$ is of exponential order no greater than $\exp(h(\mathbf{z}^{(i)}))$. It turns out (Theorem 2.9 and the formulae of Section 2.4) that for smooth and multiple points, this is a lower bound on the exponential order as well, provided, in the multiple point case, that $G(\mathbf{z}^{(i)}) \neq 0$. Thus if $n_i \neq 0$, one may ignore any contributions from $\mathbf{z}^{(j)}$ with $h(\mathbf{z}^{(j)}) < h(\mathbf{z}^{(i)})$ and still obtain an asymptotic expansion containing all terms not exponentially smaller than the leading term.

Definition 2 (contributing critical points)**.** *The set* $\mathsf{contrib} = \mathsf{contrib}_{\overline{\mathbf{r}}}$ *of contributing critical points is defined to be the set of* $\mathbf{z}^{(i)}$ *such that* $n_i \neq 0$ *and* $n_j = 0$ *for all* j *with* $h(\mathbf{z}^{(j)}) > h(\mathbf{z}^{(i)})$. *In other words, the contributing critical point(s) are just the highest ones with nonzero coefficients,* n_i. *Note that if there is more than one contributing critical point, all must have the same height.*

The problem of computing the topological decomposition may in general be difficult and no complete solution is known. In two dimensions, in the case where $\mathcal{V}$ is globally smooth, an effective algorithm is known for finding the contributing critical points. The algorithm follows approximate steepest descent paths; the details are not yet published [vdHP01]. In any dimension, if the variety $\mathcal{V}$ is the union of hyperplanes, it is shown in [BP04] how to evaluate all the n_i; see also some preliminary work on this in [BM93]. In the case where $\mathcal{V}$ is not smooth, while no general solution is known, we may still state a sufficient condition for the critical point $\mathbf{z}^{(i)}$ to be a contributing critical point. It is to these geometric sufficient conditions that we turn next.

2.2. **Geometric criteria.** Let F, G, H and $a_{\mathbf{r}}$ be as in (1.2). If $a_{\mathbf{r}} \geq 0$ for all $\mathbf{r}$, we say we are in the **combinatorial case**. We assume throughout that G and H are relatively prime. We wish to compute the function $\mathsf{contrib}$ which maps directions in $\overline{\mathcal{O}}$ to finite subsets of $\mathcal{V}$ (often singletons). The importance of **minimal points**, defined below, is that when $\mathsf{contrib}_{\overline{\mathbf{r}}}$ contains a minimal point, this point may be identified by a variational principle.

Definition 3 (notation for polydisks and tori)**.** *Let* $\mathbf{D}(\mathbf{z})$ *and* $\mathbf{T}(\mathbf{z})$ *denote respectively the polydisk and torus defined by*

$$\mathbf{D}(\mathbf{z}) := \{\mathbf{z}' : |z_i'| \le |z_i| \text{ for all } 1 \le i \le d\} ;$$
$$\mathbf{T}(\mathbf{z}) := \{\mathbf{z}' : |z_i'| = |z_i| \text{ for all } 1 \le i \le d\} .$$

The open domain of convergence of F *is denoted* $\mathcal{D}$ *and is the union of tori* $\mathbf{T}(\mathbf{z})$*. The logarithmic domain of convergence, namely those* $\mathbf{x} \in \mathbb{R}^d$ *with* $(e^{x_1}, \ldots, e^{x_d}) \in \mathcal{D}$*, is denoted* $\log\mathcal{D}$ *and is always convex [Hör90]. The image* $\{\log|\mathbf{z}| : \mathbf{z} \in \mathcal{V}\}$ *of* $\mathcal{V}$ *under the coordinatewise log modulus map is denoted* $\log\mathcal{V}$ *(this is sometimes called the* ***amoeba*** *of* $\mathcal{V}$ *[GKZ94]).*

Definition 4 (minimality)**.** *A point* $\mathbf{z} \in \mathcal{V}$ *is* ***minimal*** *if all coordinates are nonzero and the relative interior of its polydisk contains no element of* $\mathcal{V}$*, that is,* $\mathcal{V} \cap \mathbf{D}(\mathbf{z}) \subseteq \partial\mathbf{D}(\mathbf{z})$*. More explicitly,* $z_i \neq 0$ *for all* i *and there is no* $\mathbf{z}' \in \mathcal{V}$ *with* $|z_i'| < |z_i|$ *for all* i*. The minimal point* $\mathbf{z}$ *is said to be* ***strictly minimal*** *if it is the only point of* $\mathcal{V}$ *in the closed polydisk:* $\mathcal{V} \cap \mathbf{D}(\mathbf{z}) = \{\mathbf{z}\}$*.*

Proposition 2.1 (minimal points in contrib)**.**

(i) *A point* $\mathbf{z} \in \mathcal{V}$ *with nonzero coordinates is minimal if and only if* $\mathbf{z} \in \partial\mathcal{D}$*, the boundary of the domain of convergence.*
(ii) *All minimal points in* $\mathsf{contrib}_{\overline{\mathbf{r}}}$ *must lie on the same torus and the height* $h_{\hat{\mathbf{r}}}(\mathbf{z})$ *of the point(s) in* contrib *are at most the minimum,* c*, of the height function on* $\partial\mathcal{D}$*.*
(iii) *If* $\mathbf{z} \in \mathsf{contrib}_{\overline{\mathbf{r}}} \cap \partial\mathcal{D}$*, then the hyperplane normal to* $\overline{\mathbf{r}}$ *at* $\log|\mathbf{z}|$ *is a support hyperplane to* $\log\mathcal{D}$*.*

To summarize, if $\mathsf{contrib}_{\overline{\mathbf{r}}}$ *contains any minimal points, then these points minimize* $h_{\mathbf{r}}$ *on* $\mathcal{D}$ *and they all project, via coordinatewise log modulus, to a single point of* $\partial\log\mathcal{D}$ *where the support hyperplane is normal to* $\mathbf{r}$*.*

Proof. A power series converges absolutely on any open polydisk about the origin in which it is analytic; this can be seen by using Cauchy's formula (2.2) to estimate $a_{\mathbf{r}}$. Thus any minimal point is in the closure of $\mathcal{D}$. A series cannot converge at a pole, so $\mathbf{z} \notin \mathcal{D}$ and the first assertion follows.

To prove the second assertion, let $\mathbf{z} \in \partial\mathcal{D}$ minimize h and let $\mathbf{H}$ be the homotopy $\{t\mathbf{T}(\mathbf{z}) : \epsilon \leq t < 1 - \epsilon\}$. This may be extended to a homotopy $\mathbf{H}'$ that pushes the height below $c - \epsilon$ except in small neighborhoods of $\mathcal{V} \cap \mathbf{T}(\mathbf{z})$. Thus the small torus T in (2.2) is homotopic to a cycle in the union of $\mathcal{M}^{c-\epsilon}$ and a neighborhood of $\mathcal{V} \cap \mathbf{T}(\mathbf{z})$. Sending $\epsilon \to 0$, it follows that $\mathsf{contrib}$ has height at most c. If there is a line segment in $\partial \log \mathcal{D}$, it is possible that the minimum height on $\mathcal{D}$ occurs on more than one torus, but in this case, if $\mathbf{z}$ and $\mathbf{w}$ are on different such tori, the above argument shows that neither $\mathbf{z}$ nor $\mathbf{w}$ can be in $\mathsf{contrib}$.

The third assertion is immediate from the fact that $\log|\mathbf{z}|$ minimizes the linear function $h_{\hat{\mathbf{r}}}$ on the convex set $\log\mathcal{D}$. □

Having characterized $\mathsf{contrib}$ when it is in $\partial\mathcal{D}$, via a variational principle, we now relate this to the algebraic definition of crit which is better for symbolic computation. It is easy to write down the inverse of crit, which we will denote by $\mathbf{L}$.

Definition 5 (geometry of minimal points)**.** *Say that* $\mathbf{z}$ *is a* ***smooth point*** *of* $\mathcal{V}$ *if* $\mathcal{V}$ *is a manifold in a neighborhood of* $\mathbf{z}$*. Say that* $\mathbf{z}$ *is a* ***multiple point*** *of* $\mathcal{V}$ *if* $\mathcal{V}$ *is the union of finitely many manifolds near* $\mathbf{z}$ *intersecting transversely (that is, normals to any* k *of these manifolds span a space of dimension* $\min\{k, d\}$*). Points that are neither smooth nor multiple are informally called bad points.*

Definition 6 (the linear space $\mathbf{L}$)**.** *Let* $\mathbf{z} \in \mathcal{V}$ *be in a stratum* $\mathcal{S}$ *of dimension* k*. Let* $\mathbf{L}(\mathbf{z}) \subseteq \mathbb{CP}^d$ *denote the span of the projections of vectors* $(z_1 v_1, \ldots, z_d v_d)$ *as* $\mathbf{v}$ *ranges over vectors orthogonal to the tangent space of* $\mathcal{S}$ *at* $\mathbf{z}$*.*

The following figure gives a pictorial example of the foregoing definitions. The singular variety $\mathcal{V}$ for the function $F(x, y) = 1/[(3 - x - 2y)(3 - 2x - y)]$ is composed of two complex lines meeting at $(1, 1)$, and arrows are drawn depicting $\mathbf{L}(\mathbf{z})$ for smooth points $\mathbf{z}$ in the positive real quadrant as well as for the crossing point $(1, 1)$. The following proposition, showing why $\mathbf{L}$ is important, follows directly from the definitions.

Proposition 2.2 ($\mathbf{L}$ inverts crit)**.** *The point* $\mathbf{z}$ *is in* $\mathsf{crit}_{\overline{\mathbf{r}}}$ *if and only if* $\overline{\mathbf{r}} \in \mathbf{L}(\mathbf{z})$*. If* $\mathbf{z}$ *is a smooth point, then (replacing* H *by its radical if needed),* $\mathbf{L}(\mathbf{z})$ *is the singleton set*

$$\mathbf{L}(\mathbf{z}) = \{\nabla_{\log} H\} := \left\{ \left(z_1 \frac{\partial H}{\partial z_1}, \ldots, z_d \frac{\partial H}{\partial z_d} \right) \right\}$$

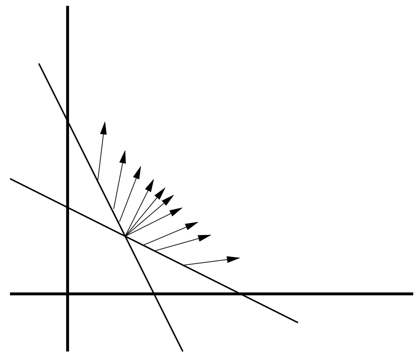

FIGURE 3. direction $\mathbf{L}(\mathbf{z})$ for positive real points of $\mathcal{V}$ when $1/F = (3-x-2y)(3-2x-y)$

while the set $\mathsf{crit}_{\overline{\mathbf{r}}}$ *is a zero-dimensional variety given by the* d *equations*

$$\begin{aligned} H &= 0 \\ r_d z_j \frac{\partial H}{\partial z_j} &= r_j z_d \frac{\partial H}{\partial z_d} \qquad (1 \le j \le d-1) \end{aligned} \tag{2.5}$$

If $\mathbf{z}$ *is a multiple point, then* $\mathbf{L}(\mathbf{z})$ *is the span of the vectors*

$$\nabla_{\log} H_k := \left(z_1 \frac{\partial H_k}{\partial z_1}, \ldots, z_d \frac{\partial H_k}{\partial z_d} \right)$$

as H_k *ranges over annihilators of the locally smooth components of* $\mathcal{V}$ *at* $\mathbf{z}$*.* □

Connecting this up to the variational principle, we have:

Proposition 2.3 (smooth minimal points are critical)**.** *If* $\mathbf{z}$ *is minimal and smooth then* $\mathbf{L}(\mathbf{z}) \in \overline{\mathcal{O}}$ *and is normal to a support hyperplane to* $\log\mathcal{D}$ *at* $\log|\mathbf{z}|$*. Consequently, a minimal smooth point* $\mathbf{z}$ *is a critical point for some outward normal direction to* $\log\mathcal{D}$ *at* $\log|\mathbf{z}|$*.*

Proof. Assume first that none of the partial derivatives $h_j := dH/dz_j$ vanishes at $\mathbf{z}$. Suppose the arguments of two of the partials h_k and h_l are not equal. Since we have assumed no h_j vanishes, there is a tangent vector $\mathbf{v}$ to $\mathcal{V}$ with $v_j = 0$ for $j \notin \{k, l\}$ and $v_k \neq 0 \neq v_l$. We may choose a multiple of this so that $\overline{v_k} h_k$ and $\overline{v_l} h_l$ both have negative real parts. Perturbing slightly, we may choose a tangent vector $\mathbf{v}$ to $\mathcal{V}$ at $\mathbf{z}$ with all nonzero coordinates so that $\overline{v_j} h_j$ has negative real part for all j. This implies there is a path from $\mathbf{z}$ on $\mathcal{V}$ such that the moduli of all coordinates strictly decrease, which contradicts minimality. It follows that all arguments of h_j are equal, and therefore that $\mathbf{L}(\mathbf{z})$ is a point of $\overline{\mathcal{O}} \subseteq \mathbb{CP}^d$; hence it is normal to a support hyperplane to $\log\mathcal{D}$ at $\log|\mathbf{z}|$.

Now, suppose $h_j = 0$ for j in some nonempty set, J. Let $\mathbf{x} = \log|\mathbf{z}|$. Vary $\{z_j : j \notin J\}$ by $\Theta(\epsilon)$ so as to stay on $\mathcal{V}$, varying $\{z_j : j \in J\}$ by $O(\epsilon^2)$. We see that the complement of $\log\mathcal{D}$ contains planes arbitarily close to the $|J|$-dimensional real plane through $\mathbf{x}$ in directions $\{e_j : j \in J\}$, whence the closure of the complement contains this plane. But by minimality, the orthant of this plane in the $-e_j$ directions must be in the closure of $\log\mathcal{D}$. Thus this orthant is in $\partial\log\mathcal{D}$, whence there is a lifting of this orthant to $\mathcal{V}$. Now the argument by contradiction in the previous paragraph shows we may move on $\mathcal{V}$ so the moduli of the z_j for $j \notin J$ strictly decrease, while moving down the lifting of the orthant allows us also to decrease the moduli of the z_j for $j \in J$, and we have again contradicted minimality. □

Having related critical points to $\mathbf{L}(\mathbf{z}) \subseteq \mathbb{CP}^{d-1}$, we now relate contributing critical points to a set $\mathbf{K}(\mathbf{z}) \subseteq \mathbb{RP}^{d-1}$. An open problem is to find the right definition of $\mathbf{K}(\mathbf{z})$ when $\mathbf{z}$ is not minimal.

Definition 7 (the cone $\mathbf{K}$ of a minimal point)**.** *If $\mathbf{z}$ is a smooth point of $\mathbf{z}$ in $\partial\mathcal{D}$, let $\mathbf{dir}(\mathbf{z})$ be defined as $\mathbf{L}(\mathbf{z})$, which is in $\overline{\mathcal{O}}$ by Proposition 2.3. For minimal $\mathbf{z} \in \mathcal{V}$, not necessarily smooth, define $\mathbf{K}(\mathbf{z})$ to be the convex hull of the set of limit points of $\mathbf{dir}(\mathbf{w})$ as $\mathbf{w} \to \mathbf{z}$ through smooth points.*

The next proposition then follows from Proposition 2.2 and basic properties of convex sets.

Proposition 2.4 (description of $\mathbf{K}$)**.** *Let $\mathbf{z}$ be a minimal point.*

(i) $\mathbf{K}(\mathbf{z}) \subseteq \mathbf{L}(\mathbf{z}) \cap \overline{\mathcal{O}}$.
(ii) *If $\mathbf{z}$ is a smooth or multiple point, then $\mathbf{K}(\mathbf{z})$ is the cone spanned by the vectors $\nabla_{\log} H_k$ of Proposition 2.2.*
(iii) *If a neighborhood of $\mathbf{z}$ in $\mathcal{V}$ covers a neighborhood of $\log|\mathbf{z}|$ in $\partial\log\mathcal{D}$, then $\mathbf{K}(\mathbf{z})$ is the set of outward normals to support hyperplanes to $\log\mathcal{D}$ at $\log|\mathbf{z}|$.*

Example 1. *The highest critical point may be outside the closure of $\mathcal{D}$, while the highest contributing critical point is in $\partial\mathcal{D}$. For an example where there are critical points $\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$ with $h(\mathbf{z}^{(1)}) > h(\mathbf{z}^{(2)})$ but $\mathbf{z}^{(2)}$ minimal and $\mathbf{z}^{(1)}$ not minimal, the reader is referred to [PW02, Example 3.4]. There, $F = 1/(3 - 3z - w + z^2)$ and for generic $\mathbf{r}$, there are exactly two minimal points. For directions above the diagonal, that is $s/r > 1$, one of the points is higher but only the other is minimal.*

The first nontrivial result we need is:

Theorem 2.5 ($\mathbf{K}$ inverts $\mathsf{contrib}$)**.** *Let $\mathbf{z}$ be a minimal point and either smooth or multiple. Then $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$ if and only if $\mathbf{z} \in \mathsf{contrib}_{\overline{\mathbf{r}}}$, provided, in the multiple point case, that $G(\mathbf{z}) \neq 0$.*

Remarks:

(1) Here, rather than global meromorphicity, it is only required that F be meromorphic in a neighborhood of $\mathbf{D}(\mathbf{z})$; see [PW02] and [PW04] for details.
(2) We believe the theorem to be valid for bad points as well, but this is still under investigation.
(3) In the multiple point case, when $G(\mathbf{z}) = 0$, there is a local partial fraction decomposition of F, while the condition for $\mathbf{z} \in \mathsf{contrib}_{\overline{\mathbf{r}}}$ becomes $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$ together with nonmembership of G in a certain ideal; see [BP04, Definition 8] or [Pem00] for details.

Proof. Assume first that $\mathbf{z}$ is strictly minimal, that is, $\mathbf{T}(\mathbf{z}) \cap \mathcal{V} = \{\mathbf{z}\}$.

Suppose for now that $\overline{\mathbf{r}}$ is in the relative interior of $\mathbf{K}(\mathbf{z})$. Theorems [PW02, Theorem 3.5] and [PW04, Theorems 3.6 and 3.9] give expressions for $a_{\mathbf{r}}$ which are of the order $|\mathbf{r}|^{\beta}\mathbf{z}^{-\mathbf{r}}$, relying, in the multiple point case, on the assumption of transverse intersection, which we have built into our definition of multiple point, and on $G(\mathbf{z}) \neq 0$.

On one hand, it is evident from (2.2) that

$$a_{\mathbf{r}} = O\left(\exp\left[|\mathbf{r}|(h_{\hat{\mathbf{r}}}(\mathbf{w}) + \epsilon)\right]\right)$$

for any $\epsilon > 0$, where $\mathbf{w} \in \mathsf{contrib}_{\overline{\mathbf{r}}}$. On the other hand, we know by Proposition 2.1 that no points higher than $\mathbf{z}$ are in $\mathsf{contrib}_{\overline{\mathbf{r}}}$. Since $\mathsf{contrib}_{\overline{\mathbf{r}}}$ is nonempty, we conclude that it has points at height $h_{\hat{\mathbf{r}}}(\mathbf{z})$, whence from the expressions for $a_{\mathbf{r}}$ again, we see that $\mathbf{z} \in \mathsf{contrib}_{\overline{\mathbf{r}}}$.

In the case $\overline{\mathbf{r}} \in \partial\mathbf{K}(\mathbf{z})$ consider $\overline{\mathbf{r}}' \to \overline{\mathbf{r}}$ through the interior of $\mathbf{K}(\mathbf{z})$ (here we mean relative boundary and relative interior). There will be a contributing point $\mathbf{z}'$ which converges to $\mathbf{z}$. The coefficient of the quasi-local cycle at $\mathbf{z}$ in the limit must be nonzero (see footnote 2). The theorem is now proven for strictly minimal points.

To remove the assumption of strict minimality, one must verify that this was not necessary for the formulae we quoted from [PW02; PW04]. These formulae were proved by reducing the Cauchy integral to an integral over a neighborhood $\mathcal{N}$ of a $(d-1)$-dimensional subset Θ of $\mathcal{V}$. It is pointed out [PW02, Corollary 3.7] that it is sufficient to assume **finite minimality**, that is, finiteness of $\mathbf{T}(\mathbf{z}) \cap \mathcal{V}$. In fact, one needs only finiteness of $\mathbf{T}(\mathbf{z}) \cap \Theta$, since the truncation of the integral to $\mathcal{N}$ incurs a boundary term that is sufficiently small as long as Θ avoids $\mathbf{T}(\mathbf{z})$. The remaining case, where $\Theta \cap \mathbf{T}(\mathbf{z})$ is infinite can be handled by contour rotation arguments, but since that work is not yet published, we point out here that in the case $d = 2$ (the only case needed in the applications we review), the set Θ is one-dimensional. Thus when $\Theta \cap \mathbf{T}(\mathbf{z})$ is infinite, the set Θ must be a subset of $\mathbf{T}(\mathbf{z})$; the integrals in [PW02; PW04] may then be taken over all of Θ with no truncation. $\square$

We have remarked earlier that the main challenge in computing asymptotics is to identify $\mathsf{contrib}$. Our progress to this point is that we may find all strictly minimal smooth or multiple points in $\mathsf{contrib}$ by solving the equation $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$ for $\mathbf{z}$. This equation may be solved automatically, at least when F is a rational function. To elaborate, we have seen that the critical points are the zero-dimensional variety (finite set) solving $\overline{\mathbf{r}} \in \mathbf{L}(\mathbf{z})$. Narrowing down to the subset solving $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$ is an easy computation. Checking these for minimality is a semi-algebraic computation. This computation can be difficult in practice and we do not know whether it is in principle automatic.

