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Asymptotics of the minimum manipulating coalition size for positional voting rules under IC behaviour

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1. Introduction

In 1973–75 Gibbard and Satterthwaite published a fundamental impossibility theorem which states that every non-dictatorial social choice function, whose range contains at least three alternatives, at certain profiles can be manipulated by a single individual voter [**6, 15**]. After that, the natural question arose: if there are no perfect rules, which ones are the best, i.e. least manipulable? To this question there can be no absolute answer – it depends both on the behaviour of the voters, and on the measure used to quantify the term "manipulability".

Among models of voter behaviour, the following two have gained the most attention ([**3,14**]). The Impartial Culture (IC) model assumes that voters are independent, and that each voter is equally likely to vote for any candidate. The Impartial Anonymous Culture (IAC) model assumes some degree of dependency. This paper concerns itself with the IC model.

Among measures of manipulability, the most popular is the probability that the votes fall in such a way as to create the (coalitional or individual) "logical possibility of manipulation" ([**4,7,8,9,12,13,14**]). This means that some coalition of voters (or individual voter) with incentive to do so can change the election result by voting insincerely. Counterthreats are not considered – we assume that the manipulator(s) are not opposed by the other, naive voters. The probability of manipulability has been especially well-studied for the important class of positional (scoring) voting rules, and significant progress has been made in comparing them. In his seminal paper [**14**], Saari showed that in his "geometric" model, Borda's rule is the least manipulable for the three-alternative case in relation to micro manipulation, but that this does not extend to the case of four alternatives.

In this paper, we further refine this notion of manipulability by considering the sizes of the coalitions involved. Intuitively, a rule is more resistant to manipulation if many voters must be recruited to assemble the manipulating coalition, and less resistant if only a few voters are required. We may thus consider the probability that a coalition *of at most k voters* can manipulate ($k = 1, 2, \ldots$). Equivalently, we study the probability distribution of the size of the smallest manipulating coalition (a random variable). Similar ideas are explored, in a more limited way, in [**12**] and [**13**].

We use the following notation and assumptions throughout. An election is held to choose one from among m candidates ($m \geq 3$). There are n voters, who hold opinions according to the IC model. That is, each voter (independently) chooses one of the $m!$ possible types (preference orders on the candidates), each type being equally likely to be chosen. The election uses the positional voting rule with score vector $w = (w_1, \ldots, w_m)$, where $1 = w_1 \geq w_2 \geq \cdots \geq w_m = 0$. That is, a vote ranking candidate α in ith place contributes w_i to the score of α, and the candidate with the greatest total score is declared the winner. The possibility of a tie for first place will not be considered in this paper, as Proposition 3 makes it largely irrelevant; it is discussed in detail in [**13**]. We aim to describe the limiting probability distribution of the minimum manipulating coalition size as $n \to \infty$, and to use this as a criterion for comparing the rules.

The remainder of this paper is organized as follows. The next section records some basic results regarding IC behaviour in large populations. The long section 3 is the theoretical core of the paper: it defines the manipulation problem as an integer linear program and then, through a series of simplifications, shows how this may be replaced by a much simpler linear program. Readers impatient to reach the main results may wish to skip much of this section at a first reading. The desired limiting probability distributions are derived in section 4, and used in section 5 to compare the rules. Section 6 contains some conclusions.

2. Asymptotic results for large electorates of IC voters

Let C be the set of candidates, and T the set of all voter types (i.e. all permutations of C). Let $N = (N_t)_{t \in T}$ be the random vector giving the number of voters of each type (so $\sum_{t \in T} N_t = n$). For IC voter behaviour, N has a multinomial probability distribution with mean $(n/m!)\mathbf{1}$. (Here and subsequently, the notation $\mathbf{1}$ is used to denote a vector whose entries are all 1.) Under the asymptotic conditions of interest to us, this may be approximated by a multivariate normal distribution.

Proposition 1.

$$\frac{N - np\mathbf{1}}{\sqrt{n}} \xrightarrow{D} N(0, \Sigma),$$

where $p = 1/m!$, and Σ is the matrix with entries

$$\Sigma_{st} = \begin{cases} p(1-p), & \text{if } s = t \\ -p^2, & \text{if } s \neq t \ . \end{cases}$$

Remark. Here and in the rest of this paper, the notation $\xrightarrow{D}$ denotes convergence in distribution (see [**5, Ch. 2**]). Note that the limiting multivariate normal distribution is degenerate, being supported on $\left\{(q_t)_{t\in T} : \sum_{t\in T} q_t = 0\right\}$.

Proof. Since IC voters choose their types at random, and independently, we have $N = \sum_{i=1}^n X_i$, where $X_1, \ldots, X_n$ are independent and have probability distribution assigning probability $1/m!$ to each of the unit vectors of $\mathbf{R}^T$. Note that $E\left[X_1\right] = p\mathbf{1}$ and the covariance matrix of X_1 is Σ. The result then follows by the central limit theorem ([**5, p.170**]). ∎

We use the notation $\sigma_t(\alpha)$ for the contribution to candidate α's score made by a vote of type t (so if t ranks α in ith place, then $\sigma_t(\alpha) = w_i$). The total score of α is then

$$|\alpha| = \sum_{t\in T} N_t \sigma_t(\alpha).$$

Let $S = (|\alpha|)_{\alpha\in C}$ be the vector of candidates' scores (the "scoreboard"). Proposition 1 immediately gives a central limit result for S, too.

Proposition 2.

$$\frac{S - n\bar{w}\mathbf{1}}{\sqrt{n}} \xrightarrow{D} \sigma_w \left(\frac{m}{m-1}\right)^{1/2} (Z - \bar{Z}\mathbf{1}),$$

where $\bar{w}$, σ_w are the mean and standard deviation of the score vector w (i.e. $\bar{w} = (w_1 + \cdots + w_m)/m$ and $\sigma_w^2 = (w_1^2 + \cdots + w_m^2)/m - (\bar{w})^2$); Z is a vector, indexed by C, of independent standard normal random variables; and $\bar{Z} = \frac{1}{m}\sum_\alpha Z_\alpha$.

Proof. Let $Y = (Y_t)_{t\in T} \sim N(0, \Sigma)$. From Proposition 1 we have

$$\frac{S - n\bar{w}\mathbf{1}}{\sqrt{n}} \xrightarrow{D} U,$$

where $U_\alpha = \sum_{t\in T} Y_t \sigma_t(\alpha)$. It only remains to show that U and $\sigma_w \left(\frac{m}{m-1}\right)^{1/2} (Z - \bar{Z}\mathbf{1})$ have the same multivariate normal distribution. For this, it suffices to observe that they have the same mean (zero), variances, and covariances. It is routine to check that

$$\operatorname{Var}\left(\sum_{t\in T} Y_t \sigma_t(\alpha)\right) = \sigma_w^2 = \sigma_w^2 \left(\frac{m}{m-1}\right) \operatorname{Var}\left(Z_\alpha - \bar{Z}\right)$$

for $\alpha \in C$, and for distinct $\alpha, \beta \in C$

$$\operatorname{Cov}\left(\sum_{t\in T} Y_t \sigma_t(\alpha), \sum_{t\in T} Y_t \sigma_t(\beta)\right) = \frac{-\sigma_w^2}{m-1} = \sigma_w^2 \left(\frac{m}{m-1}\right) \operatorname{Cov}\left(Z_\alpha - \bar{Z}, Z_\beta - \bar{Z}\right).$$

∎

Proposition 2 implies that under IC behaviour in large electorates, the average candidate's score will be of order n, but the variability among the scores will be of order only $\sqrt{n}$. Consequently, most elections will result in all candidates receiving relatively similar scores, and the margin of victory will be small. However, exact ties in the scores become increasingly rare as the number of voters increases. The following result establishes this formally.

Proposition 3.

$$P\left(\text{all candidates' scores are numerically distinct}\right) \to 1 \qquad \text{as } n \to \infty.$$

Proof. For two distinct candidates α and β, we have

$$\{|\alpha| \neq |\beta|\} = \left\{\frac{S - n\bar{w}1}{\sqrt{n}} \in G\right\},$$

where $G = \left\{s \in \mathbf{R}^C : s_\alpha \neq s_\beta\right\}$. By Proposition 2 (and **[5, p.87]**), since G is an open set we have

$$\underline{\lim}_n P\left(\frac{S - n\bar{w}1}{\sqrt{n}} \in G\right) = P\left(\sigma_w\left(\frac{m}{m-1}\right)^{1/2}(Z - \bar{Z}\mathbf{1}) \in G\right) = P\left(Z_\alpha \neq Z_\beta\right) = 1.$$

That is, $P\left(|\alpha| = |\beta|\right) \to 0$ as $n \to \infty$. The result follows since

$$P\left(\bigcup_{\alpha\neq\beta}\{|\alpha| = |\beta|\}\right) \leq \sum_{\alpha\neq\beta} P\left(|\alpha| = |\beta|\right).$$

∎

3. Approximations of the minimum manipulating coalition size.

In this section, we formulate the coalitional manipulation problem as an integer linear program. We then show, via a series of simplifying steps, that this is well approximated by a much simpler linear program in which there are only two variables, and in which the constraint set does not depend on the voting situation, but only on the voting rule. Since the proofs involved are fairly lengthy, we summarize the steps involved before embarking on them:

- The problem of assembling the smallest possible manipulating coalition can be expressed as an integer linear program, in which the variables are the numbers of voters of each type (x_t) to recruit;
- The integrality and upper bound ($x_t \leq N_t$) constraints of this program may be ignored (Propositions 4 and 5).
- Only manipulations in favour of the second-placegetter need be considered (Proposition 7).
- Coalition recruiting may be limited to those voters who rank the two leading candidates a and b adjacent, as these voters are best able to manipulate (Proposition 8).
- We may find the minimum coalition size by considering only the members' (sincere) rankings of a and b, without regard to how they rank other candidates (Propositions 9 and 10). This reduces the problem to a mere linear program with $m-1$ variables and two constraints.
- Replacing this linear program with its dual gives us two variables and $m-1$ constraints.

