Open and closed questions in decision-making

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1. Open and closed questions
   - Introduction
   - Questions as dynamic ceteris paribus operators
   - Some examples
   - Axiomatisation
2. Making Decisions
   - A decision operator
   - An example of questions and decisions working together
   - Characterizing transitivity of preferences
3. Complex questions
   - Cordorcet's paradox
   - The arithmetic of ceteris paribus conditions

Outline

No time for an introduction

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But we'll have some examples later.

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And this is very well-motivated; trust me!

locus classicus (sic)

G.H. von Wright. *The Logic of Preference*. Edinburgh, 1963.

Patrick Girard. *Modal Logics for Belief and Preference Change*. PhD thesis, Stanford University, 2008.

Johan van Benthem, Patrick Girard, and Olivier Roy. Everything else being equal: a modal logic for *ceteris paribus* preferences. *Journal of philosophical logic*, August 2008.

Jeremy Seligman and Patrick Girard. Being flexible about ceteris paribus, *Australasian Journal of Logic*, to appear, 2011.

Outline

Start with a standard modal semantics

$$p \quad | \quad \neg \quad | \quad \wedge \quad | \quad \Box$$

$M = \langle W, R, V \rangle$

- W = a set of possible states
- R = a relation between states
- V = a propositional valuation

Semantics

$M, u \models p$	iff	$u \in V(p)$
$M, u \models \Box\varphi$	iff	if Ruv then $M, v \models \varphi$ for each $v \in W$

Then add *ceteris paribus* conditions

$M = \langle W, R, V, \Gamma \rangle$

- Γ = a set of formulas

u is equivalent to v *ceteris paribus* when the formulas in Γ have the same value at u as at v:

$$u \approx v \quad \text{iff} \quad \text{for all } \varphi \in \Gamma \; u \models \varphi \text{ iff } v \models \varphi$$

Semantics

$M, u \models p$	iff	$u \in V(p)$
$M, u \models \Box\varphi$	iff	if $u \approx v$ and Ruv then $M, v \models \varphi$ for each $v \in W$

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$M, u \models p$	iff	$u \in V(p)$
$M, u \models \Box\varphi$	iff	if $u \approx v$ and Ruv then $M, v \models \varphi$ for each $v \in W$

Now add dynamic operators to change the *ceteris paribus* conditions

(For now: assume Γ contains only propositional variables.)

$$[!p] \quad | \quad [?p]$$

$$[!p]M = \langle W, R, V, \Gamma \cup \{p\} \rangle$$
$$[?p]M = \langle W, R, V, \Gamma \setminus \{p\} \rangle$$

Semantics

$M, u \models [!p]\varphi$ iff $[!p]M, u \models \varphi$

$M, u \models [?p]\varphi$ iff $[?p]M, u \models \varphi$

Now add dynamic operators to change the *ceteris paribus* conditions

(For now: assume Γ contains only propositional variables.)

$$[!p] \quad | \quad [?p]$$

$[!p]M$	=	$\langle W, R, V, \Gamma \cup \{p\}\rangle$	fix the value of p to the current value
$[?p]M$	=	$\langle W, R, V, \Gamma \setminus \{p\}\rangle$	allow the value of p to vary

Semantics

$M, u \models [!p]\varphi$ iff $[!p]M, u \models \varphi$

$M, u \models [?p]\varphi$ iff $[?p]M, u \models \varphi$

and generalise to sets of questions

For Q a set of propositional variables:

$$[!Q] \quad | \quad [?Q]$$

$[!Q]M$	=	$\langle W, R, V, \Gamma \cup Q\rangle$	fix the value of every variable in Q to its current value
$[?Q]M$	=	$\langle W, R, V, \Gamma \setminus Q\rangle$	allow the value of variables in Q to vary

Semantics

$M, u \models [!Q]\varphi$ iff $[!Q]M, u \models \varphi$
$M, u \models [?Q]\varphi$ iff $[?Q]M, u \models \varphi$

Outline

When we start with epistemic logic…

$[!p]M$ is the result of discovering the answer to “is p true?”

