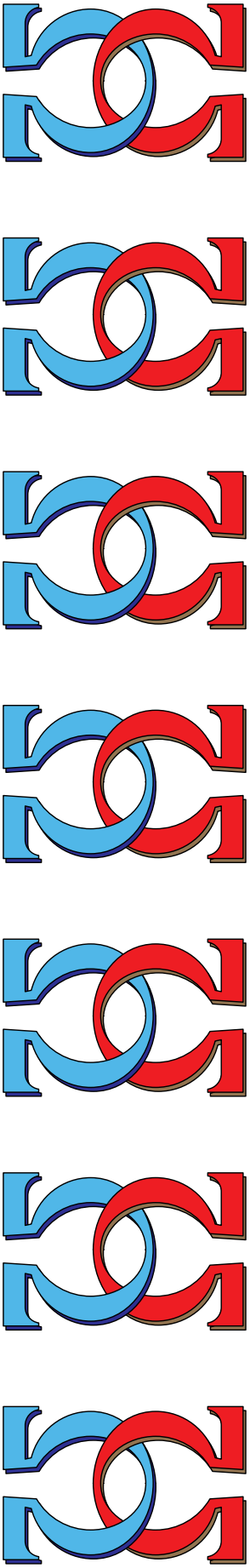


CDMTCS
Research
Report
Series

Quantum Scholasticism: On Quantum Contexts, Counterfactuals, and the Absurdities of Quantum Omniscience

Karl Svozil
Vienna University of Technology

CDMTCS-349
January 2009

Centre for Discrete Mathematics and
Theoretical Computer Science

Quantum Scholasticism: On Quantum Contexts, Counterfactuals, and the Absurdities of Quantum Omniscience

Karl Svozil

Institut für Theoretische Physik, University of Technology Vienna, Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria

Abstract

Unlike classical information, quantum knowledge is restricted to the outcome of measurements of maximal observables corresponding to single contexts.

Key words: maximal observables, context, quantum information, omniscience
PACS: 01.70.+w, 01.65.+g, 03.67.-a,02.50.-r

1 Introduction

The violation of classical bounds [1,2] on joint quantum probabilities enumerated by Bell [3,4], Clauser-Horn-Shimony-Holt (CHSH) [5,6] and others [7–10], the Kochen-Specker (KS) [11–19] as well as the Greenberger-Horn-Zeilinger (GHZ) [20–22] theorems provide constructive, finite proofs that observables outside of a single quantum context cannot consistently co-exist. Here, the term *context* refers to a maximal collection of co-measurable observables associated with commuting operators. Every context can also be characterized by a single (but nonunique) maximal operator. All operators within a context are functions thereof (see Ref. [23], Sec. II.10, p. 90, English translation p. 173 and Ref. [24], Sec. 84). In quantum logic [25–28], contexts are represented by Boolean subalgebras or blocks pasted together to form the Hilbert lattice. (For the sake of nontriviality, Hilbert spaces of dimension higher than two

Email address: svozil@tuwien.ac.at (Karl Svozil).
URL: http://tph.tuwien.ac.at/~svozil/ (Karl Svozil).

are considered.) In an algebraic sense, a context represents a "classical mini–universe," which is distributive and allows for as many two–valued states — interpretable as classical truth assignments — as there are atoms.

By definition, no direct measurement of observables "outside" of a single context is possible. Therefore, any assumption about the physical existence of such observables results in the invocation of counterfactuals. For example, Einstein, Podolsky and Rosen (EPR) [29] suggested to measure and counterfactually infer two contexts simultaneously by considering elements of physical reality which cannot be measured simultaneously on the same quantum. In this respect, quantum physics relates to scholastic philosophy. Indeed, in an informal paper [11] announcing KS, Specker explicitly referred to the "infuturabilities" of scholastic philosophy.

Related to counterfactuals is the idea of a (divine) *omniscience* "knowing" all the factuals and counterfactuals in the naive sense that "if a proposition is true, then an omniscient agent (such as God) knows that it is true." Already Thomas Aquinas considered questions such as whether God has knowledge of non-existing things (Ref. [30], Part 1, Question 14, Article 9) or things that are not yet (Ref. [30], Part 1, Question 14, Article 13).

In classical physics, there is just one global context which is trivially constituted by all conceivable observables. Hence, there is no conceptual or principal reason to assume counterfactuals; sometimes they are just considered for convenience (saving the experimenter from measuring redundant observables). The empirical sciences implement classical omniscience by assuming that in principle all observables of classical physics are (co-)measurable without any restrictions. No distinction is made between an observable obtained by an "actual" and a "potential" measurement. Precision and (co-)measurability are limited only by the technical capacities of the experimenter. The principle of empirical classical omniscience has given rise to the realistic believe that all observables "exist," regardless of the state preparations and observations. Physical (co-)existence is thereby related to the realistic assumption [31] (sometimes referred to as the "ontic" [32] viewpoint) that such physical entities exist even without being experienced by any finite mind.

Formally, counterfactuals and classical omniscience are supported by the following two properties.

