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Abstract

Let $\sum_{\mathbf{n}\in\mathbb{N}^d} f_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$ be a multivariate generating function that converges in a neighborhood of the origin of $\mathbb{C}^d$. We present a general theorem for computing the asymptotics of the diagonal coefficients $f_{a_1n,\ldots,a_dn}$.

1 Introduction

This article presents some recent progress in the asymptotics of diagonal coefficients of multivariate generating functions and can be seen as an extension of [RW]. Before beginning, let us set some notation. Boldface letters denote row vectors, with the symbols $\mathbf{0}$ and $\mathbf{1}$ denoting the vectors of all zeros and all ones, respectively. Component i of a vector $\mathbf{x}$ is denoted by x_i. We also use multi-index notation, so that $\mathbf{x}^{\mathbf{n}} = x_1^{n_1}\cdots x_d^{n_d}$ and $\partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$, where $\partial_i = \partial/\partial x_i$.

Let $F(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}^d} f_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$ be a complex power series that converges in a neighborhood of the origin but not on all of $\mathbb{C}^d$. We wish to compute asymptotics for the diagonal coefficients $f_{a_1n,\ldots,a_dn}$ for fixed positive integers $a_1,\ldots,a_d$, a task often useful in enumerative combinatorics. To this end, we apply multivariate singularity analysis (in the style of [PW02], [PW04], [BP]) directly to $F(\mathbf{x})$ to compute the asymptotics. For simplicity of presentation, we suppose that $f_{\mathbf{n}} \geq 0$ and that F is rational, although much greater generality is possible.

2 Results

Let $\mathcal{D} \subset \mathbb{C}^d$ be the open domain of convergence of F and write $F(\mathbf{x}) = I(\mathbf{x})/J(\mathbf{x})^p$ for some I and J holomorphic on an open domain $\mathcal{D}'$ containing the closure of $\mathcal{D}$

and relatively prime in the ring of holomorphic functions on $\mathcal{D}'$ and for some positive integer p. Let $\mathcal{V}$ be the complex variety $\{\mathbf{x} \in \mathbb{C}^d : J^p(\mathbf{x}) = 0\} = \{\mathbf{x} \in \mathbb{C}^d : J(\mathbf{x}) = 0\}$.

A **critical point** of F for $\mathbf{n} \in (\mathbb{N}^+)^d$ is a solution of

$$\begin{aligned} J(\mathbf{x}) &= 0 \\ n_d x_i \partial_i J(\mathbf{x}) &= n_i x_d \partial_d J(\mathbf{x}) \quad (i < d). \end{aligned}$$

Let $\textsc{Crit}(\mathbf{n})$ denote the set of all critical points of F for $\mathbf{n}$. For generic directions $\mathbf{n}$, this set is finite, being a zero-dimensional complex variety. The main situation in which $\textsc{Crit}(\mathbf{n})$ is infinite occurs when J defines a binomial variety $\{\mathbf{x} \mid \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}\}$, in which case $\textsc{Crit}(\mathbf{n})$ is empty for all but one direction and uncountable otherwise. Such examples can be analysed by a variant of the methods shown here.

A **contributing point** of F for $\mathbf{n}$ is a critical point that influences the asymptotics of the coefficients of F in the direction of $\mathbf{n}$. Let $\textsc{Contrib}(\mathbf{n})$ denote the set of all such points. While $\textsc{Contrib}(\mathbf{n})$ is ill-defined here, its functional role will become clear from the next two theorems.

Theorem 2.1 ([PW]). If $\textsc{Crit}(\mathbf{n})$ is finite, then

- $\textsc{Crit}(\mathbf{n})$ contains exactly one point, call it $\mathbf{c}$, that lies in the positive orthant of $\mathbb{R}^d$, and $\mathbf{c} \in \textsc{Contrib}(\mathbf{n})$;
- all other members of $\textsc{Contrib}(\mathbf{n})$ lie on $T(\mathbf{c})$;
- all members $\mathbf{p}$ of $\textsc{Contrib}(\mathbf{n})$ satisfy $\mathcal{V} \cap D(\mathbf{p}) \subseteq T(\mathbf{p})$;
- in the case where $J = 1 - P$ for some aperiodic power series with nonnegative coefficients P, $\textsc{Contrib}(\mathbf{n}) = \{\mathbf{c}\}$.