$[?p]M$ is the result of calling p into question: “what about p?”

So:

$[!p]K\varphi$	=	If I knew whether p then I would know that φ
$[?p]K\varphi$	=	I would know that φ even if I doubt whether p

and with preference logic…

$[!p]M$ is the result of accepting the answer "is p true?" as the *status quo*

$[?p]M$ is the result of calling p into question: "what about p?"

So:

$[!p]P\varphi$ = If I just accept the current status of p then I would prefer that φ.

$[?p]P\varphi$ = I would still prefer that φ if I were to consider alternatives in which p differs from its present value.

Outline

As is typical in dynamic logic, we can eliminate closed question operators easily

Closed Question Distribution

$$\begin{array}{lcl}
\vdash [!Q]p & \leftrightarrow & p \\
\vdash [!Q]\neg\varphi & \leftrightarrow & \neg[!Q]\varphi \\
\vdash [!Q](\varphi \wedge \psi) & \leftrightarrow & ([!Q]\varphi \wedge [!Q]\psi) \\
\vdash [!Q_1][!Q_2]\varphi & \leftrightarrow & [!Q_1 \cup Q_2]\varphi \\
\vdash [!Q_1][?Q_2]\varphi & \leftrightarrow & [?Q_2 \setminus Q_1][!Q_1 \setminus Q_2]\varphi
\end{array}$$

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\vdash [!Q_1][?Q_2]\varphi & \leftrightarrow & [?Q_2 \setminus Q_1][!Q_1 \setminus Q_2]\varphi
\end{array}
$$

Closed Question Reduction

$$\vdash [!p, Q]\Diamond\varphi \quad \leftrightarrow \quad (p \wedge [!Q]\Diamond(p \wedge \varphi)) \vee (\neg p \wedge [!Q]\Diamond(\neg p \wedge \varphi))$$

and open question operators are not much more trouble

Open Question Distribution

$$\begin{array}{lll} \vdash [?Q]p & \leftrightarrow & p \\ \vdash [?Q]\neg\varphi & \leftrightarrow & \neg[?Q]\varphi \\ \vdash [?Q](\varphi \wedge \psi) & \leftrightarrow & ([?Q]\varphi \wedge [?Q]\psi) \\ \vdash [?Q_1][?Q_2]\varphi & \leftrightarrow & [?Q_1 \cup Q_2]\varphi \end{array}$$

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\end{array}
$$

Open Question Expansion

$$
\vdash [?Q]\Diamond\varphi \quad \leftrightarrow \quad (p \wedge [?p, Q]\Diamond(p \wedge \varphi)) \vee (\neg p \wedge [?p, Q]\Diamond(\neg p \wedge \varphi))
$$

and open question operators are not much more trouble

Open Question Distribution

$$\begin{array}{lll} \vdash [?Q]p & \leftrightarrow & p \\ \vdash [?Q]\neg\varphi & \leftrightarrow & \neg[?Q]\varphi \\ \vdash [?Q](\varphi \wedge \psi) & \leftrightarrow & ([?Q]\varphi \wedge [?Q]\psi) \\ \vdash [?Q_1][?Q_2]\varphi & \leftrightarrow & [?Q_1 \cup Q_2]\varphi \end{array}$$

Open Question Expansion

$$\vdash [?Q]\Diamond\varphi \quad \leftrightarrow \quad (p \wedge [?p, Q]\Diamond(p \wedge \varphi)) \vee (\neg p \wedge [?p, Q]\Diamond(\neg p \wedge \varphi))$$

Open Question Generalisation

if $\vdash \varphi$ *then* $\vdash [?Q]\varphi$

Outline

Introduce a decision operator

Preference logic:

Interpret Ruv as 'I am indifferent between or prefer v to u'

So R is reflexive. But it may not be transitive.