To complement this, we would like to know when solving $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$ and checking for minimality does indeed find all points of $\mathsf{contrib}$. It cannot, for instance, do so if there are no minimal contributing points. Thus we ask (a) are there any minimal points $\mathbf{z}$ with $\mathbf{K}(\mathbf{z}) = \overline{\mathbf{r}}$, (b) are all the contributing points minimal, and (c) is there more than one minimal point? It turns out that the answers are, roughly: (a) yes, in the combinatorial case, (b) yes as long as (a) is true, by part 2 of Proposition 2.1, and (c) rarely (we know they are never on different tori).

Let $\Xi \subseteq \overline{\mathcal{O}}$ denote the set of all normals to support hyperplanes of $\log\mathcal{D}$. If the restriction of F to each coordinate hyperplane is not entire, then Ξ is the entire nonnegative orthant. This is because $\log\mathcal{D}$ has support hyperplanes parallel to each coordinate hyperplane. When Ξ is not the whole orthant, then in directions $\overline{\mathbf{r}} \notin \overline{\mathcal{O}}$, the quantity $a_{\mathbf{r}}$ is either identically zero or decays faster than exponentially (this follows, for example, from [PW04, (1.2)], since $-\hat{\mathbf{r}} \cdot \log|\mathbf{z}|$ is not bounded from below on $\mathcal{D}$).

Theorem 2.6 (existence of minimal points in the combinatorial case)**.** *Suppose that we are in the combinatorial case, $a_{\mathbf{r}} \geq 0$. Then for every $\overline{\mathbf{r}}$ in the interior of Ξ, there is a minimal $\mathbf{z} \in \mathcal{V}$ which lies in the positive orthant of $\mathbb{R}^d$ and has $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$.*

Remark. *Given a compact set $K \subseteq \Xi$, it is sufficient that F be meromorphic in a neighborhood of $\mathcal{D}(K)$, where $\mathcal{D}(K)$ is the inverse image under the coordinatewise log modulus map of the set $\log\mathcal{D}(K)$, and $\log\mathcal{D}(K)$ is the set of all $\mathbf{x}$ such that $\mathbf{x} \leq \mathbf{y}$ coordinatewise for some $\mathbf{y} \in \partial\log\mathcal{D}$ whose normal direction is in K.*

Proof. We follow the proof of [PW02, Theorem 6.3]. For any $\overline{\mathbf{r}}$ in the interior of Ξ there is a point $\mathbf{x} \in \partial\log\mathcal{D}$ with a support hyperplane normal to $\overline{\mathbf{r}}$. Let $z_i = e^{x_i}$ so $\mathbf{z}$ is a real point in $\partial\mathcal{D}$. We claim that $\mathbf{z} \in \mathcal{V}$. To see this, note that there is a singularity on $\mathbf{T}(\mathbf{z})$, which must be a pole by the assumption of meromorphicity on a neighborhood of $\mathbf{D}(\mathbf{z})$. Together, $a_{\mathbf{r}} \geq 0$ and lack of absolute convergence of the series on $\mathbf{T}(\mathbf{z})$ imply that $F(\mathbf{w})$ converges to $+\infty$ as $\mathbf{w} \to \mathbf{z}$ from beneath. By meromorphicity, $\mathbf{z}$ is therefore a pole of F, so $\mathbf{z} \in \mathcal{V}$.

We conclude that there is a lifting of $\partial\log\mathcal{D}$ to the real points of $\mathcal{V}$, that is, a subset of the real points of $\mathcal{V}$ maps properly and one to one onto $\partial\log\mathcal{D}$. By the last part of Proposition 2.4, it follows that $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$, and hence from Theorem 2.5 that $\mathbf{z} \in \mathsf{contrib}_{\overline{\mathbf{r}}}$. □

Example 2 (in general, critical points are not minimal)**.** *Theorem 2.6 can fail without the assumption $a_{\mathbf{r}} \geq 0$. For example, it does not hold when $d = 2$ and*

$$F(x, y) = \frac{1}{(3 - 2x - y)(3 + x + 2y)} .$$

In this case, $\mathcal{V}$ does have some minimal points, namely the line segment satisfying $w+2z=3$ between $(1,1)$ and $(3/2,0)$ together with the line segment satisfying $x+2y=-3$ between $(-1,-1)$ and $(0,-3/2)$, in each case containing the endpoint not on a coordinate axis. The boundary of $\log\mathcal{D}$ *is the image of these two segments under coordinatewise logarithms. When $\hat{\mathbf{r}}$ has $1/2 < s/r < 2$, the support hyperplane to* $\log\mathcal{D}$ *in direction $\hat{\mathbf{r}}$ meets $\partial\log\mathcal{D}$ at $(0,0)$, but there is no critical point on $T(1,1)$. In fact in this case there are two critical points in* contrib*, one satisfying $y+2x=3$ and one satisfying $x+2y=-3$, but neither is on the line segment of minimal such points.*

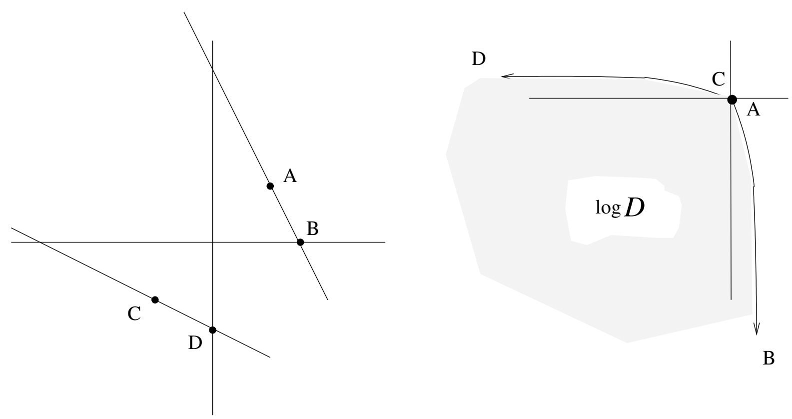

FIGURE 4. the boundary of $\log\mathcal{D}$ is the image of $\overline{AB}$ and $\overline{CD}$

This example also shows that $\mathbf{K}(\mathbf{z})$ can be a proper subset of the set of normals to support hyperplanes of $\log\mathcal{D}$ *at* $\log|\mathbf{z}|$*. When $s/r=2$, there are two minimal points $(1,1)$ and $(-1,-1)$, mapping to $(0,0)\in\log\mathcal{D}$. We have $\mathbf{K}(1,1)=\overline{(1,2)}$, while $\mathbf{K}(-1,-1)=\overline{(2,1)}$, and both of these are proper subsets of the cone $\{\overline{(1,\lambda)} : 1/2 \le \lambda \le 2\}$ of normals to support hyperplanes of* $\log\mathcal{D}$ *at $(0,0)$.*

We say that a power series P is **aperiodic** if the sublattice of $\mathbb{Z}^d$ of integer combinations of exponent vectors of the monomials of P is all of $\mathbb{Z}^d$. By a change of variables, we lose no generality from the point of view of generating functions if we assume in the following proposition that P is aperiodic.

Proposition 2.7 (often, every minimal point is strictly minimal)**.** *If $H=1-P(\mathbf{z})$ where P is aperiodic with nonnegative coefficients, then every minimal point is strictly minimal and lies in the positive orthant.*

Proof. Suppose that $\mathbf{z} = \mathbf{x}e^{i\theta}$, with $\theta \neq \mathbf{0}$, is minimal. Since $\mathbf{z} \in \mathcal{V}$, we have

$$\begin{aligned} 1 &= \left| \sum_{\mathbf{r} \neq \mathbf{0}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} \exp(i\mathbf{r} \cdot \theta) \right| \\ &\leq \sum_{\mathbf{r} \neq \mathbf{0}} a_{\mathbf{r}} \mathbf{x}^{\mathbf{r}} . \end{aligned}$$

Thus $\mathbf{z}$ can be strictly minimal if and only if $\theta = \mathbf{0}$.

Equality holds in the above inequality if and only if either there is only one term in the latter sum, or, in case there is more than one such term, if $\exp(i\mathbf{r} \cdot \theta) = \mathbf{1}$ whenever $a_{\mathbf{r}} \neq 0$. The former case occurs precisely when P has the form $c\mathbf{x}^{\mathbf{r}}$ and the latter can occur on when P has the form $P(\mathbf{z}) = g(\mathbf{z}^{\mathbf{b}})$ for some $\mathbf{b} \neq 0$, both of which are ruled out by aperiodicity. □

We restate the gist of Theorem 2.6 and Proposition 2.7 as the following corollary.

Corollary 2.8. *If $H = 1 - P$ with P aperiodic and having nonnegative coefficients, then the contributing critical points as $\overline{\mathbf{r}}$ varies are precisely the points of $\mathcal{V} \cap \mathcal{O}^d$ that are minimal in the coordinatewise partial order. The point $\mathbf{z}$ is in $\mathsf{contrib}_{\overline{\mathbf{r}}}$ exactly when $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z})$.*

Without the assumption $H = 1 - P$, assuming only $a_{\mathbf{r}} \geq 0$, the same holds except that there might be more contributing critical points on the torus $\mathbf{T}(\mathbf{z})$. □

2.3. **Asymptotic formulae for minimal smooth points.** It is shown in [PW02] how to compute the integral (2.2) when $\overline{\mathbf{r}}$ is fixed and $\mathsf{contrib}_{\overline{\mathbf{r}}}$ is a finite set of smooth points on a torus. Given a smooth point $\mathbf{z} \in \mathcal{V}$, let $\tilde{f} = \tilde{f}_{\mathbf{z}}$ be the map on a neighborhood of the origin in $\mathbb{R}^d$ taking the origin to zero and taking $(\theta_1, \ldots, \theta_{d-1})$ to $\log w$ for w such that

$$\left(z_1 e^{i\theta_1}, \ldots, z_{d-1} e^{i\theta_{d-1}}, z_d w\right) \in \mathcal{V} .$$

The following somewhat general result is shown in [PW02, Theorem 3.5].

Theorem 2.9 (smooth point asymptotics)**.** *Let $K \subset \Xi$ be compact, and suppose that for $\overline{\mathbf{r}} \in K$, the set $\mathsf{contrib}_{\overline{\mathbf{r}}}$ is a single smooth point $\mathbf{z}(\overline{\mathbf{r}})$ and that $\tilde{f}_{\mathbf{z}}$ has nonvanishing*

Hessian (determinant of the matrix of second partial derivatives). Then there are effectively computable functions $b_l(\overline{\mathbf{r}})$ *such that*

$$a_{\mathbf{r}} \sim \mathbf{z}(\overline{\mathbf{r}})^{-\mathbf{r}} \sum_{l \geq 0} b_l(\overline{\mathbf{r}}) |\mathbf{r}|^{-(l+d-1)/2} \tag{2.6}$$

as an asymptotic expansion when $|\mathbf{r}| \to \infty$*, uniformly for* $\overline{\mathbf{r}} \in K$*.* □

Remarks:

(1) The coefficients b_l depend on the derivatives of G and H to order $k + l - 1$ at $\mathbf{z}(\mathbf{r})$, and $b_0(\overline{\mathbf{r}}) = 0$ if and only if $G(\mathbf{z}(\overline{\mathbf{r}})) = 0$.

(2) For finitely many minimal points on the same torus, one sums the contribution from each, corresponding to finitely many coefficients of $+1$ in (2.4) or the meta-formula (2.1).

Theorem 2.9 is somewhat messy: even when $l = 1$, the combination of partial derivatives of G and H is cumbersome, though the prescription for each b_l in terms of partial derivatives of G and H is completely algorithmic for any l and d. In the applications herein, we will confine our asymptotic computations to the leading term. The following expression for the leading coefficient is more explicit than (2.6) and is proved in [PW02, Theorems 3.5 and 3.1].

Theorem 2.10 (smooth point leading term)**.** *When* $G(\mathbf{z}(\overline{\mathbf{r}})) \neq 0$*, the leading coefficient is*

$$b_0 = (2\pi)^{(1-d)/2} \mathcal{H}^{-1/2} \frac{G(\mathbf{z}(\overline{\mathbf{r}}))}{z_d \partial H / \partial z_d} \tag{2.7}$$

where $\mathcal{H}$ *denotes the Hessian of the function* $\tilde{f}_{\mathbf{z}}$ *at the origin.*

Theorem 2.11. *In particular, when* $d = 2$ *and* $G(\mathbf{z}(\overline{\mathbf{r}})) \neq 0$*, we have*

$$a_{rs} \sim \frac{G(x,y)}{\sqrt{2\pi}} x^{-r} y^{-s} \sqrt{\frac{-y \partial H / \partial y}{s Q(x,y)}} \tag{2.8}$$

where $(x, y) = \mathbf{z}(\overline{(r,s)})$ *and* $Q(x, y)$ *is defined to be the expression*

(2.9)

$$-(xH_x)(yH_y)^2 - (yH_y)(xH_x)^2 - \left[(yH_y)^2 x^2 H_{xx} + (xH_x)^2 y^2 H_{yy} - 2(xH_x)(yH_y) xy H_{xy}\right].$$

□

Remark. *In the combinatorial case, the expression in the radical will be positive real (this is true for more general F with correct choice of radical). The condition $ryH_y = sxH_x$, which we know to hold because $(x, y) = \mathsf{contrib}_{\overline{\mathbf{r}}}$, shows that the given expression for a_{rs}, though at first sight asymmetric in x and y, has the expected symmetry.*

2.4. **Multiple point asymptotics.** Throughout this section, $\mathbf{z}$ will denote a minimal multiple point. We will let m denote the number of sheets of $\mathcal{V}$ intersecting at $\mathbf{z}$ and let

$$H = \prod_{j=1}^{m} H_k^{n_k}$$

denote a local representation of H as a product of functions whose zero set is locally smooth. For $1 \leq k \leq m$, let $\mathbf{b}^{(m)}$ denote the vector whose k^{th} component is $z_k \partial H_k / \partial z_k$.

We divide the asymptotic analysis of $a_{\mathbf{r}}$ into two cases, namely $m \geq d$ and $m < d$. In the former case, the stratum of $\mathbf{z}$ is just the singleton $\{\mathbf{z}\}$. The asymptotics of $a_{\mathbf{r}}$ are simpler in this case. In fact it is shown in [Pem00, Theorem 3.1] that

$$a_{\mathbf{r}} = \mathbf{z}^{-\mathbf{r}} \left(P(\mathbf{r}) + E(\mathbf{r}) \right) \tag{2.10}$$

where P is piecewise polynomial and E decays exponentially on compact subcones of the interior of $\mathbf{K}(\mathbf{z})$. We begin with this case.

We will state three results in decreasing order of generality. The first is completely general, the second holds in the special case $m = d$ and $n_k = 1$ for all $k \leq m$, and the last holds for $m = d = 2$ and $n_1 = n_2 = 1$. Our version of the most general result provides a relatively simple formula from [BP04] but under the more general scope of [PW04].

Definition 8. *Let $M = \sum_{k=1}^{m} n_k$ and for $1 \leq j \leq M$ let t_j be integers so that the multiset $\{t_1, \ldots, t_M\}$ contains n_k occurrences of k for $1 \leq k \leq m$. Define a map $\Psi : \mathbb{R}^M \to \mathbb{R}^d$ by $\Psi(e_j) = \mathbf{b}^{(t_j)}$. Let λ^M be Lebesgue measure on $\overline{\mathcal{O}}^M$ and let $P(\mathbf{x})$ be the density at $\mathbf{x}$ of the pushforward $\lambda^M \circ \Psi$ of Lebesgue measure under Ψ. The function P is a piecewise polynomial of degree $M - d$, the regions of polynomiality being no finer than the common refinement of triangulations of the set $\{\mathbf{b}^{(k)} : 1 \leq k \leq m\}$ in $\mathbb{RP}^{d-1}$ [BP04, definition 5].*

The above definition of P is a little involved but is made clear by the worked example in Section 3.10. Armed with this definition, we may say what happens when there are d or more sheets of $\mathcal{V}$ intersecting at a single point.

Theorem 2.12 (isolated point)**.** *If* $\mathbf{z}$ *is a minimal point in a singleton stratum, then uniformly over compact subcones of* $\mathbf{K}(\mathbf{z})$*,*

$$a_{\mathbf{r}} \sim G(\mathbf{z}) P\left(\frac{r_1}{z_1}, \ldots, \frac{r_d}{z_d}\right) \mathbf{z}^{-\mathbf{r}}\,, \tag{2.11}$$

provided that $\mathsf{contrib}_{\overline{\mathbf{r}}}$ *contains only* $\mathbf{z}$*. This formula may be summed over* $\mathsf{contrib}_{\overline{\mathbf{r}}}$ *if* $\mathsf{contrib}_{\overline{\mathbf{r}}}$ *has finite cardinality greater than 1.*

In the case $\mathbf{z} = \mathbf{1}$*, (2.11) reduces further to*

$$a_{\mathbf{r}} \sim G(\mathbf{1}) P(\mathbf{r})\,.$$

If, furthermore, $a_{\mathbf{r}}$ *are integers and* $G(\mathbf{1})$ *is an integer, then* $a_{\mathbf{r}}$ *is actually a piecewise polynomial whose leading term coincides with that of* $P(\mathbf{r})$*.*

Proof. First assume $\mathbf{z}$ is strictly minimal. Theorem 3.6 of [PW04] gives an asymptotic expression for $a_{\mathbf{r}}$ valid whenever $\mathbf{z}$ is a strictly minimal multiple point in a singleton stratum. The formula, while not awful, is not as useful as the later formula given in [BP04, equation (3.8)]. This latter equation was derived under the assumption that H is a product of linear polynomials. Noting that the formula in [PW04, Theorem 3.6] depends on H only through the vectors $\{\mathbf{b}^{(k)} : 1 \leq k \leq m\}$, we see it must agree with [BP04, equation (3.8)] which is (2.11). The last statement of the theorem follows from (2.10).

To remove the assumption of strict minimality, suppose that $\mathbf{z}$ is minimal but not strictly minimal and that there are finitely many points $\mathbf{z} = \mathbf{z}^{(1)}, \ldots, \mathbf{z}^{(k)}$ in $\mathsf{contrib}_{\overline{\mathbf{r}}}$, necessarily all on a torus. The torus T in (2.2) may be pushed out to $(1+\epsilon)\mathbf{T}(\mathbf{z})$ except in neighborhoods of each $\mathbf{z}^{(j)}$. The same sequence of surgeries and residue computation in the proofs of [PW04, Proposition 4.1, Corollary 4.3 and Theorem 4.6] now establish Theorem 3.6 of [PW04] without the hypothesis of strict minimality, and as before one obtains (2.11) from (3.8) of [BP04]. □

While this gives quite a compact representation of the leading term, it may not be straightforward to compute P from its definition as a density. When $M = m = d$, P is a constant and the computation may be reduced to the following formula.

Corollary 2.13 (simple isolated point). *If $m = d$ and each $n_k = 1$, then if $G(\mathbf{z}) \neq 0$,*

$$a_{\mathbf{r}} \sim G(\mathbf{z})\mathbf{z}^{-\mathbf{r}} |J|^{-1}$$

where J is the Jacobian matrix $(\partial H_i / \partial z_j)$. □

Occasionally it is useful to have a result that does not depend on finding an explicit factorization of H. The matrix J can be recovered from the partial derivatives of H. The result of this in the case $m = d = 2$ is given by the following formula [PW04, Theorem 3.1].

Corollary 2.14 (simple isolated point, $d = 2$). *If $M = m = d = 2$, and if $G(\mathbf{z}) \neq 0$, then setting $\mathbf{z} = (x, y)$,*

$$a_{rs} \sim x^{-r} y^{-s} \frac{G(x,y)}{\sqrt{-x^2 y^2 \mathcal{H}}}$$

where $\mathcal{H} := H_{xx}H_{yy} - H_{xy}^2$ is the Hessian of H at the point (x, y). In the special but frequent case $x = y = 1$ we have simply

$$a_{rs} \sim \frac{G(1,1)}{\sqrt{-\mathcal{H}}} .$$

□

Finally, we turn to the case $m < d$. In this case, $\mathbf{z}$ is in a stratum containing more than just one point. The leading term of $a_{\mathbf{r}}$ is obtained by doing a saddle point integral of a formula such as (2.11) over a patch of the same dimension as the stratum. Stating the outcome takes two pages in [PW04]. Rather than give a formula for the resulting constant here, we state the asymptotic form and refer the reader to [PW04, Theorem 3.9] for evaluation of the constant, b_0.