(i) Boolean logics and absence of complementarity: Historically, the discovery of the uncertainty principle and quantum complementarity marked a first departure from classical omniscience. A formal expression of complementarity is the nondistributive algebra of quantum observables. Alas, nondistributivity of the empirical logical structure is no sufficient condition for the impossibility of omniscience. For example, both generalized urn models

[33,34] as well as equivalent [35] finite automata [36–39] exhibit complementarity, yet they possess two–valued states interpretable as omniscience; i.e., as global truth assignments with a consistent value "0" (*false*) or "1" (*true*) for every observable.

(ii) "Abundance" of two–valued states interpretable as omniscience of the system: Thereby, any such "dispersionless" two–valued state — associated with a classical "truth table" — can be defined on all observables, regardless of whether they have been actually observed or not.

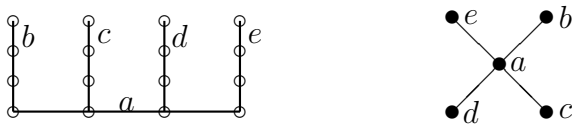
In contrast, quantum propositions neither satisfy distributivity, nor do they support two–valued states. Recall Schrödinger's interpretation of the quantum wave function (in §7 of Ref. [40]) in terms of a "catalogue of expectations." Every page of this catalogue of expectations is represented by a single context. In quantum mechanics, (as well as in quasi-classical models [35]), due to complementarity, contexts are not global, and the structure of contexts as well as the probability measures on them [41,42] pose many challenging questions.

2 "Scarcity" of two–valued states

Gleason's theorem [41,42] states that the quantum probabilities can be derived from the assumption that classical probability theory holds within contexts. Yet, unlike classical systems, they are no convex combination of global two–valued states. Formally, this is due to the fact that the quantum propositions do not support globally defined two–valued states.

What happens if one insist in the use of two–valued states outside of a single context by considering quantum propositional structures still allowing "a few" two–valued states? In this case, the invocation of counterfactuals and the "scarcity" of two–valued states accounts for some consequences which, depending on the disposition of the recipient, appear "mindboggling" to absurd.

By bundling together propositional structures giving rise to such "mindboggling" properties, one arrives at the KS conclusion. For such finite compositions of observables, the mere assumption of a globally defined truth table results in a complete contradiction. Alas, by contemplating the situation not bottom–up as usually, but top–down; i.e., from the point of view of KS, it is not too difficult to derive "mindboggling" statements from absurdities. Indeed, the *principle of explosion* (stating that *ex falso quodlibet,* or *contradictione sequitur quodlibet*) which, due to the pasting construction of Hilbert lattices, holds also in quantum logic, implies that "anything follows from a contradiction."



a) b)

Fig. 1. Four-star configuration in four-dimensional Hilbert space a) Greechie diagram representing atoms by points, and contexts by maximal smooth, unbroken curves. b) Dual Tkadlec diagram representing contexts by filled points, and interconnected contexts by lines.

2.1 Dual Greechie and Tkadlec diagrams

For a proof of the "scarcity" of two–valued states, Greechie diagrams symbolizing one-dimensional projectors by points and contexts by maximal smooth unbroken curves are considered. The "dual" Tkadlec diagrams [43] symbolize entire contexts by points, and links between contexts by lines joining them.

Tkadlec diagrams suggest the most compact representation of a context in terms of a single maximal operator. Note that, for the n-dimensional Hilbert space, an n-star configuration represents n different contexts joined in n different atoms of the center context; see Fig. 1.

2.2 The "one–zero" rule

For the sake of presentation of such properties, consider the proof that, for the observables depicted in Fig. 2, the occurrence of an outcome corresponding to K (abbreviated by "K occurs") implies that E cannot occur. This property, which has been already exploited by KS [12, Γ_1], will be called the "one-zero rule."

2.3 The "one–one/zero–zero" rule

For another example, consider two collections of observables as above, which are combined by "gluing" them together in two contexts. The geometry based upon the Γ_3-configuration in KS [12] is depicted in Fig. 3. In this case one obtains the "one-one" and "zero-zero rules," stating that K occurs if and only if K' occurs.

For a quantum falsification of the one-zero and the one-one/zero-zero rules it suffices to record a single pair of outcomes which does not obey these classical predictions. This can for instance been demonstrated in an EPR-type setup