To specify an attempted coalitional manipulation, we must specify for each $t \in T$ the number x_t of coalition members of (sincere preference) type t, as well as the number y_t of coalition members who will insincerely vote t. Of course, we must have $\sum_{t\in T} x_t = \sum_{t\in T} y_t$.

Proposition 3 suggests that it will be enough to consider manipulation only at profiles (or voting situations) for which there is a clear winner a (i.e. a candidate with $|a| > |\alpha|$ for each $\alpha \neq a$). For such profiles, we will consider a manipulation attempt successful if it results in the score of another candidate (or tied group of candidates) matching or exceeding the score of a.

For a coalition to successfully manipulate in favour of a candidate β, its members must all be of types which prefer β to a. Let $\bar{T}_{\beta a} \subseteq T$ be the set of such types.

Additionally, it is clear that the coalition members need only consider insincere votes of types which rank β in first place; this is a dominant strategy for such manipulations. Let T_β be the set of such types.

Let MCS be the minimum size of a successful manipulating coalition (or ∞ if no manipulation is possible). Then in the light of Proposition 3 and the above remarks we have

$$P\left(MCS = \min_{\beta\neq a} Q_1(\beta)\right) \to 1$$

where $Q_1(\beta)$ is the optimal value of the linear program

$$\begin{aligned}
\min \ & \sum_{t\in\bar{T}_{\beta a}} x_t \\
\text{s.t.} \ & \sum_{t\in T_\beta} y_t(1-\sigma_t(\alpha)) - \sum_{t\in\bar{T}_{\beta a}} x_t(\sigma_t(\beta)-\sigma_t(\alpha)) \geq |\alpha|-|\beta| \qquad \forall \alpha\neq\beta \\
& \sum_{t\in T_\beta} y_t = \sum_{t\in\bar{T}_{\beta a}} x_t, \\
& 0\leq x_t\leq N_t \qquad \forall t\in\bar{T}_{\beta a} \\
& y_t\geq 0 \qquad \forall t\in T_\beta \\
& x_t, y_t\in\mathbf{Z}
\end{aligned} \tag{1}$$

(or ∞ if (1) is infeasible).

We now aim to show that, for IC asymptotic purposes, we may drop the constraints $x_t \leq N_t$ and $x_t, y_t \in \mathbf{Z}$ in (1). This is as we should expect: each N_t will be about $n/m!$, while the differences between candidates' scores (and hence, presumably, coalition sizes) are only of order $\sqrt{n}$. Similarly, the requirement that x_t and y_t be integral should not be much of a hindrance when dealing with large numbers of voters.

To this end, consider $Q_2(\beta)$, defined in the same way as $Q_1(\beta)$, except that we drop the integrality constraints $x_t, y_t \in \mathbf{Z}$ in (1), and replace the constraint $x_t \leq N_t$ by $x_t \leq N_t - K$, where K is a constant that depends only on the voting rule. It will be convenient to choose

$$K = \begin{cases} 2m!(1-w_{m-1})^{-1}, & \text{if } w_{m-1}<1 \\ 0, & \text{if } w_{m-1}=1. \end{cases}$$

Define $Q_3(\beta)$ in the same way as $Q_2(\beta)$, except that the upper bound constraint on x_t is dropped entirely. Note that we have $Q_3(\beta) \leq Q_1(\beta)$ and $Q_3(\beta) \leq Q_2(\beta)$.

Proposition 4.

$$P\left(Q_2(\beta)=Q_3(\beta)\right)\to 1 \qquad \text{as } n\to\infty.$$

Proof. Define $f_0 : \mathbf{R}^T \to [0,\infty]$ as follows: for $q \in \mathbf{R}^T$, $f_0(q)$ is the optimal value of the linear program

$$\begin{aligned}
\min \ & \sum_{t\in\bar{T}_{\beta a}} x_t \\
\text{s.t.} \ & \sum_{t\in T_\beta} y_t(1-\sigma_t(\alpha)) - \sum_{t\in\bar{T}_{\beta a}} x_t(\sigma_t(\beta)-\sigma_t(\alpha)) \geq \sum_{t\in T} q_t(\sigma_t(\alpha)-\sigma_t(\beta)) \qquad \forall \alpha\neq\beta \\
& \sum_{t\in T_\beta} y_t = \sum_{t\in\bar{T}_{\beta a}} x_t, \\
& x_t\geq 0 \qquad \forall t\in\bar{T}_{\beta a} \\
& y_t\geq 0 \qquad \forall t\in T_\beta
\end{aligned}$$

Then $Q_3(\beta) = f_0(N)$. Note that $f_0(\lambda q + \mu e) = \lambda f_0(q)$ for any $q \in \mathbf{R}^T$, $\lambda \geq 0$, and $\mu \in \mathbf{R}$. Also, f_0 is continuous on the closed subset $K = \{q : f_0(q) < \infty\}$ of $\mathbf{R}^T$. Let $\pi : \mathbf{R}^T \to K$ be the projection which maps each point $q \in \mathbf{R}^T$ to the nearest point of K to q; then π is continuous and $\pi(\lambda q + \mu e) = \lambda\pi(q) + \mu e$ for any $q \in \mathbf{R}^T$, $\lambda \geq 0$, and $\mu \in \mathbf{R}$.

Let $f : \mathbf{R}^T \to [0,\infty)$ be given by $f(q) = f_0(\pi(q))$. Then f is continuous; has $f(\lambda q+\mu e) = \lambda f(q)$ for any $q \in \mathbf{R}^T$, $\lambda \geq 0$, and $\mu \in \mathbf{R}$; and $Q_3(\beta) = f(N)$ whenever $Q_3(\beta) < \infty$.

When $Q_3(\beta) = \infty$, we have $Q_2(\beta) = \infty$ too. On the other hand, when $Q_3(\beta) \leq \min_t N_t - K$, the corresponding optimal point of the linear program for $Q_3(\beta)$ is also feasible for the linear program for $Q_2(\beta)$, so $Q_2(\beta) = Q_3(\beta)$. Hence

$$\{Q_2(\beta)\neq Q_3(\beta)\} \subseteq \{\min_t N_t - K < Q_3(\beta) < \infty\} \subseteq \{\min_t N_t - K < f(N)\},$$

and so it suffices to show that this last event has probability converging to 0.

Now define $h : \mathbf{R}^T \to (-\infty, \infty)$ by $h(q) = (\min_t q_t) - f(q)$. By Proposition 1 and the continuity of h, we have

$$h\left(\frac{N - ne}{\sqrt{n}}\right) \xrightarrow{D} h(X), \qquad \text{where } X \sim N(0, \Sigma).$$

This yields

$$\frac{(\min_t N_t) - f(N) - n/m!}{\sqrt{n}} \xrightarrow{D} h(X),$$

from which

$$\overline{\lim_n} P\left(\frac{(\min_t N_t) - f(N) - n/m!}{\sqrt{n}} \leq -\lambda\right) \leq P\left(h(X) \leq -\lambda\right)$$

for any $\lambda > 0$. That is,

$$\overline{\lim_n} P\left((\min_t N_t) - K - f(N) \leq n/m! - \lambda\sqrt{n} - K\right) \leq P\left(h(X) \leq -\lambda\right).$$

We have $n/m! - \lambda\sqrt{n} - K > 0$ for sufficiently large n, so

$$\overline{\lim_n} P\left((\min_t N_t) - K \leq f(N)\right) \leq P\left(h(X) \leq -\lambda\right).$$

Since λ was arbitrary,

$$P\left((\min_t N_t) - K \leq f(N)\right) \to 0 \qquad \text{as } n \to \infty,$$

and the result follows. ∎

Proposition 5. With probability 1,

$$Q_1(\beta) \leq Q_2(\beta) + K.$$

Proof of Proposition 5 for the case $w_{m-1} < 1$. Let $(x_t)_{t \in \bar{T}_{\beta a}}$, $(y_t)_{t \in T_\beta}$ be optimal for the problem defining $Q_2(\beta)$. Then

$$\begin{aligned}
&\sum_{t \in T_\beta} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x_t(\sigma_t(\beta) - \sigma_t(\alpha)) \geq |\alpha| - |\beta| \qquad \forall \alpha \neq \beta \\
&\sum_{t \in T_\beta} y_t = \sum_{t \in \bar{T}_{\beta a}} x_t, \\
&0 \leq x_t \leq N_t - K \qquad \forall t \in \bar{T}_{\beta a} \\
&y_t \geq 0. \qquad \forall t \in T_\beta
\end{aligned}$$

Choose types $t_0 \in \bar{T}_{\beta a}$, $t_1 \in T_\beta$ such that t_0 ranks a last, β next-to-last, and some γ first, while t_1 is obtained from t_0 by transposing the rankings of β and γ. Define $(x'_t)_{t \in \bar{T}_{\beta a}}$, $(y'_t)_{t \in T_\beta}$ by $x'_{t_0} = x_{t_0} + K$, $y'_{t_1} = y_{t_1} + K$, and $x'_t = x_t$ and $y'_t = y_t$ for all other t. Then (x'_t), (y'_t) satisfy

$$\begin{aligned}
&\sum_{t \in T_\beta} y'_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x'_t(\sigma_t(\beta) - \sigma_t(\alpha)) \\
&\qquad = \sum_{t \in T_\beta} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x_t(\sigma_t(\beta) - \sigma_t(\alpha)) + K(1 - w_{m-1}) + K(\sigma_{t_0}(\alpha) - \sigma_{t_1}(\alpha)) \\
&\qquad \geq |\alpha| - |\beta| + 2m!
\end{aligned}$$

for each $\alpha \neq \beta$. Also, $\sum_{t \in T_\beta} y'_t = \sum_{t \in \bar{T}_{\beta a}} x'_t$ and $0 \leq x'_t \leq N_t$ for each t.