Theorem 2.15. *Suppose that as $\overline{\mathbf{r}}$ varies over a compact subset K of $\overline{\mathcal{O}}$, the set $\mathsf{contrib}_{\overline{\mathbf{r}}}$ is always a singleton varying over a fixed stratum of codimension $m < d$. If also $\mathbf{z}(\overline{\mathbf{r}})$ remains a strictly minimal multiple point, with each smooth sheet being a simple pole of F, and if $G(\mathbf{z}) \neq 0$ on K, then*

$$a_{\mathbf{r}} \sim \mathbf{z}(\overline{\mathbf{r}})^{-\mathbf{r}} b_0(\overline{\mathbf{r}}) |\mathbf{r}|^{\frac{m-d}{2}} \tag{2.12}$$

uniformly as $|\mathbf{r}| \to \infty$ *in* K. □

2.5. **Distributional limits.** Of the small body of existing work on multivariate asymptotics, a good portion focuses on limit theorems. Consider, for example, the point of view taken in [Ben73; BR83] and the sequels to those papers [GR92; BR99], where one thinks of the the numbers $a_{\mathbf{r}}$ as defining a sequence of $(d-1)$-dimensional arrays, the k^{th} of which is the **horizontal slice** $\{a_{\mathbf{r}} : r_d = k\}$ (cf. Section 1.2). Often this point of view is justified by the combinatorial application, in which the last coordinate, r_d, is a size parameter, and one wishes to understand rescaled limits of the horizontal slices. Distributional limit theory assumes nonnegative weights $a_{\mathbf{r}}$, so for the remainder of this section we assume we are in the combinatorial case, $a_{\mathbf{r}} \geq 0$.

If $C_k := \sum_{\mathbf{r}\,:\,r_d=k} a_{\mathbf{r}} < \infty$ for all k, we may define the slice distribution μ_k on $(d-1)$-vectors to be the probability measure giving mass $a_{\mathbf{r}}/C_k$ to the vector $(r_1, \ldots, r_{d-1})$. There are several levels of limit theorem that may be of interest. A weak law of large numbers (WLLN) is said to hold if the measures $A \mapsto \mu_k(\frac{1}{k}A)$ converge to a point mass at some vector $\mathbf{m}$ (here, division of A by k means division of each element by k). Equivalently, a WLLN holds if and only if:

$$(\exists \mathbf{m})\,(\forall \epsilon > 0)\ \lim_{k\to\infty} \mu_k\{\mathbf{r} : |\frac{\mathbf{r}}{k} - \mathbf{m}| > \epsilon\} = 0\,. \tag{2.13}$$

Stronger than this is Gaussian limit behavior. As $\mathbf{r}$ varies over a neighborhood of size $\sqrt{k}$ about $k\mathbf{m}$ with r_d held constant at k, formula (2.6) takes on the following form:

$$a_{\mathbf{r}} \sim C \exp\left[\phi(\mathbf{r})\right]\,.$$

Here the constant C varies only by $o(1)$ when $\mathbf{r}$ varies by $O(\sqrt{k})$, and

$$\phi(\mathbf{r}) := h_{\mathbf{r}}(\mathbf{z}(\mathbf{r})) = -\mathbf{r} \cdot \log|\mathbf{z}(\mathbf{r}) = \mathbf{r} \cdot \log \mathbf{z}(\mathbf{r})\,,$$

the last equality holding in the common case that $\mathbf{z}(\mathbf{r})$ is real. The WLLN identifies the location $\mathbf{r}_{\max}$ at which ϕ takes its maximum for r_d held constant at k. A Taylor expansion then gives

$$\exp\left[\phi(\mathbf{r}_{\max} + \mathbf{s}) - \phi(\mathbf{r}_{\max})\right] = \exp\left[-\frac{B(\mathbf{s})}{2k}\right] \tag{2.14}$$

for some nonnegative quadratic form, B. Typically B will be positive definite, resulting in a local central limit theorem for $\{\mu_k\}$.

Unfortunately, the computation of B, though straightforward from the first two Taylor terms of H, is very messy. Furthermore, it can happen that B is degenerate, and there does not seem to be a good test for this. As a result, existing limit theorems such as [BR83, Theorems 1 and 2] and [GR92, Theorem 2] all contain the hypothesis "if B is nonsingular". Because of the great attention that has been paid to Gaussian limit results, we will state a local central limit result at the end of this section. But since we cannot provide a better method for determining nonsingularity of B, we will not develop this in the examples.

Distributional limit theory requires the normalizing constants C_k be finite. We make the slightly stronger assumption that $a_{\mathbf{r}} = 0$ when $\mathbf{r}/r_d$ is outside of a compact set K. We define the map

$$\mathbf{r} \mapsto \mathbf{r}^* := \left(\frac{r_1}{r_d}, \ldots, \frac{r_{d-1}}{r_d} \right) ,$$

which will be more useful for studying horizontal slices than was the the previously defined projection $\mathbf{r} \mapsto \hat{\mathbf{r}}$.

Theorem 2.16 (WLLN)**.** *Let F be a d-variate generating function with nonnegative coefficients, and with $a_{\mathbf{r}} = 0$ for $\mathbf{r}/r_d$ outside some compact set K. Let $f(x) = F(1, \ldots, 1, x)$. Suppose f has a unique singularity x_0 of minimal modulus. Let $\overline{\mathbf{r}} = \mathbf{dir}(\mathbf{x})$, where $\mathbf{x} = (1, \ldots, 1, x_0)$. If F meromorphic in a neighborhood of $\mathbf{D}(\mathbf{x})$, then the measures $\{\mu_k\}$ satisfy a WLLN with mean vector $\mathbf{r}^* := (r_1/r_d, \ldots, r_{d-1}/r_d)$.*

Proof. We begin by showing that $\log\mathcal{D}$ has a unique normal at $\log|\mathbf{x}|$, which is in the direction $\overline{\mathbf{r}}$. We know that x_0 is positive real, since f has nonnegative coefficients. The proof of Theorem 2.6 showed that a neighborhood $\mathcal{N}$ of $\mathbf{x}$ in $\mathcal{V}$ maps, via log moduli, onto a neighborhood of $\log|\mathbf{x}|$ in $\partial\log\mathcal{D}$. Since x_0 is a simple pole of f, we see that $\mathcal{N}$ is smooth with a unique normal $\nabla H(\mathbf{x})$, whence $\log\mathcal{D}$ has the unique normal direction $\overline{\mathbf{r}}$ at $\log|\mathbf{x}|$.

We observe next, via the well known theory of univariate generating functions (see, e.g., [FO90a]), that the normalizing constants satisfy $C_k \sim c x_0^{-k}$. Thus Theorem 2.16 will follow once we establish:

$$\limsup_k \frac{1}{k} \log \left(\sum_{\mathbf{s}: s_d = k, \mathbf{s}^* \in K, |\mathbf{s}^* - \mathbf{r}^*| > \epsilon} a_{\mathbf{s}} \right) < -\log x_0 \, . \tag{2.15}$$

Each $\mathbf{s}^* \in K$ with $|\mathbf{s}^* - \mathbf{r}^*| \geq \epsilon$ is not normal to a support hyperplane at $\log|\mathbf{x}|$, so for each such $\mathbf{s}^*$ there is a $\mathbf{y} \in \partial\log\mathcal{D}$ for which

$$h_{\mathbf{s}^*}(\mathbf{y}) < h_{\mathbf{s}^*}(\mathbf{x}) = -\log x_0 \,. \tag{2.16}$$

By compactness of $\{\mathbf{s}^* \in K : |\mathbf{s}^* - \mathbf{r}^*| \geq \epsilon\}$ we may find finitely many $\mathbf{y}$ such that one of these, denoted $\mathbf{y}(\mathbf{s}^*)$, satisfies (2.16) for any $\mathbf{s}^*$. By compactness again,

$$\sup_{\mathbf{s}^*} h_{\mathbf{s}^*}(\mathbf{y}(\mathbf{s}^*)) < -\log x_0 \,.$$

It follows from the representation (2.2), choosing a torus just inside $\mathbf{T}(\mathbf{y})$ for $\mathbf{y}$ as chosen above depending on $\mathbf{s}$, that $|\mathbf{s}|^{-1}\log a_{\mathbf{s}}$ is at most $-\log x_0 - \delta$ for some $\delta > 0$ and all sufficiently large $\mathbf{s}$ with $|\mathbf{s}^* - \mathbf{r}^*| \geq \epsilon$. Summing over the polynomially many such $\mathbf{s}$ with $s_d = k$ proves (2.15) and the theorem. □

Coordinate slices are not the only natural sections on which to study limit theory. One might, for example study limits of the sections $\{a_{\mathbf{r}} : |\mathbf{r}| = k\}$, that is, over a foliation of $(d-1)$-simplices of increasing size. The following version of the WLLN is adapted to this situation. the proof is exactly the same when slicing by $|\mathbf{r}|$ or any other linear function as it was for slicing by r_d.

Theorem 2.17 (WLLN for simplices)**.** *Let F be a d-variate generating function with nonnegative coefficients. Let $f(x) = F(x,\ldots,x)$. Suppose f has a unique singularity x_0 of minimal modulus. Let $\overline{\mathbf{r}} = \mathbf{dir}(\mathbf{x})$, where $\mathbf{x} = (x_0,\ldots,x_0)$. If F is meromorphic in a neighborhood of $\mathbf{D}(\mathbf{x})$, then the measures $\{\mu'_k\}$, defined analgously to μ_k but over the simplices $\{|\mathbf{r}| = k\}$, satisfy a WLLN with mean vector $\hat{\mathbf{r}}$.* □

We end with a statement of a local central limit theorem.

Theorem 2.18 (LCLT)**.** *Let $F, f, x_0, \mathbf{x}, \mathbf{r}, \mathbf{r}^*$ and B be as in Theorem 2.16. Suppose further that $\mathsf{contrib}_{\overline{\mathbf{r}}}$ is the singleton, $\{\mathbf{x}\}$. If the quadratic form B from (2.14) is nonsingular, then*

$$\lim_{k\to\infty} k^{(d-1)/2} \sup_{\mathbf{r}} |\mu_k(\mathbf{r}) - n(0,B)(\mathbf{r})| = 0 \tag{2.17}$$

where

$$n(0,B)(\mathbf{r}) := (2\pi)^{-(d-1)/2} \det(B)^{-1/2} \exp\left(-\frac{1}{2}B(\mathbf{r})\right) . \tag{2.18}$$

is the discrete normal density. □

3. Detailed examples

3.1. **Horizontally convex polyominoes.** A **horizontally convex polyomino** (HCP) is a union of cells $[a, a+1] \times [b, b+1]$ in the two-dimensional integer lattice such that the interior of the figure is connected and every row is connected. Formally, if $S \subseteq \mathbb{Z}^2$ and $P = \bigcup_{(a,b)\in S}[a, a+1] \times [b, b+1]$ then P is an HCP if and only if the following three conditions hold: $B := \{b : \exists a, (a, b) \in S\}$ is an interval; the set $A_b := \{a : (a, b) \in S\}$ is an interval for each $b \in \mathbb{Z}$; and whenever $b, b+1 \in B$, the sets A_b and A_{b+1} intersect.

Let a_n be the number of HCP's with n cells, counting two as the same if they are translates of one another. Pólya [Pol69] proved that

$$\sum a_n x^n = \frac{x(1-x)^3}{1 - 5x + 7x^2 - 4x^3} . \tag{3.1}$$

Further discussion of the origins of this formula and its accompanying recursion may be found in [Odl95] and [Sta97]. The proof in [Wil94, pages 150–153] shows in fact that

$$F(x, y) = \sum a_{n,k} x^n y^k = \frac{xy(1-x)^3}{(1-x)^4 - xy(1 - x - x^2 + x^3 + x^2 y)} , \tag{3.2}$$

where $a_{n,k}$ is the number of HCP's with n cells and k rows. Let us find an asymptotic formula for $a_{r,s}$.

Since this is the first worked example, we will spell out the use along the way of all the theoretical results. All the coefficients of $F(x, y)$ are nonnegative; they vanish when $s > r$ but otherwise are at least 1. By Corollary 2.8 (the last part, which requires only $a_{\mathbf{r}} \geq 0$), we know that all points of $\mathcal{V}$ in the first quadrant that are on the southwest facing part of the graph (that is, that are minimal in the coordinatewise partial order) are contributing critical points. We do not know yet but will see later that there are no other critical points on each torus. As $\overline{\mathbf{r}}$ varies over $\Xi = \{\overline{\mathbf{r}} : 0 < s/r < 1\}$ from the horizontal to the diagonal, the point $\mathsf{contrib}_{\overline{\mathbf{r}}}$ moves along this graph from $(1, 0)$ to $(0, \infty)$.

To make the mapping from $\overline{\mathbf{r}}$ to $\mathbf{z}$ explicit, we use the fact that $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{z}) \subseteq \mathbf{L}(\mathbf{z})$ (Theorem 2.6 and part (i) of Proposition 2.4), so $\mathbf{z}$ may be gotten from $\mathbf{r}$ by (2.5). It is readily computed that $\nabla H \neq \mathbf{0}$ except at $(1, 0)$. Thus all minimal points are smooth. The only solution to $G = H = 0$ is at $(1, 0)$, so the numerator is

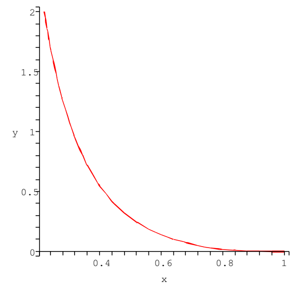

FIGURE 5. minimal points of $\mathcal{V}$ in the positive real quadrant

nonvanishing at any minimal point as well. Checking whether the quantity Q defined in equation (2.9) of Theorem 2.11 ever vanishes, we find that the only solutions to $H = Q = 0$ are at $(1,0)$ and at complex locations that are not minimal because $\mathbf{L}(\mathbf{z})$ is not real there (Proposition 2.3). Lastly, we must check that the minimal points in the positive quadrant are strictly minimal. We can ascertain that they are by checking the "extraneous" critical points, which will lie on three other branches of a quartic (see below), and seeing that their moduli are too large. We then know that the asymptotics for a_{rs} are uniform as s/r varies over a compact subset of $(0,1)$ and given by

$$a_{rs} \sim C x^{-r} y^{-s} r^{-1/2} .$$

Below, we will use a computer algebra system to determine x, y and C as explicitly functions of $\lambda := s/r$. This gives asymptotics for the number of HCP's whose shape is not asymptotically vertical or horizontal. A crude approximation at the logarithmic level is given by

$$a_{rs} \approx \exp\left[r(-\log x - (s/r)\log y\right]$$

where x and y of course still depend on s/r.

Let us compute the average row length of a typical HCP. We may apply Theorem 2.16 to find the limit in probability of h_k/k, where h_k is the height of an HCP chosen uniformly from among all HCP's of size k. Setting $y = 1$ in the bivariate generating function recovers the univariate generating function (3.1). The point $(x,1)$, where $x = x_0$ is the smallest root of the denominator of (3.1) controls asymptotics in this direction; we compute $\mathbf{dir}(x,1)$ there to be $(xH_x, yH_y)(x_0,1)$, which simplifies to $r/s = 4(5 - 14x_0 + 12x_0^2)/(5 - 9x_0 + 11x_0^2)$ or still further to

$\alpha := \frac{1}{47}(147 - 246x_0 + 344x_0^2) \approx 2.207$. We conclude that for large k, $k/h_k \to \alpha$. Thus we see that the average row length in a typical large HCP is around 2.2.

Although it is not particularly enlightening, we do the computer algebra for general $\overline{\mathbf{r}}$ to show how it turns out. First, we solve the pair of equations $H = 0, sxH_x = ryH_y$, to find points on $\mathcal{V}$ with $\overline{\mathbf{r}} = \overline{(r,s)} \in \mathbf{L}(x,y)$. Explicitly, we set $\lambda = s/r$ and ask Maple for a Gröbner basis for the ideal generated by H and $\lambda x H_x - yH_y$. We find that the elimination polynomial for x is the quartic

$$(1+\lambda)x^4 + 4(1+\lambda)^2x^3 + 10(\lambda^2+\lambda-1)x^2 + 4(2\lambda-1)^2x + (\lambda-1)(2\lambda-1)\,.$$

Looking next at the polynomial for y in terms of x we find it is linear in y, so y is a rational function of x and λ. It is messy, however, to write y this way and easier to go back and find the elimination polynomial for y which is the quartic

$$\begin{aligned} & (4\lambda^4 - 4\lambda^3 - 3\lambda^2 + 4\lambda - 1)y^4 + (40\lambda^4 - 44\lambda^3 - 20\lambda^2 + 48\lambda - 16)y^3 \\ + \;& (-172\lambda^4 + 128\lambda^3 + 160\lambda^2 - 256\lambda + 64)y^2 + (1152\lambda^4 - 1024\lambda^3 - 512\lambda^2)y - 1024\lambda^4\,. \end{aligned}$$

Although there are four root pairs (x,y) for each λ, there can only be one that is minimal, and we find it by selecting the one with the least magnitude coordinates (verifying in the process that (x,y) is strictly minimal).

Finally, from Theorem 2.11 we see that

$$C = \frac{xy(1-x)^3}{\sqrt{2\pi}}\sqrt{\frac{y\,(-x\,(1-x-x^2+x^3+x^2y)-x^3y)}{Q}}$$

with Q given by

$$\begin{aligned} &-112\,x^{11}y - 448\,x^9y + 16\,x^3y - 112\,x^4y + 4\,x^3y^2 + 16\,x^{12}y + x^7y^5 - 9\,x^9y^5 + 25\,x^{10}y^5 + 25\,x^{11}y^4 \\ &-105\,x^{11}y^3 - 98\,x^{10}y^4 + 320\,x^{10}y - 16\,x^{12}y^2 - 345\,x^9y^3 - 11\,x^8y^3 - 464\,x^{10}y^2 + 132\,x^{11}y^2 + 349\,x^{10}y^3 \\ &-1104\,x^8y^2 + 912\,x^9y^2 + 127\,x^9y^4 - x^8y^5 - 60\,x^8y^4 + 7\,x^7y^4 - 2\,x^6y^4 + 189\,x^7y^3 + 224\,x^8y + x^5y^4 \\ &+856\,x^7y^2 - 81\,x^6y^3 + 5\,x^5y^3 - x^4y^3 - 32\,x^4y^2 + 224\,x^7y - 432\,x^6y^2 - 448\,x^6y + 144\,x^5y^2 + 320\,x^5y. \end{aligned}$$

3.2. **Trace monoid enumeration.** Let D be a connected graph on the symbols $[n] = \{1, \ldots, n\}$. Define the **trace monoid**, Σ, to be the quotient of the free monoid on n symbols by the commutation relations that allow i and j to commute whenever

$\{i, j\} \notin D$. These were introduced by Cartier and Foata [CF69] and have connections to formal languages, automata and concurrent systems.

If I is an independent set in D then the product w_I of symbols in I is independent of order. This may be used to define the **Cartier-Foata decomposition**, which is a canonical form for words in Σ. In particular, there exactly one way to write a word as $w_{I_1} \cdots w_{I_k}$ in such a way that the set $I_i \cup \{j\}$ is not independent for any i and any $j \in I_{i+1}$. More constructively, if $(I_1, \ldots, I_k)$ gives the Cartier-Foata decomposition of a word w then wj has decomposition $(I_1, \ldots, I_k, \{j\})$ if $j \in I_k$, and $(I_1, \ldots, I_s \cup \{j\}, I_{s+1}, \ldots, I_k)$ otherwise, where s is minimal such that j is independent of each I_t for $t \geq s$.

Let $|w|$ denote the number of symbols of a word w, and $h(w)$ its **height**, that is, the number of sets in its decomposition. Let $F(x, y) = \sum_{w \in \Sigma} x^{h(w)} y^{|w|}$ count words in Σ by height and length. The paper [KMM03] considers several notions of sampling from words of size k, in particular, uniform sampling from words of length k and uniform sampling from words of height k. They investigate in each case the typical behavior of $|w|/h(w)$ as $k \to \infty$. They call this the average parallelism.

It is well known that the univariate generating function $L(y) = F(1, y) = \sum_{w \in \Sigma} y^{|w|}$ is rational. In fact, it was shown in [GS00] that L has a unique pole of minimal modulus which is simple (this relies on the connectedness hypothesis on D). Krob et al [KMM03] show that in fact F is rational and find a formula via what is essentially the transfer-matrix method. They use this to obtain their main result on sampling Σ by length, namely is a formula for the average parallelism (which shows it to be an algebraic number).

It follows from Theorem 2.16 and the known fact that L has a strictly minimal simple pole that the average parallelism of a uniform word of length k converges as $k \to \infty$ to the ratio $x(\partial F/\partial x)/[y(\partial F/\partial y)]$ evaluated at the point $(1, y_0)$, where y_0 is the dominant pole of L.