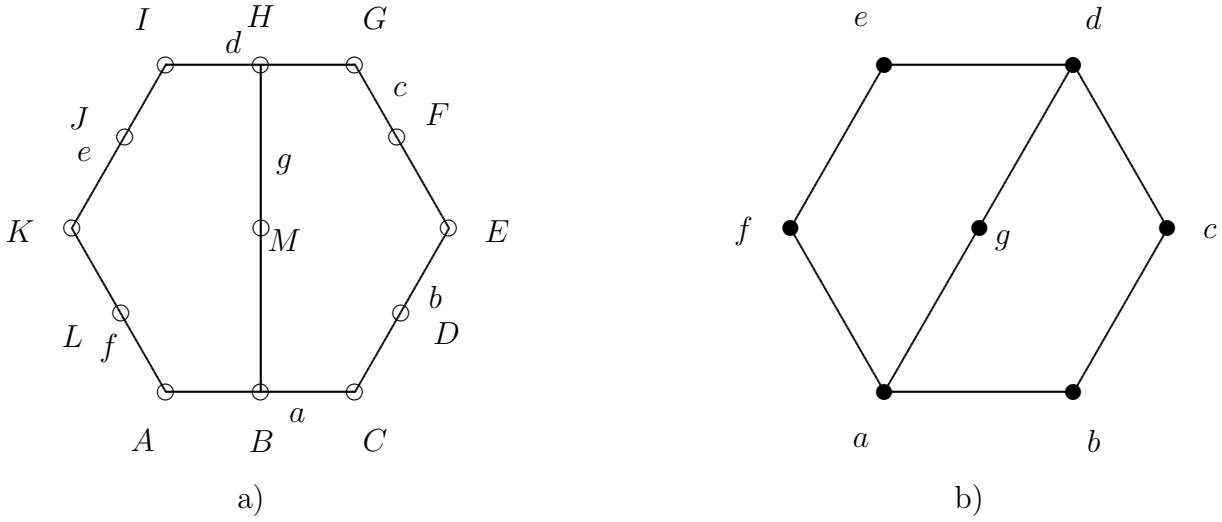


Fig. 2. Configuration of observables in three-dimensional Hilbert space implying that whenever K is true, E must be false. The seven interconnected contexts $a = \{A, B, C\}$, $b = \{C, D, E\}$, $c = \{E, F, G\}$, $d = \{G, H, I\}$, $e = \{I, J, K\}$, $f = \{K, L, A\}$, $g = \{B, H, M\}$, consist of the 13 projectors associated with the one dimensional subspaces spanned by [18] $A = (1, \sqrt{2}, -1)$, $B = (1, 0, 1)$, $C = (-1, \sqrt{2}, 1)$, $D = (-1, \sqrt{2}, -3)$, $E = (\sqrt{2}, 1, 0)$, $F = (1, -\sqrt{2}, -3)$, $G = (-1, \sqrt{2}, -1)$, $H = (1, 0, -1)$, $I = (1, \sqrt{2}, 1)$, $J = (1, \sqrt{2}, -3)$, $K = (\sqrt{2}, -1, 0)$, $L = (1, \sqrt{2}, 3)$, $M = (0, 1, 0)$. a) Greechie diagram representing atoms by points, and contexts by maximal smooth, unbroken curves. Only a single observable per context can be true. Noncontextuality requests that link observables in different contexts are either true or false in all of these context. Then, whenever K is true, E cannot be true, since then at least one of the two contexts a and d would contain only outcomes which do not occur. b) Dual Tkadlec diagram representing contexts by filled points, and interconnected contexts by lines.

of two spin one particles in a singlet state

$$\frac{1}{\sqrt{3}}(-|0,0\rangle + |-1,1\rangle + |1,-1\rangle),$$

and observables corresponding to E and K, or to K and K'. Generalized beam splitters are possible realizations [44–46]. This adds to the evidence accumulated already by Bell, KS and GHZ, that we are not living in a classical world.

2.4 The absence of two–valued states

The simplest known proof [19,47] of KS is in four-dimensional real Hilbert space and requires nine intricately interwoven contexts — every observable is in exactly two different contexts — drawn in Fig. 4. In order to appreciate the proofs (by contradiction), note that