Here $T(\mathbf{p})$ and $D(\mathbf{p})$ denote respectively the **polytorus** and the **polydisk** of a point $\mathbf{p} \in \mathbb{C}^d$, that is, respectively the sets $\{\mathbf{x} \in \mathbb{C}^d : \forall i \leq d \ |x_i| = |c_i|\}$ and and $\{\mathbf{x} \in \mathbb{C}^d : \forall i \leq d \ |x_i| \leq |c_i|\}$, and a power series is **aperiodic** if the $\mathbb{Z}$-span of its monomial vectors is all of $\mathbb{Z}^d$.

A point $\mathbf{c} \in \mathcal{V}$ is a **smooth point** if $\mathcal{V}$ is a smooth complex manifold in a neighborhood of $\mathbf{c}$, or equivalently, if $\partial_i J(\mathbf{c}) \neq 0$ for some i (see [BK86, page 363] for instance). For simplicity of presentation, we deal only with smooth points in this article. This is the generic case, although interesting examples are not always generic. For more on the case of non-smooth points, see [PW04].

Since J is holomorphic, given a smooth point $\mathbf{c}$ of $\mathcal{V}$ with $\partial_d J(\mathbf{c}) \neq 0$, say, there exist, by the implicit function theorem, a holomorphic function g and an open ball around $\mathbf{c}$ such that $J(\widehat{\mathbf{x}}, g(\widehat{\mathbf{x}})) = 0$. We will refer to this function g throughout.

We now state a technical lemma on the asymptotics of oscillatory integrals, a simplification of [Hör83, Theorem 7.7.5]. Because this lemma gives an explicit, albeit complicated, formula for the coefficients of the asymptotic series, it allows one to calculate straightforwardly coefficients beyond the leading term. In contrast, the more common, nonconstructive-Morse-lemma approximations found in the literature do not provide this.

Lemma 2.2. Let $K \subset \mathbb{R}^{d-1}$ be a compact set, X an open neighborhood of K, and N a positive integer. If $\psi : K \to \mathbb{C}$ is $2N$-continuously differentiable with compact support, $h : X \to \mathbb{C}$ is $(3N+1)$-continuously differentiable, $\Re h \geq 0$ in X, $\Re h(\boldsymbol{\theta}_0) = \mathbf{0}$, $\mathbf{h}'(\boldsymbol{\theta}_0) = \mathbf{0}$, and $\mathbf{h}' \neq \mathbf{0}$ in $K \setminus \boldsymbol{\theta}_0$, then

$$\int \psi(\boldsymbol{\theta}) \mathrm{e}^{-\omega h(\boldsymbol{\theta})} d\boldsymbol{\theta} = \mathrm{e}^{-\omega h(\boldsymbol{\theta}_0)} \left[\det(\omega \mathbf{h}''(\boldsymbol{\theta}_0)/2\pi)\right]^{-1/2} \sum_{0 \leq k < N} \omega^{-k} L_k \psi + O(\omega^{-N}),$$

as $\omega \to \infty$. With

$$\underline{h}(\boldsymbol{\theta}) = h(\boldsymbol{\theta}) - h(\boldsymbol{\theta}_0) - \langle (\boldsymbol{\theta} - \boldsymbol{\theta}_0)\mathbf{h}''(\boldsymbol{\theta}_0), \boldsymbol{\theta} - \boldsymbol{\theta}_0 \rangle / 2,$$

which vanishes to order three at $\boldsymbol{\theta}_0$, we have

$$L_k \psi = (-1)^k \sum_{0 \leq m \leq 2k} \frac{\langle \boldsymbol{\partial} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}, \boldsymbol{\partial} \rangle^{m+k} (\underline{h}^m \psi)(\boldsymbol{\theta}_0)}{2^{m+k}(m+k)! m!}.$$

Here $\langle \mathbf{a}, \mathbf{b} \rangle$ is the inner product $\mathbf{a}\mathbf{b}^T$, and $\boldsymbol{\partial}$ is the vector of partial differential operators $\boldsymbol{\partial}_j = \partial_j$.