This is one of the two main results in [KMM03]. Their proof involves reducing F back to one variable before extracting the mean, via the observation that the k^{th} coefficient of the univariate function $\frac{\partial F}{\partial x}(1, y)$ counts the total height of words of length k. They then extract relevant properties of $\frac{\partial F}{\partial x}(1, y)$ by a somewhat involved analysis. Similarly, their other main result, on sampling words of a given height, follows from our Theorem 2.16 as well, once it is verified that the height generating function $H(x) = F(x,1)$ has a strictly minimal simple pole; this appeared to be unknown before it was proved in [KMM03], their argument also requiring this step.

3.3. **Generalized Riordan arrays.** Suppose that the bivariate generating function F may be written in the form

$$F(x,y) = \frac{\phi(x,y)}{1 - xv(y)} \tag{3.3}$$

where v and ϕ are formal power series. When v and ϕ are analytic in the right region, one may apply the methodology presented in this paper, and the formulae simplify. If we define the quantities

$$\mu(v;y) \;:=\; \frac{yv'(y)}{v(y)} \tag{3.4}$$

$$\sigma^2(v;y) \;:=\; \frac{y^2v''(y)}{v(y)} + \mu(v;y) - \mu(v;y)^2 = y\frac{d\mu(v;y)}{dy} \tag{3.5}$$

then it is readily established that $(1/v(y), y) \in \mathsf{crit}_{(r,s)}$ if and only if $r\mu(v;y) = s$, and when this holds, then $Q(1/v(y), y) = \sigma^2(v;y)$. Provided that ϕ and σ^2 are nonzero at a minimal point, the leading term of its asymptotic contribution in (2.8) then becomes

$$a_{rs} \sim v(y)^{-r} y^s \frac{\phi(1/v(y), y)}{2\pi r\sigma^2(v;y)} \tag{3.6}$$

where $\mu(v;y) = s/r$.

The notations μ and σ^2 are of course drawn from probability theory. These quantities are always nonnegative when v has nonnegative coefficients. To relate this to the limit theorems in Section 2.5, observe that setting $y = 1$ gives

$$\begin{aligned} \mu(v;1) &= \frac{v'(1)}{v(1)}\,; \\ \sigma^2(v;1) &= \frac{vv'' - (v')^2 + vv'}{v^2}(1)\,. \end{aligned}$$

Thus, under the hypotheses of Theorem 2.16, a WLLN will hold with mean $\mathbf{m} = \mu(v;1)$. Of course we see here that $\mu(v;1)$ is simply the mean of the renormalized distribution on the nonnegative integers with probability generating function v. Similarly, we see in Theorem 2.18 that $B(r,s) = (s - \mu(v;1)r)^2/\sigma^2(v;1)$ is the Gaussian term corresponding to the variance $\sigma^2(v;1)$ of this renormalized distribution.

A very common special case to which the foregoing discussion applies is that of **Riordan arrays** (we give a brief discussion here; see [Wil05] for more details and references). Here by definition ϕ depends on x only, and $\phi(0) \neq 0$. If also $v'(0) \neq 0$, the array (a_{rs}) is called a **proper** Riordan array and is lower triangular when written as a matrix in the usual way. The set of proper Riordan arrays forms a group under matrix multiplication, and this fact allows for many combinatorial sums to be simplified [Spr94]. There is also a strong connection with Lagrange inversion, since there is a 1–1 correspondence between the series $v(y)$ of a proper Riordan array and the series $A(t)$ such that $v(y) = yA(v(y))$ as formal power series. This allows for (3.7) to be stated in terms of A in many cases, without explicit mention of v, as we show, for example, in Proposition 5.3.

Now it is easily seen that if v is combinatorial and aperiodic, the minimal points of $\mathcal{V}$ are precisely those with x positive real, they are all strictly minimal, and σ^2 is always nonzero at such points. Also $\mu(v;y)$ increases strictly since its derivative is $\sigma^2(v;y)/y$. This leads quickly to the following result.

Proposition 3.1. *Let $(v(y), \phi(y))$ determine a Riordan array. Suppose that $v(y)$ has radius of convergence $R > 0$ and is aperiodic with nonnegative coefficients, and that ϕ has radius of convergence at least R. Then*

$$a_{rs} \sim v(y)^r y^{-s} r^{-1/2} \sum_{k=0}^{\infty} b_k(s/r) r^{-k} \tag{3.7}$$

where y is the unique positive real solution to $\mu(v;y) = s/r$. Here $b_0 = \frac{\phi(y)}{\sqrt{2\pi\sigma^2(v;y)}} \neq 0$. The asymptotic approximation is uniform as s/r varies within a compact subset of (A,B), where A is the order of v at 0 and B its order at infinity. □

Note that by definition, $a_{rs} = 0$ outside the interval $[A,B]$, so all nontrivial diagonals are covered.

If v is periodic with period b, say $v(y) = y^a g(y^b)$ with g aperiodic, then we obtain b minimal points on the same torus, and their contributions must be summed.

This yields periodicity in the asymptotics, as expected. See Section 5 for a sample calculation along these lines.

We note that if the combinatorial restriction is lifted, much more complicated behaviour can occur. The generating function of Example 1 is of Riordan type with $\phi(y) = v(y) = (3 - 3y + y^2)^{-1}$, and even though v is aperiodic, $\mathsf{contrib}_{\mathbf{r}}$ usually has cardinality 2. Furthermore, at the unique contributing point for the diagonal direction, σ^2 vanishes.

3.4. **Maximum number of distinct subsequences.** We shall see several examples to which Proposition 3.1 applies in Section 5 and Section 6. Here is another one with a slightly different flavour. Flaxman, Harrow and Sorkin [FHS04] show that the generating function whose (n, k) coefficient is the maximum number of distinct subsequences of length k contained in a length n string over an alphabet of size d is

$$F(x, y) = \frac{1}{1 - x - xy(1 - x^d)}$$

(note that there is an error in the formula given in [FHS04]). This is of Riordan type with $\phi(x) = (1-x)^{-1}$ and $v(x) = x + x^2 + \cdots + x^d$, and Proposition 3.1 applies when $d \geq 2$. For each slope $d > n/k > 1$ the value $x = x_d$ in (3.7) lies in $(1/d, 1)$ and x_d tends to $1 - k/n$ as $d \to \infty$. We note in passing that that the limiting case $d = \infty$ above corresponds to Pascal's triangle, with $\phi(x) = (1 - x)^{-1}, v(x) = x/(1 - x)$ and that the leading term asymptotic approximation is uniform in d.

3.5. **Smirnov words.** Given an integer $d \geq 3$ we define a Smirnov word in the alphabet $\{1, \ldots, d\}$ to be a word in which no letter repeats consecutively. The number of Smirnov words of length n is of course easily seen by a direct counting argument to be $d(d-1)^{n-1}$. If we count these words according to the number of occurrences of each symbol, we get the multivariate generating function

$$F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = \frac{1}{1 - \sum_{j=1}^{d} \frac{z_j}{1+z_j}} \tag{3.8}$$

where $a_{\mathbf{r}}$ is the number of Smirnov words with r_j occurrences of the symbol j for all $1 \leq j \leq d$. This generating function may be derived from the generating function

$$G(\mathbf{y}) = \frac{1}{1 - \sum_{j=1}^{d} y_j}$$

for all words as follows. Note that collapsing all consecutive occurrences of each symbol in an arbitrary word yields a Smirnov word; this may be inverted by expanding each symbol of a Smirnov word into an arbitary positive number of identical symbols, whence $F(\frac{y_1}{1-y_1}, \dots, \frac{y_d}{1-y_d}) = G(\mathbf{y})$. Substituting $y_j = z_j/(1+z_j)$ yields the formula (3.8) for F; see [FS05, p. 168] for more on this old result.

When we write F as a rational function, we get the denominator

$$H := \sum_{j=0}^{d} (1-j) e_j(\mathbf{z})$$

where e_j is the j^{th} elementary symmetric function of $z_1, \dots, z_d$ and $e_0 = 1$ by definition. Thus when $d = 3$ for example, one obtains $H = 1 - (xy + yz + xz) - 2xyz$. The expression for H may be denoted quite compactly by

$$H = g - u \frac{\partial g}{\partial u}\bigg|_{u=1}$$

where $g(\mathbf{z}, u) := \prod_{j=1}^{d} (1 + u z_j)$ is the polynomial in u whose coefficients are the elementary symmetric functions in $\mathbf{z}$. The denominator is of the form $1 - P$ for an aperiodic polynomial P with nonnegative coefficients, so by Proposition 2.7, $\mathsf{contrib}_{\overline{\mathbf{r}}}$ will always consist of one strictly minimal point in the positive orthant.

The typical statistics of a Smirnov word of length n are not in doubt, since it is clear that each letter will appear with frequency $1/d$. This may be formally deduced from Theorem 2.17, which also shows the variation around the mean to be Gaussian. We may, perhaps, be more interested in the so-called large-deviation probabilities: the exponential rate at which the number of words decreases if we alter the statistics. Let $\hat{\mathbf{s}}$ denote some frequency vector. Then as $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}} \to \hat{\mathbf{s}}$, we have

$$\frac{1}{|\mathbf{r}|} \log a_{\mathbf{r}} \to -\mathbf{r} \cdot \log |\mathbf{z}|$$

where $\mathbf{z} \in \overline{\mathcal{O}}^d$ satisfies $\mathbf{dir}(\mathbf{z}) = \overline{\mathbf{r}}$.

To solve for $\mathbf{z}$ in terms of $\overline{\mathbf{r}}$, we use the symmetries of the problem. It is evident that z_j is symmetric in the variables $\{r_i : i \neq j\}$. For example, when $d = 3$, if we refer to $\mathbf{z}$ as (x, y, z) and $\hat{\mathbf{r}}$ as (r, s, t), then, using $r + s + t = 1$, the equations (2.5) have the solution

$$x = \frac{r^2 - (s-t)^2}{2r(1-2r)}$$

where the values of y and z are given by the same equation with r, s and t permuted. When $d \geq 4$, the solution is not a rational function and it is difficult to get Maple to halt on a Gröbner basis computation. This points to the need for computational algebraic tools better suited to working with symmetric functions.

3.6. **Hamiltonian tournaments.** A **tournament** is an orientation of a complete graph. A Hamilton cycle in a tournament is an oriented cycle passing through each vertex exactly once. The maximum possible number of Hamilton cycles on a tournament with n vertices is not known, but one way to bound this from below is to find a class of tournaments with a high average number of cycles. The average over all tournaments is easy to compute and gives a crude lower bound on the maximum, namely $g(n) := 2^{-n}(n-1)!$. Wormald [Wor04] considers a random tournament T on dn vertices obtained from a specific tournament T^* on d vertices by replacing each vertex v of T^* by a uniformly chosen random tournament $T(v)$ on n vertices, and each edge vw of T^* by n^2 edges going between every pair (x, y) of vertices $x \in T(v)$ and $y \in T(w)$. He then shows [Wor04, Theorem 4] that this achieves an improvement by a constant factor asymptotically, for any choice of T^* that is **regular**, that is, every vertex has equal in and out degree (for which d must be odd). His analysis rests on a computation which we now replicate, involving a multivariate generating function.

Let A be the adjacency matrix of T^* and let I_d denote the $d \times d$ identity matrix. Define a function B of d variables by

$$B(\mathbf{z}) = I_d - \left(\frac{1}{2} I_d + A\right) \operatorname{diag}(\mathbf{z}) .$$

In a somewhat involved computation, Wormald shows that ratio, $c_n = c_n(T^*)$ of the expected number of Hamilton cycles in T to the expectation for a uniformly chosen tournament of size dn is asympotically given by

$$c_n \sim \left(\frac{2}{d}\right)^{dn} (2\pi n)^{(d-1)/2} \frac{d^{3/2}}{d-1} [z_1^n \cdots z_d^n] \frac{(1 - \frac{1}{2} z_1) \det(B : 1, 1)(\mathbf{z})}{\det B(\mathbf{z})} .$$

It remains then asymptotically to evaluate the n^{th} main diagonal coefficient of $\mathrm{F}(\mathbf{z}) := (1 - z_1/2) \det(B : 1, 1)(\mathbf{z}) / \det B(\mathbf{z})$. Here $(B : 1, 1)$ denotes the $(1, 1)$-cofactor of B, namely B with the first row and column stricken.

Let $\mathcal{V}$ denote the variety where the polynomial $\det B(\mathbf{z})$ vanishes. When $\mathbf{z} = \mathbf{z}_* := (2/d)\mathbf{1} = (2/d, \ldots, 2/d)$, all the row sums of B vanish, and hence $(2/d)\mathbf{1} \in \mathcal{V}$. Using regularity of T^*, it is next shown that at $\mathbf{z}_*$, the partial derivatives are all equal and nonzero, whence $\mathbf{dir}(\mathbf{z}^*) = \mathbf{1}$. At this point, Wormald applies a version of the formula (2.7), verifying the nonvanishing of G and Q there, to conclude that

$$[z_1^n \cdots z_d^n]F(\mathbf{z}) = (2\pi)^{(1-d)/2} \left(\frac{2}{d}\right)^{-dn} \frac{\det(B:1,1)(\mathbf{z}_*)}{(z_d \partial \det B(\mathbf{z})/\partial z_d)_{\mathbf{z}=\mathbf{z}_*}\sqrt{\mathcal{H}}} .$$

It turns out that the determinant in the numerator cancels with the partial derivative in the denominator, and one is left with

$$c_n \sim \sqrt{\frac{d}{\mathcal{H}}} .$$

Finally, to show that an improvement of a constant factor has been obtained, the Hessian $\mathcal{H}$ must be evaluated. For each of the first fifty odd values of d, Wormald evaluated $\mathcal{H}$ in Maple for $T^* = R_d$, the tournament in which i beats j if and only if the cyclic arc from i to j mod d is shorter than the one from j to i. Some comparisons indicated that this gives the best value of $c(T^*)$, which is stated by Wormald as a conjecture. The corresponding values of $\sqrt{d/\mathcal{H}}$ are all rational, beginning at 2 for $d = 3$ and apparently approaching a limit of around 2.855958. As a byproduct of this computation it was also conjectured that $d/\mathcal{H}$ is always the square of a rational number.

3.7. **Symmetric Eulerian numbers.** The symmetric Eulerian numbers $\hat{A}(r,s)$ [Com74, page 246] count the number of permutations of the set $[r+s+1] := \{1, 2, \ldots, r+s+1\}$ with precisely r descents. By reading backwards, this equals the number with exactly s descents, hence the symmetry in r and s. The symmetric Eulerian numbers have exponential generating function [GJ83, 2.4.21]

$$F(x,y) = \frac{e^x - e^y}{xe^y - ye^x} = \sum_{r,s} \frac{\hat{A}(r,s)}{r!\,s!} := \sum_{r,s} a_{rs} x^r y^s . \tag{3.9}$$

The denominator in this representation is singular at the origin and has a factor of $(x - y)$ in common with the numerator. We factor this out of the numerator and denominator, writing $F = G/H$ with $G = (e^x - e^y)/(x - y)$ and $H = (xe^y - ye^x)/(x - y)$. Both the numerator and the denominator in this representation are entire functions.

The symmetric Eulerian numbers are nonnegative, so we look at the lowest piece of $\mathcal{V}$ in the first quadrant. Here H is increasing in each argument, whence $\mathcal{V}$ has a single component which is roughly hyperbola-shaped. It is easy to check that the gradient of $\mathcal{V}$ never vanishes, so $\mathcal{V}$ is smooth. At the point $x = y = 1$, we have $\partial H/\partial x = \partial H/\partial y = 1$, so $\mathbf{dir}(1,1) = \mathbf{1}$. For any other point of $\mathcal{V}$, we may work with the representation of H having $(x-y)$ in the denominator. We set

$$\alpha = \frac{x\partial H/\partial x}{x\partial H/\partial x + y\partial H/\partial y}$$

so that $(x,y) \in \overline{\mathbf{r}} \Leftrightarrow \hat{\mathbf{r}} = (\alpha, 1-\alpha)$. The expression for α does not look all that neat but on $\mathcal{V}$ we may substitute $(y/x)e^x$ for e^y, after which the expression for α reduces to $(1-x)/(y-x)$. We see, therefore, from Theorem 2.9, that $\hat{A}(r,s)$ is asymptotic to $C(r+s)^{-1/2}r!s!\gamma^{r+s}$, where for a given value of $\alpha = r/(r+s)$, the value of γ is given by $x^{-\alpha}y^{-(1-\alpha)}$ after solving the following transcendental equations for (x,y):

$$\begin{aligned} \frac{1-x}{y-x} &= \alpha\,; \\ xe^y &= ye^x\,. \end{aligned}$$

A more symmetric formulation of the first displayed equation above is simply $rx+sy = r+s$.

3.8. **Alignments of sequences.** The problem of **sequence alignment** is of interest in molecular biology [Wat95; RT05], since a given string may evolve via substitutions, insertions, and deletions. We seek to place several strings of varying lengths in parallel, which may necessitate adding some spaces. A basic problem underlying design of algorithms which seek the best alignment in the sense of minimizing some score function is simply to count such alignments.

Mathematically, given positive integers k and a k-tuple $\mathbf{n} = (n_1, \ldots, n_k)$, we may define a $(k,\mathbf{n})$-alignment as a $k \times n$ binary matrix for some n, such that no columns are identically zero and the ith row sum is n_i. Column j is **aligned** if it has no zero entry, and a **block** of size b is a $k \times b$ submatrix, with contiguous columns, all of which are aligned. See Figure 3.8 for a graphical representation of an alignment and its corresponding matrix; note that for this problem the elements of the string are irrelevant — we only care whether there is a letter or a space.

The generating function giving the number of $(k, \mathbf{n})$-alignments with the ith row having sum n_i is given by

$$F(z_1, \ldots, z_k) = \frac{1}{2 - p(\mathbf{z})}$$

where $p(\mathbf{z}) = \prod_{j=1}^k (1 + z_j)$.

The special case where all sequences involved have the same length (all rows have the same sum) is easily dealt with. By Proposition 2.7, there is a single strictly minimal point in the positive orthant that controls asymptotics in the diagonal direction. By symmetry of F and of the equation $\mathbf{dir}(\mathbf{z}) = \mathbf{1}$, this point has the form $\mathbf{z} = z\mathbf{1}$ for some positive z. Thus we have $z = 2^{1/k} - 1$ and hence the asymptotic has the form $C(z^{-k})^n n^{-\frac{k-1}{2}}$. This result was derived by Griggs, Hanlon, Odlyzko and Waterman in [GHOW90] using a saddle point analysis. Note that for large k, $z^{-k} \sim 2^{-1/2} k^k (\log 2)^{-k}$. To compute the constant C, we use the formula of Theorem 2.10. It is readily computed that $-z_k \partial H / \partial z_k = 2z/(1 + z)$. To compute the Hessian determinant, we need only consider the Hessian in $\theta_1, \ldots, \theta_{d-1}$ of $-\log \prod_{j<k} (1 + z \exp(i\theta_j)) + \log(2 - \prod_{j<k} (1 + z \exp(i\theta_j)))$. Direct computation using the fact that $\prod_{j<k} (1 + z \exp(i\theta_j)$ takes the value $2/(1 + z)$ when $\theta = \mathbf{0}$ shows that the Hessian matrix has diagonal elements $2/(1 + z)$ and off-diagonal elements $1/(1 + z)$. Thus $\mathcal{H} = k(1 + z)^{-(k-1)} = k2^{1/k}/2$ and this yields

$$C = \frac{2^{(1-k^2)/2k}}{(2^{1/k} - 1)\sqrt{k\pi^{k-1}}}.$$

This yields the result of [GHOW90] (note that there is an error in the displayed formula on [Wat95, p. 1989] – the factor r^k should read r, and the factor $2^{(k^2-1)/(2k)}$ should be $2^{(1-k^2)/(2k)}$).

A more important case for biological applications is when the minimum block size is bounded below, by b, say. The generating function in this case is shown in [RT05], by means of standard operations on generating functions of formal languages, to have the form

$$F(z_1, \ldots, z_k) = \frac{A}{1 + (1 - p)A + (A - 1)t} = \frac{1 - t + t^b}{(1 - t)(1 - g) - t^b g} = \frac{1 + \frac{t^b}{1-t}}{1 - g\left(1 + \frac{t^b}{1-t}\right)}$$

where $t = \prod_j z_i$, $g = p - 1 - t$ and $A = 1 - t + t^b$. Note that when $b = 1$ then $A = 1$, and we recover the unrestricted case analysed above. In this case, asymptotics for the case where all sequences have equal length have been derived (and as far as we know) only for the case $k = 2$, using the "diagonal method" discussed in Section 7. We can deal with general k using the methods of this paper.

Again, by symmetry we need only look for a contributing minimal point of the form $\mathbf{z} = z\mathbf{1}$. Let $H = 1 + (1 - p)A + (A - 1)t$. Since $t = z^k$ and $p = (1 + z)^k$ we must find the root(s) ρ of smallest modulus of $h(z) := H(z, z, \ldots, z)$. Note that the third formula above shows that we may take $h = 1 - P$ where P is aperiodic with nonnegative coefficients, so there will be a unique root ρ of smallest modulus, and it will be positive real. The exponential growth rate is then ρ^{-k} with polynomial correction of order $n^{(1-k)/2}$ as above.