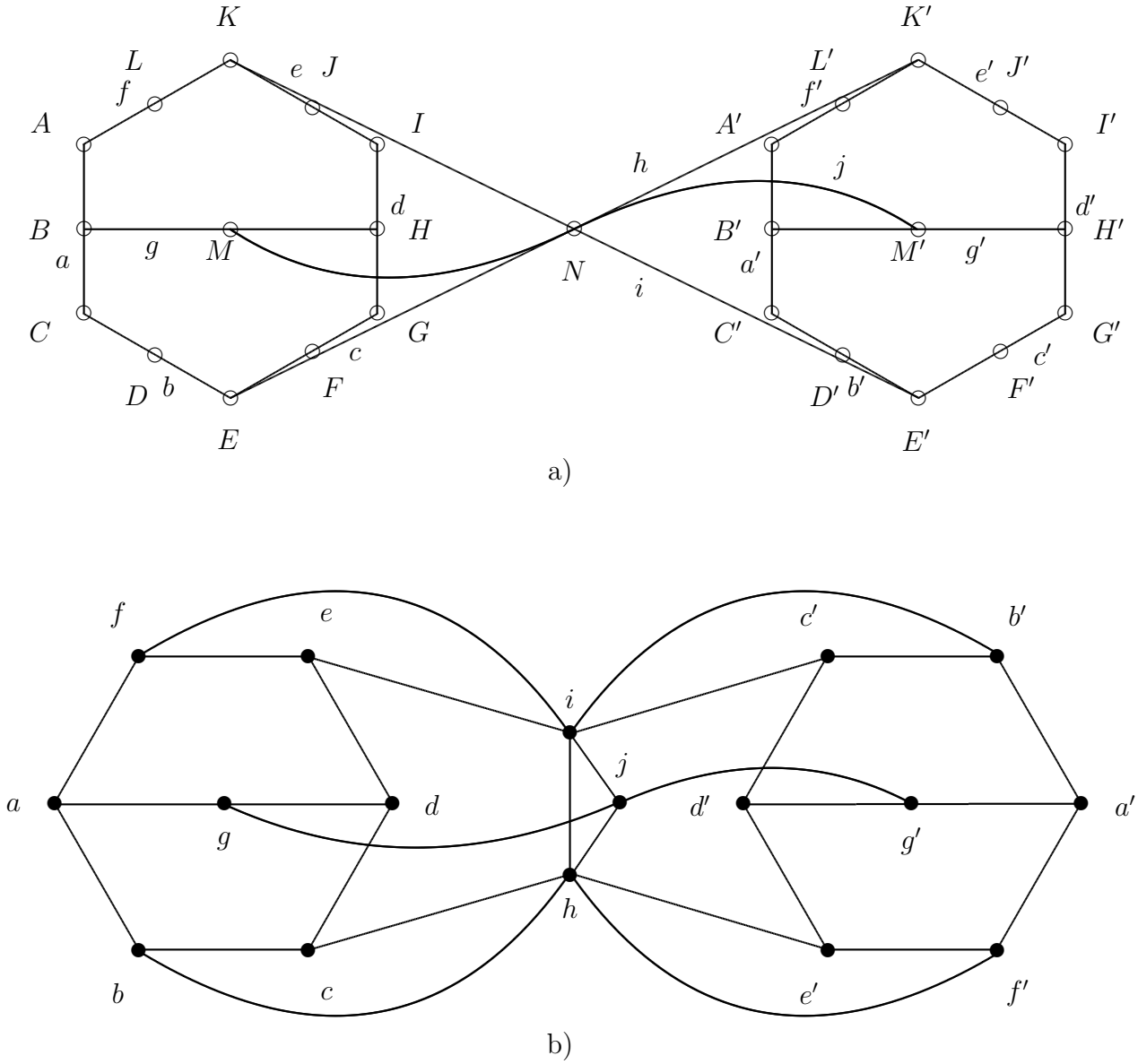


Fig. 3. Configuration of observables implying that the occurrences of K and K' coincide. a) Greechie diagram representing atoms by points, and contexts by maximal smooth, unbroken curves. The coordinates of the "primed" points A'–M' are obtained by interchanging the first and the second components of the unprimed coordinates A–M enumerated in Fig. 2; and $N = (0, 0, 1)$. The two contexts h and i linking the primed with the unprimed observables allow the following argument: Whenever K occurs, then by the one-zero rule E cannot occur; moreover N cannot occur, hence K' must occur. Conversely, by symmetry whenever K' occurs, K must occur. b) Dual Tkadlec diagram representing contexts by filled points, and interconnected contexts by lines.

(i) the proofs require the assumption of counterfactuals; i.e., of "potential" observables which, due to quantum complementarity, are incompatible with the "actual" measurement context; yet could have been measured if the measurement apparatus were different. These counterfactuals are organized