Now obtain $(x''_t)_{t \in \bar{T}_{\beta a}}$, $(y''_t)_{t \in T_\beta}$ by rounding each x'_t, y'_t to an integral value. The choice between rounding up and rounding down (i.e. between $x''_t = \lfloor x'_t \rfloor$ and $x''_t = \lceil x'_t \rceil$) can be made arbitrarily, but should be done in such a way that $\sum_{t \in T_\beta} y''_t = \sum_{t \in \bar{T}_{\beta a}} x''_t$. Since $\left|x''_t - x'_t\right| \leq 1$ and $\left|y''_t - y'_t\right| \leq 1$, we obtain

$$\sum_{t \in T_\beta} y''_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x''_t(\sigma_t(\beta) - \sigma_t(\alpha)) \geq |\alpha| - |\beta|,$$

and so (x''_t), (y''_t) are feasible for (1). We have $\sum_{t \in \bar{T}_{\beta a}} x''_t = \sum_{t \in \bar{T}_{\beta a}} x_t + K$, from which it follows that $Q_1(\beta) \leq Q_2(\beta) + K$. ∎

Proof of Proposition 5 for the case $w_{m-1} = 1$. A separate proof is required for this case (the anti-plurality rule $w = (1, \ldots, 1, 0)$). We can show that $Q_1(\beta) \leq Q_2(\beta)$ by showing that the optimal (x_t), (y_t) for the problem defining $Q_2(\beta)$ are always integral (and hence give a feasible solution to (1)). To establish this, we will use a well-known result in linear programming (see, e.g. **[10]** or **[11]**), which assures us that the optimal solution of a linear program will always be integral when the constraint coefficient matrix A is totally unimodular (i.e. every square submatrix has determinant 1, -1, or 0).

A useful sufficient condition for total unimodularity is given in **[10]** (Theorem 13.3) as follows: a matrix whose entries are all 1, -1, or 0 is totally unimodular if each column has at most two non-zero entries, and if the rows can be partitioned into two sets I_1 and I_2 such that: (i) if a column has two entries of the same sign, their rows are in different sets; (ii) if a column has two entries of different signs, their rows are in the same set.

For this problem A has columns corresponding to the variables x_t ($t \in \bar{T}_{\beta a}$) and y_t ($t \in T_\beta$). There is one row corresponding to each candidate $\alpha \neq \beta$, in which the entry in the column corresponding to x_t is -1 if t ranks α last, and 0 otherwise; the entry in the column corresponding to y_t is 1 if t ranks α last, and 0 otherwise. Let these rows constitute the set I_1. There is also a further row corresponding to the constraint $\sum_{t \in T_\beta} y_t - \sum_{t \in \bar{T}_{\beta a}} x_t = 0$; in this row, the entries in the columns corresponding to the x_t are all -1, and those corresponding to the y_t are all 1. Let this row constitute the set I_2. It is clear that this matrix satisfies the sufficient condition above; the result follows. ∎

Corollary 6.

$$P\left(|MCS - \min_{\beta \neq a} Q_3(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty$$

Proof. From Propositions 4 and 5 we have

$$P\left(Q_1(\beta) - K \leq Q_2(\beta) \leq Q_3(\beta) \leq Q_1(\beta)\right) \to 1 \qquad \text{as } n \to \infty,$$

from which

$$P\left(|Q_1(\beta) - Q_3(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty,$$

and so

$$P\left(|\min_{\beta \neq a} Q_1(\beta) - \min_{\beta \neq a} Q_3(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty.$$

The result then follows from the earlier observation that $P\left(MCS = \min_{\beta \neq a} Q_1(\beta)\right) \to 1$. ∎

Now that we have effectively eliminated, for our purposes, the integrality and upper-bound constraints in (1), we can further simplify the description of the minimum manipulating coalition size.

Suppose b is the candidate with second-highest score after a. The next result consists of the observation that only manipulations in favour of b need now be considered.

Proposition 7. $\min_{\beta \neq a} Q_3(\beta) = Q_3(b)$.

Proof. Let $(x_t)_{t \in \bar{T}_{\beta a}}$, $(y_t)_{t \in T_\beta}$ be optimal for the problem defining $Q_3(\beta)$, some $\beta \neq b$. Form $(x'_t)_{t \in \bar{T}_{\beta a}}$, $(y'_t)_{t \in T_\beta}$ by transposing β and b in all the voter types involved. (So if types s and t are related by transposition of the ranks of β and b, then $x'_s = x_t$.) We have, for any α,

$$\sum_{t \in T_b} y'_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{ba}} x'_t(\sigma_t(b) - \sigma_t(\alpha)) = \sum_{t \in T_\beta} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x_t(\sigma_t(\beta) - \sigma_t(\alpha)).$$

Since $|\beta| \leq |b|$, we see that (x'_t), (y'_t) are feasible for the problem defining $Q_3(b)$, and give the same objective value. The result follows. ∎

We next show that a manipulating coalition may always be formed by recruiting only voters who sincerely rank b and a adjacent. Let $T_i \subset T$ consist of those types which rank b in ith place and a in $(i+1)$st place, and $T_{ba} = \cup_{i=1}^{m-1} T_i \subset \bar{T}_{ba}$. If we replace $\bar{T}_{ba}$ by T_{ba} in the linear program defining $Q_3(b)$, we obtain the linear program

$$\begin{aligned}
\min \; & \sum_{t \in T_{ba}} x_t \\
\text{s.t.} \; & \sum_{t \in T_b} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in T_{ba}} x_t(\sigma_t(b) - \sigma_t(\alpha)) \geq |\alpha| - |b| \qquad \forall \alpha \neq b \\
& \sum_{t \in T_b} y_t = \sum_{t \in T_{ba}} x_t, \\
& x_t \geq 0 \qquad \forall t \in T_{ba} \\
& y_t \geq 0 \qquad \forall t \in T_b
\end{aligned} \tag{2}$$

Let Q denote the optimal value of (2) (or ∞ if (2) is infeasible).

Proposition 8. $Q_3(b) = Q$.

Proof. For $t \in T_i$, let $\tau(t) \subseteq \bar{T}_{ba}$ consist of those types which (i) agree with t in ranking positions $i+1, \ldots, m$; and (ii) rank the remaining candidates, other than b, in either the same position as t does, or one place lower. Observe that $\{\tau(t) : t \in T_{ba}\}$ is a partition of $\bar{T}_{ba}$, and that for $s \in \tau(t)$, $\alpha \neq b$

$$\sigma_t(b) - \sigma_t(\alpha) \leq \sigma_s(b) - \sigma_s(\alpha).$$

(This implies that a voter of type $s \in \tau(t)$ can always be dismissed from the manipulating coalition and replaced with one of type t.)

Let $(x_t)_{t \in \bar{T}_{ba}}$, $(y_t)_{t \in T_b}$ be optimal for the problem defining $Q_3(b)$. Form $(x'_t)_{t \in T_{ba}}$ by $x'_t = \sum_{s \in \tau(t)} x_s$. Then $\sum_{t \in T_{ba}} x'_t = \sum_{s \in \bar{T}_{ba}} x_s$, and

$$\sum_{t \in T_{ba}} x'_t(\sigma_t(b) - \sigma_t(\alpha) \leq \sum_{s \in \bar{T}_{ba}} x_s(\sigma_s(b) - \sigma_s(\alpha)).$$

Hence (x'_t) is feasible for (2). The result follows. ∎

Our efforts thus far have established that $P\left(|MCS - Q| \leq K\right) \to 1$ as $n \to \infty$. This will ensure that for IC asymptotic purposes, we may replace our original description of the minimum manipulating coalition size with the more tractable problem (2). Our next step results in a considerable further simplification of the linear program.

Theorem 9. Suppose non-negative numbers $z_1, \ldots, z_{m-1}$ satisfy

$$\begin{aligned} \sum_{i=1}^{m-1} z_i(1 - w_i + w_{i+1}) &\geq |a| - |b| \\ \sum_{i=1}^{m-1} z_i(1 - w_i) &\geq n\bar{w} - |b| \end{aligned} \tag{3}$$

where $\bar{w} = (w_1 + \cdots + w_m)/m$. Then there exists $(x_t)_{t \in T_{ba}}$, $(y_t)_{t \in T_b}$ feasible for (2), with $\sum_{t \in T_i} x_t = z_i$.

Remark. The conclusion of this result asserts that we may arrange a successful manipulation in which z_i members of the coalition are of types in T_i, for each $i = 1, \ldots, m-1$.

Note that if z_i voters of types in T_i all cast insincere votes which rank b first, the score of b will be increased by $z_i(1 - w_i)$. If they all cast insincere votes which rank a last, the score of a will be decreased by $z_i w_{i+1}$. Thus, the first inequality of (3) makes it possible for b to catch up to a. The second inequality of (3) makes it possible for b to catch up to the average candidate's score. It is remarkable that these two (apparently rather weak) linear conditions are sufficient to make manipulation possible.

Proof. Let D be the set of candidates other than a or b. For any type $t \in T$, we denote by $t(i)$ the candidate ranked in ith place by t.

We will write our proposed (x_t), (y_t) in terms of parameters r and $(u_\alpha)_{\alpha \in D}$, which are to be determined later in such a way that $0 \leq r \leq 1$, $u_\alpha \geq 0\ \forall \alpha$, and $\sum_{\alpha \in D} u_\alpha = 1$.

For each $\alpha \in D$, let

$$v_\alpha = \begin{cases} \frac{1-u_\alpha}{m-3} & \text{, if } m \geq 4 \\ 1 & \text{, if } m = 3; \end{cases}$$

note $\sum_{\alpha \in D} v_\alpha = 1$.

