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A statistical mechanical interpretation of algorithmic information theory III: Composite systems and fixed points

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***Abstract*—The statistical mechanical interpretation of algorithmic information theory (AIT, for short) was introduced and developed by our former works [K. Tadaki, Local Proceedings of CiE 2008, pp. 425–434, 2008] and [K. Tadaki, Proceedings of LFCS'09, Springer's LNCS, vol. 5407, pp. 422–440, 2009], where we introduced the notion of thermodynamic quantities, such as partition function $Z(T)$, free energy $F(T)$, energy $E(T)$, and statistical mechanical entropy $S(T)$, into AIT. We then discovered that, in the interpretation, the temperature T equals to the partial randomness of the values of all these thermodynamic quantities, where the notion of partial randomness is a stronger representation of the compression rate by means of program-size complexity. Furthermore, we showed that this situation holds for the temperature itself as a thermodynamic quantity, namely, for each of all the thermodynamic quantities above, the computability of its value at temperature T gives a sufficient condition for $T \in (0, 1)$ to be a fixed point on partial randomness. In this paper, we develop the statistical mechanical interpretation of AIT further and pursue its formal correspondence to normal statistical mechanics. The thermodynamic quantities in AIT are defined based on the halting set of an optimal computer, which is a universal decoding algorithm used to define the notion of program-size complexity. We show that there are infinitely many optimal computers which give completely different sufficient conditions in each of the thermodynamic quantities in AIT. We do this by introducing the notion of composition of computers to AIT, which corresponds to the notion of composition of systems in normal statistical mechanics.**

I. Introduction

Algorithmic information theory (AIT, for short) is a framework for applying information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of AIT is the *program-size complexity* (or *Kolmogorov complexity*) $H(s)$ of a finite binary string s, which is defined as the length of the shortest binary program for an optimal computer U to output s. Here an optimal computer is a universal decoding algorithm. By the definition, $H(s)$ is thought to represent the degree of randomness of a finite binary string s. In particular, the notion of program-size complexity plays a crucial role in characterizing the *randomness* of an infinite binary string, or equivalently, a real.

In [14] we introduced and developed a statistical mechanical interpretation of AIT. We there introduced the notion of *thermodynamic quantities*, such as partition function $Z(T)$, free energy $F(T)$, energy $E(T)$, and statistical mechanical entropy $S(T)$, into AIT. These quantities are reals which depend only on temperature T, any positive real. We then proved that if the temperature T is a computable real with $0 < T < 1$ then, for each of these thermodynamic quantities, the partial randomness of its value equals to T, where the notion of *partial randomness* is a stronger representation of the compression rate by means of program-size complexity. Thus, the temperature T plays a role as the partial randomness of all the thermodynamic quantities in the statistical mechanical interpretation of AIT. In [14] we further showed that the temperature T plays a role as the partial randomness of the temperature T itself, which is a thermodynamic quantity of itself. Namely, we proved *the fixed point theorem on partial randomness*,[1] which states that, for every $T \in (0, 1)$, if the value of partition function $Z(T)$ at temperature T is a computable real, then the partial randomness of T equals to T, and therefore the compression rate of T equals to T, i.e., $\lim_{n\to\infty} H(T_n)/n = T$, where T_n is the first n bits of the base-two expansion of T.

In our second work [15] on this interpretation, we showed that a fixed point theorem of the same form as for $Z(T)$ holds also for each of free energy $F(T)$, energy $E(T)$, and statistical mechanical entropy $S(T)$. Moreover, based on the statistical mechanical relation $F(T) = -T \log_2 Z(T)$, we showed that the computability of $F(T)$ gives completely different fixed points from the computability of $Z(T)$.

In this paper, we develop the statistical mechanical interpretation of AIT further and pursue its formal correspondence to normal statistical mechanics. As a result, we unlock the properties of the sufficient conditions further. The thermodynamic quantities in AIT are defined based on the halting set of an optimal computer. In this paper, we show in Theorem 9 below that there are infinitely many optimal computers which give completely different sufficient conditions in each of the thermodynamic quantities in AIT. We do this by introducing

[1] The fixed point theorem on partial randomness is called a fixed point theorem on compression rate in [14].

the notion of composition of computers to AIT, which corresponds to the notion of composition of systems in normal statistical mechanics.

