# The logic of a priori and a posteriori rationality in strategic games

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# Outline



- The facts and norms of decision-making

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- Games as social decision frame

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- A logic of rational choice
- Games as social decision frame
- Conclusion and further directions

# Decision Frame





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- $\leqslant$ is a binary relation, $u \leqslant v$ represents that you regard situation $v$ as at least as good as situation $u$.

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- $u \# u$
- $u < v$, $v < w$ and $w < u$ (the same with $\leqslant$)
- $u \# v$ (hence $u \leqslant v \vee v \leqslant u$ fails)

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• Suppose that the actual situation is $u = u(3, 2)$, you stupidly chose envelope 3 (green) and the money is in envelope 2 (red). ( Situation $u(n, m)$ represents you chose envelope $n$ (1,2 or 3) and the money is in envelope $m$ (1 or 2).

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- Why choosing envelope 3 (green) was stupid?

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- ⋆ $u$ is *priori* rational iff $\neg\exists v(u \sim v \wedge (u <_F v))$
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- ⋆ $u$ is *posterio* rational iff $\neg\exists v(u \sim v \wedge (u <_{F'} v))$
- Back to the example: Choosing envelope 3 (green) was both *priori* irrational and *posterio* irrational.

# The language of CPDL



## Definition

The language of CPDLover sets Prop of *propositional variables*, Nom of *nominals* and AtProg of *atomic programs* consists of the sets Form of *formulas* and Prog of *programs* given by

$$\begin{array}{lll} \varphi \in \textsc{Form} & ::= & i \mid [\pi]\varphi \mid \neg\varphi \mid (\varphi \wedge \varphi) \\ \pi \in \textsc{Prog} & ::= & a \mid \varphi? \mid \overline{\pi} \mid \pi^{\circ} \mid \pi^{*} \mid (\pi;\pi) \mid (\varphi \cup \varphi) \end{array}$$

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for $i \in \text{NOM}$, $p \in \text{PROP}$ and $a \in \text{ATPROG}$.
Abbreviations: $U = a \cup \overline{a}$ (universal), $I = \top?$ (identity), $\langle\pi\rangle = \neg[\pi]\neg$, $(\pi \cap \rho) = \overline{(\overline{\pi} \cup \overline{\rho})}$, $(\pi \subset \rho) = [\pi \cap \overline{\rho}]\bot$.
Abbreviate $(\pi \cap \rho)$ as $\pi\rho$.

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A formula $\varphi$ is *pure* iff it contains no propositional variables.

# Model



## Definition

*A structure* $F = \langle W, R \rangle$ *is a* CPDL *frame if* $R(a) \subseteq W^2$ *for each* $a \in \text{AtProg}$. *A structure* $M = \langle W, R, V \rangle$ *is a* CPDL *model if* $\langle W, R \rangle$ *is a* CPDL *frame,* $V(i) \in W$ *for each* $i \in \text{Nom}$, *and* $V(p) \subseteq W$ *for each* $p \in \text{Prop}$.

# Semantics



## Definition

*Given a* CPDL *model* $M = \langle W, R, V \rangle$, *and a state* $u \in W$, *we define* $[\![\varphi]\!]^M \subseteq W$ *for each* $\varphi \in$ FORM *and* $[\![\pi]\!]^M \subseteq W^2$ *for each* $\pi \in$ PROG *as follows:*

$$
\begin{array}{lcl}
[\![i]\!]^M & = & \{V(i)\} \\
[\![p]\!]^M & = & V(p) \\
[\![[\pi]\varphi]\!]^M & = & \{u \in W \mid v \in [\![\varphi]\!]^M \text{ for each } v \in W \\
 & & \;\text{such that } \langle u, v \rangle \in [\![\pi]\!]^M\} \\
[\![\neg\varphi]\!]^M & = & W \backslash [\![\varphi]\!]^M \\
[\![(\varphi \wedge \psi)]\!]^M & = & [\![\varphi]\!]^M \cap [\![\psi]\!]^M \\
[\![a]\!]^M & = & R(a) \\
[\![\varphi?]\!]^M & = & \{\langle u, u \rangle \mid u \in [\![\varphi]\!]^M\} \\
[\![\overline{\pi}]\!]^M & = & \{\langle u, v \rangle \mid \langle u, v \rangle \notin [\![\pi]\!]^M\} \\
[\![\pi^\circ]\!]^M & = & \{\langle u, v \rangle \mid \langle v, u \rangle \in [\![\pi]\!]^M\}
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# Semantics(cotn.)