L_k is a differential operator of order $2k$ acting on ψ at $\mathbf{x}_0$ (considering the order $3m$ zero of $\underline{h}^m$). The coefficients are rational homogeneous functions of degree $-k$ in $\mathbf{h}''(\boldsymbol{\theta}_0), \ldots, \mathbf{h}^{(2k+2)}$ with denominator $(\det \mathbf{h}''(\boldsymbol{\theta}_0))^{3k}$ (which comes from the observation that $\langle \boldsymbol{\partial} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}, \boldsymbol{\partial} \rangle = \langle \boldsymbol{\partial}, \boldsymbol{\partial} \frac{\operatorname{adj} \mathbf{h}''(\boldsymbol{\theta}_0)}{\det \mathbf{h}''(\boldsymbol{\theta}_0)} \rangle$). In every term the total number of derivatives of ψ and of $\mathbf{h}''$ is at most $2k$.

Remark 2.3. In Lemma 2.2, since ψ and h are sufficiently continuously differentiable, the order of partial differentiation is inessential when taking their derivatives. Also, $\langle \boldsymbol{\partial} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}, \boldsymbol{\partial} \rangle = \sum_{1 \leq j,k < d} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}_{jk} \partial_j \partial_k$, a weighted sum of all possible second partial derivatives. More generally, $\langle \boldsymbol{\partial} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}, \boldsymbol{\partial} \rangle^j$ is a weighted sum of all possible $(2j)$th partial derivatives, the weight of a partial derivative being the product of the weights of the second partial derivatives (read from left to right, say) that compose it. For example, $\partial^{(3,2,7,7,1,5)}$ has weight $\mathbf{h}''(\boldsymbol{\theta}_0)^{-1}_{3,2} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}_{7,7} \mathbf{h}''(\boldsymbol{\theta}_0)^{-1}_{1,5}$. Thus calculating $L_k \psi$ requires calculating for each $j \in \{0, \ldots, 3k\}$ the weights of the $(2j)$th partial derivatives involved. For each j there are at most $\binom{d}{2}^j / j!$ such weights since each $(2j)$th partial derivative has j pairs of second partial derivatives and since order of differentiation within and among pairs is inessential for calculating weights. Thus calculating $L_k \psi$, having already calculated $L_{k-1} \psi$, requires calculating at most $\sum_{0 \leq j \leq 3k} \binom{d}{2}^j / j = \frac{d^{6k}}{(3k)!} + O(d^{3k})$ weights.

In the special case where ψ and h are symmetric and $\boldsymbol{\theta}_0$ is a multiple of $\mathbf{1}$, it is easy to see that all $(2j)$th partial derivatives of $\underline{h}^m \psi$ within the same partition class are equal when evaluated at $\boldsymbol{\theta}_0$. Here the partition class $[p_1, \ldots, p_l]$ of a $(2j)$th partial derivative $\partial^{\boldsymbol{\alpha}}$ is defined by p_1 equals the number of occurrences of the greatest entry of $\boldsymbol{\alpha}$, p_2 equals the number of occurrences of the second greatest entry of $\boldsymbol{\alpha}$, etc.; thus $(p_1, \ldots, p_l)$ is a partition of $(2j)$. Moreover, by symmetry $\mathbf{h}''(\boldsymbol{\theta}_0)$ has its diagonal entries all equal and its off-diagonal entries all equal, so that there are only two different weights for second partial derivatives, one for repeated derivatives and one for mixed derivatives. Since there are $O((2j)^{-1/4} \mathrm{e}^{\pi \sqrt{2(2j)/3}}$ partitions of $2j$

(see [FS] for instance) and at most $j+1$ possible weightings for each partition, the number of binary (repeated vs. mixed partials pairs) words of length j disregarding order, calculating $L_k\psi$, having already calculated $L_{k-1}\psi$, requires calculating at most $\sum_{3k-1\le j\le 3k} O(j^{3/4}\mathrm{e}^{2\pi\sqrt{j/3}}) = O(k^{3/4}\mathrm{e}^{2\pi\sqrt{k}})$ weights.

We come now to our main theorem on diagonal asymptotics. This is a generalization of [PW02, Theorem 3.5] (please note the typo therein: the $z_d H_d$ should be a $-z_d H_d$) in that we employ the constructive Lemma 2.2 and address the case $p>1$, which arises often when computing statistics, such as expectation and variance, with generating functions. See Section 3 for such an example.