For each t, let $x_t = \sum_{k=1}^{4} x_t^{(k)}$ and $y_t = \sum_{k=1}^{4} t_t^{(k)}$, where

$$x_t^{(1)} = \begin{cases} \frac{r u_{t(m)} z_i}{(m-3)!} & \text{, if } t \in T_i,\ i = 1, \ldots, m-2 \\ 0 & \text{, if } t \in T_{m-1} \end{cases}$$

$$y_t^{(1)} = \begin{cases} \sum_{i=1}^{m-2} \frac{r u_{t(i+1)} z_i}{(m-3)!} & \text{, if } t \in T_b \text{ with } t(m) = a \\ 0 & \text{, otherwise} \end{cases}$$

(this corresponds to $ru_\alpha z_i$ voters of types in T_i changing sincere votes of form $\ldots ba \ldots \alpha$ to insincere ones $b \ldots \alpha \ldots a$, for each $i = 1, \ldots, m-2$);

$$x_t^{(2)} = \begin{cases} \frac{(1-r)v_{t(1)} z_i}{(m-3)!} & \text{, if } t \in T_i,\ i = 2, \ldots, m-2 \\ 0 & \text{, if } t \in T_1 \cup T_{m-1} \end{cases}$$

$$y_t^{(2)} = \begin{cases} \frac{(1-r)v_{t(i)} z_i}{(m-3)!} & \text{, if } t \in T_b \text{ with } t(i+1) = a,\ i = 2, \ldots, m-2 \\ 0 & \text{, otherwise} \end{cases}$$

(this corresponds to $(1-r)v_\alpha z_i$ voters of types in T_i changing sincere votes of form $\alpha \ldots ba \ldots$ to insincere ones $b \ldots \alpha a \ldots$, for each $i = 2, \ldots, m-2$);

$$x_t^{(3)} = y_t^{(3)} = \begin{cases} \frac{(1-r)z_1}{(m-2)!} & \text{, if } t \in T_1 \\ 0 & \text{, otherwise;} \end{cases}$$

(this corresponds to $(1-r)z_1$ voters of types in T_1 leaving their votes unchanged);

$$x_t^{(4)} = \begin{cases} \frac{v_{t(1)} z_{m-1}}{(m-3)!} & \text{, if } t \in T_{m-1} \\ 0 & \text{, if } t \notin T_{m-1} \end{cases}$$

$$y_t^{(4)} = \begin{cases} \frac{v_{t(m-1)} z_{m-1}}{(m-3)!} & \text{, if } t \in T_b \text{ with } t(m) = a \\ 0 & \text{, otherwise} \end{cases}$$

(this corresponds to $v_\alpha z_{m-1}$ voters of types in T_{m-1} changing sincere votes of form $\alpha \ldots ba$ to insincere ones $b \ldots \alpha a$).

The reader may verify that

$$\sum_{t \in T_i} x_t = \sum_{k=1}^{4} \sum_{t \in T_i} x_t^{(k)} = z_i \qquad \text{and} \qquad \sum_{t \in T_b} y_t = \sum_{k=1}^{4} \sum_{t \in T_b} y_t^{(k)} = \sum_{i=1}^{m-1} z_i.$$

We now verify that the inequality constraints of (2) hold. Note that

$$\sum_{t \in T_{ba}} x_t \sigma_t(b) = \sum_{i=1}^{m-1} z_i w_i, \qquad \sum_{t \in T_{ba}} x_t \sigma_t(a) = \sum_{i=1}^{m-1} z_i w_{i+1},$$

and

$$\begin{aligned} \sum_{t \in T_b} y_t \sigma_t(a) &= \sum_{k=1}^{4} \sum_{t \in T_b} y_t^{(k)} \sigma_t(a) \\ &= 0 \ + \ (1-r) \sum_{i=2}^{m-2} z_i w_{i+1} \ + \ (1-r) z_1 w_2 \ + \ 0 \\ &= (1-r) \sum_{i=1}^{m-1} z_i w_{i+1}. \end{aligned}$$

Hence

$$\sum_{t \in T_b} y_t (1 - \sigma_t(a)) - \sum_{t \in T_{ba}} x_t (\sigma_t(b) - \sigma_t(a)) = B + rA,$$

where $A = \sum_{i=1}^{m-1} z_i w_{i+1}$ and $B = \sum_{i=1}^{m-1} z_i (1 - w_i)$. The inequality constraint for $\alpha = a$ in (2) thus reduces to

$$B + rA \geq |a| - |b|.$$

Now consider the other inequality constraints. We use the notation $\bar{w}_{-i,j,k}$ to denote the average of all elements of w other than the ith, jth, and kth (i.e. $(-w_i - w_j - w_k + \sum_{\ell=1}^{m} w_\ell)/(m-3)$), or 0 if $m = 3$. Similarly

$\bar{w}_{-i,j} = (-w_i - w_j + \sum_{\ell=1}^m w_\ell)/(m-2)$. For $\alpha \in D$ we have

$$\begin{aligned}
\sum_{t \in T_{ba}} x_t \sigma_t(\alpha) &= \sum_{k=1}^{4} \sum_{t \in T_{ba}} x_t^{(k)} \sigma_t(\alpha) \\
&= r \sum_{i=1}^{m-2} z_i \left(u_\alpha \cdot 0 + \left(\sum_{\gamma \neq \alpha} u_\gamma \right) \bar{w}_{-i,i+1,m} \right) \\
&\quad + (1-r) \sum_{i=2}^{m-2} z_i \left(v_\alpha \cdot 1 + \left(\sum_{\gamma \neq \alpha} v_\gamma \right) \bar{w}_{-1,i,i+1} \right) \\
&\quad + (1-r) z_1 \bar{w}_{-1,2} \\
&\quad + z_{m-1} \left(v_\alpha \cdot 1 + \left(\sum_{\gamma \neq \alpha} v_\gamma \right) \bar{w}_{-1,m-1,m} \right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t \in T_b} y_t \sigma_t(\alpha) &= \sum_{k=1}^{4} \sum_{t \in T_b} y_t^{(k)} \sigma_t(\alpha) \\
&= r \sum_{i=1}^{m-2} z_i \left(u_\alpha w_{i+1} + \left(\sum_{\gamma \neq \alpha} u_\gamma \right) \bar{w}_{-1,i+1,m} \right) \\
&\quad + (1-r) \sum_{i=2}^{m-2} z_i \left(v_\alpha w_i + \left(\sum_{\gamma \neq \alpha} v_\gamma \right) \bar{w}_{-1,i,i+1} \right) \\
&\quad + (1-r) z_1 \bar{w}_{-1,2} \\
&\quad + z_{m-1} \left(v_\alpha w_{m-1} + \left(\sum_{\gamma \neq \alpha} v_\gamma \right) \bar{w}_{-1,m-1,m} \right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{t \in T_b} y_t (1 - \sigma_t(\alpha)) - \sum_{t \in T_{ba}} x_t (\sigma_t(b) - \sigma_t(\alpha)) = \\
&\qquad \sum_{i=1}^{m-1} z_i - r u_\alpha \sum_{i=1}^{m-2} z_i w_{i+1} + r(1-u_\alpha) \sum_{i=1}^{m-2} z_i (\bar{w}_{-i,i+1,m} - \bar{w}_{-1,i+1,m}) \\
&\qquad\qquad + (1-r) v_\alpha \sum_{i=2}^{m-2} z_i (1 - w_i) + z_{m-1} v_\alpha (1 - w_{m-1}) - \sum_{i=1}^{m-1} z_i w_i \\
&\qquad = \sum_{i=1}^{m-1} z_i (1 - w_i) - r u_\alpha \sum_{i=1}^{m-1} z_i w_{i+1} + r v_\alpha \sum_{i=1}^{m-2} z_i (1 - w_i) \\
&\qquad\qquad + (1-r) v_\alpha \sum_{i=1}^{m-2} z_i (1 - w_i) + z_{m-1} v_\alpha (1 - w_{m-1}) \\
&\qquad = (1 + v_\alpha) B - r u_\alpha A.
\end{aligned}$$

The $(x_t), (y_t)$ that we have constructed will thus satisfy the constraints required to be feasible for (2), provided r

and (u_α) can be chosen in such a way that

$$\begin{aligned}
B + rA &\geq |a| - |b| \\
(1+v_\alpha)B - ru_\alpha A &\geq |\alpha| - |b| \qquad\qquad \forall \alpha \in D \\
\sum_{\alpha \in D} u_\alpha &= 1 \\
u_\alpha &\geq 0 \qquad\qquad \forall \alpha \in D \\
0 \leq r &\leq 1.
\end{aligned}$$

Suppose for the moment that $m \geq 4$. Then the inequality required of u_α may be written

$$\left(1 + \frac{1-u_\alpha}{m-3}\right) B - ru_\alpha A \geq |\alpha| - |b|\,,$$

or

$$\left(rA + \frac{B}{m-3}\right) u_\alpha \leq |b| - |\alpha| + \left(\frac{m-2}{m-3}\right) B,$$

a simple upper bound on u_α. Note that the upper bound is non-negative. For given r, it will be possible to choose non-negative (u_α) such that $\sum_{\alpha \in D} u_\alpha = 1$ while complying with these upper bounds if and only if

$$\sum_{\alpha \in D} \left(|b| - |\alpha| + \left(\frac{m-2}{m-3}\right) B\right) \geq rA + \frac{B}{m-3},$$

or

$$rA \leq (m-2)\,|b| - \sum_{\alpha \in D} |\alpha| + \left(\frac{(m-2)^2 - 1}{m-3}\right) B,$$

that is

$$rA \leq (m-1)(|b| + B - n\bar{w}) + |a| - n\bar{w}, \tag{$*$}$$

using the fact that the sum of all the candidates' scores is $mn\bar{w}$.

Consider now the case $m = 3$. Then the sole $\alpha \in D$ has $u_\alpha = v_\alpha = 1$, and the inequality required of u_α reduces to

$$2B - rA \geq |\alpha| - |b|\,,$$

which is $(*)$.