II. Preliminaries

A. Basic notation

We start with some notation about numbers and strings which will be used in this paper. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rationals, and $\mathbb{Q}^+$ is the set of positive rationals. $\mathbb{R}$ is the set of reals. Let $f\colon S \to \mathbb{R}$ with $S \subset \mathbb{R}$. We say that f is *increasing* (resp., *decreasing*) if $f(x) < f(y)$ (resp., $f(x) > f(y)$) for all $x, y \in S$ with $x < y$.

Normally, $o(n)$ denotes any function $f\colon \mathbb{N}^+ \to \mathbb{R}$ such that $\lim_{n\to\infty} f(n)/n = 0$.

$\{0,1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, \dots\}$ is the set of finite binary strings, where λ denotes the *empty string*. For any $s \in \{0,1\}^*$, $|s|$ is the *length* of s. A subset S of $\{0,1\}^*$ is called *prefix-free* if no string in S is a prefix of another string in S. For any partial function f, the domain of definition of f is denoted by $\operatorname{dom} f$.

Let α be an arbitrary real. We denote by $\alpha_n \in \{0,1\}^*$ the first n bits of the base-two expansion of $\alpha - \lfloor\alpha\rfloor$ with infinitely many zeros, where $\lfloor\alpha\rfloor$ is the greatest integer less than or equal to α. For example, in the case of $\alpha = 5/8$, $\alpha_6 = 101000$.

We say that a real α is *computable* if there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $|\alpha - f(n)| < 1/n$ for all $n \in \mathbb{N}^+$. See e.g. Weihrauch [17] for the detail of the treatment of the computability of reals.

B. Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [4], [5], [7], [6]. A *computer* is a partial recursive function $C\colon \{0,1\}^* \to \{0,1\}^*$ such that $\operatorname{dom} C$ is a nonempty prefix-free set. For each computer C and each $s \in \{0,1\}^*$, $H_C(s)$ is defined by $H_C(s) = \min\left\{\, |p| \;\middle|\; p \in \{0,1\}^* \,\&\, C(p) = s \,\right\}$ (may be ∞). A computer U is said to be *optimal* if for each computer C there exists $d \in \mathbb{N}$ with the following property; for every $p \in \operatorname{dom} C$ there exists $q \in \{0,1\}^*$ for which $U(q) = C(p)$ and $|q| \le |p| + d$. It is easy to see that there exists an optimal computer. We choose a particular optimal computer U as the standard one for use, and define $H(s)$ as $H_U(s)$, which is referred to as the *program-size complexity* of s or the *Kolmogorov complexity* of s. It follows that for every computer C there exists $d \in \mathbb{N}$ such that, every $s \in \{0,1\}^*$, $H(s) \le H_C(s) + d$.

For any $\alpha \in \mathbb{R}$, we say that α is *weakly Chaitin random* if there exists $c \in \mathbb{N}$ such that $n - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$ [4], [5]. On the other hand, for any $\alpha \in \mathbb{R}$, we say that α is *Chaitin random* if $\lim_{n\to\infty} H(\alpha_n) - n = \infty$ [4], [5]. It is then shown that, for every $\alpha \in \mathbb{R}$, α is weakly Chaitin random if and only if α is Chaitin random (see Chaitin [5] for the proof and historical detail).

C. Partial randomness

In the works [11], [12], we generalized the notion of the randomness of a real so that *the degree of the randomness*, which is often referred to as *the partial randomness* recently [2], [9], [3], can be characterized by a real T with $0 \le T \le 1$ as follows.