$D\varphi$ = after any rational choice that I can make, *ceteris paribus*, φ holds.

v is *stable* iff there is no $w \approx v$ such that Rvw and not Rwv

(I.e., stable = $R \cap \approx$-maximal)

with some support from hybrid logic

$$i \quad | \quad D \quad | \quad U$$

i is a *nominal* with $V(i)$ restricted to be a singleton

U is the modal operator for $\approx$

Semantics

$M, u \models U\varphi$ iff if $u \approx v$ then $M, v \models \varphi$ for each $v \in W$

$M, u \models D\varphi$ iff if $u \approx v$ and v is stable then $M, v \models \varphi$ for each $v \in W$

and get the following axiomatisation

$\mathcal{PC}$

Universal Equivalence	$i \to Ei$
	$i \to UEi$
	$EEi \to Ei$
Reflexivity	$i \to \Diamond i$
Inclusion	$\Diamond i \to Ei$
Preferential Choice	$\langle D \rangle i \quad \leftrightarrow \quad E(i \wedge \Box\Diamond i)$

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Inclusion	$\Diamond i \to Ei$
Preferential Choice	$\langle D \rangle i \quad \leftrightarrow \quad E(i \wedge \Box \Diamond i)$

Note: stability is not preserved under bisimulation, so the use of nominals is necessary.

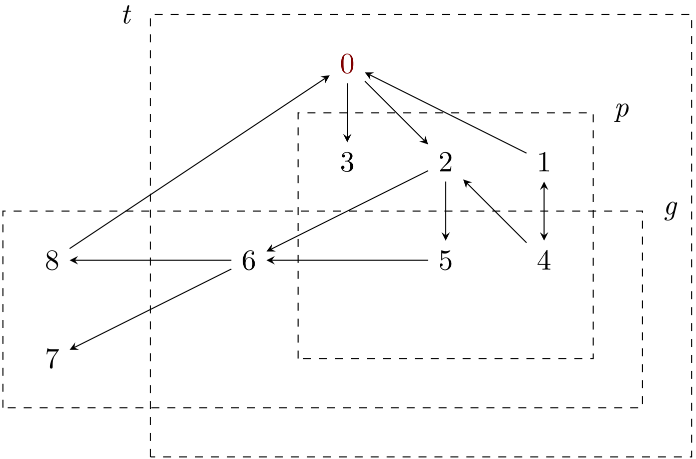
Outline

Alice's story

Alice is considering moving house. She searches the listings for a house that is better located and sees several that she likes better. She goes to visit the one of them with Betty, her good friend. When Betty sees the house, she says 'what about a garden?' This is not a question that Alice had considered before. Her own house doesn't have one, but she is influenced by Betty to go back to the listings and check out houses with gardens. Eventually, she finds a house and moves. It has a nice big garden. But a few months later, she visits Chandra, a friend of Betty's who lives in a concrete house. Alice finds in quite charming. Her new house is timber-framed, like her old house and every house she has ever lives in. That night, she goes back to the listings...

Diagram

$$M, 0 \models [?p][?g]\langle D\rangle[!p, g]6$$



Diagram

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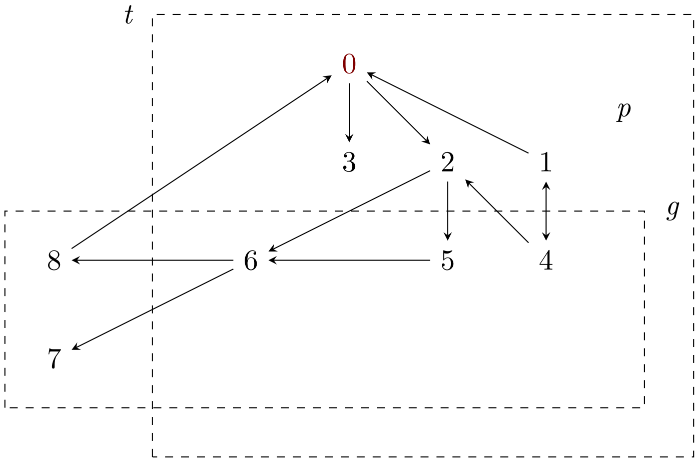