Theorem 2.4. Let $\mathbf{n} = (a_1 n, \dots, a_d n)$ for some $a_1, \dots, a_d \in \mathbb{N}^+$. If $\mathrm{C\scriptstyle ONTRIB}(\mathbf{n})$ consists of a single smooth point $\mathbf{c}$ such that $c_d \partial_d J(\mathbf{c}) \neq 0$ and c_d is a zero of order one of $x_d \mapsto J(c_1, \dots, c_{d-1}, x_d)$, then

$$f_{\mathbf{n}} = \mathbf{c}^{-\mathbf{n}} \left(\sum_{0\le j<p} \frac{(n_d+1)^{\underline{p-1-j}}}{(p-1-j)!j!} \left[(2\pi n_d)^{d-1} \det \widetilde{\mathbf{h}}''(\mathbf{0}) \right]^{-1/2} \sum_{0\le k<N} n_d^{-k} L_k \widetilde{\psi}_j + O\left(n_d^{p-1-N} \right) \right)$$

as $n \to \infty$. Here

$$\begin{aligned}
\psi_j(\widehat{\mathbf{x}}) &= \lim_{x_d \to g(\widehat{\mathbf{x}})} (-x_d)^{-p+j} \partial_d^j [(x_d - g(\widehat{\mathbf{x}}))^p F(\widehat{\mathbf{x}}, x_d)], \\
h(\mathbf{x}) &= \log g(\widehat{\mathbf{x}}) + \mathrm{i} \sum_{j<d} \frac{n_j}{n_d} x_j - \log g(\mathbf{c}), \\
\mathbf{E}(\widehat{\boldsymbol{\theta}}) &= (c_1 \mathrm{e}^{\mathrm{i}\theta_1}, \dots, c_{d-1} \mathrm{e}^{\mathrm{i}\theta_{d-1}}), \\
\widetilde{\psi}_j &= \psi \circ \mathbf{E}, \\
\widetilde{h} &= h \circ \mathbf{E},
\end{aligned}$$

and $L_k \widetilde{\psi}$ is as in Lemma 2.2 (but with tildes). In particular,

$$\psi_0(\widehat{\mathbf{x}}) = \frac{I(\widehat{\mathbf{x}}, g(\widehat{\mathbf{x}}))}{(-g(\widehat{\mathbf{x}}) \partial_d J(\widehat{\mathbf{x}}, g(\widehat{\mathbf{x}})))^p},$$

which is often useful in calculations.

Proof. From the proof of [PW02, Lemma 1.4], for all $\epsilon > 0$ there exists a neighborhood $\mathcal{N} \subseteq T(\widehat{\mathbf{c}}) \cap \mathcal{V} \cap \mathbb{C}^{d-1}$ of $\widehat{\mathbf{c}}$ such that

$$f_{\mathbf{n}} = \mathbf{c}^{-\mathbf{n}} \left((2\pi \mathrm{i})^{1-d} \int_{\mathcal{N}} \widehat{\mathbf{x}}^{-\widehat{\mathbf{n}}-\mathbf{1}} R(\widehat{\mathbf{x}}) d\widehat{\mathbf{x}} + O((1+\epsilon)^{n_d}) \right),$$

as $n_d \to \infty$, where $R(\widehat{\mathbf{x}}) = -\operatorname{Res}(F(\widehat{\mathbf{x}}, x_d) x_d^{-n_d-1}; x_d = g(\widehat{\mathbf{x}}))$.