Definition 1 (weak Chaitin T-randomness)**.** *Let $T \in \mathbb{R}$ with $T \ge 0$. For any $\alpha \in \mathbb{R}$, we say that α is weakly Chaitin T-random if there exists $c \in \mathbb{N}$ such that $Tn - c \le H(\alpha_n)$ for all $n \in \mathbb{N}^+$.* ■

Definition 2 (T-compressibility)**.** *Let $T \in \mathbb{R}$ with $T \ge 0$. For any $\alpha \in \mathbb{R}$, we say that α is T-compressible if $H(\alpha_n) \le Tn + o(n)$, which is equivalent to $\limsup_{n\to\infty} H(\alpha_n)/n \le T$.* ■

In the case of $T = 1$, the weak Chaitin T-randomness results in the weak Chaitin randomness. For every $T \in [0,1]$ and every $\alpha \in \mathbb{R}$, if α is weakly Chaitin T-random and T-compressible, then

$$\lim_{n\to\infty} \frac{H(\alpha_n)}{n} = T. \tag{1}$$

The left-hand side of (1) is referred to as the *compression rate* of a real α in general. Note, however, that (1) does not necessarily imply that α is weakly Chaitin T-random. Thus, the notion of partial randomness is a stronger representation of compression rate.

Definition 3 (Chaitin T-randomness, Tadaki [11], [12])**.** *Let $T \in \mathbb{R}$ with $T \ge 0$. For any $\alpha \in \mathbb{R}$, we say that α is Chaitin T-random if $\lim_{n\to\infty} H(\alpha_n) - Tn = \infty$.* ■

In the case of $T = 1$, the Chaitin T-randomness results in the Chaitin randomness. Obviously, for every $T \in [0,1]$ and every $\alpha \in \mathbb{R}$, if α is Chaitin T-random, then α is weakly Chaitin T-random. However, in 2005 Reimann and Stephan [9] showed that, in the case of $T < 1$, the converse does not necessarily hold. This contrasts with the equivalence between the weak Chaitin randomness and the Chaitin randomness, each of which corresponds to the case of $T = 1$.

III. The previous results

In this section, we review some results of the statistical mechanical interpretation of AIT, developed by our former works [14], [15]. We first introduce the notion of thermodynamic quantities into AIT in the following manner.

In statistical mechanics, the partition function $Z_{\mathrm{sm}}(T)$, free energy $F_{\mathrm{sm}}(T)$, energy $E_{\mathrm{sm}}(T)$, and entropy $S_{\mathrm{sm}}(T)$ at temperature T are given as follows:

$$\begin{aligned}
Z_{\mathrm{sm}}(T) &= \sum_{x\in X} e^{-\frac{E_x}{k_{\mathrm{B}}T}}, \\
F_{\mathrm{sm}}(T) &= -k_{\mathrm{B}}T \ln Z_{\mathrm{sm}}(T), \\
E_{\mathrm{sm}}(T) &= \frac{1}{Z_{\mathrm{sm}}(T)} \sum_{x\in X} E_x e^{-\frac{E_x}{k_{\mathrm{B}}T}}, \\
S_{\mathrm{sm}}(T) &= \frac{E_{\mathrm{sm}}(T) - F_{\mathrm{sm}}(T)}{T},
\end{aligned} \tag{2}$$

where X is a complete set of energy eigenstates of a quantum system and E_x is the energy of an energy eigenstate x. The constant k_{B} is called the Boltzmann Constant, and the $\ln$ denotes the natural logarithm.[2]

Let C be an arbitrary computer. We introduce the notion of thermodynamic quantities into AIT by performing Replacements 1 below for the thermodynamic quantities (2) in statistical mechanics.

Replacements 1.

(i) *Replace the complete set X of energy eigenstates x by the set $\operatorname{dom} C$ of all programs p for C.*

(ii) *Replace the energy E_x of an energy eigenstate x by the length $|p|$ of a program p.*

(iii) *Set the Boltzmann Constant k_{B} to $1/\ln 2$.* ■

Thus, motivated by the formulae (2) and taking into account Replacements 1, we introduce the notion of thermodynamic quantities into AIT as follows.