We have thus reduced our requirements to a condition on $r \in [0,1]$:

$$|a| - |b| - B \;\leq\; rA \;\leq\; (m-1)(|b| + B - n\bar{w}) + (|a| - n\bar{w}).$$

To see that there exists $r \in [0,1]$ satisfying this condition we note that, firstly,

$$(m-1)(|b| + B - n\bar{w}) + (|a| - n\bar{w}) \geq 0;$$

secondly,

$$|a| - |b| - B \leq A$$

(from the first condition of (3)); and thirdly,

$$|a| - |b| - B \leq (m-1)(|b| + B - n\bar{w}) + (|a| - n\bar{w}).$$

This last condition can be simplified to

$$m(|b| + B - n\bar{w}) \geq 0,$$

the second condition of (3). ∎

Corollary 10. The linear program (2) has the same optimal value as the following one.

$$
\begin{aligned}
&\min \sum_{i=1}^{m-1} z_i \\
&\text{s.t.} \sum_{i=1}^{m-1} (1 - w_i + w_{i+1}) z_i \geq |a| - |b| \\
&\qquad \sum_{i=1}^{m-1} (1 - w_i) z_i \geq n\bar{w} - |b| \\
&\qquad z_i \geq 0 \qquad \text{for } i = 1, \ldots, m-1
\end{aligned}
\tag{4}
$$

Proof. Theorem 9 shows that for any feasible point of (4), there corresponds a feasible point of (2) with the same objective value. Hence the optimal value of (2) cannot be greater than that of (4).

To show that the optimal value of (2) cannot be less than that of (4), suppose we have $(x_t)_{t \in T_{ba}}$, $(y_t)_{t \in T_b}$ feasible for (2). Let $z_i = \sum_{t \in T_i} x_t$. Then $\sum_{i=1}^{m-1} z_i = \sum_{t \in T_{ba}} x_t$, and we will show that $(z_i)_{i=1}^{m-1}$ is feasible for (4). The inequality for $\alpha = a$ in (2) says that

$$\sum_{t \in T_b} y_t (1 - \sigma_t(a)) \geq |a| - |b| + \sum_{t \in T_{ba}} x_t (\sigma_t(b) - \sigma_t(a)).$$

Noting that $\sum_{t \in T_b} y_t \sigma_t(a) \geq 0$, we obtain

$$\sum_{t \in T_b} y_t \geq |a| - |b| + \sum_{i=1}^{m-1} \sum_{t \in T_i} x_t (w_i - w_{i+1})$$

and so

$$\sum_{i=1}^{m-1} z_i \geq |a| - |b| + \sum_{i=1}^{m-1} (w_i - w_{i+1}) z_i$$

from which the first constraint of (4) can be seen to hold.

If we add the inequalities in (2) for all $\alpha \neq b$, we obtain

$$\sum_{t \in T_b} y_t \sum_{\alpha \neq b} (1 - \sigma_t(\alpha)) \geq \left(\sum_{\alpha \neq b} |\alpha| \right) - (m-1)|b| + \sum_{t \in T_{ba}} x_t \sum_{\alpha \neq b} (\sigma_t(b) - \sigma_t(\alpha)),$$

or

$$\sum_{t \in T_b} y_t \sum_{i=1}^{m-1} (1 - w_i) \geq nm\bar{w} - m|b| + \sum_{i=1}^{m-1} \sum_{t \in T_i} x_t \sum_{j \neq i} (w_i - w_j),$$

which gives

$$(1 - \bar{w}) \sum_{i=1}^{m-1} z_i \geq n\bar{w} - |b| + \sum_{i=1}^{m-1} z_i (w_i - \bar{w}),$$

from which the second constraint of (4) can be seen to hold. ∎

Theorem 11. The linear programs (2) and (4) have the same objective value as the following one.

$$
\begin{aligned}
&\max\ (|a| - n\bar{w})\lambda + (n\bar{w} - |b|)\mu \\
&\text{s.t. } w_{i+1}\lambda + (1 - w_i)\mu \leq 1 \qquad \text{for } i = 1, \ldots, m-1 \\
&\qquad 0 \leq \lambda \leq \mu.
\end{aligned}
\tag{5}
$$

Also, (5) is unbounded if and only if (2) and (4) are infeasible.

Proof. The linear program (4) has the same objective value as its dual program ([**1, p. 251**]):

$$\begin{aligned}&\max\ (|a|-|b|)\lambda+(n\bar{w}-|b|)\lambda' \\ &\text{s.t. } (1-w_i+w_{i+1})\lambda+(1-w_i)\lambda'\le 1 \qquad \text{for } i=1,\dots,m-1 \\ &\qquad \lambda\ge 0,\quad \lambda'\ge 0.\end{aligned}$$

Substituting $\mu=\lambda+\lambda'$ yields (5). ∎

Remark. We have now replaced our original description of the minimum manipulating coalition size with the simple two-variable linear program (5). This will be exploited in the remaining sections of the paper.

4. IC asymptotics of the minimum manipulating coalition size

The results of the previous section have established that

$$P\left(|MCS-Q|\le K\right)\to 1 \qquad \text{as } n\to\infty,$$

where

$$Q=\max\left\{\lambda(|a|-n\bar{w})+\mu(n\bar{w}-|b|):(\lambda,\mu)\in M_w\right\}$$

and

$$M_w=\left\{(\lambda,\mu):0\le\lambda\le\mu \text{ and } w_{i+1}\lambda+(1-w_i)\mu\le 1 \text{ for } i=1,\dots,m-1\right\}.$$

Note in particular that the constraint set M_w does not depend on the voting situation, but only on the voting rule. For a given rule, it is possible to identify the corresponding M_w (a two-dimensional linear polytope), and then to identify the (finitely many) vertices of M_w which may achieve the optimum Q. This often leads to an explicit expression for Q in terms of $|a|$ and $|b|$.

Furthermore, we have $(|a|,|b|)=(\rho_1(S),\rho_2(S))$, where $\rho_j(x)$ denotes the jth largest element of a vector x. Proposition 2 gives us

$$\frac{(\rho_1(S)-n\bar{w},n\bar{w}-\rho_2(S))}{\sqrt{n}}\xrightarrow{D}\sigma_w\left(\frac{m}{m-1}\right)^{1/2}(\rho_1(Z)-\bar{Z},\bar{Z}-\rho_2(Z)),$$

and so it follows that

$$\frac{Q}{\sqrt{n}}\xrightarrow{D}V_w,$$

where

$$V_w=\max\left\{\lambda(\rho_1(Z)-\bar{Z})+\mu(\bar{Z}-\rho_2(Z)):(\lambda,\mu)\in\sigma_w\left(\frac{m}{m-1}\right)^{1/2}M_w\right\}.$$

Hence, too,

$$\frac{MCS}{\sqrt{n}}\xrightarrow{D}V_w,$$

by the converging-together lemma ([**5, p.91**]). Consequently,

$$P\left(MCS\le v\sqrt{n}\right)\to g_w(v):=P\left(V_w\le v\right) \qquad \text{as } n\to\infty.$$

That is, the asymptotic probability that the voting situation is manipulable by a coalition of $v\sqrt{n}$ or fewer voters is computable as a (non-decreasing) function of v. This function depends only on the voting rule.

A further observation will be helpful in determining which vertices of M_w may achieve the above maximum. If $x\in\mathbf{R}^m$ has mean element $\bar{x}=(x_1+\cdots+x_m)/m$, then

$$\rho_1(x)-\bar{x}\ge 0 \qquad \text{and} \qquad -\left(\rho_1(x)-\bar{x}\right)\le\bar{x}-\rho_2(x)\le\frac{1}{m-1}(\rho_1(x)-\bar{x}).$$

We note in passing that for all rules other than the anti-plurality rule $w=(1,\dots,1,0)$, M_w is a bounded set, since the constraints defining it include $(1-w_{m-1})\mu\le 1$. Hence V_w is a finite-valued random variable. It follows that $P\left(MCS=\infty\right)\to 0$ as $n\to\infty$. For the anti-plurality rule itself, M_w is unbounded and V_w may take ∞ as a value; see below for further detail.

The remainder of this section will be devoted to carrying out the above analysis for some common positional voting rules.

Borda's rule. $w_i = (m-i)/(m-1)$ for $i = 1, \ldots, m$. The constraints defining M_w for this rule are

$$\begin{aligned} (m-2)\lambda &\leq m-1 \\ (m-3)\lambda + \mu &\leq m-1 \\ &\vdots \\ \lambda + (m-3)\mu &\leq m-1 \\ (m-2)\mu &\leq m-1 \\ 0 \leq \lambda &\leq \mu \end{aligned}$$

Note that the point with $\lambda = \mu = (m-1)/(m-2)$ satisfies each of the constraints with equality. So M_w is the triangle with vertices at $(0,0)$, $(0, \frac{m-1}{m-2})$, and $(\frac{m-1}{m-2}, \frac{m-1}{m-2})$; the last of these always achieves the optimum Q. Thus, a good approximation to the minimum manipulating coalition size is

$$Q = \left(\frac{m-1}{m-2}\right)(|a| - |b|).$$

For the corresponding asymptotic result, note that

$$\sigma_w^2 = \frac{m+1}{12(m-1)}.$$

It then transpires that

$$\frac{MCS}{\sqrt{n}} \overset{D}{\to} \left(\frac{m(m+1)}{12(m-2)^2}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Anti-plurality rule: $w = (1, \ldots, 1, 0)$. The constraints defining M_w reduce to $0 \leq \lambda \leq 1$, $\mu \geq \lambda$. Thus

$$Q = \begin{cases} |a| - |b|, & \text{if } |b| \geq n\bar{w} \\ \infty, & \text{otherwise.} \end{cases}$$

The corresponding asymptotic result is

$$\frac{MCS}{\sqrt{n}} \overset{D}{\to} V_w = \begin{cases} m^{-1/2}(\rho_1(Z) - \rho_2(Z)), & \text{if } \rho_2(Z) \geq \bar{Z} \\ \infty, & \text{otherwise.} \end{cases}$$

As a consequence of this, we can find the limiting probability that an anti-plurality election is invulnerable to manipulation:

$$\lim_n P(MCS = \infty) = P(V_w = \infty) = P\left(\rho_2(Z) < \bar{Z}\right).$$

The limit is positive, a property unique to anti-plurality. For all other positional rules, $\lim_n P(MCS = \infty) = 0$, a result essentially contained in **[7]**.