For easy reading, let $x = x_d$. Since $g(\widehat{\mathbf{x}})$ is a a simple zero of $J(\widehat{\mathbf{x}}, \cdot)$, it is a pole of order p of $F(\widehat{\mathbf{x}}, \cdot)$. Therefore we can compute the above residue:

$$\begin{aligned}
R(\widehat{\mathbf{x}}) =& - \lim_{x \to g(\widehat{\mathbf{x}})} \frac{1}{(p-1)!} \partial_d^{p-1} \left[(x - g(\widehat{\mathbf{x}}))^p F(\widehat{\mathbf{x}}, x) x^{-n_d - 1} \right] \\
=& - \frac{1}{(p-1)!} \lim_{x \to g(\widehat{\mathbf{x}})} \sum_{0 \le j < p} \binom{p-1}{j} \partial_d^j \left[(x - g(\widehat{\mathbf{x}}))^p F(\widehat{\mathbf{x}}, x) \right] \partial_d^{p-1-j} x^{-n_d - 1} \\
& \text{(by the Leibniz rule)} \\
=& \frac{1}{(p-1)!} \lim_{x \to g(\widehat{\mathbf{x}})} \sum_{0 \le j < p} \binom{p-1}{j} \partial_d^j \left[(x - g(\widehat{\mathbf{x}}))^p F(\widehat{\mathbf{x}}, x) \right] (-1)^{p-j} \cdot \\
& (n_d + 1)^{\underline{p-1-j}} \, x^{-n_d - p + j} \\
=& \sum_{0 \le j < p} \frac{(n_d + 1)^{\underline{p-1-j}}}{(p-1-j)! j!} g(\widehat{\mathbf{x}})^{-n_d} \underbrace{\lim_{x \to g(\widehat{\mathbf{x}})} (-x)^{-p+j} \partial_d^j \left[(x - g(\widehat{\mathbf{x}}))^p F(\widehat{\mathbf{x}}, x) \right]}_{\psi_j(\widehat{\mathbf{x}}) :=}.
\end{aligned}$$

Here $m^{\underline{a}}$ denotes $m(m-1)\cdots(m-a+1)$, the ath falling factorial power of m (with the convention that $m^{\underline{0}} = 1$).

Proceeding as in the proof of [PW02, Lemma 4.1], we have

$$\begin{aligned}
& (2\pi\mathrm{i})^{1-d} \int_{\mathcal{N}} \widehat{\mathbf{x}}^{-\widehat{\mathbf{n}} - \mathbf{1}} R(\widehat{\mathbf{x}}) d\widehat{\mathbf{x}} \\
=& (2\pi\mathrm{i})^{1-d} \int_{\mathcal{N}} \widehat{\mathbf{x}}^{-\widehat{\mathbf{n}} - \mathbf{1}} \sum_{0 \le j < p} \frac{(n_d + 1)^{\underline{p-1-j}}}{(p-1-j)! j!} g(\widehat{\mathbf{x}})^{-n_d} \psi_j(\widehat{\mathbf{x}}) d\widehat{\mathbf{x}} \\
=& \mathbf{c}^{-\mathbf{n}} (2\pi\mathrm{i})^{1-d} \sum_{0 \le j < p} \frac{(n_d + 1)^{\underline{p-1-j}}}{(p-1-j)! j!} \int_{\mathcal{N}} \frac{\widehat{\mathbf{x}}^{-\widehat{\mathbf{n}}}}{\widehat{\mathbf{c}}^{-\mathbf{n}}} \psi_j(\widehat{\mathbf{x}}) \left(\frac{g(\widehat{\mathbf{x}})}{g(\widehat{\mathbf{c}})} \right)^{-n_d} \frac{d\widehat{\mathbf{x}}}{\prod_{1 \le k < d} x_k} \\
=& \mathbf{c}^{-\mathbf{n}} (2\pi)^{1-d} \sum_{0 \le j < p} \frac{(n_d + 1)^{\underline{p-1-j}}}{(p-1-j)! j!} \int_{\widetilde{\mathcal{N}}} \prod_{1 \le k < d} \mathrm{e}^{-\mathrm{i} n_k \theta_k} \widetilde{\psi}_j(\widehat{\boldsymbol{\theta}}) \left(\frac{g(\widehat{\mathbf{x}})}{g(\widehat{\mathbf{c}})} \right)^{-n_d} d\widehat{\boldsymbol{\theta}} \\
& \text{(via the change of variables } \widehat{\mathbf{x}} = \mathbf{E}(\widehat{\boldsymbol{\theta}})) \\
=& \mathbf{c}^{-\mathbf{n}} (2\pi)^{1-d} \sum_{0 \le j < p} \frac{(n_d + 1)^{\underline{p-1-j}}}{(p-1-j)! j!} \underbrace{\int_{\widetilde{\mathcal{N}}} \mathrm{e}^{-n_d \widetilde{h}(\widehat{\boldsymbol{\theta}})} \widetilde{\psi}_j(\widehat{\boldsymbol{\theta}}) d\widehat{\boldsymbol{\theta}}}_{\mathcal{I}_j :=}.
\end{aligned}$$

