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Bayesian nonparametric approaches to multivariate time series analysis

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Abstract

While there is an increasing amount of literature about Bayesian time series analysis, only few nonparametric approaches to multivariate time series exist. Most notably, [1] and [2] rely on Whittle's likelihood, involving the second order structure of the time series by means of the spectral density matrix. The latter is modeled with a smoothing splines prior for the components of the Cholesky decomposition. While these approaches are shown to perform well in many applications, their theoretical asymptotic behavior in terms of posterior consistency is not known. Consistency results are typically restricted to parameters outside a prior null set, which is unsatisfactory in infinite dimensions, or even fail entirely [3]. Having a posterior consistency result in mind, we investigate multivariate extensions of the Bernstein-Dirichlet prior from [4], for which consistency under the Gaussianity assumption has been shown in the univariate case. We also consider a multivariate extension of the corrected parametric likelihood, a generalization of Whittle's likelihood, which has recently been developed by Kirch et al [5].

Whittle's likelihood

Let $\{\underline{X}_n : n = 0, \pm 1, \ldots\}$ be a stationary and centered real d-dimensional time series with absolute summable autocovariance matrix function $\mathbf{\Gamma}(h) = \mathbb{E}[\underline{X}_t \underline{X}_{t+h}^\top]$ and spectral density

$$\mathbf{f}(\lambda) = \frac{1}{2\pi} \sum_{h\in\mathbb{Z}} \mathbf{\Gamma}(h) e^{-ih\lambda}, \quad 0 \le \lambda \le 2\pi.$$

The Fourier coefficients of $\mathbf{X}_n = (\underline{X}_1, \ldots, \underline{X}_n)^\top$ are given by

$$\tilde{\underline{X}}(\lambda_k) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \underline{X}_j e^{-ij\lambda_k}, \quad \lambda_k = 2\pi k/n$$

and the periodogram matrix is defined as

$$\mathbf{I}_{n,\lambda_k}(\mathbf{X}_n) = \frac{1}{2\pi} \tilde{\underline{X}}(\lambda_k) \tilde{\underline{X}}(\lambda_k)^*.$$

Asymptotically, the periodogram ordinates are independent Wishart distributed with asymptotic mean $\mathbf{f}(\lambda_k)$, motivating Whittle's likelihood:

$$p^W(\mathbf{x}_n|\mathbf{f}) \propto \prod_{k=1}^{n} |\det \mathbf{f}(\lambda_k)|^{-1/2} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{n} \operatorname{tr}\left(\mathbf{I}_{n,\lambda_k}(\mathbf{x}_n) \mathbf{f}(\lambda_k)^{-1} \right) \right\}$$

Full Bayesian model

Use the **Bernstein-Dirichlet** scheme [4]:

$$B(\omega|k,G) = \sum_{j=1}^{k} G\left(\left(\frac{j-1}{k}, \frac{j}{k} \right] \right) \beta(\omega|j, k-j+1), \quad 0 \le \omega \le 1,$$
$$G \sim \mathrm{DP}(\alpha G_0).$$

Prior on the **Cholesky compoments** of inverse spectral density matrix [1]:

$$\mathbf{f}(\lambda)^{-1/2} = \begin{pmatrix} 1 & & & \\ g_{2,1}(\lambda) & 1 & & \\ \vdots & & \ddots & \\ g_{p,1}(\lambda) & \ldots & g_{p,p-1}(\lambda) & 1 \end{pmatrix} \begin{pmatrix} \delta_1(\lambda) & & & \\ & \delta_2(\lambda) & & \\ & & \ddots & \\ & & & \delta_p(\lambda) \end{pmatrix}, \quad 0 \le \lambda \le \pi,$$

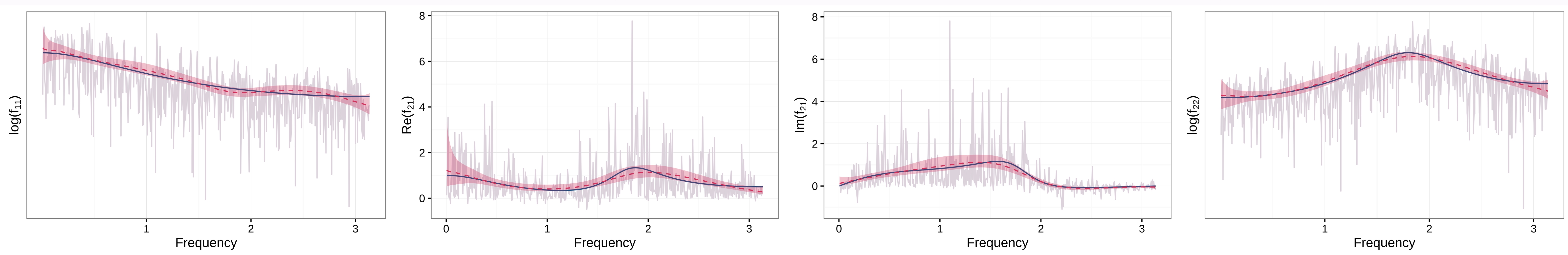
with **positive diagonals** and **unconstrained off-diagonals**:

$$\delta_r(\pi\omega) = \tau_r B_r(\omega|k_r, G_r),$$
$$g_{r,s}(\pi\omega) = \log\left\{ \tau_{r,s}^{(1)} B_{r,s}^{(1)}\left(\omega|k_{r,s}^{(1)}, G_{r,s}^{(1)}\right) \right\} + i \log\left\{ \tau_{r,s}^{(2)} B_{r,s}^{(2)}\left(\omega|k_{r,s}^{(2)}, G_{r,s}^{(2)}\right) \right\},$$

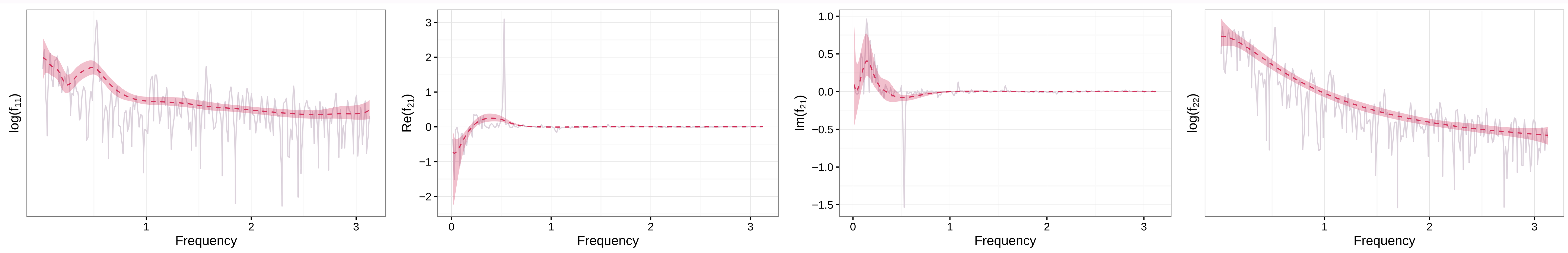
all a priori independent.

Numerical experiments

- Simulated **bivariate VAR(2)** process $\underline{X}_t = \begin{pmatrix} 0.5 & 0 \\ 0 & -0.3 \end{pmatrix} \underline{X}_{t-1} + \begin{pmatrix} 0 & 0 \\ 0 & -0.5 \end{pmatrix} \underline{X}_{t-2} + \underline{e}_t$, with normal innovations $\underline{e}_t \sim \mathrm{N}\left(\underline{\mu} = 0, \mathbf{\Sigma} = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix} \right)$ of length $n = 1024$:



- **Southern Oscillation Index** and Recruitment series [1]:



Beyond Whittle: The corrected parametric likelihood

Idea: **Nonparametric correction** of a **parametric likelihood**

time domain: $\mathbf{X}_n = (\underline{X}_1, \ldots, \underline{X}_n) \sim p_{\text{param}}$ —mDFT→ frequency domain: $(\tilde{\underline{X}}(\lambda_1), \ldots, \tilde{\underline{X}}(\lambda_n))^\top =: \mathcal{F}_n \mathbf{X}_n$

Correction $\mathbf{Q}(\lambda_k) = \mathbf{f}(\lambda_k)^{1/2} \mathbf{f}_{\text{param}}(\lambda_k)^{-1/2}$

$\mathcal{F}_n^{-1} \mathcal{Q}_n \mathcal{F}_n \mathbf{X}_n \sim p_{\text{param}}^C$ ←mDFT^{-1}— $(\mathbf{Q}(\lambda_1)\tilde{\underline{X}}(\lambda_1), \ldots, \mathbf{Q}(\lambda_n)\tilde{\underline{X}}(\lambda_n))^\top =: \mathcal{Q}_n \mathcal{F}_n \mathbf{X}_n$

$$p_{\text{param}}^C(\mathbf{x}_n|\mathbf{f}) \propto \prod_{k=1}^{n} |\det \mathbf{Q}(\lambda_k)|^{-1} p_{\text{param}}\left(\mathcal{F}_n^{-1} \mathcal{Q}_n^{-1} \mathcal{F}_n \mathbf{x}_n \right)$$

Outlook

- **Show consistency:** For **Gaussian** time series with Whittle's likelihood [4] and corrected parametric likelihood [5]
- **Other priors:** **B-splines** [6], **Smoothing splines** [1].
- **Shrinkage approaches:** Cope with the curse of dimensionality.
- **Show consistency:** **Beyond Gaussianity**.

References

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