Diagram

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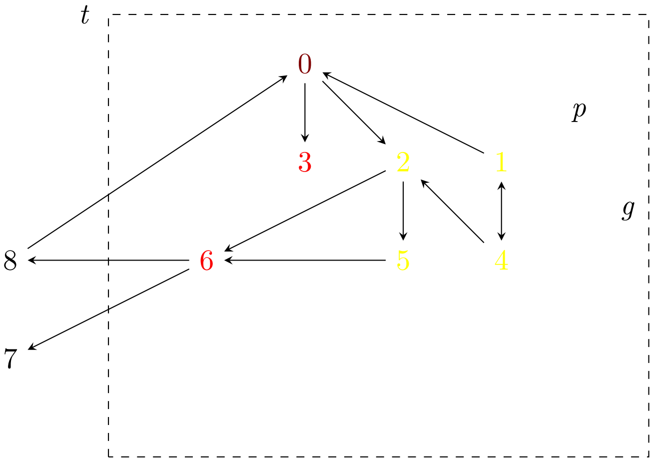

Diagram

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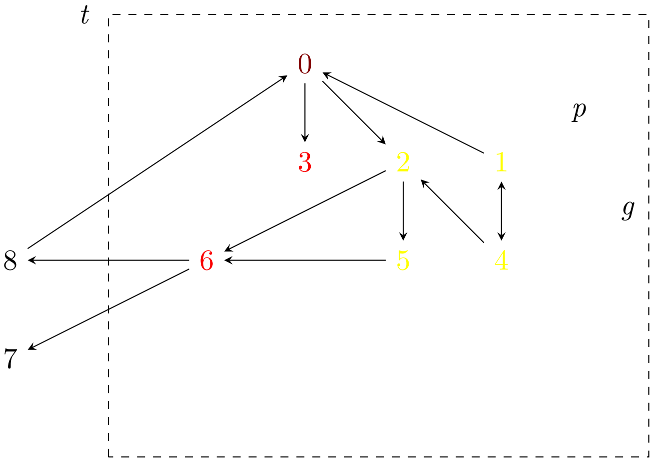

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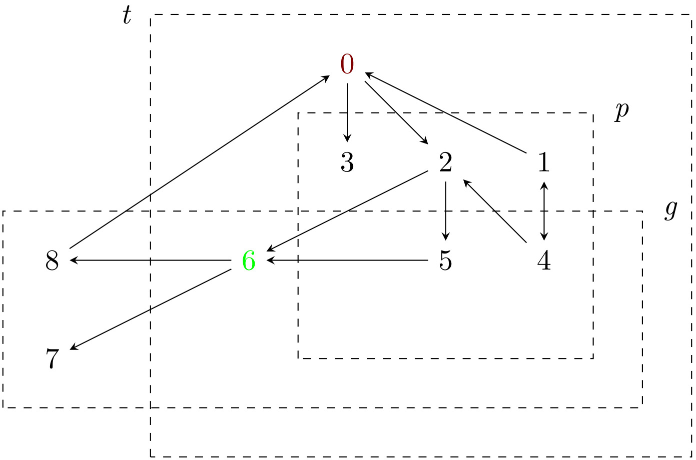
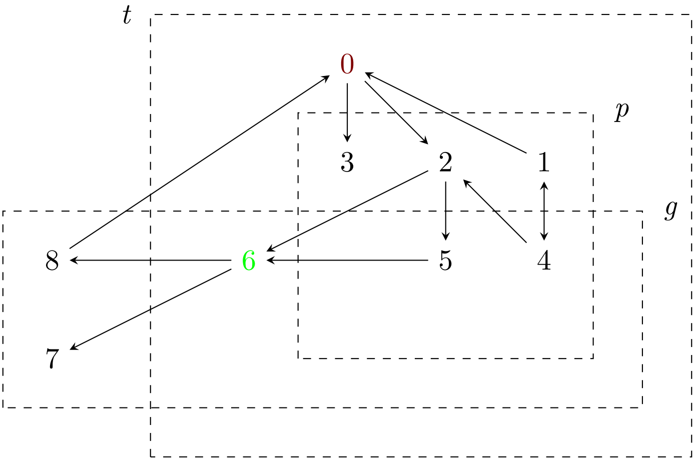


Alice's story continued

Alice is so impressed with the houses she looks at. Why had she never thought of houses being made out of anything but wood? Next month, she has moved into her new plaster house, which looks very modern and stylish.