Now $\Re \widetilde{h} \ge 0$ on $\widetilde{\mathcal{N}}$ iff $\log \left| \frac{g(\mathbf{E}(\widehat{\boldsymbol{\theta}}))}{g(\widehat{\mathbf{c}})} \right| \ge 0$ on $\widetilde{\mathcal{N}}$ iff $|g(\widehat{\mathbf{x}})| \ge |g(\widehat{\mathbf{c}})|$ on $\mathcal{N}$. The last inequality does indeed hold, for if $|g(\widehat{\mathbf{x}})| < |g(\widehat{\mathbf{c}})|$ with $x \in \mathcal{N} \subseteq T(\widehat{\mathbf{c}}) \cap \mathcal{V} \cap \mathbb{C}^{d-1}$, then $(\widehat{\mathbf{x}}, g(\widehat{\mathbf{x}})) \in \mathcal{V} \cap D(\mathbf{c}) \subseteq T(\mathbf{c})$ by Theorem 2.1, so that $|g(\widehat{\mathbf{x}})| = |g(\widehat{\mathbf{c}})|$, a contradiction.

So by Lemma 2.2, for each $j \in \{0, \ldots, p-1\}$ we have

$$\begin{aligned}
\mathcal{I}_j &= \mathrm{e}^{-n_d \widetilde{h}(\mathbf{0})} \left[\det(n_d \widetilde{\mathbf{h}}''(\mathbf{0}) / 2\pi) \right]^{-1/2} \sum_{k < N} n_d^{-k} L_k \widetilde{\psi}_j + O(n_d^{-N}) \\
&= \mathrm{e}^{-n_d \widetilde{h}(\mathbf{0})} \left[(\frac{n_d}{2\pi})^{d-1} \det \widetilde{\mathbf{h}}''(\mathbf{0}) \right]^{-1/2} \sum_{k < N} n_d^{-k} L_k \widetilde{\psi}_j + O(n_d^{-N}).
\end{aligned}$$

Therefore

$$
\begin{aligned}
f_{\mathbf{n}} &= \mathbf{c}^{-\mathbf{n}}(2\pi)^{1-d} \sum_{0\le j<p} \frac{(n_d+1)^{p-1-j}}{(p-1-j)!j!}\mathcal{I}_j + O((1+\epsilon)^{n_d}) \\
&= \mathbf{c}^{-\mathbf{n}} \left(\sum_{j<p} \frac{(n_d+1)^{p-1-j}}{(p-1-j)!j!} \left[(2\pi n_d)^{d-1} \det \widetilde{\mathbf{h}}''(\mathbf{0})\right]^{-1/2} \sum_{k<N} n_d^{-k} L_k \widetilde{\psi}_j + O\left(n_d^{p-1-N}\right)\right),
\end{aligned}
$$

as desired.

Lastly,

$$
\begin{aligned}
\psi_0(\widehat{\mathbf{x}}) &= \lim_{x\to g(\widehat{\mathbf{x}})} (-x)^{-p}(x-g(\widehat{\mathbf{x}}))^p F(\widehat{\mathbf{x}}, x) \\
&= \lim_{x\to g(\widehat{\mathbf{x}})} \frac{I(\widehat{\mathbf{x}}, x)}{\left(-x\frac{J(\widehat{\mathbf{x}},x)-0}{x-g(\widehat{\mathbf{x}})}\right)^p} \\
&= \frac{I(\widehat{\mathbf{x}}, g(\widehat{\mathbf{x}}))}{(-g(\widehat{\mathbf{x}})\partial_d J(\widehat{\mathbf{x}}, g(\widehat{\mathbf{x}})))^p}.
\end{aligned}
$$

□

Proposition 2.5 ([RW]).