We note that in the case $k = 2$, such a result was proved in [GHW86], but stated slightly differently. The value $\tau = \rho^2$ is given as the minimal positive root of the polynomial $(1 - x)^2 - 4x(1 - x + x^b)^2$. Setting $x = z^2$ and factoring the difference of squares yields ρ as the minimal positive root of $H(z, z) = 1 - 2z - z^2 + 2z^3 - 2z^{2b+1}$.

To better estimate $\rho := \rho_b$, note that $\rho_b < 1$ so it is reasonable to consider the approximation obtained by setting all powers involving z^b to zero. The minimal real zero ρ_1 of $(1 - t)(1 - g)$ is asymptotically of order $1/k$ (and equals $2^{1/k} - 1$). Now $\rho_b > \rho_1$ but ρ_b should be close to ρ_1. Indeed, it appears that an iteration scheme based on the fixed point equation

$$z = \left(1 + z^k + \frac{1 - z^k}{1 - z^k + z^{kb}}\right)^{1/k} - 1$$

given by $h(z) = 0$ converges rapidly to ρ_b from starting point ρ_1.

3.9. **Probability that there is an edge in an induced subgraph.** From the n-set $[n] := \{1, \ldots, n\}$, a collection of t disjoint pairs is named. Then a k element subset, $S \subseteq [n]$, is chosen uniformly at random. What is the probability $p(n, k, t)$ that S fails to contain as a subset any of the t pairs? This question is posed in [PSS05] as a step in computing the diameter of a random Cayley graph of a group of cardinality n when k elements are chosen at random (the diameter is infinite if the k elements do not generate G).

There are a number of ways of evaluating $p(n,k,t)$, one of which is by inclusion-exclusion on the number of pairs contained. This leads to

$$p(n,k,t) = \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n-2i}{n-k} \binom{n}{k}^{-1} . \tag{3.10}$$

The numbers $a(n,k,t) := \binom{n}{k} p(n,k,t)$ have the trivariate generating function

$$F(x,y,z) = \sum_{n,k,t} a(n,k,t) x^n y^k z^t = \frac{1}{1 - z(1 - x^2y^2)} \frac{1}{1 - x(1+y)} . \tag{3.11}$$

To see this, following [PSS05], we write

$$h(x,y) = \sum_{w,s \geq 0} \binom{w}{s} x^w y^{w-s} = \frac{1}{1 - x(1+y)}$$

and

$$g(y,z) = \sum_{i,b \geq 0} (-1)^i \binom{b}{i} z^b (xy)^{2i} = \frac{1}{1 - z(1 - x^2y^2)} .$$

The (n,k,t)-coefficient of $g \cdot h$ is the product of the (b,i)-term of g and the (w,s)-term of h, summed over $b = t, s = n-k, w + 2i = n$, which is just the (b,i) term of g times the $(n-2i, n-k)$-term of h summed over i and is evidently $a(n,k,t)$.

An asymptotic analysis in [PSS05] in the case $k = 2t$ and $n = 2ct$ with c an integer at least 2, shows that the exponential growth rate of $a(n,k,t)$ is a quadratic algebraic number and is less than that of $\binom{n}{k}$, implying that $p(n,k,t)$ goes to zero and certain Cayley graphs have diameter 2 with high probability. Here we give asymptotics for $a(n,k,t)$ in directions other than $\overline{(2c,2,1)}$. Such an analysis was posed as an open question at the end of [PSS05].

One may easily compute $\mathbf{\Xi}$ to be the above-diagonal $\{\mathbf{r} : r_1 \geq r_2\}$, corresponding to the vanishing of the second binomial coefficient in (3.10) when $k > n$. The alternating sum defining $p(n,k,t)$ does not, however, vanish when $2t > n$, even though it no longer defines the probability of any event. In fact, $a(n,k,t)$ can be negative for large t, meaning we will be unable to use Theorem 2.6. Nevertheless, we will be able to find $\mathsf{contrib}$ and thereby determine the asymptotic behavior of $a(n,k,t)$.

The denominator of F factors into two smooth pieces, call them $h_1 := 1 - x(1+y)$ and $h_2 := 1 = z(1 - x^2y^2)$. There is a corresponding stratificaton of $\mathcal{V}$ into two surfaces, $\mathcal{V}_1 := \{h_1 = 0 \neq h_2\}$ and $\mathcal{V}_2 = \{h_2 = 0 \neq h_1\}$, and a curve, $\mathcal{V}_0 := \{h_1 =$

$h_2 = 0\}$. For $\mathbf{z} \in \mathcal{V}_1$, lack of dependence on z means that $\mathbf{dir}(\mathbf{z}) \perp (0,0,1)$, so $\mathbf{dir}$ cannot be in the strictly positive orthant; hence for $\overline{\mathbf{r}} \in \overline{\mathcal{O}}$, there are no points of $\mathsf{crit}_{\overline{\mathbf{r}}}$ in $\mathcal{V}_1$. It turns out there are no points of $\mathsf{crit}_{\overline{\mathbf{r}}}$ in $\mathcal{V}_2$ either. This is discovered by computation. If $\mathbf{z} \in \mathcal{V}_2 \cap \mathsf{crit}_{\overline{\mathbf{r}}}$ and $\overline{\mathbf{r}} = \overline{(r,s,1)}$, then $\mathbf{z}$ satisfies the equations

$$\begin{aligned} h_2(\mathbf{z}) &= 0\,; \\ x\frac{\partial h_2}{\partial x} - rz\frac{\partial h_2}{\partial z} &= 0\,; \\ y\frac{\partial h_2}{\partial y} - sz\frac{\partial h_2}{\partial z} &= 0\,. \end{aligned}$$

These equations turn out to have no solutions: Maple tells us in an instant that the ideal generated by the left-hand sides of the three above equations in $\mathbb{C}[x,y,z,r,s]$ contains $r-s$; thus $\mathcal{V}_2$ may contain points of $\mathsf{crit}_{\overline{\mathbf{r}}}$ only when $r=s$, that is, only governing asymptotics of $a(n,k,t)$ from which $n=k$, which are not interesting.

Evidently, $\mathsf{crit}_{\overline{\mathbf{r}}} \subseteq \mathcal{V}_0$. Two equations are $h_1 = h_2 = 0$, that is,

$$(x,y,z) = \left(\frac{1}{1+y}\,,\, y\,,\, \frac{(1+y)^2}{1+2y}\right)\,.$$

The last equation is that $\overline{\mathbf{r}}$ is in the linear space spanned by $\nabla_{\log} h_1$ and $\nabla_{\log} h_2$. Setting the determinant of $(\overline{\mathbf{r}}, \nabla_{\log} h_1, \nabla_{\log} h_2)$ equal to zero gives the equation

$$2(r-s-1)y^2 + (r-3s)y - s = 0$$

which has one positive and one negative root, the positive root being

$$y_+ := \frac{\sqrt{(r+s)^2 - 8s} - (r-3s)}{4(r-s-1)}\,.$$

Plugging into for x_+ and z_+ yields expressions for these which are also quadratic over $\mathbb{Z}[r,s]$. The expression for (x_-, y_-, z_-) is just algebraic conjugate of (x_+, y_+, z_+), but is negative in the second and third coordinates.

Having identified $\mathsf{crit}_{\overline{\mathbf{r}}}$ as these two conjugate points, which we will call $\mathbf{z}_+$ and $\mathbf{z}_-$, it remains to find $\mathsf{contrib}_{\overline{\mathbf{r}}}$. But this is now easy by process of elimination. The set $\mathsf{contrib}_{\overline{\mathbf{r}}}$ consists of exactly one of $\mathbf{z}_+$ and $\mathbf{z}_-$: $\mathsf{contrib}_{\overline{\mathbf{r}}}$ cannot be empty nor contain two points on different tori. It is not possible to have $(x_-, y_-, z_-) \in \mathsf{contrib}_{\overline{\mathbf{r}}}$ both because of the negative coordinates, which would force alternations of signs in $a(n,k,r)$, and because the exponential growth rate is less than zero, which would force the leading term asymptotic for an integer to be exponentially small.

We conclude that $\mathsf{contrib}_{\overline{\mathbf{r}}} = \{\mathbf{z}_+\}$. The form of the leading term asymptotic is then given by Theorem 2.15:

$$a(n,k,t) \sim C\left(\frac{k}{n}, \frac{t}{n}\right) n^{-1/2} x_+^{-n} y_+^{-k} z_+^{-t} .$$

For an explicit, if not enlightening, evaluation of the function C, see [PSS05].

3.10. **Integer solutions to linear equations.** Let $a_{\mathbf{r}}$ be the number of nonnegative integer solutions to $A\mathbf{x} = \mathbf{r}$ where A is a $d \times m$ integer matrix. Denote by $\mathbf{b}^{(k)}$ the k^{th} column of A. Then

$$F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = \prod_{k=1}^{m} \frac{1}{1 - \mathbf{z}^{\mathbf{b}^{(k)}}}$$

This enumeration problem has a long history. We learned of it from [dLS03] (see also [Sta97, Section 4.6]). One special case is when $A = A_{m,n}$ is the incidence matrix for a complete bipartite graph. Solutions to $A\mathbf{x} = \mathbf{r}$ count nonnegative integer $m \times n$ matrices with row and column sums prescribed by $\mathbf{r}$; enumerating these is important when constructing statistical tests for contingency tables. Another special case is when $A = A_n$ is the incidence matrix of a complete directed graph on n vertices, directed via a linear order on the vertices. Here solutions are counted in various cones other than the postive orthant and the function enumerating them is known as Kostant's partition function.

It is known that $a_{\mathbf{r}}$ is piecewise polynomial, and it has been a benchmark problem in computation to determine these polynomials, and the regions or **chambers** of polynomiality, explicitly. Indeed, several subproblems merit benchmark status. Counting the chambers for Kostant's partition function is one such problem. Another is to evaluate the leading term for the diagonal polynomial $a_{n\mathbf{1}}$ in the case where $A = A_{k,k}$, the so-called **Erhart polynomial**, which counts $k \times k$ nonnegative integer matrices with all rows and columns summing to n. Equivalently, this counts integer points in the n-fold dilation of the **Birkhoff polytope**, defined as the set in $\mathbb{R}^{\binom{k}{2}}$ of all doubly stochastic $k \times k$ matrices; the leading term of $a_{n\mathbf{1}}$ is the volume of the Birkhoff polytope.

Our methods do not improve on the computational efficiency of previous researchers: our ending point is a well known representation, from which other researchers have attempted to find efficient means of computing. Our methods do, however, give an effective answer, which illustrates that this class of problems can be put in the framework for which our methods give an automatic solution. The the remainder of this section is devoted to such an analysis.

Our examination of the pole variety $\mathcal{V}$ begins with an answer rather than a question: we see that $\mathbf{1} \in \mathcal{V}$ and realize that life would be easy if $\mathsf{contrib}_{\overline{\mathbf{r}}} = \{\mathbf{1}\}$ for all $\overline{\mathbf{r}}$. In order for $\mathbf{1}$ to be a singleton stratum it is necessary and sufficient that the columns $\mathbf{b}^{(k)}$ span all of $\mathbb{R}^d$. For the remainder of the section we assume this to be the case.

The variety $\mathcal{V}$ is the union of the smooth sheets $\mathcal{V}_k$, where for $1 \leq k \leq m$, $\mathcal{V}_k$ is the binomial variety $\{\mathbf{z} : H_k(\mathbf{z}) := \mathbf{z}^{\mathbf{b}^{(k)}} - 1 = 0\}$. On each of these varieties, $\mathbf{dir}_k(\mathbf{z}) = \nabla_{\log}(H_k)(\mathbf{z})$ is constant and equal to $\mathbf{b}^{(k)}$. If a stratum S is bigger than a singleton, then at any $\mathbf{z} \in S$, the vectors $\nabla_{\log} H_k$ span a proper subspace of $\mathbb{R}^d$; but these vectors do not vary as $\mathbf{z}$ varies in S, so the union over $\mathbf{z} \in S$ of $\mathbf{L}(\mathbf{z})$ is this same proper subspace of $\mathbb{R}^d$. Consequently, the union over all points of all non-singleton strata of $\mathbf{L}(\mathbf{z})$ is a union of proper subspaces, which we denote Ξ'. For $\overline{\mathbf{r}} \notin \Xi'$, then, $\mathsf{contrib}_{\overline{\mathbf{r}}}$ consists of one or more singleton strata.

Taking logs, we see that $\log \mathcal{V}_k$ is a hyperplane normal to $\mathbf{b}^{(k)}$ and is central (passes through the origin). We see that $\mathbf{K}(\mathbf{0})$ is the positive hull of the vectors $\mathbf{b}^{(k)}$; this hull is Ξ and outside of the closure of this, $a_{\mathbf{r}}$ vanishes. Forgetting about the logs, we see that $\mathbf{1} \in \mathsf{contrib}_{\overline{\mathbf{r}}}$ for all $\overline{\mathbf{r}} \in \Xi_0 := \Xi \setminus \Xi'$: this follows from Theorem 2.12 since $\overline{\mathbf{r}} \in \mathbf{K}(\mathbf{1})$.

If the intersection of all the surfaces $\mathcal{V}_k$ contains any other points on the unit torus $\mathbf{T}(\mathbf{1})$ then these too are in $\mathsf{contrib}_{\overline{\mathbf{r}}}$ for all $\overline{\mathbf{r}} \in \Xi_0$. This is easy to check, since it is equivalent to the integer combinations of the columns of A spanning a proper sublattice of $\mathbb{Z}^d$. For instance, in the example of counting matrices with constrained row and column sums, the columns of A span the alternating sublattice of $\mathbb{Z}^d$, corresponding to the fact that $\mathsf{contrib}_{\overline{\mathbf{r}}} = \{\mathbf{1}, -\mathbf{1}\}$ for all $\overline{\mathbf{r}} \in \Xi_0$.

There may as well be singleton strata given by intersections of subfamilies of $\{\mathcal{V}_k : 1 \leq k \leq m\}$, that lie on the unit torus, but in computing the leading term asymptotics these may be ignored because they yield polynomials in $\mathbf{r}$ of lower degree.

In summary, for any $\overline{\mathbf{r}} \in \Xi_0$, the leading term asymptotics are given by summing (2.11) over a set containing $\mathbf{1}$, and isomorphic to the quotient of $\mathbb{Z}^d$ by the integer span of the columns of A. Our *a priori* knowledge that $a_{\mathbf{r}}$ are integers, teamed with (2.10) as in the last part of Theorem 2.12, shows that in fact $a_{\mathbf{r}}$ are piecewise polynomial, at least away from Ξ'. Thus, except on a set of codimension 1, we recover the well known piecewise polynomiality of $a_{\mathbf{r}}$.

In their paper, de Loera and Sturmfels use as a running example the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

In this example we see that the columns of A span $\mathbb{Z}^3$ over $\mathbb{Z}$, so the only contributing point is $\mathbf{1}$. The first three $\mathbf{b}$ vectors, in the order given, are the standard basis vectors, so the cone Ξ is the whole positive orthant of $\mathbb{RP}^2$, which is a 2-simplex. The other two $\mathbf{b}$ vectors are two of the three face diagonals of this cone, which are two midpoints of edges of the 2-simplex in $\mathbb{RP}^2$. In addition to the boundary of Ξ, there are three projective line segments in Ξ', corresponding to the one-dimensional strata $\mathcal{V}_4 \cap \mathcal{V}_5$, $\mathcal{V}_4 \cap \mathcal{V}_3$ and $\mathcal{V}_5 \cap \mathcal{V}_2$. The complement of these three line segments (two medians of the simplex $\mathbb{RP}_2$ and the line segment connecting two midpoints of edges) divides $\mathbb{RP}^2$ into five chambers, and on each of these chambers $a_{\mathbf{r}}$ is a quadratic polynomial. An algorithm is given in Section 5 of [BP04] to compute these polynomials, but since it does not improve on the algorithms given in [dLS03], we do not repeat it here.

4. Transfer matrices

Suppose we have a class $\mathcal{C}$ of combinatorial objects that may be put in bijective correspondence with the collection of paths in a finite directed graph. For example, it is shown in [Sta97, Example 4.7.7] that the class $\mathcal{C}$ of permutations $\pi \in \bigcup_{n=0}^{\infty} \mathcal{S}_n$ such that $|\pi(j) - j| \leq 1$ for all j in the domain of π is of this type. Specifically, the digraph has vertices $V := \{-1, 0, 1\}^2 \setminus \{(0, -1), (1, 0)\}$, with a visit to the vertex (a, b) corresponding to j for which $\pi(j) = a$ and $\pi(j+1) = b$; the vertices $(0, -1)$ and $(1, 0)$ do not occur because this would require $\pi(j) = \pi(j+1)$, which cannot happen; the edge set E is all elements of V^2 except those connecting $(1, x)$ to $(y, -1)$ for some

x and y, since this would require $\pi(j) = \pi(j+2)$; the correspondence is complete because once $\pi(j), \pi(j+1)$ and $\pi(j+2)$ are distinct, no more collisions are possible.

The **transfer-matrix method** is a general device for producing a generating function which counts paths from a vertex i to a vertex j according to a vector weight function w such that $w(v_1, \ldots, v_n) = \sum_{j=1}^n w(v_i, v_j)$ for some function $w : E \to \mathbb{N}^d$.

Proposition 4.1 (Theorem 4.7.2 of [Sta97])**.** *Let A be the weight matrix, that is, the matrix defined by $A_{ij} = \mathbf{1}_{(i,j)\in E}\mathbf{z}^{w(i,j)}$, where $\mathbf{z} = (z_1, \ldots, z_d)$. Let $\mathcal{C}_{ij}$ be the subclass of paths starting at i and ending at j and define $F(\mathbf{z}) = \sum_{\gamma\in\mathcal{C}_{ij}} \mathbf{z}^{w(\gamma)}$. Then*

$$F(\mathbf{z}) = \left((I-A)^{-1}\right)_{ij} = \frac{(-1)^{i+j}\det(I - A : i, j)}{\det(I-A)}$$

where $(M : i, j)$ denotes (i, j)-cofactor of M, that is, M with row i and column j removed. □

Thus any class to which the transfer-matrix method applies will have a multivariate rational generating function. It is obvious, for example, that the transfer-matrix method applies to the class of words in a finite alphabet with certain transitions forbidden. The examples discussed in [Sta97, pages 241–260] range from restricted permutations to forbidden transition problems to the derivation of polyomino identities including (3.1), and finally to a very general result about counting classes that factor when viewed as monoids. Probably because techniques for extracting asymptotics were not widely known in the multivariate case, the discussion in [Sta97] centers around counting by a single variable: in the end all weights are set equal to a single variable, x, reducing reducing multivariate formulae such as (3.2) to univariate ones such as (3.1). But the methods in the present paper allow us to handle multivariate rational functions almost as easily as univariate functions, so we are able to derive joint asymptotics for several statistics at once, which is useful to the degree that we care about joint statistics.

4.1. **Restricted switching.** Some of the examples we have already seen, such as trace monoid enumeration, are essentially transfer-matrix computations. A simpler example of the method is the following path counting problem.

Let G be the graph on $K+L+2$ vertices which is the union of two complete graphs K_{K+1} and K_{L+1} and one edge $\overline{xy}$ between them. Paths on this graph correspond perhaps to a message or task being passed around two workgroups, with communication between the workgroups not allowed except between the bosses. If we sample uniformly among paths of length n, how much time does it spend, say, among the common members of group 1 (excluding the boss)?

We can model this efficiently by a four-vertex graph, with vertices C_1, B_1, B_2 and C_2, where B stands for boss (so $x = B_1$ and $y = B_2$) and the number refers to the workgroup. We have collapsed K vertices to form C_1 and L to form C_2, so a path whose number of visits to C_1 is r and to C_2 is s will count for $r^K s^L$ actual paths.

Let $F(u,v,z) = \sum_\omega u^{N(\omega,C_1)} v^{N(\omega,C_2)} z^{|\omega|}$ where $N(\omega, C_j)$ denotes the number of visits by ω to C_j and $|\omega|$ is the length of the path ω. By Proposition 4.1, the function F is rational with denominator $H = \det(I - A)$, where

$$A = \begin{bmatrix} uz & uz & 0 & 0 \\ z & z & z & 0 \\ 0 & z & z & z \\ 0 & 0 & vz & vz \end{bmatrix}.$$

Subtracting from the identity and taking the determinant yields

$$H = uz^2 + uz^2v - uz - uz^4v + z^2v - 2\,z - zv + 1 + z^3v + uz^3\,.$$

Giving each visit to C_1 weight K corresponds to the substitution $u = Ku'$ and similarly, $v = Lv'$. Applying Theorem 2.16 and reframing the results in terms of the original u and v shows that the proportion of time a long path spends among C_1 tends as $n \to \infty$ to

$$\left.\frac{u\partial H/\partial u}{z\partial H/\partial z}\right|_{(K,L,z_0)}$$

where z_0 is the minimum modulus root of $H(K,L,z)$. In the present example, this yields

$$\left.\frac{K\left(-z - zL + 1 + z^3L - z^2\right)}{-2\,Kz - 2\,KzL + K + 4\,Kz^3L - 2\,zL + 2 + L - 3\,z^2L - 3\,Kz^2}\right|_{z=z_0}$$

for the proportion of time spent in C_1, where z_0 is the minimal modulus root of $H(K,L,z)$. Trying $K = 1$, $L = 1$ as an example, we find that $H(1,1,z) = 1 - 4z + 2z^2 + 3z^3 - z^4$, leading to $z_0 \approx 0.381966$ and a proportion of just over $1/8$ for the time spent in C_1. There are 4 vertices, so the portion of time the task spends at the isolated employee is just over half what it would have been had bosses and employees had equal access to communication. This effect is more marked when the workgroups have different sizes. Increasing the size of the second group to 2, we plug in $K = 1, L = 2$ and find that $z_0 \approx 0.311108$ and that the fraction of time spent in C_1 has plummetted to just under 0.024.