Diagram

$$M, 0 \models [?p][?g]\langle D\rangle[!p, g][?t]\langle D\rangle[!t]8$$



Diagram

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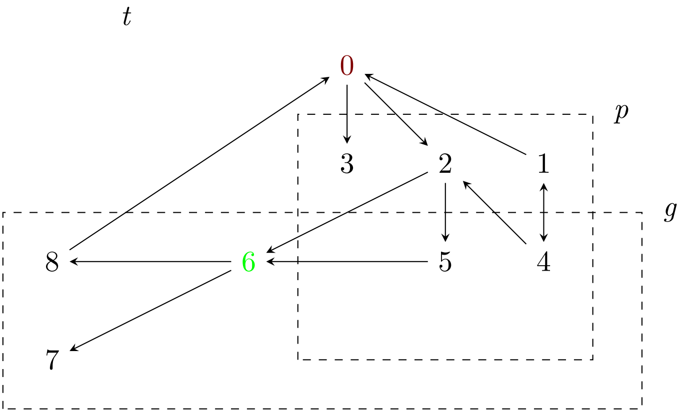

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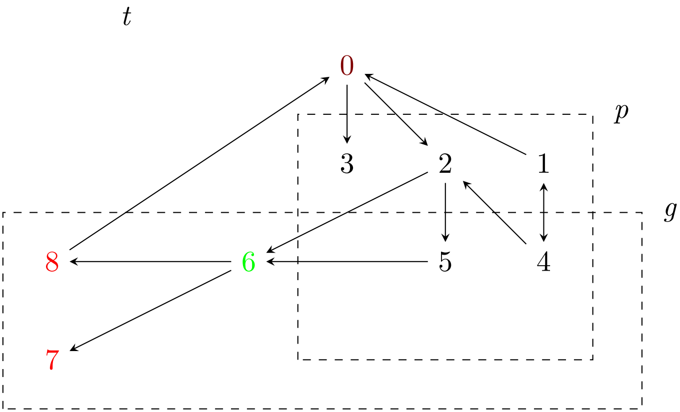
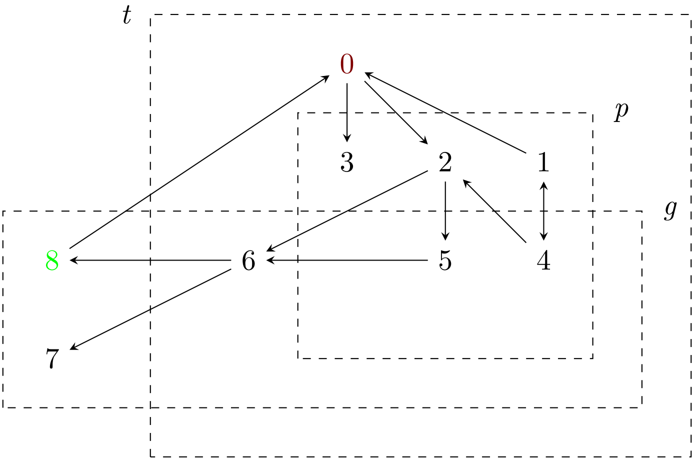