## Definition

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\begin{aligned}
[\![\pi^*]\!]^M &= \textit{the smallest transitive, reflexive relation} \\
&\quad \textit{containing } [\![\pi]\!]^M \\
[\![\pi;\rho]\!]^M &= \{\langle u,v\rangle \mid \langle v,w\rangle \in [\![\pi]\!]^M \text{ and } \langle w,v\rangle \in [\![\rho]\!]^M \\
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*When $M$ is clear from the context, we write $[\![\varphi]\!]^M$ as $[\![\varphi]\!]$. Note that $[\![(\pi \cap \rho)]\!]^M = [\![\pi]\!]^M \cap [\![\rho]\!]^M$ and $[\![(\pi \subset \rho)]\!]^M = W$ if $[\![\pi]\!]^M \subseteq [\![\rho]\!]^M$, otherwise $\varnothing$.*

# *Priori*free preference and *posterio*preference in object language



Take $\textsc{AtProg} = \{\mathsf{p}, \mathsf{k}, \mathsf{c}\}$ with $R(\mathsf{p}) = \leqslant$, $R(\mathsf{k}) = \approx$ and $R(\mathsf{c}) = \sim$. Then the relations of *a priori* and *a posteriori* free preference can be defined as

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\mathsf{f}_a & = & \overline{\mathsf{k}_a ; \mathsf{c}_a \overline{\mathsf{p}_a} ; \mathsf{k}_a} & \text{(priori free preference)} \\
\mathsf{f}'_a & = & \overline{\mathsf{c}_a \mathsf{k}_a ; \mathsf{c}_a \overline{\mathsf{p}_a} ; \mathsf{c}_a \mathsf{k}_a} & \text{(posterio free preference)}
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We write $(u)_G$ for the $\sim_G$-equivalence class of $u$, which is the the set of situations $v$ that could have occurred, given the capacities of all the agents in $G$, had they chosen otherwise.

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| **F** is deterministic | iff | $(u)[u] = \{u\}$ for every $u \in W$ |
| **F** is linear | iff | $\leqslant$ is reflexive, transitive, and total. |

# Games as social decision frame



## Definition

*Given a set $A$ of agents, sets $D_a$ of strategies (for each $a \in A$), and utility functions*

$$U_a : \prod_{a \in A} D_a \to \mathbb{R}$$

# Games as social decision frame

## Definition

*Given a set $A$ of agents, sets $D_a$ of strategies (for each $a \in A$), and utility functions*

$$U_a : \prod_{a \in A} D_a \to \mathbb{R}$$

*the strategic game frame $G(A, D, U)$ is the social decision frame $\langle W, \sim, \approx, \leqslant \rangle$ given by*

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$$\begin{aligned} &W = \textstyle\prod_{a \in A} D_a \\ &w \sim_a v \text{ iff } w_b = v_b \text{ for all } b \neq a \text{ in } A \\ &w \approx_a v \text{ iff } w_a = v_a \\ &w \leqslant_a v \quad \text{iff} \quad U_a(w) \leqslant U_a(v) \end{aligned}$$

# Representation theorem



## Theorem

*A social decision frame is isomorphic to a strategic game frame iff it is connected, isolated, unordered, deterministic, linear, with a value size* $\leqslant 2^{\aleph_0}$ *and a finite number of agents.*

$$\bowtie_a(u) \quad = \quad \{v \in W \mid u \bowtie_a v\} \quad \text{Value}$$

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The *value size* of a social decision frame is the cardinality of the set of its values.

# Completeness

**Theorem (Areces and ten Cate,2006)**

*There is an axiomatisation* $\mathsf{K}$ *of* CPDL *which is sound and such that for every extension* $\mathsf{K}\Gamma$ *of* $\mathsf{K}$ *with pure formulas* $\Gamma$ *as axioms, if a formula is consistent in* $\mathsf{K}\Gamma$ *then it has a countable model on a frame in which all the formulas in* $\Gamma$ *are valid. The system* $\mathsf{K}\Gamma$ *is therefore also complete for that class of frame.*

# Completeness



## Theorem (Areces and ten Cate,2006)

*There is an axiomatisation* K *of* CPDL *which is sound and such that for every extension* KΓ *of* K *with pure formulas* Γ *as axioms, if a formula is consistent in* KΓ *then it has a countable model on a frame in which all the formulas in* Γ *are valid. The system* KΓ *is therefore also complete for that class of frame.*

## Theorem

*A frame is a decision frame iff the following pure formulas* D *are valid on* $F$*:*

| | | |
|---|---|---|
| $\neg @_i K \neg i$ | $\neg @_i C \neg i$ | *(reflexivity of* $\approx$ *and* $\sim$*)* |
| $\neg(KK\neg i \wedge K\neg i)$ | $\neg(CC\neg i \wedge C\neg i)$ | *(transitivity of* $\approx$ *and* $\sim$*)* |
| $@_i K \neg j \to @j K \neg i$ | $@_i C \neg j \to @j C \neg i$ | *(symmetry of* $\approx$ *and* $\sim$*)* |