4.2. **Connector matrices.** For a more substantial example, we have chosen a further refinement of the transfer-matrix method, called the **connector matrix method**. The specific problem we will look at is the enumeration of sequences with forbidden substrings, enumerated by the composition of the sequence. This is a problem in which we are guaranteed (by the general transfer matrix methodology) to get a multivariate rational generating function, but for which a more clever analysis greatly reduces the complexity of the computations. Our discussion of this method is distilled from Goulden and Jackson's exposition [GJ83].

Let S be a finite alphabet and let T be a finite set of words (elements of $\bigcup_n S^n$). Let $\mathcal{C}$ be the class of words with no substrings in T, that is, those $(v_0, \ldots, v_n) \in \mathcal{C}$ such that $(v_j, \ldots, v_{j+k}) \notin T$ for all $j, k \geq 0$. We may reduce this to a forbidden transition problem with vertex set $V := S^k$ with k the maximum length of a word in T. This proves the generating function will be rational, but is a very unwieldy way to compute it, involving a $|S|^k$ by $|S|^k$ determinant.

The connector matrix method of [GO81], as presented in [GJ83, pages 135–136], finds a much more efficient way to solve the forbidden substring problem (see also a more recent and elementary proof of these results via martingales [Li80]). They discover that it is easy to enumerate sequences containing at least v_i occurrences of the forbidden substring ω_i. By inclusion-exclusion, one can then determine the number containing precisely y_i occurrences of ω_i, and then set $\mathbf{y} = \mathbf{0}$ to count sequences entirely avoiding the forbidden substrings.

To state their result, let $T = \{\omega_1, \ldots, \omega_m\}$ be a finite set of forbidden substrings in the alphabet $[d] := \{1, \ldots, d\}$. We wish to count words according to how many occurrence of each symbol $1, \ldots, d$ they contain and how many occurrences of each forbidden word they contain. Thus for a word η, we define $\tau(\eta)$ to be the d-vector counting how many occurrences of each letter and we let $\sigma(\eta)$ be the m-vector counting how many occurrences of each forbidden substring (possibly overlapping) occur in η. Let F be the $(d+m)$-variate generating function with variables $x_1, \ldots, x_m, y_1, \ldots, y_m$ defined by

$$F(\mathbf{x}, \mathbf{y}) := \sum_{\eta} \mathbf{x}^{\tau(\eta)} \mathbf{y}^{\sigma(\eta)} .$$

Proposition 4.2 (Lemma 2.8.10 of [GJ83]). *Given a pair of finite words, ω and ω', let*

$\mathsf{connect}(\omega, \omega')$ *denote the sum of the weight of α over all words α such that some initial segment β of ω' is equal to a final segment of ω and α is the initial unused segment of ω. Formally,*

$$\mathsf{connect}(\omega, \omega') = \sum_{\alpha : (\exists \beta, \gamma)\, \omega = \alpha\beta, \omega' = \beta\gamma} \tau(\alpha) .$$

Let $\mathbf{V}$ be the square matrix of size m defined by $\mathbf{V}_{ij} = \mathsf{connect}(\omega_i, \omega_j)$, denote the diagonal matrices $\mathbf{I} := \operatorname{diag}(y_1, \ldots, y_m)$, $\mathbf{L} := \operatorname{diag}(\mathbf{x}^{\tau(\omega_1)}, \ldots, \mathbf{x}^{\tau(\omega_m)})$, and let $\mathbf{J}$ to be the m by m matrix of ones. Then

$$F(\mathbf{x}, \mathbf{y}) = [1 - (x_1 + \cdots + x_d) - C(\mathbf{x}, \mathbf{y} - \mathbf{1})]^{-1}$$

where

$$C(\mathbf{x}, \mathbf{y}) = \operatorname{trace}\left((\mathbf{I} - \mathbf{Y}\mathbf{V})^{-1} \mathbf{Y}\mathbf{L}\mathbf{J}\right) .$$

In particular, setting $\mathbf{y} = \mathbf{0}$, the generating function for the words with no occurrences of any forbidden substring is

$$F(\mathbf{x}, \mathbf{0}) = \left[1 - (x_1 + \cdots + x_d) - \operatorname{trace}\left((\mathbf{I} + \mathbf{V})^{-1} \mathbf{L}\mathbf{J}\right)\right]^{-1} . \tag{4.12}$$

□

Remark. *The function C is the so-called cluster generating function, whose $(\mathbf{r}, \mathbf{s})$-coefficient counts strings of composition $\mathbf{r}$ once for each way that the collection of s_j occurrence of the substring ω_j may be found in the string.*

4.3. **Forbidden substring example.** Goulden and Jackson apply this to an example. Let $S = \{0, 1\}$ be the binary alphabet, and let $T = \{\omega_1, \omega_2\} = \{10101101, 1110101\}$. The final substrings of length 1 and 3 of ω_1 are initial substrings of ω_1 with corresponding leftover pieces 1010110 and 10101. Thus $V_{11} = x_1^3 x_2^4 + x_1^2 x_2^3$. Computing the other three entries similarly, we get

$$\mathbf{V} = \begin{bmatrix} x_1^3 x_2^4 + x_1^2 x_2^3 & x_1^3 x_2^4 \\ x_1^2 x_2^4 + x_1 x_2^3 + x_2^2 & x_1^2 x_2^4 \end{bmatrix} .$$

We also obtain

$$\mathbf{L} = \begin{bmatrix} x_1^3 x_2^5 & 0 \\ 0 & x_1^2 x_2^5 \end{bmatrix} .$$

Plugging into (4.12), denoting (x_1, x_2) by (x, y), finally yields

$$F(x,y) = \frac{1 + x^2y^3 + x^2y^4 + x^3y^4 - x^3y^6}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6} . \tag{4.13}$$

We make the usual preliminary computations on F. Write G and H for the numerator and denominator in (4.13). A Gröbner basis computation in Maple quickly tells us that H has no singularities and that $H = G = 0$ does not occur in the positive quadrant of $\mathbb{R}^2$:

```
> gbasis([H , diff(H,x) , diff(H,y)] , [x,y] , tdeg);

                                        [1]
> gbasis([G,H],[x,y],plex);

                   3    2              3      2        4
          [x + y  + y  - y + 1, y  - 2 y  + 1 + y ]      .
```

Similarly, we see that Q and H vanish simultaneously only at $(1,0)$ and at $(0,1)$. Using our *a priori* knowledge that the coefficients of F are nonnegative, we apply Corollary 2.8 and look for minimal points on the lowest arc of the graph of $\mathcal{V}$ in the first quadrant. The plot of this is visually indistinguishable from the line segment $x + y = 1$, which is not surprising because the forbidden substrings only affect the terms of the generating function of order 7 and higher. More computer algebra shows the point $(x, y) \in \mathsf{contrib}_{\overline{\mathbf{r}}}$ to be algebraic of degree 21.

One question we might ask next is to what degree the forbidden substrings bias the typical word to contain an unequal number of zeros and ones. One might imagine there will be a slight preference for zeros since the forbidden substrings contain mostly ones. To find out the composition of the typical word, we apply Theorem 2.17. Setting $y = x$ gives the univariate generating function

$$f(x) = F(x,x) = \frac{1 + x^5 + x^6 + x^7 - x^9}{1 - 2x + x^5 - x^6 - x^8 - x^9 + x^{10}} = \sum_n N(n) x^n$$

for the number $N(n)$ of words of length n with no occurrences of forbidden substrings. The root of the denominator of minimum modulus is $x_0 = 0.505496\ldots$, whence $N(n) \sim C(1/x_0)^n$.

The direction of the point (x_0, x_0) is the projective point

$$\bar{\mathbf{r}} = \left(x\frac{\partial H}{\partial x}, y\frac{\partial H}{\partial y} \right)(x_0, x_0)\ . \tag{4.14}$$

The ratio $\lambda_0 = r/s$ of zeros to ones for this $\bar{\mathbf{r}}$ is a rational function of x_0. We may evaluate x_0 numerically and plug it into this rational function, but the numerics will be more accurate if we reduce algebraically first in $\mathbb{Q}[x_0]$. Specifically, we may first reduce the rational function to a polynomial by inverting the denominator modulo the minimal polynomial for x_0 (Maple's `gcdex` function) then multiplying by the numerator and reducing again. Then, from the representation $\lambda_0 = P(x_0)$, we may produce the minimal polynomial for λ_0 by writing the powers of λ_0 all as polynomials of degree 9 in x_0 and determining the linear relation holding among the powers $\lambda_0^0, \ldots, \lambda_0^{10}$. We find in the end that $\lambda_0 = 1.059834\ldots$. Thus indeed, there is a slight bias toward zeros.

Suppose we wish to know how long a string must be before the count of of zeros and ones tells us whether the string was sampled from the subsequence-avoiding measure versus the uniform measure. We may answer this by means of the local central limit behavior described in Theorem 2.17. We may verify that the proportion of zeros is distributed as

$$\frac{\lambda_0}{\lambda_0 + 1} + cn^{-1/2} N(0,1)$$

where $N(0,1)$ denotes a standard normal. Once we have computed the constant c, we will know how big n must be before $\frac{\lambda_0}{\lambda_0+1} - \frac{1}{2} \gg cn^{-1/2}$ and the count of zeros and ones will tip us off as to which of the two measures we are seeing.

5. Lagrange inversion

Suppose that a univariate generating function $f(z)$ satisfies the functional equation $f(z) = z\phi(f(z))$ for some function ϕ analytic at the origin and not vanishing there. Such functions often arise, among other places, in graph and tree enumeration problems. If ϕ is a polynomial, then f is algebraic, but even in this case it may not be possible to solve for f explicitly. A better way to analyse f is via the Lagrange inversion formula. One common formulation states [GJ83, Thm 1.2.4] that

$$[z^n]f(z) = \frac{1}{n}\left[y^{n-1}\right]\phi(y)^n \tag{5.1}$$

where $[y^n]$ denotes the coefficient of y^n.

To evaluate the right side of (5.1), we look at the generating function

$$\frac{1}{1 - x\phi(y)} = \sum_{n=0}^{\infty} x^n \phi(y)^n$$

which generates the powers of ϕ. The $x^n y^{n-1}$ term of this is the same as the y^{n-1} term of of $\phi(y)^n$. In other words,

$$[z^n]f(z) = \frac{1}{n}\left[x^n y^n\right]\frac{y}{1 - x\phi(y)}\,. \tag{5.2}$$

This is a special case of the more general formula for $\psi(f(z))$:

$$[z^n]\psi(f(z)) = \frac{1}{n}\left[x^n y^n\right]\frac{y\psi'(y)}{1 - x\phi(y)}\,. \tag{5.3}$$

These formulae hold at the level of formal power series, and, if ψ and ϕ have nonzero radius of convergence, at the level of analytic functions.

We can now apply the analysis leading to (3.7) (taking note that the function v in that equation is presently called ϕ, while the function ϕ there is here called $y\psi'(y)$). Recall the definitions of μ and σ^2 from Section 3.3. We are interested only in coefficients of $x^n y^n$, that is, the diagonal direction. Since ϕ vanishes to order 0 at 0, we can obtain such a diagonal asymptotic using Proposition 3.1 only if ϕ is of order at least 2 at ∞.

The complementary case requires different methods. Suppose that $\phi(z) = az + b$ with $a, b \geq 0$. Then $f(z) = az/(1 - bz)$ and $[z^n]f(z) = ab^{n-1}$. The coefficients of $\psi(az/(1 - bz))$ can usually be computed easily for a given ψ.

On the other hand, if ϕ does have order at ∞ at least 2, we set $\mu(\phi; y) := y\phi'/\phi$ equal to 1; geometrically, we graph $\phi(y)$ against y and ask that the secant line from the origin to the point $(y, \phi(y))$ be tangent to the graph there. Letting y_0 denote a solution to this, we then have a point $(1/\phi(y_0), y_0)$ in the set crit_1 and at this point the quantity $x^{-n}y^{-n}$ is equal to $(\phi(y_0)/y_0)^n = \phi'(y_0)^n$. In equations (3.4) and (3.5) we have $\mu(\phi; y_0) = 1$ and consequently $\sigma^2(\phi; y_0)$ simplifies to $y_0^2\phi''(y_0)/\phi(y_0)$. Putting this together with the asymptotic formula (3.7), setting $r = s = n$ and simplifying leads to the following proposition. Note that it is easily shown that f is aperiodic if and only if ϕ is.

Proposition 5.1. *Let ϕ be analytic and nonvanishing at the origin, aperiodic with nonnegative coefficients, with degree at least 2. Let f be the positive series satisfying $f(z) = z\phi(f(z))$. Let $y = y_0$ be the positive solution of $y\phi'(y) = \phi(y)$. Then if ψ has radius of convergence strictly greater than y_0, we have*

$$[z^n]\psi(f(z)) \sim \phi'(y_0)^n n^{-3/2} \sum_{k\geq 0} b_k n^{-k} \tag{5.4}$$

where $b_0 = \dfrac{y_0\psi'(y_0)}{\sqrt{2\pi\phi''(y_0)/\phi(y_0)}}$. □

A variant of Proposition 5.1, proved by other means, is found in [FS05, Thm VI.4]. That result is stronger than Proposition 5.1 in some ways, which is not surprising as the former is based on dedicated univariate methods. For example, it can handle the estimation of $[z^n]T(z)(1 - T(z))^{-2}$ where $T(z) = z\exp(T(z))$, which occurs in the study of random mappings, whereas an attempt to use Proposition 5.1 directly runs into the problem that $y_0 = 1$ and ψ has radius of convergence 1.

We can however derive the following result easily. A **combinatorial class** is a set stratified as $X = \bigcup_n X_n$ where each X_n is finite; the enumerating generating function is $\sum_n |X_n| z^n$ in the unlabelled case or $\sum_n |X_n| z^n/n!$ in the labelled case. The size of an element of X_n is defined to be n, and this notion extends in a consistent way to sequences or sets of elements from X.

Proposition 5.2. *Let X be a combinatorial class with enumerating generating function $f(z) = \sum a_n z^n$, satisfying the Lagrange inversion equation $f(z) = z\phi(f(z))$ as above. Let $y = y_0$ be the positive solution of $\mu(\phi; y) = 1$.*

Let b_n denote, respectively, the number of sequences (unlabelled case) or sets (labelled case) constructed from X and having total size n. Then we have $b_n/a_n \sim C$ where

$$\begin{cases} C = (1-y_0)^{-2} & \text{for sequences;} \\ C = e^{y_0} & \text{for sets.} \end{cases}$$

Proof. Let $\psi(z)$ denote the ordinary generating function $(1-z)^{-1}$ in the case of sequences of unlabelled objects, and the exponential generating function e^z in the case of sets of labelled objects. We need to show that y_0 does not exceed the radius of convergence of ψ. The set case is trivial. In the sequence case, note that the radius of convergence of ϕ is $y_0/\phi(y_0) < 1$. Since there is a solution inside the disk of convergence of ϕ, it follows that $y_0 < 1$. Applying Proposition 5.1 we can estimate the number (possibly adjusted by a factor of $1/n!$) of sequences (respectively sets) of total size n. It results that asymptotically, the ratio of the number of sequences (respectively sets) of size n to the number of elements of the original class of size n is simply $\psi'(y_0)$. This yields the stated formulae in the respective cases. □

For example, the number of ordered forests of general plane trees is asymptotically 4 times the number of plane trees of the same size, in accordance with the exact ratio C_n/C_{n-1}.

We now discuss bivariate asymptotics. In the special case $\psi(y) = y^k$, for fixed k, the above results on Lagrange inversion yield the first order asymptotic

$$[z^n]f(z)^k \sim \frac{k}{n}\phi'(y_0)^n \frac{y_0^k}{\sqrt{2\pi n \phi''(y_0)/\phi(y_0)}}$$

where $y_0\phi'(y_0) = \phi'(y_0)$. We can also derive an asymptotic as both n and k approach ∞. We sketch an argument here (see [Wil05] for details). The formula (3.7) supplies asymptotics whenever n/k belongs to a compact set of the interior of (A, B), where A is the order of f at 0 and B the order at ∞. In the present case we have $A = 1$ and $B = \infty$. We may then use the defining relation for $f(z)$ to express the formula in terms of ϕ only, leading to the following result.

Proposition 5.3. *Let ϕ be analytic and nonvanishing at the origin, with nonnegative coefficients, aperiodic and of order at least 2 at infinity. Let f be the positive series* *satisfying* $f(z) = z\phi(f(z))$. *For each* n, k, *set* $\lambda = k/n$ *and let* $y = y_\lambda$ *be the positive real solution of the equation* $\mu(\phi; y) = (1 - \lambda)$. *Then*

$$(5.5)\quad [z^n]f(z)^k \sim \lambda\phi(y_\lambda)^n y_\lambda^{k-n}\frac{1}{\sqrt{2\pi n\sigma^2(\phi; y_\lambda)}} = \lambda(1-\lambda)^{-n}\phi'(y_\lambda)^n \frac{y_\lambda^k}{\sqrt{2\pi n\sigma^2(\phi; y_\lambda)}}.$$

Here μ *and* σ^2 *are given by equations (3.4) and (3.5) respectively. The asymptotic approximation holds uniformly provided that* λ *lies in a compact subset of* $(0, 1)$. □

The similarity between (5.4) and (5.5) seems to indicate that a version of Proposition 5.3 that holds uniformly for $0 \le k/n \le 1-\varepsilon$ should apply. This is consistent with a result of Drmota [Drm94] but is seemingly more general, and warrants further study. Such a result can very likely be obtained using the extension by Lladser [Lla03] of the methods of [PW02], as can results of Meir and Moon [MM90] and Gardy [Gar95] related to Proposition 5.3.

5.1. **Trees in a simple variety.** We now discuss a well-known situation [FS05, VII.2] in which the foregoing results can be applied.

Consider the class of ordered (plane) unlabelled trees belonging to a so-called **simple variety**, namely a class $\mathcal{W}$ defined by the restriction that each node may have a number of children belonging to a fixed subset Ω of $\mathbb{N}$. Some commonly used simple varieties are: regular d-ary trees, $\Omega = \{0, d\}$; unary-binary trees, $\Omega = \{0, 1, 2\}$; general plane trees, $\Omega = \mathbb{N}$. The generating function $f(z)$ counting trees by nodes satisfies $f(z) = z\omega(f(z))$ where $\omega(y) = \sum_{t\in\Omega} y^t$.

Provided that ω is aperiodic, Proposition 5.1 applies. The form of ω shows that the equation $y\omega'(y) = \omega(y)$ always has a unique solution strictly between 0 and 1. As a simple example, we compute the asymptotics for the number of general plane trees with n nodes (the exact answer being the Catalan number C_{n-1}). The equation $y\phi'(y) = \phi(y)$ has solution $y = 1/2$, corresponding to $\phi'(y) = 1/4$. Thus we obtain

$$[z^n]f(z) \sim 4^{n-1}\frac{1}{\sqrt{\pi n^3}}$$

in accordance with Stirling's approximation applied to the expression of C_{n-1} in terms of factorials.

Proposition 5.1 does not directly apply to the case of regular d-ary trees. Rather, we use a version where the contribution from several minimal points on the same torus must be added. The details are as follows. The equation $\mu(\phi; y) = 1$ has solutions $\omega\rho$ where $\rho = (d-1)^{-1/d}$ and $\omega^d = 1$. Each of these is a contributing critical point and the corresponding leading term asymptotic contribution is $C\omega^{(n-1)}\alpha^n n^{-3/2}$ where $C = \rho^2(2\pi(d-1))^{-1/2}$ and $\alpha^d = d^d/((d-1)^{d-1})$. Thus the asymptotic leading term is 0 unless d divides $n-1$, in which case it is $dC\alpha^n n^{-3/2}$.

We note in passing that one can avoid the last computation as follows [FS05, p. 57]. There is a bijection between the class of d-ary trees and the class $\mathcal{C}$ of trees of degree at most d with $\binom{d}{j}$ types of nodes of degree j. The pruning map removes all external nodes (nodes of degree 0) from a d-ary tree. The number m of internal nodes and total number of nodes n are easily seen to satisfy $n = 1 + dm$. The extension of the above asymptotics to a multiset of degrees (allowing for different types of children) is straightforward. Thus we can compute using the degree enumerator $g(z) = (1+z)^d$ of $\mathcal{C}$ in Proposition 5.1, and $[z^n]f(z) = [z^m]g(z)$.