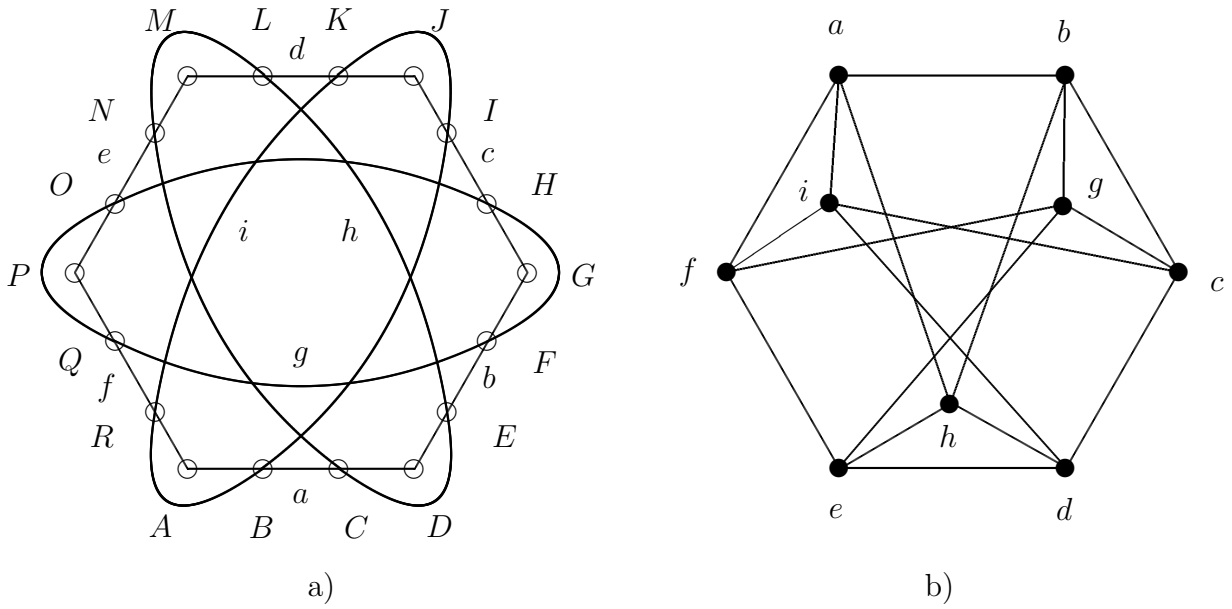


Fig. 4. Proof of the Kochen-Specker theorem [19,47] in four-dimensional real vector space. The nine tightly interconnected contexts $a = \{A, B, C, D\}$, $b = \{D, E, F, G\}$, $c = \{G, H, I, J\}$, $d = \{J, K, L, M\}$, $e = \{M, N, O, P\}$, $f = \{P, Q, R, A\}$, $g = \{B, I, K, R\}$, $h = \{C, E, L, N\}$, $i = \{F, H, O, Q\}$ consist of the 18 projectors associated with the one dimensional subspaces spanned by $A = (0, 0, 1, -1)$, $B = (1, -1, 0, 0)$, $C = (1, 1, -1, -1)$, $D = (1, 1, 1, 1)$, $E = (1, -1, 1, -1)$, $F = (1, 0, -1, 0)$, $G = (0, 1, 0, -1)$, $H = (1, 0, 1, 0)$, $I = (1, 1, -1, 1)$, $J = (-1, 1, 1, 1)$, $K = (1, 1, 1, -1)$, $L = (1, 0, 0, 1)$, $M = (0, 1, -1, 0)$, $N = (0, 1, 1, 0)$, $O = (0, 0, 0, 1)$, $P = (1, 0, 0, 0)$, $Q = (0, 1, 0, 0)$, $R = (0, 0, 1, 1)$. a) Greechie diagram representing atoms by points, and contexts by maximal smooth, unbroken curves. b) Dual Tkadlec diagram representing contexts by filled points, and interconnected contexts are connected by lines. (Duality means that points represent blocks and maximal smooth curves represent atoms.) The nine contexts in four dimensional Hilbert space are interlinked in a four-star form; hence every observable proposition occurs in exactly two contexts. Thus, in an enumeration of the four observable propositions of each of the nine contexts, there appears to be an *even* number of true propositions. Yet, as there is an odd number of contexts, there should be an *odd* number (actually nine) of true propositions.

into groups of interconnected contexts which, due to quantum complementarity, are incompatible and therefore cannot be measured simultaneously; not even in Einstein-Podolsky-Rosen (EPR) [29] type setups [48].

(ii) The proofs by contradiction have no direct experimental realizations. As has already been pointedly stated by Robert Clifton [49], "how can you measure a contradiction?"

(iii) So–called "experimental tests" inspired by Bell-type inequalities [50–52], KS [53,54] as well as GHZ [55] measure the incompatible contexts which are considered in the proofs one after another; i.e., temporally sequentially, and not simultaneously. Hence, different contexts can only be measured on

different particles.

3 Alternatives

The following alternatives present some ways to cope with these findings:

(i) abandonment of classical omniscience: in this view, it is wrong to assume that all observables which could in principle ("potentially") have been measured also co–exist, irrespective of whether or not they have or even could have been actually measured. Realism might still be assumed for a *single* context, in particular the one in which the system was prepared;

(ii) abandonment of realism: in this view, it is wrong to assume that physical entities exist even without being experienced by any finite mind. Quite literary, with this assumption, the proofs of KS and similar decay into thin air because there are no counterfactuals or unobserved physical observables or inferred (rather than measured) elements of physical reality.

(iii) contextuality; i.e., the abandonment of context independence of measurement outcomes [56–58]: it is wrong to assume (cf. Ref. [56], Sec. 5) that the result of an observation is independent not only of the state of the system but also of the complete disposition of the apparatus. Compare also Bohr's remarks [59] about *"the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."*

It should come as no surprise that realists such as Bell favor contextuality rather than giving up realism or classical omniscience. Nonetheless, to this date there does not exist a single experimental finding to support contextuality, and, as pointed out above, contextuality is only one of at least three possibilities to interpret quantum probability theory.

The simplest configuration testing contextuality corresponds to an arrangement of five observables A, B, C, D, K with two comeasurable, mutually commuting, contexts $\{A, B, C\}$ and $\{A, D, K\}$ interconnected at A. This propositional structure L_{12} can be represented in three-dimensional Hilbert space by two tripods with a single common leg. Indeed, if contextuality is a physically meaningful principle for the finite systems of observables employed in proofs of KS, then contextuality should already be detectable in this simple system of observables. It would be a challenging task to realize the L_{12} quantum logical structure experimentally in an EPR-type setup, and falsify contextuality there.