# Completeness(cntn.)



## Corollary

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## Theorem

*The extension KG of KDA with the following axioms is complete for the class of strategic game frames:*

| | | |
|---|---|---|
| $\mathbf{G_1}$ | $\vdash U \subset c_A$ | *(connected)* |
| $\mathbf{G_2}$ | $\vdash c_{\bar{a}} \subset k_a$ | *(isolated)* |
| $\mathbf{G_3}$ | $\vdash (c_a; c_b) \subset (c_b; c_a)$ | *(unordered)* |
| $\mathbf{G_4}$ | $\vdash (c_a \cap k_a) \subset \top?$ | *(deterministic)* |
| $\mathbf{G_5}$ | $\vdash ((p_a^* \subset p_a) \wedge (U \subset p_a \cup p_a{}^\circ))$ | *(linear)* |

# Weakly dominated strategy

## Definition

*Given a strategic game frame $G(A, D, U)$, agent $a$'s strategy $d \in D_a$ is (weakly) dominated by another strategy $d' \in D_a$ iff*

1. *$d'$ is sure to be at least as good as $d$: $w[^a_{d'}] \geqslant_a w[^a_d]$ for all $w \in W(A, D)$, and*
2. *$d'$ may be better than $d$: $w[^a_{d'}] >_a w[^a_d]$ for some $w \in W(A, D)$.*

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2. *$d'$ may be better than $d$: $w[^a_{d'}] >_a w[^a_d]$ for some $w \in W(A, D)$.*

$w^a_d$ is the strategy profile obtained by replacing $a$'s strategy in $w$ by $d$, i.e.

$$w^a_d(b) = \begin{cases} d & \text{if } b = a \\ w_b & \text{otherwise} \end{cases}$$

# Weakly dominated strategy and *priori rational*

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**Theorem**

*In a model $M$ based on a strategic game frame $G(D, U, A)$, a strategy $w_a$ is dominated iff $M, w \models \neg R_a$.*

# Best response and *posterio rational*



## Definition

*A decision situation $w$ in a strategic game frame $G(A, D, U)$ is a best response for agent $a$ iff there is no strategy $d \in D_a$ such that $w <_a w[{}^a_d]$*

# Best response and *posterio rational*



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*A decision situation $w$ in a strategic game frame $G(A, D, U)$ is a best response for agent $a$ iff there is no strategy $d \in D_a$ such that $w <_a w[{}^a_d]$*
*It is a Nash equilibrium iff it is a best response for all agents.*

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## Theorem

*In a model $M$ based on a strategic game frame $G(D, U, A)$, a situation $w$ is Nash equilibrium iff $M, w \models \bigwedge_{a \in A} R'_a$.*

# Extend to mixed strategic games

## Definition

*Given a strategic game frame* $G(A, D, U)$ *with finite* $D$*, the mixed-strategy extension of* $G$ *is the strategic game frame* $G^*(A, D^*, U^*)$ *in which* $D^*_a$ *is the set of probability functions* $\delta : D_a \to [0, 1]$ *and for each* $\delta \in D^*$*,*

$$U^*_a(\delta) = \sum_{s \in \prod_{b \in A} D_b} u_a(s) \prod_{b \in A} \delta_b(s_b)$$

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$$U_a^*(\delta) = \sum_{s \in \prod_{b \in A} D_b} u_a(s) \prod_{b \in A} \delta_b(s_b)$$

The restriction to finite $D$ is essential, because $U^*$ is calculated as a finite sum. This restriction prevent us from forming $G^{**}$ because $D^*$ is always infinite.

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**Theorem**

$\mathsf{K}G \nvdash \langle E\rangle \bigwedge_{a\in A} R'_a$ *but this formula is valid on all mixed-strategy strategic game frames.*

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- It has a number of well-known decidable fragments, include PDL itself and its extension to allow $^\circ$ and either $\cap$ or $\overline{a}$, i.e., $^-$ restricted to atomic programs, but not both.
- Hybrid PDL namely PDL with nominals is also decidable but even non-hybrid PDL with unrestricted $^-$ is not.
- Frame consequence is undecidable even for PDL with premises restricted to pure formulas, and so the decidability of validity for these fragments of CPDL cannot be automatically extended to specific classes of frames defined by pure formulas, despite the existence of a complete axiomatisation.

# Conclusion



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- To provide a sharp distinction between descriptive and normative aspects of decision making in both the individual and social settings, as is the distinction between *a priori* and *a posteriori* rationality.
- We gives a useful characterisation of the assumptions of strategic game theory and presented a complete logic that formalises our approach to rational decision-making.
- We apply our account to the theory of strategic games with both pure and mixed (probabilistic) strategies, showing that the concept of a dominated strategy and Nash equilibrium are correctly predicted by more general norms.

# Future work

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- To characterize the procedure of finding game solution by public announcement of rationality .
- To axiomatise of the class of mixed-strategy games in probabilistic DEL.
- To use this framework to explore other concepts in game theory and other games, such as those of imperfect information, incomplete games, sequential games, etc.

# Thank you!