Diagram

$$M, 0 \models [?p][?g]\langle D\rangle[!p, g][?t]\langle D\rangle[!t]8$$

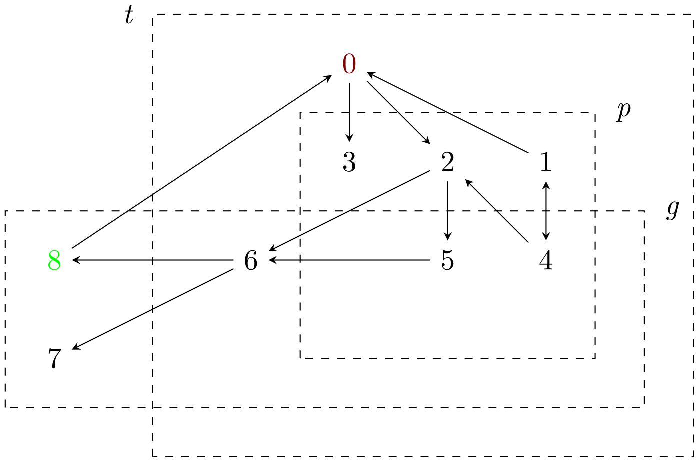


Alice's story brought to a conclusion

After a while, Alice happens to walk past her old house - the first one. Taken by it's quaint charm, worn woodwork and cottage garden, she realises that she prefers it to her new house.

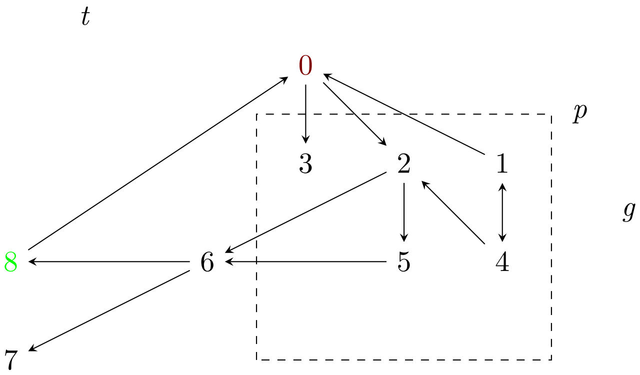
Diagram

$$M, 0 \models [?p][?g]\langle D\rangle[!p, g][?t]\langle D\rangle[!t][?t, g]\langle D\rangle 0$$



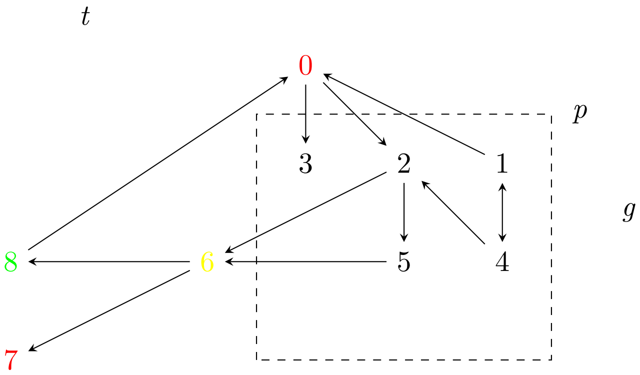
Diagram

$$M, 0 \models [?p][?g]\langle D\rangle[!p, g][?t]\langle D\rangle[!t][?t, g]\langle D\rangle 0$$



Diagram

$$M, 0 \models [?p][?g]\langle D\rangle[!p, g][?t]\langle D\rangle[!t][?t, g]\langle D\rangle 0$$



Outline

Characterizing transitivity of preferences

Question Permutation $[?p]\Diamond[?q]\Diamond i \leftrightarrow [?q]\Diamond[?p]\Diamond i$

Outline

Cordorcet's drivers

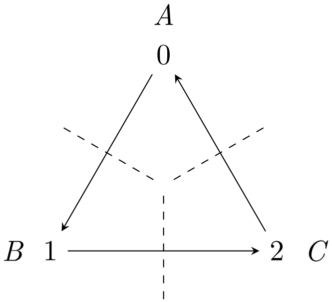
Three friends are going out together for the evening and need to pick a designated driver. First they assume that Andrew will drive, but he objects and suggest Barry instead. They agree that Andrew will drive unless the majority prefer that Barry drives. But they vote for Barry to drive. This time Barry objects because Charles has not even been considered. They decide that Barry should drive unless the majority prefer that Charles drives, in which case Charles is picked as the driver. This happens and Charles is selected, despite the fact that the majority prefer Andrew (the original choice) as the designated driver.