Definition 4 (thermodynamic quantities in AIT, [14])**.** *Let C be any computer, and let T be any real with $T > 0$.*

First consider the case where $\operatorname{dom} C$ is an infinite set. In this case, we choose a particular enumeration $p_1, p_2, p_3, p_4, \dots$ of the countably infinite set $\operatorname{dom} C$.[3]

(i) *The partition function $Z_C(T)$ at temperature T is defined as $\lim_{k\to\infty} Z_k(T)$ where*

$$Z_k(T) = \sum_{i=1}^{k} 2^{-\frac{|p_i|}{T}}. \tag{3}$$

(ii) *The free energy $F_C(T)$ at temperature T is defined as $\lim_{k\to\infty} F_k(T)$ where*

$$F_k(T) = -T \log_2 Z_k(T). \tag{4}$$

(iii) *The energy $E_C(T)$ at temperature T is defined as $\lim_{k\to\infty} E_k(T)$ where*

$$E_k(T) = \frac{1}{Z_k(T)} \sum_{i=1}^{k} |p_i|\, 2^{-\frac{|p_i|}{T}}. \tag{5}$$

(iv) *The statistical mechanical entropy $S_C(T)$ at temperature T is defined as $\lim_{k\to\infty} S_k(T)$ where*

$$S_k(T) = \frac{E_k(T) - F_k(T)}{T}. \tag{6}$$

In the case where $\operatorname{dom} C$ is a nonempty finite set, the quantities $Z_C(T)$, $F_C(T)$, $E_C(T)$, and $S_C(T)$ are just defined as (3)*,* (4)*,* (5)*, and* (6)*, respectively, where $p_1, \dots, p_k$ is an enumeration of the finite set $\operatorname{dom} C$.* ■

[2] For the thermodynamic quantities in statistical mechanics, see e.g. Chapter 16 of [1] and Chapter 2 of [16]. To be precise, the partition function is not a thermodynamic quantity but a statistical mechanical quantity.

[3] The enumeration $\{p_i\}$ can be chosen quite arbitrarily, and the results of this paper are independent of the choice of $\{p_i\}$. This is because the sum $\sum_{i=1}^{k} 2^{-|p_i|/T}$ and $\sum_{i=1}^{k} |p_i|\, 2^{-|p_i|/T}$ in Definition 4 are positive term series and converge as $k \to \infty$ for every $T \in (0,1)$.

Note that $Z_V(1)$ is precisely a Chaitin Ω number for every optimal computer V. Then Theorems 5 and 6 below hold for these thermodynamic quantities in AIT.

Theorem 5 (properties of $Z(T)$ and $F(T)$, [11], [12], [14])**.** *Let V be an optimal computer, and let $T \in \mathbb{R}$.*

(i) *If $0 < T \le 1$ and T is computable, then each of $Z_V(T)$ and $F_V(T)$ converges and is weakly Chaitin T-random and T-compressible.*

(ii) *If $1 < T$, then $Z_V(T)$ and $F_V(T)$ diverge to ∞ and $-\infty$, respectively.* ■

Theorem 6 (properties of $E(T)$ and $S(T)$, [14])**.** *Let V be an optimal computer, and let $T \in \mathbb{R}$.*

(i) *If $0 < T < 1$ and T is computable, then each of $E_V(T)$ and $S_V(T)$ converges and is Chaitin T-random and T-compressible.*

(ii) *If $1 \le T$, then both $E_V(T)$ and $S_V(T)$ diverge to ∞.* ■

The above two theorems show that if T is a computable real with $T \in (0,1)$ then the temperature T equals to the partial randomness (and therefore the compression rate) of the values of all the thermodynamic quantities in Definition 4 for an optimal computer.

These theorems also show that the values of all the thermodynamic quantities diverge when the temperature T gets across 1. This phenomenon might be regarded as some sort of phase transition in statistical mechanics. Note here that the weak Chaitin T-randomness in Theorem 5 is replaced by the Chaitin T-randomness in Theorem 6 in exchange for the divergence at $T = 1$.