Plurality and k-approval rules: $w = (1, \ldots, 1, 0, \ldots 0)$ (with k 1s), where $1 \leq k \leq m-2$. The simple plurality rule is included here as the case $k = 1$. The constraints defining M_w reduce to $0 \leq \lambda \leq \mu \leq 1$, so M_w is the triangle with vertices at $(0,0)$, $(0,1)$, and $(1,1)$; the last of these always achieves the optimum. Thus, our approximation of the minimum manipulating coalition size is simply

$$Q = |a| - |b|.$$

While this expression is valid for all k-approval rules ($1 \leq k \leq m-2$), different values of k will give rise to different probability distributions for $(|a|, |b|)$, and so different asymptotic results for MCS. We have $\bar{w} = k/m$ and $\sigma_w^2 = k(m-k)/m^2$, giving

$$\frac{MCS}{\sqrt{n}} \overset{D}{\to} \left(\frac{k(m-k)}{m(m-1)}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Three-candidate "easy case" rules: $w = (1, 1-p, 0)$ where $1/2 \le p \le 1$. This family includes the 3-candidate versions of plurality voting ($p = 1$) and Borda's rule ($p = 1/2$). The constraints defining M_w are $0 \le \lambda \le \mu$, $(1-p)\lambda \le 1$, and $p\mu \le 1$. Thus M_w is the triangle with vertices at $(0,0)$, $(0, p^{-1})$, and (p^{-1}, p^{-1}); the last of these always achieves the optimum. Thus

$$Q = p^{-1}(|a| - |b|).$$

We have $\sigma_w^2 = 2(1 - p + p^2)/9$; it follows from this that

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} \left(\frac{1 - p + p^2}{3p^2}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Three-candidate "hard case" rules: $w = (1, 1-p, 0)$ where $0 < p \le 1/2$. This family includes the remaining 3-candidate positional rules not already considered. For these rules, M_w is the quadrilateral with vertices at $(0,0)$, $(0, p^{-1})$, $((1-p)^{-1}, (1-p)^{-1})$, and $(1-p)^{-1}, p^{-1})$. In voting situations with $|b| \ge n\bar{w}$, the optimum for Q is achieved at $((1-p)^{-1}, (1-p)^{-1})$; otherwise, $(1-p)^{-1}, p^{-1})$ is optimal. Thus

$$Q = \begin{cases} (1-p)^{-1}(|a| - |b|), & \text{if } |b| \ge n\bar{w} \\ (1-p)^{-1}(|a| - n\bar{w}) + p^{-1}(n\bar{w} - |b|), & \text{if } |b| \le n\bar{w} \end{cases}$$

that is

$$Q = (1-p)^{-1}(|a| - |b|) + \left(\frac{1}{p} - \frac{1}{1-p}\right)(n\bar{w} - |b|)_+,$$

where x_+ denotes $\max(x, 0)$. The corresponding asymptotic result is

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} \left(\frac{1 - p + p^2}{3}\right)^{1/2} \left(\left(\frac{1}{1-p}\right)(\rho_1(Z) - \rho_2(Z)) + \left(\frac{1}{p} - \frac{1}{1-p}\right)(\bar{Z} - \rho_2(Z))_+\right).$$

Four-candidate rules. For $m \ge 4$ it would be tedious to reduce all possible cases to asymptotic expressions of the kind above. Instead, we have simply illustrated the sets $\sigma_w M_w$ for a variety of rules w in Figure 1. Note that $\sigma_w M_w$ may have up to 3 optimal vertices. (For general m, $\sigma_w M_w$ might have up to $m-1$ optimal vertices.)

The functions $g_w(v) = P(V_w \le v)$ for some common voting rules are shown in Figures 2, 3, and 4.

5. Comparisons between positional voting rules.

In this section we compare the various positional voting rules with respect to their manipulability under IC asymptotic conditions.

It is apparent from Figure 2 that the susceptibility of voting rules to coalitional manipulation depends on the size of the coalition involved. The graph for $m = 3$, for example, shows that elections using the plurality or Borda rules are highly likely to be manipulable by a large coalition (at least $2\sqrt{n}$ voters), whereas only about half of anti-plurality elections are so manipulable. But, if one is more concerned about manipulation by small groups, the graph shows that plurality and anti-plurality elections are about equally susceptible to manipulation by coalitions of less than $0.25\sqrt{n}$ voters, while Borda is rather less susceptible.

This suggests that there will be no single rule which is clearly superior to all others with respect to IC coalitional manipulation. However, some are clearly inferior to others. The asymptotic manipulation probability $g_w(v)$ for the plurality rule, for example, is greater than for the other two 3-candidate rules shown for all values of v.

More formally, given two m-candidate positional voting rules w, w', we will say that w *dominates* w' (and write $w' \preceq w$) if $g_w(v) \le g_{w'}(v)$ for all $v \ge 0$. That is, w is less susceptible than w' to manipulation by coalitions of any given size. Alternatively, the asymptotic minimum coalition size V_w is larger than $V_{w'}$ in the sense of first-order stochastic dominance.

Note that $\preceq$ gives a partial order on the rules. Although the present paper is concerned only with positional rules, the partial order $\preceq$ could be defined in the same way for any voting rules.

If $w' \preceq w$, then the rule w is to be preferred to w'. More generally, the best rules to use (at least from the point of view of manipulation by IC populations) are those not dominated by any other.

Proposition 12. The plurality rule is always dominated.

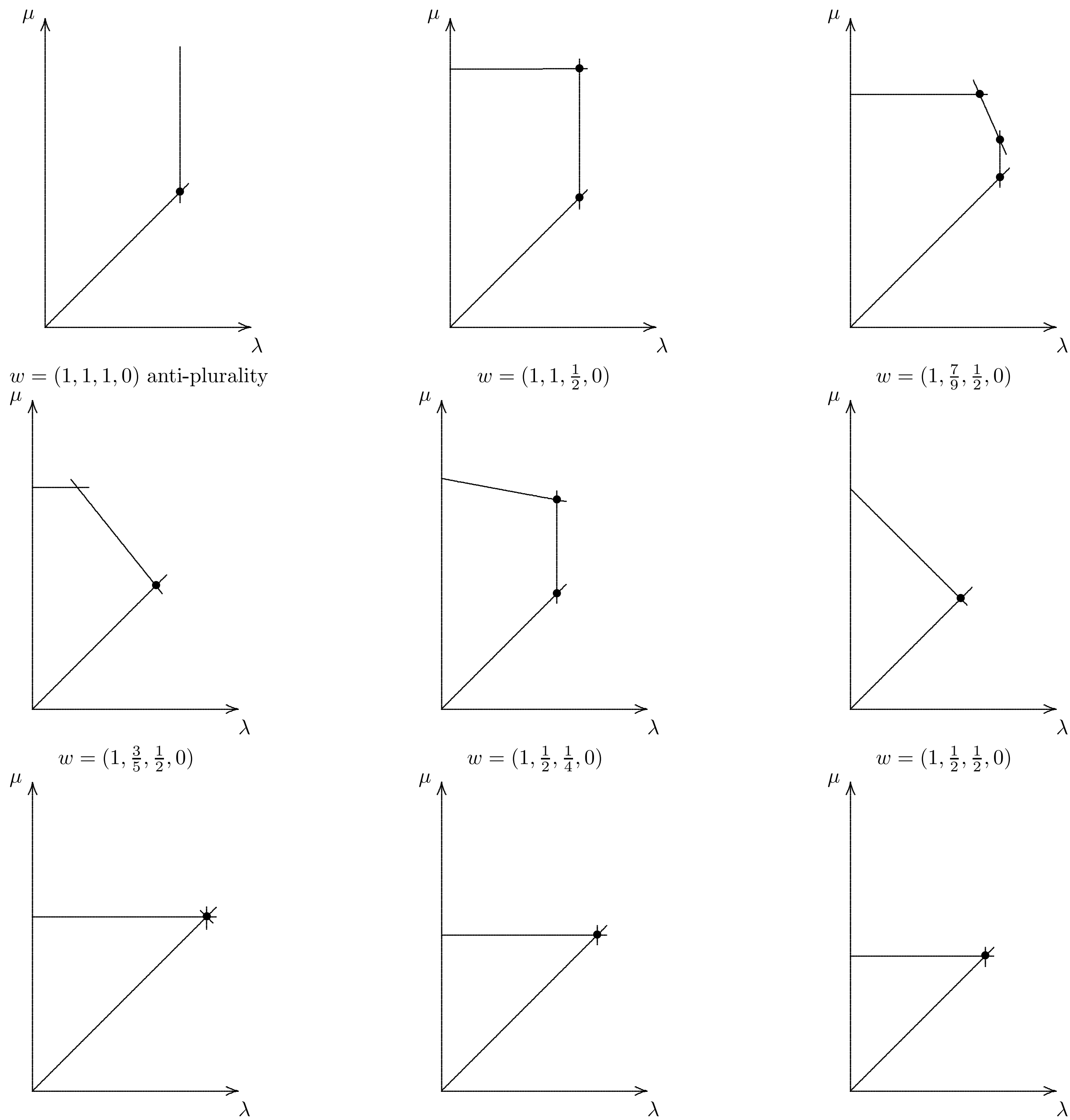

Figure 1: the sets $\sigma_w M_w$ for several four-candidate positional rules, depicted at consistent scales. The black dots show which points may be optimal.