$$
\begin{aligned}
\widetilde{\mathbf{h}}''(\mathbf{0})_{lm} &= \frac{c_l c_m}{c_d^2(\partial_d J)^2}\left(\partial_m J\partial_l J + c_d(\partial_d J\partial_m\partial_l J - \partial_m J\partial_d\partial_l J - \partial_l J\partial_m\partial_d J + \frac{\partial_l J\partial_m J}{\partial_d J}\partial_d^2 J)\right)\Big|_{\mathbf{x}=\mathbf{c}}; \\
\widetilde{\mathbf{h}}''(\mathbf{0})_{ll} &= \frac{c_l\partial_l J}{c_d\partial_d J} + \frac{c_l^2}{c_d^2(\partial_d J)^2}\left((\partial_l J)^2 + c_d(\partial_d J\partial_l^2 J - 2\partial_l J\partial_d\partial_l J + \frac{(\partial_l J)^2}{\partial_d J}\partial_d^2 J)\right)\Big|_{\mathbf{x}=\mathbf{c}},
\end{aligned}
$$

where $l, m < d$ and $l \neq m$.

In the often-encountered case of symmetric functions, Proposition 2.5 simplifies greatly.

Proposition 2.6 ([RW]). If $\textsc{Crit}(\mathbf{n})$ is finite, $\mathbf{n} = (n, \dots, n)$, $J(\mathbf{x})$ is symmetric in $\mathbf{x}$, and $\partial_d J(\mathbf{c}) \neq 0$, where $\mathbf{c}$ is the contributing point that lies in the positive orthant of $\mathbb{R}^d$, then $\mathbf{c} = (c, \dots, c)$ for some positive real c,

$$
\begin{aligned}
\widetilde{\mathbf{h}}''(\mathbf{0})_{lm} &= a, \\
\widetilde{\mathbf{h}}''(\mathbf{0})_{ll} &= 2a, \quad \text{and} \\
\det \widetilde{\mathbf{h}}''(\mathbf{0}) &= da^{d-1},
\end{aligned}
$$

where $a = 1 + \frac{c}{\partial_d J}(\partial_d^2 J - \partial_1\partial_d J)\Big|_{\mathbf{x}=\mathbf{c}}$, $l, m < d$, and $l \neq m$.

3 Example

Example 3.1. Consider the $(d+1)$-variate function

$$
W(x_1, \dots, x_d, y) = \frac{N(\mathbf{x})}{1 - yE(\mathbf{x})}.
$$

Using the symbolic method (as presented in [FS]) it is not very difficult to see that W counts words over a d-ary alphabet $\mathcal{A}$, where x_i marks occurrences of letter i of $\mathcal{A}$ and y marks occurrences of *snaps*, nonoverlapping pairs of duplicate letters; here $N(\mathbf{x}) = 1/[1 - \sum_{i=1}^{d} x_i/(x_i + 1)]$, which counts snapless words over $\mathcal{A}$ (the so-called Smirnov words), $E(\mathbf{x}) = 1 - (1 - e_1(\mathbf{x}))N(\mathbf{x})$, and $e_1(\mathbf{x}) = \sum_{i=1}^{d} x_i$.

The diagonal coefficient $[x_1^n \dots x_d^n, y^s]W(\mathbf{x}, y)$ is then the number of words with n occurrences of each letter and s snaps. Let us compute the expected number of snaps as n tends to infinity.

$$\mathbb{E}(\chi) = \frac{[\mathbf{x^n}]\frac{\partial W}{\partial y}(\mathbf{x}, 1)}{[\mathbf{x^n}]W(\mathbf{x}, 1)} = \frac{[\mathbf{x^n}]N(\mathbf{x})^{-1}E(\mathbf{x})(1 - e_1(\mathbf{x}))^{-2}}{[\mathbf{x^n}](1 - e_1(\mathbf{x}))^{-1}},$$

where χ is the random variable counting snaps and $\mathbf{n} = (n, \dots, n)$. Applying Theorem 2.4 and Proposition 2.6 with $J(\mathbf{x}) := 1 - e_1(\mathbf{x})$ to $W(\mathbf{x}, 1)$ (with $p = 1$) and $\partial W/\partial y(\mathbf{x}, 1)$ (with $p = 2$) and noting that $\textsc{Contrib}(\mathbf{n})$ consists of a single smooth point $\mathbf{c} := (1/d, \dots, 1/d)$, we get that

$$\mathbb{E}(\chi) \sim \frac{\mathbf{c}^{-\mathbf{n}} n[(2\pi n)^{d-1}d]^{-1/2}d^2(d + 1)^{-1}}{\mathbf{c}^{-\mathbf{n}}[(2\pi n)^{d-1}d]^{-1/2}d} = \frac{d}{d + 1}n,$$

as $n \to \infty$.