Now we consider the **mean degree profile** of trees in $\mathcal{W}$. Let $\xi_k(t)$ be the number of nodes of degree k in the tree t, and $|t|$ the total number of nodes in t. A standard calculation [FS05, p. 356] shows that the cumulative generating function is

$$\sum_{t \in \mathcal{W}} z^{|t|}\xi_k(t) = z^2\phi_k f(z)^{k-1}\phi'(f(z))$$

where $\phi_k = [y^k]\phi(y)$. Thus we have

$$F(z,u) := \sum_{k \geq 0}\sum_{t \in \mathcal{W}} \xi_k(t) = \mu(f; z)z\phi(uf(z))\,.$$

The mean number of nodes of degree k in a uniformly randomly chosen tree of size n from $\mathcal{W}$ is then given by

$$M_{nk} = \frac{[z^n u^k]F(z,u)}{[z^n]f(z)}.$$

Consider again the simplest case, general plane trees, with $\phi(y) = (1-y)^{-1}$. Then $F(z,u)$ corresponds to a Riordan array. A simple variant of Proposition 5.3 shows that

$$[z^n u^k]F(z,u) \sim y^k\phi'(y)^{n-1}\frac{1}{\sqrt{2\pi n\sigma^2(\phi; y)}}$$

where $\mu(\phi; y) = 1 - k/n$. This is easily solved to obtain $y = (n-k)/(2n-k)$ and hence we obtain routinely:

$$M_{nk} \sim n \left(\frac{2n-k}{2n}\right)^{2n-2} \left(\frac{n-k}{2n-k}\right)^{k} \sqrt{\frac{n^2}{2(n-k)(2n-k)}} \tag{5.6}$$

uniformly as long as k/n is in a compact subset of $(0,1)$. The mean number of leaves (nodes of degree 0) in such a tree is well known [FS05, p. 149] to be $n/2$, which is obtained by substituting $k = 0$ in the right side of (5.6). Thus it again seems likely that the approximation is in fact uniform on $[0, 1-\varepsilon]$.

5.2. **Cores of planar graphs.** A **rooted planar map** is a graph with a distinguished edge (the root) that can be embedded in the plane. A rooted planar map is completely specified by a graph, a distinguished edge, and a cyclic ordering of edges around each vertex. The **core** of a map (henceforth always a rooted planar map) is the largest 2-connected subgraph containing the root edge. The problem of the typical core size (cardinality of the core) of a map is considered in [BFSS01]. The obtain a functional equation for the generating function

$$M(u,z) = \sum_{n,k} a_{n,k} z^n u^k$$

where $a_{n,k}$ is the number of maps with n edges and core size k (that is, the core has cardinality k). They are interested in computing the probability distribution of the core size of a map sampled uniformly from among all maps of with n edges. Applying the general inversion formula (5.3) to their functional equations, they are able to compute

$$p(n,k) := \frac{a_{n,k}}{\sum_j a_{n,j}} = \frac{k}{n}[z^{n-1}]\psi'(z)\psi(z)^{k-1}\phi(z)^n$$

where $\psi(z) = (z/3)(1-z/3)^2$ and $\phi(z) = 3(1+z)^2$ are the operators that arise in the functional equation for $M(u,z)$.

To arrive directly at an asymptotic formula for $p(n,k)$ we rewrite this as

$$p(n,k) = \frac{k}{n}[x^k y^n z^n] \frac{xz\psi'(z)}{(1-x\psi(z))(1-y\phi(z))}\,.$$

The analysis of the resulting trivariate generating function is quite challenging. In particular, there is a point where Q vanishes; a generalization of Theorem 2.9 [PW02, Theorem 3.3] tells us that the asymptotics in this precise direction, but does not answer the more interesting question of asymptotics in a scaling window near that direction, which turns out to be $k = n/3 \pm n^{-2/3}$. A complete answer is given in [BFSS01] by reductions to a one-variable generating function, on which a coalescing-saddle approximation is used. Another answer, in the framework of [PW02], is given in the as yet unpublished doctoral work of M. Lladser [Lla03]. We will not go into details here.

6. The kernel method

The kernel method is a means of producing a generating function for an array $\{a_{\mathbf{r}} : \mathbf{r} \in \mathbb{N}^d\}$ of functions satisfying a linear recurrence

$$a_{\mathbf{r}} = \sum_{\mathbf{s} \in E} c_{\mathbf{s}} a_{\mathbf{r}-\mathbf{s}} \,. \tag{6.1}$$

Here E is a finite subset of $\mathbb{Z}^d$ (but not necessarily of $\mathbb{N}^d$) not containing $\mathbf{0}$, the numbers $\{c_{\mathbf{s}} : s \in E\}$ are constants, and the relation (6.1) holds for all $\mathbf{r}$ except those in the **boundary condition**, which will be made precise below. As usual, let $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. In one variable F is always a rational function, but in more than one variable the generating function can be rational, algebraic, D-finite, or differentially transcendental (not D-finite). A classification along these lines, determined more or less by the number of coordinates in which points of E can be negative, is given in [BMP00]. This is a very nice exposition of the kernel method at an elementary level.

We are interested in the kernel method because it often produces generating functions to which Theorem 2.9 may be applied. Because the method is not all that well known, we include a detailed description, drawing heavily on [BMP00]. We begin, though, with an example.

6.1. **A random walk problem.**

Two players move their tokens toward the finish square, flipping a coin each time to see who moves forward one square. At present the distances to the finish are $1 + r$ and $1 + r + s$. If the second player passes the first player, the second player wins; if the first player reaches the finish square, the first player wins; if both players are on the square before the finish square, it is a draw. What is the probability of a draw?

Let a_{rs} be the probability of a draw, starting with initial positions $1+r$ and $1+r+s$. The recursion is

$$a_{rs} = \frac{a_{r,s-1} + a_{r-1,s+1}}{2}$$

which is valid for all $(r,s) \geq (0,0)$ except for $(0,0)$, provided that we define a_{rs} to be zero when one or more coordinate is negative. The relation $a_{rs} - (1/2)a_{r,s-1} - (1/2)a_{r-1,s+1} = 0$ suggests we multiply the generating function $F(x,y) := \sum a_{rs}x^r y^s$ by $1 - (1/2)y - (1/2)(x/y)$. To clear denominators, we multiply by $2y$: define $Q(x,y) := 2y - y^2 - x$ and compute $Q \cdot F$. We see that the coefficients of this vanish with two exceptions: the x^0y^1 coefficient corresponds is $2a_{0,0} - a_{0,-1} - a_{-1,1}$ which is equal to 2, not 0, because the recursion does not hold at $(0,0)$ (a_{00} is set equal to 1); the y^0x^j coefficients do not vanish for $j \geq 1$ because, due to clearing the denominator, these correspond to $2a_{j,-1} - a_{j,-2} - a_{j-1,0}$. This expression is nonzero since, by definition, only the third term is nonzero, but the value of the expression is not given by prescribed boundary conditions. That is, we have

$$Q(x,y)F(x,y) = 2y - h(x) \tag{6.2}$$

where $h(x) = \sum_{j\geq 1} a_{j-1,0}x^j = xF(x,0)$ will not be known until we solve for F.

This generating function is in fact a simpler variant of the one derived in [LL99] for the waiting time until the two players collide, which is needed in the analysis of a sorting algorithm. Their solution is to observe that there is an analytic curve in a neighborhood of the origin on which Q vanishes. Solving $Q = 0$ for y in fact yields two solutions, one of which, $y = \xi(x) := 1 - \sqrt{1-x}$, vanishes at the origin. Since ξ has a positive radius of convergence, we have, at the level of formal power series, that $Q(x,\xi(x)) = 0$, and substituting $\xi(x)$ for y in (6.2) gives

$$0 = Q(x,\xi(x))F(x,\xi(x)) = 2\xi(x) - h(x)\,.$$

Thus $h(x) = 2\xi(x)$ and

$$F(x,y) = 2\frac{y-\xi(x)}{Q(x,y)} = \frac{1}{1+\sqrt{1-x}-y}\,.$$

As is typical of the kernel method, the generating function F has a pole along the branch of the kernel variety $\{Q(x,y) = 0\}$ that does not pass through the origin. The function F is not meromorphic everywhere, having a branch singularity on the line $x = 1$, but it is meromorphic in neighborhoods of polydisks $\mathbf{D}(x,y)$ for minimal points $(x, 1+\sqrt{1-x})$ on the pole variety for $0 < x < 1$. For $0 < x < 1$, $\mathbf{dir}(x, 1+\sqrt{1-x}) = (2\sqrt{1-x}+2-2x)/x$. If we set this equal to λ and solve for x we find $x = 4(1+\lambda)/(2+\lambda)^2$ and $1+\sqrt{1-x} = (2+2\lambda)/(2+\lambda)$ In other words,

$$\mathsf{contrib}_{\overline{\mathbf{r}}} = \left(\frac{4(1+\lambda)}{(2+\lambda)^2}, \frac{2(1+\lambda)}{2+\lambda}\right)$$

where $\lambda = s/r$. Plugging this into (2.6) gives

$$\begin{aligned} a_{rs} &\sim C(r+s)^{-1/2}\left(\frac{4r(r+s)}{(2r+s)^2}\right)^{-r}\left(\frac{2(r+s)}{2r+s}\right)^{-s} \\ &= \frac{C}{2^{2r+s}}\frac{(2r+s)^{2r+s}}{r^r(r+s)^{(r+s)}}\,. \end{aligned}$$

One recognizes in this formula the asymptotics of the binomial coefficient $\binom{2r+s}{r}$ and indeed the binomial coefficient may be obtained via a combinatorial analysis of the random walk paths.

6.2. **Explanation of the kernel method.** Because of the applicability of Theorem 2.9 to generating functions derived from the kernel method, we now give a short explanation of this method. We adopt the notation from the first paragraph of this section.

Let $\mathbf{p}$ be the coordinatewise infimum of points in $E \cup \{\mathbf{0}\}$, that is the greatest element of $\mathbb{Z}^d$ such that $\mathbf{p} \le \mathbf{s}$ for every $\mathbf{s} \in E \cup \{\mathbf{0}\}$. Let

$$Q(\mathbf{z}) := \mathbf{z}^{-\mathbf{p}}\left(1 - \sum_{\mathbf{s}\in E} c_{\mathbf{s}}\mathbf{z}^{\mathbf{s}}\right),$$

where the normalization by $\mathbf{z}^{-\mathbf{p}}$ guarantees that Q is a polynomial but not divisible by any z_j. The relation (6.1) is assumed to hold at all points $\mathbf{r} \ge \mathbf{q}$ (except the origin if $\mathbf{q} = \mathbf{0}$), where $\mathbf{q} \ge -\mathbf{p}$ and it is assumed that the boundary conditions $\{a_{\mathbf{r}} : \mathbf{r} \not\ge \mathbf{q}\}$ are specified to be some values $\{b_{\mathbf{r}} : \mathbf{r} \not\ge \mathbf{q}\}$.

If $E \subseteq \mathbb{N}^d$ then $\mathbf{p} = \mathbf{0}$, $\mathbf{q}$ can be arbitrary, and F is a function of the form G/Q, with G rational if the boundary conditions are rational. The analysis in this case is straightforward and the kernel method yields only what may be derived directly from the recursion for $a_{\mathbf{r}}$ in terms of $\{a_{\mathbf{s}} : \mathbf{s} \le \mathbf{r}\}$. We concentrate instead on the case where $d = 2$ and the second coordinate of points in E may be negative. It is known in this case [BMP00, Theorem 13] that if the generating function for the boundary conditions is algebraic then F is algebraic. On the other hand, we shall see that an outcome of the kernel method is that F will have a pole variety, and will usually satisfy the meromorphicity condition in the remark after Theorem 2.6.

To apply the kernel method, one examines the product QF_q, where for convenience we have let $F_q := \sum_{\mathbf{r}\geq\mathbf{q}} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}-\mathbf{q}}$ be the generating function for those values for which the recursion (6.1) holds. There are two kinds of contribution to QF_q. Firstly, for every pair $(\mathbf{r},\mathbf{s})$ with $\mathbf{s}\in E$, $\mathbf{r}\geq\mathbf{q}$ and $\mathbf{r}-\mathbf{s}\not\geq\mathbf{q}$, there is a term $c_{\mathbf{s}}b_{\mathbf{r}-\mathbf{s}}\mathbf{z}^{\mathbf{r}-\mathbf{p}}$. Secondly, for every pair $(\mathbf{r},\mathbf{s})$ with $\mathbf{s}\in E, \mathbf{r}-\mathbf{s}\geq\mathbf{q}$ and $\mathbf{r}\not\geq\mathbf{q}$ there is a term $-c_{\mathbf{s}}a_{\mathbf{r}-\mathbf{s}}\mathbf{z}^{\mathbf{r}-\mathbf{p}}$. Let $K(\mathbf{z})$ denote the sum of the first type of term and $U(\mathbf{z})$ denote the sum over the second kind of term. Then $K(\mathbf{z})$ encapsulates the boundary conditions. The function $U(\mathbf{z})$ refers to values $a_{\mathbf{r}}$ for which $\mathbf{r}\geq\mathbf{q}$ so it may not be freely assigned. In fact it is not hard to show that for any dimension d and any E whose convex hull does not contain a neighborhood of the origin, the following result holds.

Proposition 6.1 ([BMP00, Theorem 5]). *Let E, $c_{\mathbf{s}}$, $\mathbf{p}$, $\mathbf{q}$ and $\{b_{\mathbf{s}}\}$ be as above, and let*

$$K(\mathbf{z}) := \sum_{\mathbf{r}\geq\mathbf{q},\mathbf{s}\in E,\mathbf{r}-\mathbf{s}\not\geq\mathbf{q}} c_{\mathbf{s}}b_{\mathbf{r}-\mathbf{s}}\mathbf{z}^{\mathbf{r}-\mathbf{p}}$$

be determined from these data. Then there is a unique set of values $\{a_{\mathbf{r}}:\mathbf{r}\geq\mathbf{q}\}$ for such that (6.1) holds for all $\mathbf{r}\geq\mathbf{q}$. Consequently, there is a unique pair of formal power series F and U such that

$$QF_q = K - U\,.$$

□

In other words, the unknown power series U is determined from the data along with F_q. Another way of thinking about this is that F_q is trying to be the power series K/Q but since Q vanishes at the origin, one must subtract some terms from K to cancel the function Q_0 where $Q = Q_0Q_1$ and Q_0 consists of the branches of Q passing through the origin. The kernel method, as presented in [BMP00] turns this intuition into a precise statement.

Proposition 6.2 ([BMP00, equation (24)]). *Suppose $d=2$ and $\mathbf{p}=(0,-p)$. Suppose that $K(x,y)=x^p$. There will be exactly p formal power series $\xi_1,\ldots,\xi_p$ such* *that* $\xi_j(0) = 0$ *and* $Q(x, \xi_j(x)) = 0$*, and we may write* $Q(x, y) = -C(x) \prod_{j=1}^{p}(y - \xi_j(x)) \prod_{j=1}^{P}(y - \rho_j(x))$ *for some* r *and* $\rho_1, \ldots, \rho_P$*. The generating function* F_q *will then be given by*

$$F_q(x, y) = \frac{K(x, y) - U(x, y)}{Q(x, y)} = \frac{\prod_{j=1}^{p}(y - \xi_j(x))}{Q(x, y)} = \frac{1}{-C(x) \prod_{j=1}^{P}(y - \rho_j(x))}.$$

□

We turn to some examples.

6.3. **Dyck, Motzkin, Schröder, and generalized Dyck paths.** Let E be a set $\{(r_1, s_1), \ldots, (r_k, s_k)\}$ of integer vectors with $r_j > 0$ for all j and $\min_j s_j = -p < 0 < \max_j s_j = P$. The generalized Dyck paths with increments in E to the point (r, s) in the first quadrant are the paths from $(0, 0)$ to (r, s), with increments in E, which never go below the horizontal axis.

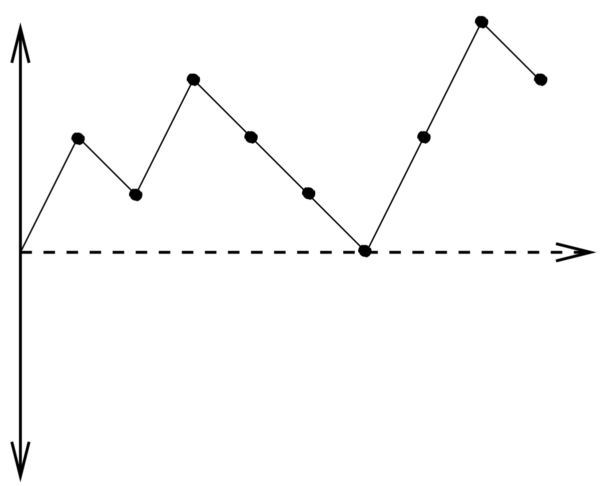

FIGURE 6. a generalized Dyck path of length nine with $E = \{(1, 2), (1, -1)\}$

Let $F(x, y) = \sum_{r,s} a_{rs} x^r y^s$ where a_{rs} is the number of generalized Dyck paths to the point (r, s); note that since $\mathbf{q} = (0, 0)$, we have $F = F_q$. Several well known instances of such paths are given in [BMP00], as described below.

Here we have $Q(x, y) = y^p(1 - \sum_i x^{r_i} y^{s_i})$ and $C(x) = \sum_{i:s_i = P} x^{r_i}$. The special case $p = P = 1$ deserves mention, since it occurs often in classical examples.

Proposition 6.3. *Let $E, p, P, C(x)$ be as above and suppose that $p = P = 1$. Then the generating function for generalized Dyck paths with steps from E is of Riordan array type. Specifically, we have*

$$F(x,y) = \frac{\xi(x)/a(x)}{1 - [C(x)\xi(x)/a(x)]y},$$

where $a(x) = \sum_{i:s_i=-1} x^{r_i}$.

Proof. Here, Q is quadratic in y and we may simplify the formula for F as follows. The product $\xi\rho$ equals $a(x)/C(x)$ where $a(x) = \sum_{i:s_i=-1} x^{r_i}$, and hence

$$F(x,y) = \frac{\xi(x)/a(x)}{1 - [C(x)\xi(x)/a(x)]y},$$

a generating function of Riordan type. □

Now consider the special case with all $r_i = 1$, so that Q has the simpler form $Q(x,y) = y^p(1 - x\sum_i y^{s_i})$ and $C(x) = x$. Each generalized Dyck path is then simply the graph of a walk on $\mathbb{Z}$ with steps s_i (the x-axis denotes time). A generalized Dyck path ending at (r, s) corresponds to a walk on $\mathbb{Z}$ of length r and final position s. If, furthermore $p = P = 1$, then we may determine ξ via the Lagrange inversion equation $\xi = x\sum_i \xi(x)^{s_i}$. See [FS05, VII.5.2] for the relation with trees from a simple variety. In addition, if the step set is symmetric with respect to the x-axis (as in the first two examples below), then $a(x) = C(x)$ since $C(x) = \sum_{i:s_i=1} x^{r_i}$. In this case F corresponds to a special case of a Riordan array, called a **renewal array**. We now discuss the three standard examples from [BMP00].

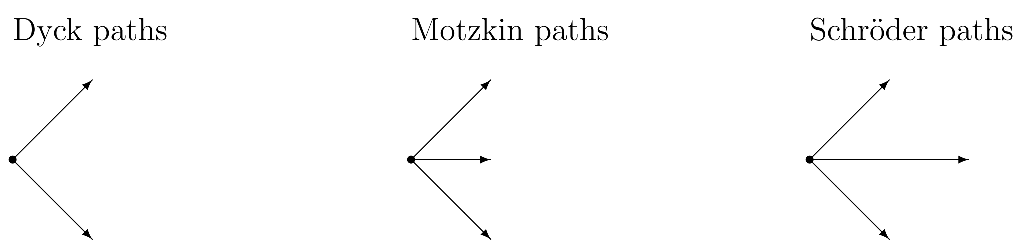

FIGURE 7. legal steps for three types of paths

Dyck paths: When $E = \{(1,1),(1,-1)\}$ we have the original Dyck paths. We have $p = 1 = P$ and $Q(x,y) = y - xy^2 - x$. Here $C(x) = x$, and $Q(x,y) = -x(y - \xi(x))(y - \rho(x))$ where $\xi(x) = (1 - \sqrt{1-4x^2})/(2x)$ and $\rho(x) = (1 + \sqrt{1-4x^2})/(2x)$ is the algebraic conjugate of ξ. Note that ρ is a formal Laurent series and $\rho\xi = 1$.

Thus we have, following the discussion above,

$$F(x,y) = \frac{1}{-x(y - \rho(x))} = \frac{\xi(x)/x}{1 - y\xi(x)}.$$

Setting $y = 0$ recovers the fact that the Dyck paths coming back to the x-axis at $(2n, 0)$ are counted by the Catalan number C_n.