have non-transitive preferences

	$A < B$	$B < C$	$A < C$	
Andrew	yes	yes	yes	$A < B < C$
Barry	no	yes	no	$B < C < A$
Charles	yes	no	no	$C < A < B$
Majority	yes	yes	no	

Diagram

$$M, 0 \models [?(A \vee B)]D[!(A \vee B)][?(B \vee C)]D[!(B \vee C)][?(A \vee C)]DA$$



Outline

First some definitions

A *state description* of given a finite set of formulas $\{\varphi_1, \ldots, \varphi_n\}$ is a formula of the form $\pm\varphi_1 \wedge \ldots \wedge \pm\varphi_n$ where $\pm\varphi_i$ is either φ_i or $\neg\varphi_i$.

A set of formulas Γ *determines* formula φ iff it contains a non-empty finite subset Γ_0 such that for every state description σ of Γ_0, either $(\sigma \rightarrow \varphi)$ or $(\sigma \rightarrow \neg\varphi)$ is a theorem of $\mathcal{PC}$.

Γ determines a set Q of formulas iff it determines each formula in Q.

for example

$\{p, q\}$ determines $(p \vee q)$ but $\{(p \wedge q), p\}$ does not.

for example

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To see this:

for example

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To see this:
$(p \vee q)$ is determined by neither $\{(p \wedge q)\}$ nor $\{p\}$ nor $\{(p \wedge q), p\}$

for example

$\{p, q\}$ determines $(p \vee q)$ but $\{(p \wedge q), p\}$ does not.

To see this:
$(p \vee q)$ is determined by neither $\{(p \wedge q)\}$ nor $\{p\}$ nor $\{(p \wedge q), p\}$
because the state descriptions $\neg(p \wedge q)$ and $\neg p$ and $\neg(p \wedge q) \wedge \neg p$
each imply neither $(p \vee q)$ nor $\neg(p \vee q)$.

then a lemma

Lemma

If Γ determines Q then in any model M and states u, v, if $u \approx v$ then for all $\varphi \in Q$, $M, u \models \varphi$ iff $M, v \models \varphi$

and some more definitions

A subset $\Gamma' \subseteq \Gamma$ is *Q-releasing* iff Γ' is a maximally non-Q-determining subset of Γ, i.e., Γ' does not determine Q and for any Γ'' that also does not determine Q, if $\Gamma' \subseteq \Gamma'' \subseteq \Gamma$ then $\Gamma'' = \Gamma'$.

for example

If P is the set of propositional variables occurring in some Boolean formula φ, then Prop $\setminus P$ does not determine φ.

for example

If P is the set of propositional variables occurring in some Boolean formula φ, then Prop $\setminus P$ does not determine φ.

But it may not be φ-releasing.

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But it may not be φ-releasing.

For example, if φ is the formula $((p \wedge q) \vee (p \wedge \neg q))$, then Prop $\setminus \{p, q\}$ is not φ-releasing, because the larger set Prop $\setminus \{p\}$ also fails to determine φ.

for example

If P is the set of propositional variables occurring in some Boolean formula φ, then Prop $\setminus P$ does not determine φ.

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For example, if φ is the formula $((p \wedge q) \vee (p \wedge \neg q))$, then Prop $\setminus \{p, q\}$ is not φ-releasing, because the larger set Prop $\setminus \{p\}$ also fails to determine φ.

Moreover, a set Γ of formulas may have more than one φ-releasing subset.

and finally, a definition

$M[?Q]M'$ iff Γ' is a Q-releasing subset of Γ (in M)

Semantics

$M, u \models [?Q]\varphi$ iff $M', u \models \varphi$ for every M' such that $M[?Q]M'$

and some remarks

- Solves a known problem about how to remove *ceteris paribus* conditions in general
- Conservative extension of the atomic case
- No axiomatisation as yet
- Suggestive of connections with belief revision/contraction

and now

Open: How about a break?

Closed: How long is the break?