Furthermore, in extension of the two-context configuration, also systems of three interlinked contexts such as $\{A, B, C\}$, $\{A, D, K\}$ and $\{K, L, M\}$ inter-

connected at A and K could be considered. Note that too tightly interconnected systems such as $\{A, B, C\}$, $\{A, D, K\}$ and $\{K, L, C\}$ have no representation in a 3-dimensional Hilbert space. However, for a greater dimension than three, we can take, e.g., $A = (1, 0, 0, 0)$, $B = (0, 1, 0, 0)$, $C = (0, 0, 1, 0)$, $D = (0, 1, 1, 0)$, $K = (0, 0, 0, 1)$, $L = (1, 1, 0, 0)$.

4 Summary

If one believes in the physical existence of counterfactuals, a lot of puzzling and mindboggling properties can be derived, bordering to mystery, if not to absurdity. Take, for example the one-zero rule discussed above: a noncontextual argument shows that certain outcomes are correlated.

Formally, this is due to the "scarcity" of two–valued states on the linear subspaces of Hilbert states. Worse yet, by considering a larger, finite group of observables, it can be shown that, with the assumption of noncontextuality, no such state exists.

Alas, it is not too difficult to derive "mindboggling" statements from absurdities. Indeed, the *principle of explosion* suggests that "anything follows from a contradiction."

It is not unreasonable to doubt the usefulness of contextuality as a resolution of the imminent inconsistencies and complete contradictions originating in the assumption of the physical (co-)existence of observables in different contexts. Contextuality might not even be measurable in the simplest cases where it could be falsified by simultaneous EPR-type measurements of two interlinked contexts. A detailed discussion on realism versus empiricism and the issues related to contextuality in EPR-type configurations can also be found in Refs. [60,61]; see also Khrennikov's findings about couterfactuals in EPR-type setups [62].

It appears most natural to abandon the notion that not all classical observables are quantum observables; that quantum omniscience is limited to a single context; that a quantized system has only observable physical properties in the context in which it was prepared; and that one should accept the obvious fact that one cannot squeeze information from an ignorant system or agent. If one tries nevertheless, then all one obtains are random, erratic outcomes. Indeed, it is not totally unreasonable to speculate that contextuality is a "red herring;" that it appears to be one of the biggest and most popular delusions in the foundations of the quantum (which is rich in mindboggling speculations), devised by Bell and other realist to retain some form of classical realistic nonsensical omniscience.

Acknowledgements

The author gratefully acknowledges the suggestions of two anonymous referees.

References

[1] G. Boole, *An investigation of the laws of thought* (Dover edition, New York, 1958).

[2] G. Boole, "On the theory of probabilities," Philosophical Transactions of the Royal Society of London **152**, 225–252 (1862).

[3] J. S. Bell, "On the Einstein Podolsky Rosen paradox," Physics **1**, 195–200 (1964), reprinted in [63, pp. 403-408] and in [64, pp. 14-21].

[4] I. Pitowsky, *Quantum Probability—Quantum Logic* (Springer, Berlin, 1989).

[5] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, "Proposed Experiment to Test Local Hidden-Variable Theories," Physical Review Letters **23**, 880–884 (1969).
http://dx.doi.org/10.1103/PhysRevLett.23.880

[6] J. F. Clauser and A. Shimony, "Bell's theorem: experimental tests and implications," Rep. Prog. Phys. **41**, 1881–1926 (1978).
http://dx.doi.org/10.1088/0034-4885/41/12/002

[7] A. Peres, "Unperformed experiments have no results," American Journal of Physics **46**, 745–747 (1978).
http://dx.doi.org/10.1119/1.11393

[8] I. Pitowsky and K. Svozil, "New optimal tests of quantum nonlocality," Physical Review A (Atomic, Molecular, and Optical Physics) **64**, 014 102 (2001).
http://dx.doi.org/10.1103/PhysRevA.64.014102

[9] D. Colins and N. Gisin, "A relevant two qbit Bell inequality inequivalent to the CHSH inequality," J. Phys. A: Math. Gen. **37**, 1775–1787 (2004).
http://dx.doi.org/10.1088/0305-4470/37/5/021

[10] S. Filipp and K. Svozil, "Generalizing Tsirelson's Bound on Bell Inequalities Using a Min-Max Principle," Physical Review Letters **93**, 130 407 (2004).
http://dx.doi.org/10.1103/PhysRevLett.93.130407

[11] E. Specker, "Die Logik nicht gleichzeitig entscheidbarer Aussagen," Dialectica **14**, 175–182 (1960), reprinted in [65, pp. 175–182]; English translation: *The logic of propositions which are not simultaneously decidable*, reprinted in [66, pp. 135-140].

[12] S. Kochen and E. P. Specker, "The Problem of Hidden Variables in Quantum Mechanics," Journal of Mathematics and Mechanics **17**, 59–87 (1967), reprinted in [65, pp. 235–263].

[13] N. Zierler and M. Schlessinger, "Boolean embeddings of orthomodular sets and quantum logic," Duke Mathematical Journal **32**, 251–262 (1965).