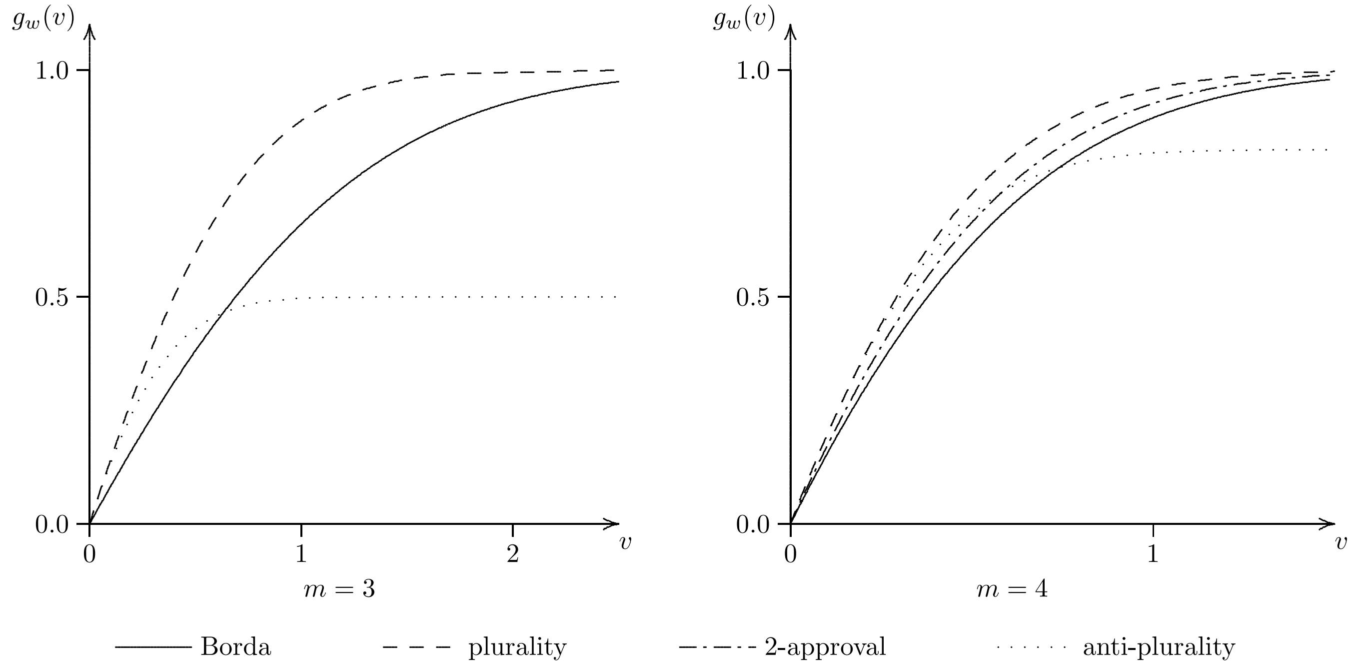

Figure 2: the functions $g_w(v) = P(V_w \leq v)$ for some three- and four-candidate voting rules.

Proof. Indeed, plurality is dominated both by anti-plurality and by Borda. This is apparent from the asymptotic results of the previous section:

$$V_{\text{antiplurality}} = \begin{cases} m^{-1/2}(\rho_1(Z) - \rho_2(Z)), & \text{if } \rho_2(Z) \geq \bar{Z} \\ \infty, & \text{otherwise} \end{cases},$$

while

$$V_{\text{plurality}} = m^{-1/2}(\rho_1(Z) - \rho_2(Z)),$$

giving $V_{\text{plurality}} \leq V_{\text{antiplurality}}$ and so (plurality) $\preceq$ (anti-plurality). Similarly for Borda. ∎

Proposition 13. The anti-plurality rule is never dominated.

Proof. No other rule w may dominate the anti-plurality rule, because $\lim_{v\to\infty} g_{\text{antiplurality}}(v) = P\left(\rho_2(Z) \geq \bar{Z}\right) < 1$, whereas $\lim_{v\to\infty} g_w(v) = 1$. ∎

Proposition 13 is true because the anti-plurality rule is resistant to manipulation by very large coalitions, in a way that every other positional rule is not. However, as we shall see later, this advantage becomes very slight in elections with 6 or more candidates.

Proposition 14. Borda's rule is undominated for $m \in \{3, 4\}$, but dominated for $m \geq 5$.

Proof. From the asymptotic results of the previous section:

$$V_{\text{Borda}} = \left(\frac{m(m+1)}{12(m-2)^2}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)),$$

while

$$V_{\lfloor m/2 \rfloor-\text{approval}} = \left(\frac{\lfloor m/2 \rfloor (m - \lfloor m/2 \rfloor)}{m(m-1)}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Note that for $m \geq 5$,

$$\frac{\lfloor m/2 \rfloor (m - \lfloor m/2 \rfloor)}{m(m-1)} \geq \frac{(m-1)(m+1)}{4m(m-1)} = 3\left(1 - \frac{2}{m}\right)^2 \frac{m(m+1)}{12(m-2)^2} \geq \frac{m(m+1)}{12(m-2)^2}.$$

This gives $V_{\text{Borda}} \leq V_{\lfloor m/2 \rfloor-\text{approval}}$ and so (Borda) $\preceq$ ($\lfloor m/2 \rfloor$-approval).

Now let $m \in \{3,4\}$ and let w be any positional rule for m-candidate elections. We may observe that

$$M_w \subseteq M'_w := \left\{(\lambda,\mu) \in \mathbf{R}^2 : 0 \leq \lambda \leq b_w, \lambda \leq \mu\right\},$$

where

$$b_w = \min_i (1 - w_i + w_{i+1})^{-1}.$$

It is shown in **[12]** that the quantity $\sigma_w b_w$ is maximized, over all m-candidate positional rules, by Borda's rule. (This is true only when $m \in \{3,4\}$; for $m \geq 5$, the maximum value is achieved by the $\lfloor m/2 \rfloor$-approval rule.) From this we obtain

$$\begin{aligned}
g_w(v) &= P\left(V_w \leq v\right) \\
&\geq P\left(V_w \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&\geq P\left(\max\left\{\lambda(\rho_1(Z) - \bar{Z}) + \mu(\bar{Z} - \rho_2(Z)) : (\lambda,\mu) \in \left(\frac{m}{m-1}\right)^{1/2} \sigma_w M_w\right\} \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&\geq P\left(\max\left\{\lambda(\rho_1(Z) - \bar{Z}) + \mu(\bar{Z} - \rho_2(Z)) : (\lambda,\mu) \in \left(\frac{m}{m-1}\right)^{1/2} \sigma_w M'_w\right\} \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&= P\left(\left(\frac{m}{m-1}\right)^{1/2} \sigma_w b_w (\rho_1(Z) - \rho_2(Z)) \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&\geq P\left(\left(\frac{m}{m-1}\right)^{1/2} \sigma_{\text{Borda}} b_{\text{Borda}} (\rho_1(Z) - \rho_2(Z)) \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&= P\left(V_{\text{Borda}} \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&= g_{\text{Borda}}(v) \;-\; P\left(V_{\text{Borda}} \leq v \text{ and } \rho_2(Z) < \bar{Z}\right).
\end{aligned}$$

Let $c = (m/(m-1))^{1/2} \sigma_{\text{Borda}} b_{\text{Borda}}$. Since all orderings of $Z = (Z_1, \ldots, Z_m)$ are equally likely,

$$\begin{aligned}
P\left(V_{\text{Borda}} \leq v \text{ and } \rho_2(Z) < \bar{Z}\right) &= m! P\left(Z_1 \geq Z_2 \geq \cdots \geq Z_m,\; c(Z_1 - Z_2) \leq v, \;\text{ and } Z_2 < \bar{Z}\right) \\
&\leq m! P\left(0 \leq \bar{Z} - Z_2 \leq Z_1 - Z_2 \leq v/c\right) \\
&= m! \int_0^{v/c} \int_0^x f(x,y)\, dx dy \\
&\leq \frac{1}{2} m! (\sup f) \left(\frac{v}{c}\right)^2 \\
&= O(v^2) \qquad\qquad \text{as } v \to 0,
\end{aligned}$$

where f is the (non-degenerate) bivariate normal probability density of $(Z_1 - Z_2, \bar{Z} - Z_2)$. It follows that

$$g'_w(0) \geq g'_{\text{Borda}}(0),$$

and hence that w cannot dominate Borda. ∎

The argument used in Proposition 14 shows that among all positional rules for three- or four- candidate elections, Borda's rule has the g function with the smallest derivative at the origin. This means that it is the most resistant to manipulation by very small coalitions. However, this property does not hold when there are five or more candidates. In that case, it is the $\lfloor m/2 \rfloor$-approval rule which enjoys maximal resistance to manipulation by very small coalitions. These results are similar to those of **[12]**, although the criterion considered there is the "average threshhold coalition size" rather than the minimum manipulating coalition size of the present paper.