Asymptotics are readily obtained either using the explicit or implicit form of ξ (noting the periodicity of ξ). Let us use the implicit form in this example, since we will be illustrating use of the explicit form in the case of Schröder paths, where Lagrange inversion does not apply. The vanishing of $Q(x,y) = y - xy^2 - x$ occurs when $y = x\phi(y)$ with $\phi(y) = 1 + y^2$. We use a slight variant of (5.5), namely

$$[z^n]\xi(z)^{k+1}/z \sim \lambda\phi(y_\lambda)^{n+1} y_\lambda^{k-n} \frac{1}{\sqrt{2\pi n\sigma^2(\phi; y_\lambda)}}$$

where $\lambda = k/n$ and $\mu(\phi; y) = 1 - \lambda$. The chain rule yields $\mu(\phi; y) = \mu(y^2; y)\mu(1+t; t)$ with $t = y^2$. Thus we solve $1 - \lambda = 2t/(1+t)$, or $y^2 = (1-\lambda)/(1+\lambda) = (r-s)/(r+s)$. The two contributing points y_λ cancel out if $r - s$ is odd and reinforce if $r - s$ is even. We compute $\phi(y_\lambda) = 2/(1+\lambda) = 2r/(r+s)$ and $\sigma^2(\phi; y_\lambda) = 1 - \lambda^2 = (r^2 - s^2)/r^2$. We obtain

$$\begin{aligned} a_{rs} &\sim 2\,\frac{s}{r}\left(\frac{2r}{r+s}\right)^{r+1}\left(\frac{r-s}{r+s}\right)^{\frac{s-r}{2}} \frac{\sqrt{r}}{\sqrt{2\pi(r-s)(r+s)}} \\ &= \frac{2s}{(r+s)} \frac{r^r}{(\frac{r+s}{2})^{\frac{r+s}{2}}(\frac{r-s}{2})^{\frac{r-s}{2}}} \frac{\sqrt{r}}{\sqrt{2\pi(\frac{r-s}{2})(\frac{r+s}{2})}} \end{aligned}$$

provided $r - s$ is even, and 0 otherwise. This is uniform for $0 < \delta \le s/r \le 1 - \varepsilon < 1$.

Of course, in this simple example Lagrange inversion also gives an exact formula involving binomial coefficients, namely

(6.3)

$$a_{rs} = [x^{r+1}]\xi(x)^{s+1} = \frac{s+1}{r+1}[y^{r-s}](1+y^2)^{r+1} = \frac{s+1}{r+1}\binom{r+1}{(r-s)/2} = \frac{2(s+1)}{r+s+2}\frac{r!}{(\frac{r-s}{2})!(\frac{r+s}{2})!}$$

when $r - s$ is even, and 0 otherwise. If we had instead computed $[z^{r+1}]\xi(z)^{s+1}$ using (5.5), we would have obtained a correct leading order asymptotic of a slightly different form (r, s replaced by $r+1, s+1$ in some places). In each case these asymptotics are consistent with what is obtained by applying Stirling's formula to the factorials in (6.3).

Motzkin paths: Let $E = \{(1,1),(1,0),(1,-1)\}$. In this case the generalized Dyck paths are known as **Motzkin paths**. Again we have case $Q(x,y) = y - xy^2 - x - xy$. Now ρ and ξ are given by $(1 - x \pm \sqrt{1-2x-3x^2})/(2x)$ and again

$$F(x,y) = \frac{\xi(x)/x}{1 - y\xi(x)} = \frac{2}{1 - x + \sqrt{1-2x-3x^2} - 2xy}\,.$$

This time ξ is given implicitly by $\xi = x(1+\xi+\xi^2)$ and the coefficients are not binomial coefficients, but the asymptotics are no harder to compute. Here, with $\lambda = s/r$, we have that $\mathsf{contrib}_{1-\lambda}$ is a singleton $\{y_0\}$ by aperiodicity. We obtain the same shape asymptotics as the previous example, with

$$y_0 = \frac{\sqrt{4-3\lambda^2} - \lambda}{2(1+\lambda)}\,.$$

Schröder paths: Here $E = \{(1,1),(2,0),(1,-1)\}$. We have $C(x) = x, Q(x,y) = y - xy^2 - x^2y - x$, and ρ and ξ are given by $(1 - x^2 \pm \sqrt{1-6x^2+x^4})/(2x)$. This time Lagrange inversion does not obviously apply. We perform an explicit computation, noting the periodicity of F. We have $\mathbf{dir}(x,y) = \overline{\mathbf{r}}$ with

$$\frac{s}{r} = \frac{\sqrt{1-6x^2+x^4}}{1+x^2}.$$

This decreases from 1 to 0 as x increases from 0 to the smaller positive root of $1 - 6x^2 + x^4$, namely $\sqrt{2} - 1$. We also have

$$x^2 = \frac{3 + \lambda^2 - 2\sqrt{(2+2\lambda^2)}}{1-\lambda^2}.$$

Choosing the positive value of x, we see that asymptotics are given by

$$a_{rs} \sim 2Cx_\lambda^{-r}y_\lambda^{-s}s^{-1/2}$$

where $y_\lambda = 1/\xi(x_\lambda) = x_\lambda\mu(x_\lambda)$, when $r+s$ is even, and 0 otherwise. Any particular diagonal (with a value of λ between 0 and 1) can be extracted easily.

6.4. **Pebble configurations.** Chung, Graham, Morrison and Odlyzlo [CGMO95] consider the following problem. Pebbles are placed on the nonnegative integer points of the plane. The pebble at (i, j) may be replaced by two pebbles, one at $(i+1, j)$ and one at $(i, j+1)$, provided this does not cause two pebbles to occupy the same point. Starting from a single pebble at the origin, it is known to be impossible to move all pebbles to infinity; in fact it is impossible to clear the region $\{(i, j) : 1 \leq i + j \leq 2\}$ [CGMO95, Lemma 2].

They consider the problem of enumerating minimal unavoidable configurations. More specifically, say that a set T is a **minimal unavoidable configuration** with respect to some starting configuration S if it is impossible starting from S to move all pebbles off of T, but pebbles may be cleared from any proper subset of T. Let S_t denote the starting configuration where (i, j) is occupied if and only if $i + j = t$. Let $f_t(k)$ denote the number of sets in the region $\{(i, j) : i, j \geq 0\,; i + j \geq t + 1\}$ that are minimal unavoidable configurations for the starting configuration S_t.

They derive the recurrence

$$f_t(k) = f_{t-1}(k) + 2f_t(k-1) + f_{t+1}(k-2)$$

which holds whenever $t \geq 3$ and $k \geq 2$. Let

$$F(x, y) = \sum_{t,k \geq 0} f_t(k) x^k y^t \,.$$

According to the kernel method, we will have $F = \eta/Q$ for some η vanishing where Q vanishes near the origin, where $Q(x, y) = x - (x + y)^2$. Using some more identities, Chung *et al.* are able to evaluate F explicitly. They state that they are primarily interested in $f_0(k)$, so they specialize to $F(x, 0)$ and compute the univariate asymptotics. It seems to us that the values $f_t(k)$ are of comparable interest, and we pursue asymptotics of the full generating function.

The formula for F is cumbersome, but its principal features are (i) a denominator of $P \cdot Q$ where P is the univariate polynomial $1 - 7x + 14x^2 = 9x^3$, and (ii) F is algebraic of degree 2 and is in $\mathcal{C}[x, y][\sqrt{1 - 4x}]$. The minimum modulus root of P is $x_0 \approx 0.2410859\ldots$. The algebraic singularity of F occurs along the line $x = 1/4$, so conveniently, the branching is completely outside the closure of the domain of convergence.

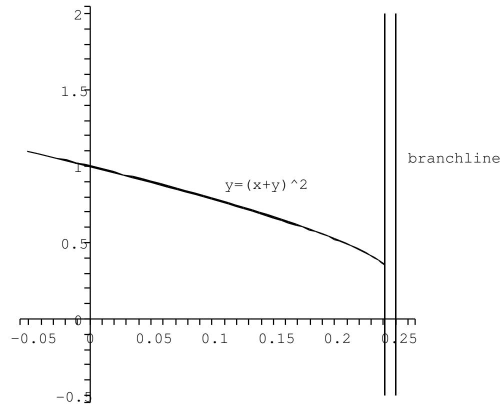

FIGURE 8. the domain of convergence and the algebraic singularity

The boundary of the domain of convergence in the first quadrant is composed of pieces of two curves, namely $x = x_0$ and $y = (x+y)^2$. These intersect at the point (x_0, y_0) where $y_0 = (1 - 2x + \sqrt{1-4x})/2 \approx 0.3533286$. We are in the combinatorial case, so we know that minimal points will be found along these curves. Along the curve $y = (x+y)^2$ the direction $\mathbf{dir}(x, y)$ is given by $\lambda = r/s = (2 - 2\sqrt{y})/(2\sqrt{y} - 1)$. As (x, y) travels from $(0, 1)$ down the curve to (x_0, y_0), λ increases from 0 to $\lambda_0 \approx 4.295798\ldots$. At the point (x_0, y_0), the cone $\mathbf{K}(x_0, y_0)$ is the convex hull of the positive x-direction the direction $\overline{(\lambda_0, 1)}$.

We then have two sorts of asymptotics for $f_t(k)$. When $t < k/\lambda_0$, the asymptotics are given by Corollary 2.14. In this case we may evaluate $G(x_0, y_0) \approx 0.00154376$ and $\sqrt{-x_0^2 y_0^2 \mathcal{H}} \approx 0.02925688$ so that

$$f_t(k) \sim C x_0^{-k} y_0^{-t} \quad \text{with } C = 0.05276\ldots$$

uniformly as t/k varies over compact subsets of $(0, 1/\lambda_0)$. It is interesting to compare to the asymptotics for $f_0(k)$. Setting $y = 0$ gives the univariate generating function [CGMO95]

$$f(x) = \sum a_n x^n = x^2 \frac{(1-4x)^{1/2}(1 - 3x + x^2) - 1 + 5x - x^2 - 6x^3}{P(x)}.$$

We may compute

$$\lim_{n\to\infty} a_n x_0^k = \lim_{x \to x_0} (x_0 - x) f(x)$$

and we find that

$$f_0(k) \sim c x_0^{-k} \quad \text{with } c = 0.016762\ldots .$$

In fact, one may calculate $\lim f_t(k) x_0^k$ for any fixed t by computing

$$f_t(x) := (t!)^{-1} \left(\frac{\partial}{\partial y} \right)^t F(x, 0) \tag{6.4}$$

and again computing $c(t) := \lim_{x \to x_0} (x_0 - x) f_t(x)$. The pole at $x = x_0$ in F is removable: replacing the denominator PQ of F by $(P/(x - x_0))Q$ we see that $g(y) := \lim_{x \to x_0} (x_0 - x) F(x, y)$ has a simple pole at $y = y_0$ and that $\lim_{t \to \infty} y_0^t c(t) = \lim_{y \to y_0} (y_0 - y) g(y)$ is equal to C. In other words, we see that the asymptotics known to hold uniformly as t/k varies over compact subsets of $(0, 1/\lambda_0)$ actually hold over $[0, 1/\lambda_0)$ as long as $t \to \infty$, while for $k \to \infty$ with t fixed we use (6.4).

On the other hand, when $t/k > 1/\lambda_0$ we may solve for x and y to get

$$\mathsf{contrib}_{\bar{\mathbf{r}}} = (x, y) := \left\{ \left(\frac{k(2t + k)}{(2t + 2r)^2}, \frac{(2t + k)^2}{(2t + 2k)^2} \right) \right\} .$$

We now use Theorem 2.9 to see that

$$f_t(k) C(\frac{t}{k}) t^{-1/2} \sim \frac{(2t + 2k)^{2t+2k}}{k^k (2t + k)^{2t+k}} .$$

The asymptotics in this case appear similar to those for the binomial coefficient $\binom{2t+2k}{k}$. As opposed to the situation with Dyck paths, one may check that $f_t(k)$ is not equal to a binomial coefficient.

7. Discussion of other methods

7.1. GF-sequence methods. The benchmark work in the area of multivariate asymptotics is still the 1983 article of Bender and Richmond [BR83]. Their main result is a local central limit theorem [BR83, Corollary 2] with the exact same conclusion as Theorem 2.18. Their hypotheses are:

(i) $a_{\mathbf{r}} \geq 0$;
(ii) F has an algebraic singularity of order $q \notin \{0, -1, -2, \ldots\}$ on the graph of a function $z_d = g(z_1, \ldots, z_{d-1})$;
(iii) F is analytic and bounded in a larger polydisk, if one excludes a neighborhood of $\mathrm{Im}\,(z_j) = 0$ for each j;

(iv) B is nonsingular.

Comparing this to the results presented herein, we find both methodological and phenomenological differences. One methodological difference is that they view a d-variate generating function F as a sequence $\{F_n\}$ of $(d-1)$-variate generating functions. Their main result on coefficients of F is derived as a corollary of a result [BR83, Theorem 2] on sequences satisfying $F_n \sim C_n g h^n$ for some appropriately smooth g and h. As we have remarked, this approach is natural for some but not all applications, and leads to some asymmetry in the hypotheses and conclusions.

A more important methodological difference is that while we always work in the analytic category, Bender and Richmond use a blend of analytic and smooth techniques[3]. This manifests itself in the hypotheses: where we require meromorphicity in a slightly enlarged polydisk, they require that the function g be in C^3, that the residue $(1 - z/g)^q F$ be in C^0, and that F be analytic away from the real coordinate planes. While our hypotheses are stronger in this regard, we know of no applications where their assumptions hold without $(1 - z/g)^q F$ being analytic. They do gain some generality by allowing q to be nonintegral. This is further exploited by Gao and Richmond, where the singularity is allowed to be algebraico-logarithmic [GR92, Corollary 3]. On the other hand, their methods entail estimates and are therefore not sufficiently robust to handle any cancellation. Consequently, they cannot leave the combinatorial case, and even there they always require strict minimality. One can see how their methods may be adapted to address periodicity (similarly to our handling a finite sum of contributions from a single torus) but the method is fundamentally unable to handle a singularity not on the boundary of the domain of convergence, such as may occur when $a_{\mathbf{r}}$ have mixed signs (see Sections 3.9 and 3.3).

Phenomenologically, there is a significant difference in generality between our methods and those of Bender, Richmond, Gao, *et al.*. Their results govern only the case where a local central limit theorem holds – indeed it seems they are interested mainly or only in this case. Other behaviors of interest, which have been analyzed in the literature by various means, include Airy-type limits (see Section 5.2), polynomial growth (see Section 3.10), and elliptic-type limits. Central limit behavior results from smooth points $\mathbf{z}(\mathbf{r})$ with nondegenerate quadratic approximations to $h_{\mathbf{r}}(\mathbf{z}(\mathbf{r}))$,

[3]Analyticity is used only once, when they rotate the quadratic form B.

while Airy-type limits result from degenerate quadratic approximations, polynomial growth or corrections result when $\mathbf{z}(\mathbf{r})$ is a multiple point, and elliptic-type limits result from bad points. By restricting our exposition to the simplest cases, we have stayed mainly within the smooth point case, where $\{a_{\mathbf{r}}\}$ obeys a LCLT and the methods of Bender *et al.* apply, in most cases equally well as the results in Section 2.3. But the advantage of multivariate analytic methods is that they can in principle be applied to any case in which F is meromorphic or algebraic. Thus, in addition to the further generality covered in Section 2.4 of this paper, the same general method has been used to produce Airy limit results [PW02; Lla03], and is being applied to algebraic generating functions and meromorphic functions with bad point singularities, the simplest of which are quadratic cones. While this work is not yet published, it appears that it will provide another proof of the elliptic limit results for tiling statistics on the Aztec Diamond [CEP96] that is capable of generalizing to any quadratic cone singularity. This would prove similar behavior in two cases where such behavior is only conjectured, namely cube groves [PS04] and quantum random walks (Chris Moore, personal communication), as well as unifying these results with the analyses of the coefficients of the generating function $1/(1 - x - y - z + 4xyz)$ that arise in the study of super ballot numbers [Ges92] and Laguerre polynomials [GRZ83].

7.2. **The diagonal method.** There is a third method for obtaining multivariate asymptotics that deserves mention, namely the so-called "diagonal method". This derives a univariate generating function $f(z) = \sum_n a_{n,n} z^n$ for the main diagonal of a bivariate generating function $F(x, y) = \sum_{rs} a_{rs} x^r y^s$. The asymptotics of $a_{n,n}$ may then be read off by standard univariate means from the function f. The method may be adapted to compute a generating function for the coefficients $a_{np,nq}$ along any line of rational slope. This method, long known in various literatures, entered the combinatorics literature in [Fur67; HK71]; our exposition is taken from [Sta99, Section 6.3].

While this elegant method produces an actual generating function, which is more informative than the diagonal asymptotics, its scope is quite limited. First, while asymptotics in any rational direction may be obtained, the complexity of the computation increases with p and q. Thus there is no continuity of complexity, and no way to obtain uniform asymptotics or asymptotics in irrational directions. Secondly, the result is strictly two-dimensional. Thirdly, even when the diagonal method may be applied, the computation is typically very unwieldy.

As an example of a diagonal computation, consider the generating function

$$F(z,w) = \sum_{m,n} w_{mn} z^m w^n = \frac{1 + zw + z^2w^2}{1 - z - w + zw - z^2w^2}$$

which [MZS04] enumerates binary words without zig-zags (a zigzag is defined to be a subword 010 or 101 — the terminology comes from the usual correspondence of such words with Dyck paths, where $0, 1$ respectively correspond to the steps $(1, 1), (1, -1)$). Here m, n respectively denote the number of 0's and 1's in the word. The main diagonal enumerates zigzag-free words with an equal number of 0's and 1's. The solutions to $H = 0, zH_z = wH_w$ are given by $z = w = 1/\phi$, $\phi = (1 + \sqrt{5})/2$, and $z = w = (1 + \sqrt{3}i)/2$. Thus `contrib` is a singleton $\{(1/\phi, 1/\phi)\}$ and the first order asymptotic is readily computed to be

$$w_{nn} \sim \phi^{2n} \frac{2\sqrt{2}}{\sqrt{n\pi\sqrt{5}}} .$$

Note that this is larger by a factor of $\sqrt{2}$ than the (incorrect) answer given in [MZS04, Sec 8].

The computation for any other diagonal is analogous, with the same amount of computational effort, and the asymptotics are uniform over any compact subset of directions keeping away from the coordinate axes.

To obtain the same result via the diagonal method requires the following steps. For each fixed t near 0, we compute the integral

$$D(z) := \sum_n w_{nn} z^n = \frac{1}{2\pi i} \int_{\mathcal{C}_t} F(z, t/z) \, \frac{dz}{z} = \frac{1}{2\pi i} \int_{\mathcal{C}_t} \frac{1 + t + t^2}{-z^2 + (1 + t - t^2)z - t}$$

where the contour is a circle that encloses all the poles of $F(z, t/z)/z$ satisfying $z(t) \to 0$ as $t \to 0$. Since $F(z, 0)/z$ has a single simple pole at $z = 0$, the same is true of $F(z, t/z)/z$ for sufficiently small t. In this simple example we can explicitly solve for the pole $z(t)$ and compute its residue so that we obtain the result

$$\sum_n w_{nn} z^n = \sqrt{\frac{1 + z + z^2}{1 - 3z + z^2}} .$$

In other cases, after some manipulation we obtain $\sum_n w_{nn} z^n$ implicitly as the solution of an algebraic equation. We are then faced with the problem of extracting asymptotics, which can probably be done using univariate techniques. However, we have left the realm of meromorphic series and this can complicate matters. In the example above, the branching occurs outside the domain of convergence, and the asymptotics are controlled by the dominant pole at ϕ^{-2} (the minimal zero of the denominator). Thus one obtains the same asymptotic as above, after some effort.

Note that if we want to repeat this computation with $\sum_n w_{pn,qn} z^n$, we are required to find all small poles of the function $F(z^q, t/z^p)$. It is unlikely that these may be found explicitly, which complicates the task of finding which ones go to zero and computing the residues there.

Finally, another serious problem faced by the diagonal method is that while the diagonal of a rational series in $d = 2$ variables is always algebraic (a fact which can itself be proved by the diagonal method), the same is not true for $d \geq 3$. Thus a description of the diagonal generating function is more challenging (the diagonal is at least D-finite). For example, consider the generating function $F(x, y, z) = (1 - x - y - z)^{-1} = \sum a_{rst} x^r y^s z^t$, whose diagonal coefficient $a_{n,n,n}$ is the multinomial coefficient $\binom{3n}{n,n,n}$. This is known not to be algebraic, since its asymptotic leading term $C\alpha^n n^{-1}$ is not of the right form for an algebraic function. It is completely routine to derive this asymptotic using the methods of the present article, but any method that relies on an exact description of the diagonal will clearly require substantial extra work.

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Department of Mathematics, 209 S. 33rd Street, Philadelphia, PA 19104 USA

E-mail address: pemantle@math.upenn.edu

Department of Computer Science, The University of Auckland, Private Bag 92019 Auckland, New Zealand

E-mail address: mcw@cs.auckland.ac.nz