[14] V. Alda, "On 0-1 measures for projectors I," Aplik. mate. **25**, 373–374 (1980).

[15] V. Alda, "On 0-1 measures for projectors II," Aplik. mate. **26**, 57–58 (1981).

[16] F. Kamber, "Die Struktur des Aussagenkalküls in einer physikalischen Theorie," Nachr. Akad. Wiss. Göttingen **10**, 103–124 (1964).

[17] F. Kamber, "Zweiwertige Wahrscheinlichkeitsfunktionen auf orthokomplementären Verbänden," Mathematische Annalen **158**, 158–196 (1965).

[18] K. Svozil and J. Tkadlec, "Greechie diagrams, nonexistence of measures in quantum logics and Kochen–Specker type constructions," Journal of Mathematical Physics **37**, 5380–5401 (1996). http://dx.doi.org/10.1063/1.531710

[19] A. Cabello, J. M. Estebaranz, and G. García-Alcaine, "Bell-Kochen-Specker theorem: A proof with 18 vectors," Physics Letters A **212**, 183–187 (1996). http://dx.doi.org/10.1016/0375-9601(96)00134-X

[20] D. M. Greenberger, M. A. Horne, and A. Zeilinger, "Going beyond Bell's theorem," in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, M. Kafatos, ed. (Kluwer Academic Publishers, Dordrecht, 1989), pp. 73–76.

[21] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities," American Journal of Physics **58**, 1131–1143 (1990).

[22] N. D. Mermin, "Hidden variables and the two theorems of John Bell," Reviews of Modern Physics **65**, 803–815 (1993). http://dx.doi.org/10.1103/RevModPhys.65.803

[23] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), English translation in [67].

[24] P. R. Halmos, *Finite-dimensional vector spaces* (Springer, New York, Heidelberg, Berlin, 1974).

[25] G. Birkhoff and J. von Neumann, "The Logic of Quantum Mechanics," Annals of Mathematics **37**, 823–843 (1936).

[26] G. W. Mackey, "Quantum mechanics and Hilbert space," Amer. Math. Monthly, Supplement **64**, 45–57 (1957).

[27] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics* (Kluwer Academic Publishers, Dordrecht, 1991).

[28] K. Svozil, *Quantum Logic* (Springer, Singapore, 1998).

[29] A. Einstein, B. Podolsky, and N. Rosen, "Can quantum-mechanical description of physical reality be considered complete?" Physical Review **47**, 777–780 (1935).
http://dx.doi.org/10.1103/PhysRev.47.777

[30] T. Aquinas, *Summa Theologica. Translated by Fathers of the English Dominican Province* (Christian Classics, USA, 1981).
http://www.ccel.org/ccel/aquinas/summa.html

[31] W. T. Stace, "The Refutation of Realism," in *Readings in philosophical analysis*, H. Feigl and W. Sellars, eds. (Appleton–Century–Crofts, New York, 1949), previously published in *Mind* **53**, 1934.

[32] H. Atmanspacher and H. Primas, "Epistemic and Ontic Quantum Realities," in *Foundations of Probability and Physics – 3, AIP Conference Proceedings Volume 750*, A. Khrennikov, ed., pp. 49–62 (2005).
http://dx.doi.org/10.1063/1.1874557

[33] R. Wright, "The state of the pentagon. A nonclassical example," in *Mathematical Foundations of Quantum Theory*, A. R. Marlow, ed. (Academic Press, New York, 1978), pp. 255–274.

[34] R. Wright, "Generalized urn models," Foundations of Physics **20**, 881–903 (1990).

[35] K. Svozil, "Logical equivalence between generalized urn models and finite automata," International Journal of Theoretical Physics **44**, 745–754 (2005).
http://dx.doi.org/10.1007/s10773-005-7052-0

[36] E. F. Moore, "Gedanken-Experiments on Sequential Machines," in *Automata Studies*, C. E. Shannon and J. McCarthy, eds. (Princeton University Press, Princeton, 1956).

[37] M. Schaller and K. Svozil, "Automaton logic," International Journal of Theoretical Physics **35**, 911–940 (1996).

[38] A. Dvurečenskij, S. Pulmannová, and K. Svozil, "Partition Logics, Orthoalgebras and Automata," Helvetica Physica Acta **68**, 407–428 (1995).

[39] C. Calude, E. Calude, K. Svozil, and S. Yu, "Physical versus Computational Complementarity I," International Journal of Theoretical Physics **36**, 1495–1523 (1997).

[40] E. Schrödinger, "Die gegenwärtige Situation in der Quantenmechanik," Naturwissenschaften **23**, 807–812, 823–828, 844–849 (1935), English translation in [68] and [63, pp. 152-167].
http://wwwthep.physik.uni-mainz.de/ matschul/rot/schroedinger.pdf

[41] A. M. Gleason, "Measures on the closed subspaces of a Hilbert space," Journal of Mathematics and Mechanics **6**, 885–893 (1957).