Figures 2–4 show the manipulability of the rules for particular numbers of candidates. We see from Figures 2 and 3 that for $m = 4, 5, 6$ there is not much difference in the manipulability of the common rules, at least by comparison with the differences evident when $m = 3$. In the graph for $m = 5$ we can see the Borda rule being dominated (by 2-approval) for the first time. Also worth noting is that for $m \geq 5$ there is very little difference between plurality and

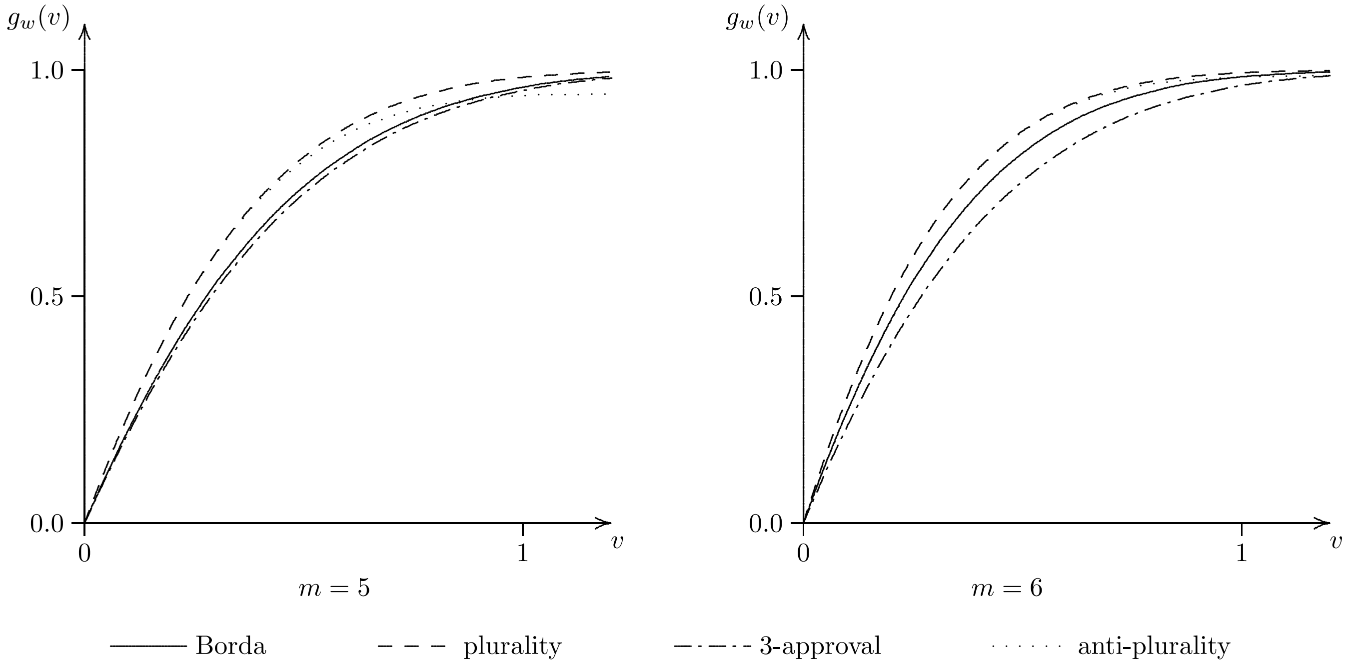

Figure 3: the functions $g_w(v) = P\left(V_w \leq v\right)$ for some five- and six-candidate voting rules.

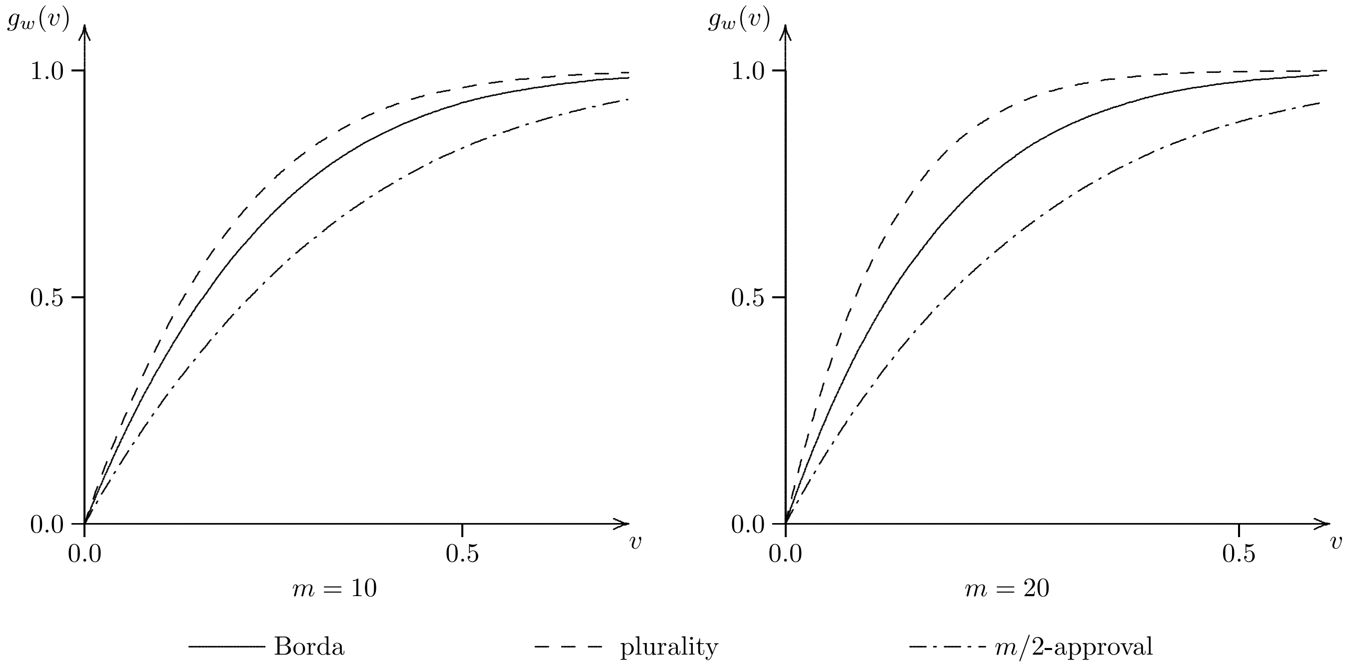

Figure 4: the functions $g_w(v) = P\left(V_w \leq v\right)$ for some ten- and twenty-candidate voting rules.

anti-plurality from a manipulation point of view. Anti-plurality has a slight additional chance of resisting attack by a large coalition − and on this basis dominates plurality − but this advantage has become almost imperceptible by $m = 6$.

Figure 4 shows the behaviour of the rules when there are many candidates. In this figure, the curves for the anti-plurality rule have been left out, as they are indistinguishable from those for plurality. We see the $\lfloor m/2 \rfloor$-approval rule dominating the others. It should be noted, though, that the IC hypothesis is possibly unconvincing when applied to elections with many candidates, as it assumes in effect that all candidates are about equally popular.

6. Conclusions.

The technique presented in this paper makes it possible to compute the (IC) limiting probability distribution of the minimum manipulating coalition size for any positional rule, with any number of candidates.

The consideration of coalition sizes is especially useful when comparing the rules. Some rules are especially resistant to manipulation by small coalitions, while others fare better with respect to manipulation by large coalitions. Previous work has made this distinction in a rather limited way, by considering "individual" and "coalitional" manipulation (the latter meaning that the coalition may be of any size). But these extremes may be somewhat uninformative. Given a large voter population, all positional rules (except anti-plurality) are highly likely to be manipulable by some coalition, and highly unlikely to be manipulable by any individual. Studying coalitions of intermediate sizes starts to reveal more differences between the rules.

The picture also changes when the number of candidates is varied. Much previous work has concentrated on the three-candidate case, for which the behaviour of the rules is quite different (Borda is least susceptible to small-coalition manipulation, anti-plurality to large-coalition manipulation). The four-candidate case is similar, except that the differences between rules are smaller. But when there are five candidates, it appears that all positional rules are about equally manipulable, across the whole range of coalition sizes, and there is not much to choose between them. With six or more candidates, the $\lfloor m/2 \rfloor$-approval rules emerge as favourites.

Another, perhaps surprising, conclusion is that for $m \geq 5$ there is very little difference between plurality and anti-plurality from a manipulation point of view. The approximate symmetry between these rules does not appear when $m = 3$, and so does not appear to have been noticed before (although it was recognized in a more limited sense already in [**14**]).

It would be possible in principle to produce results like those in this paper for the IAC voter behaviour model. However, the technique of analysis would have to be quite different. Rather than reducing the probabilities to those involving normal distributions, the calculations would entail the computation of convex volumes, as outlined in [**16**].

It would also be of interest to produce limiting distributions similar to those in this paper (or at least, graphs like those in Figures 2–4) for voting rules other than positional rules. Approval voting ([**2**]) would be an especially attractive target. However, this too would require new techniques.

References

[**1**] M. Bazaraa, J. Jarvis, and H. Sherali, *Linear programming and network flows,* Wiley, 2005.

[**2**] S. Brams and P. Fishburn, *Approval voting,* Springer, 2007.

[**3**] S. Berg and D. Lepelley, *On probability models in voting theory,* Statistica Neerlandica 48 (1994), 133–146.

[**4**] J. Chamberlin, *Investigation into the relative manipulability of four voting systems,* Behavior Science 30 (1985), 195–203.

[**5**] R. Durrett, *Probability: Theory and examples,* Duxbury, 1996.

[**6**] A. Gibbard, *Manipulation of voting schemes: a general result,* Econometrica 41 (1973), 587–601.

[**7**] K. H. Kim and F. W. Roush, *Statistical manipulability of social choice functions,* Group Decision and Negotiation 5 (1996), 263–282.

[**8**] D. Lepelley and B. Mbih, *The proportion of coalitionally unstable situations under the plurality rule,* Economic Letters 24 (1987), 311–315.

[**9**] S. Nitzan, *The vulnerability of point-voting schemes to preference variation and strategic manipulation,* Public Choice 47 (1985), 349–370.

[**10**] C. Papadimitriou and K. Steiglitz, *Combinatorial optimization,* Prentice-Hall, 1982.

[**11**] G. Parker and R. Rardin, *Discrete Optimization,* Academic Press, 1988.

[**12**] G. Pritchard and A. Slinko, *On the average minimum size of a manipulating coalition,* Social Choice and Welfare 27 (2006), 263–277.

[**13**] G. Pritchard and M. Wilson, *Exact results on manipulability of positional voting rules,* Social Choice and Welfare, in press.

[**14**] D. Saari, *Susceptibility to manipulation,* Public Choice 64 (1990), 21–41.

[**15**] M. A. Satterthwaite, *Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions,* J. Economic Theory 10 (1975), 187–217.

[16] M. Wilson and G. Pritchard, *Probability calculations under the IAC hypothesis,* Mathematical Social Sciences, in press.