To compute the variance $\mathbb{V}(\chi) = \mathbb{E}(\chi^2) - \mathbb{E}(\chi)^2$, we need

$$\begin{aligned}
\mathbb{E}(\chi^2) &= \frac{[\mathbf{x^n}]\left(\frac{\partial^2 W}{\partial y^2}(\mathbf{x}, 1) + \frac{\partial W}{\partial y}(\mathbf{x}, 1)\right)}{[\mathbf{x^n}]W(\mathbf{x}, 1)} \\
&= \frac{[\mathbf{x^n}]N(\mathbf{x})^{-2}E(\mathbf{x})(E(\mathbf{x}) + 1)(1 - e_1(\mathbf{x}))^{-3}}{[\mathbf{x^n}](1 - e_1(\mathbf{x}))^{-1}} \\
&\sim \frac{\mathbf{c}^{-\mathbf{n}} n^2 2^{-1}[(2\pi n)^{d-1}d]^{-1/2}2d^3(d + 1)^{-2}}{\mathbf{c}^{-\mathbf{n}}[(2\pi n)^{d-1}d]^{-1/2}d} \\
&= \left(\frac{d}{d + 1}n\right)^2,
\end{aligned}$$

as $n \to \infty$. Thus the variance has no n^2 term, and we need to recalculate our asymptotics to at least two terms. Actually, computer simulation suggests that the variance is linear in n and so has no $n^{3/2}$ term either, so that we need to recalculate our asymptotics to at least three terms. This is no bare-hands task, as calculating $L_2\widetilde{\psi}$ by taking advantage of the symmetry as described in Remark 2.3 requires calculating $\sum_{1 \le j \le 6}$(#partitions of $2j$) $\cdot$
(#weights for each partition) $\le 2(3) + 5(5) + 11(7) + 22(9) + 42(11) + 77(13) = 1769$ weights. We are currently trying to employ Maple to carry out this calculation.

References

[BK86] Egbert Brieskorn and Horst Knörrer, *Plane algebraic curves*, Birkhäuser Verlag, Basel, 1986, Translated from the German by John Stillwell. MR MR886476 (88a:14001)

[BP] Yuliy Baryshnikov and Robin Pemantle, *Convolutions of inverse linear functions via multivariate residues*, preprint available at `http://www.math.upenn.edu/~pemantle/papers/Preprints/hyperplanes.pdf`.

[FS] Phillipe Flajolet and Robert Sedgewick, *Analytic combinatorics*, in preparation, preprint available at `http://algo.inria.fr/flajolet/Publications/book061023.pdf`.

[GR92] Zhicheng Gao and L. Bruce Richmond, *Central and local limit theorems applied to asymptotic enumeration. IV. Multivariate generating functions*, J. Comput. Appl. Math. **41** (1992), no. 1-2, 177–186, Asymptotic methods in analysis and combinatorics. MR MR1181718 (94b:05017)

[Hör83] Lars Hörmander, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1983, Distribution theory and Fourier analysis. MR MR717035 (85g:35002a)

[PW] Robin Pemantle and Mark C. Wilson, *Twenty combinatorial examples of asymptotics derived from multivariate generating functions*, submitted, preprint available at `http://arxiv.org/pdf/math.CO/0512548`.

[PW02] ———, *Asymptotics of multivariate sequences. I. Smooth points of the singular variety*, J. Combin. Theory Ser. A **97** (2002), no. 1, 129–161. MR MR1879131 (2003a:05015)

[PW04] ———, *Asymptotics of multivariate sequences. II. Multiple points of the singular variety*, Combin. Probab. Comput. **13** (2004), no. 4-5, 735–761. MR MR2095981 (2005i:05008)

[RW] Alexanader Raichev and Mark C. Wilson, *A new diagonal method for asymptotics of multivariate sequences*, in preparation.