[42] A. Dvurečenskij, *Gleason's Theorem and Its Applications* (Kluwer Academic Publishers, Dordrecht, 1993).

[43] J. Tkadlec, "Diagrams of Kochen-Specker type constructions," International Journal of Theoretical Physics **39**, 921–926 (2000).
http://dx.doi.org/10.1023/A:1003695317353

[44] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, "Experimental realization of any discrete unitary operator," Physical Review Letters **73**, 58–61 (1994).
http://dx.doi.org/10.1103/PhysRevLett.73.58

[45] M. Zukowski, A. Zeilinger, and M. A. Horne, "Realizable higher-dimensional two-particle entanglements via multiport beam splitters," Physical Review A (Atomic, Molecular, and Optical Physics) **55**, 2564–2579 (1997).
http://dx.doi.org/10.1103/PhysRevA.55.2564

[46] K. Svozil, "Noncontextuality in multipartite entanglement," J. Phys. A: Math. Gen. **38**, 5781–5798 (2005).
http://dx.doi.org/10.1088/0305-4470/38/25/013

[47] A. Cabello, "Kochen-Specker theorem and experimental test on hidden variables," International Journal of Modern Physics **A 15**, 2813–2820 (2000).
http://dx.doi.org/10.1142/S0217751X00002020

[48] K. Svozil, "Are simultaneous Bell measurements possible?" New J. Phys. **8**, 39 (2005).
http://dx.doi.org/10.1088/1367-2630/8/3/039

[49] R. K. Clifton (1995), private communication.

[50] A. Aspect, P. Grangier, and G. Roger, "Experimental Tests of Realistic Local Theories via Bell's Theorem," Physical Review Letters **47**, 460–463 (1981).
http://dx.doi.org/10.1103/PhysRevLett.47.460

[51] A. Aspect, P. Grangier, and G. Roger, "Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell's Inequalities," Physical Review Letters **49**, 1804–1807 (1982).
http://dx.doi.org/10.1103/PhysRevLett.49.91

[52] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, "Violation of Bell's Inequality under Strict Einstein Locality Conditions," Phys. Rev. Lett. **81**, 5039–5043 (1998).
http://dx.doi.org/10.1103/PhysRevLett.81.5039

[53] C. Simon, M. Zukowski, H. Weinfurter, and A. Zeilinger, "Feasible "Kochen-Specker experiment with single particles," Physical Review Letters **85**, 1783–1786 (2000).
http://dx.doi.org/10.1103/PhysRevLett.85.1783

[54] Y. Hasegawa, R. Loidl, G. Badurek, M. Baron, and H. Rauch, "Quantum Contextuality in a Single-Neutron Optical Experiment," Physical Review

Letters **97**, 230 401 (2006).
http://dx.doi.org/10.1103/PhysRevLett.97.230401

[55] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, "Experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger entanglement," Nature **403**, 515–519 (2000).

[56] J. S. Bell, "On the Problem of hidden variables in quantum mechanics," Reviews of Modern Physics **38**, 447–452 (1966), reprinted in [64, pp. 1-13].
http://dx.doi.org/10.1103/RevModPhys.38.447

[57] P. Heywood and M. L. G. Redhead, "Nonlocality and the Kochen-Specker Paradox," Foundations of Physics **13**, 481–499 (1983).

[58] M. Redhead, *Incompleteness, Nonlocality, and Realism: A Prolegomenon to the Philosophy of Quantum Mechanics* (Clarendon Press, Oxford, 1990).

[59] N. Bohr, "Discussion with Einstein on epistemological problems in atomic physics," in *Albert Einstein: Philosopher-Scientist*, P. A. Schilpp, ed. (The Library of Living Philosophers, Evanston, Ill., 1949), pp. 200–241.
http://www.emr.hibu.no/lars/eng/schilpp/Default.html

[60] W. M. de Muynck, "Interpretations of quantum mechanics, and interpretations of violation of Bell's inequality," in *Foundations of Probability and Physics – Proceedings of the Conference, Växjö, Sweden, 25 November - 1 December 2000*, A. Khrennikov, ed., pp. 95–114 (2001).
http://www.phys.tue.nl/ktn/Wim/vaxjoproc.pdf

[61] W. M. de Muynck, *Foundations of Quantum Mechanics, an Empiricists Approach. Fundamental theories of physics, Vol. 127* (Kluwer Academic Publishers, Dordrecht, 2002).

[62] A. Khrennikov, "On the problem of completeness of QM: von Neumann against Einstein, Podolsky, and Rosen," Journal of Russian Laser Research **29**, 101 (2008).

[63] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1983).

[64] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).

[65] E. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).

[66] C. A. Hooker, *The Logico-Algebraic Approach to Quantum Mechanics. Volume I: Historical Evolution* (Reidel, Dordrecht, 1975).

[67] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).

[68] J. D. Trimmer, "The present situation in quantum mechanics: a translation of Schrödinger's "cat paradox"," Proc. Am. Phil. Soc. **124**, 323–338 (1980), reprinted in [63, pp. 152-167].