In statistical mechanics or thermodynamics, among all thermodynamic quantities one of the most typical thermodynamic quantities is temperature itself. Theorem 7 below shows that the partial randomness of the temperature T can equal to the temperature T itself in the statistical mechanical interpretation of AIT.

We denote by $\mathcal{FP}_w$ the set of all real $T \in (0,1)$ such that T is weakly Chaitin T-random and T-compressible, and denote by $\mathcal{FP}$ the set of all real $T \in (0,1)$ such that T is Chaitin T-random and T-compressible. Obviously, $\mathcal{FP} \subset \mathcal{FP}_w$. Each element T of $\mathcal{FP}_w$ is a *fixed point on partial randomness*, i.e., satisfies the property that the partial randomness of T equals to T itself, and therefore satisfies that $\lim_{n\to\infty} H(T_n)/n = T$. Let V be a computer. We define the sets $\mathcal{Z}(V)$ by

$$\mathcal{Z}(V) = \{\, T \in (0,1) \mid Z_V(T) \text{ is computable} \,\}.$$

In the same manner, we define the sets $\mathcal{F}(V)$, $\mathcal{E}(V)$, and $\mathcal{S}(V)$ based on the computability of $F_V(T)$, $E_V(T)$, and $S_V(T)$, respectively. Then we can show the following.

Theorem 7 (fixed points on partial randomness, [14], [15])**.** *Let V be an optimal computer. Then $\mathcal{Z}(V) \cup \mathcal{F}(V) \subset \mathcal{FP}_w$ and $\mathcal{E}(V) \cup \mathcal{S}(V) \subset \mathcal{FP}$.* ■

Theorem 7 is just a fixed point theorem on partial randomness, where the computability of each of the values $Z_V(T)$, $F_V(T)$, $E_V(T)$, and $S_V(T)$ gives a sufficient condition for a real $T \in (0,1)$ to be a fixed point on partial randomness. Thus, by Theorem 7, the above observation that the temperature T equals to the partial randomness of the values of the thermodynamic quantities in the statistical mechanical interpretation of AIT is further confirmed.

IV. The main result

In this paper, we investigate the properties of the sufficient conditions for T to be a fixed point on partial randomness in Theorem 7. Using the monotonicity of the functions $Z_V(T)$ and $F_V(T)$ on temperature T and the statistical mechanical relation $F_V(T) = -T \log_2 Z_V(T)$, which holds from Definition 4, we can show the following theorem for the sufficient conditions in Theorem 7.

Theorem 8 ([15])**.** *Let V be an optimal computer. Then each of the sets $\mathcal{Z}(V)$ and $\mathcal{F}(V)$ is dense in $(0,1)$ while $\mathcal{Z}(V) \cap \mathcal{F}(V) = \emptyset$.* ■

Thus, for every optimal computer V, the computability of $F_V(T)$ gives completely different fixed points from the computability of $Z_V(T)$. This implies also that $\mathcal{Z}(V) \subsetneqq \mathcal{FP}_w$ and $\mathcal{F}(V) \subsetneqq \mathcal{FP}_w$.

The aim of this paper is to investigate the structure of $\mathcal{FP}_w$ and $\mathcal{FP}$ in greater detail. Namely, we show in Theorem 9 below that there are infinitely many optimal computers which give completely different sufficient conditions in each of the thermodynamic quantities in AIT. We say that an infinite sequence $V_1, V_2, V_3, \dots$ of computers is *recursive* if there exists a partial recursive function $F \colon \mathbb{N}^+ \times \{0,1\}^* \to \{0,1\}^*$ such that for each $n \in \mathbb{N}^+$ the following two hold: (i) $p \in \operatorname{dom} V_n$ if and only if $(n,p) \in \operatorname{dom} F$, and (ii) $V_n(p) = F(n,p)$ for every $p \in \operatorname{dom} V_n$. Then the main result of this paper is given as follows.

Theorem 9 (main result)**.** *There exists a recursive infinite sequence $V_1, V_2, V_3, \dots$ of optimal computers which satisfies the following conditions:*

(i) $\mathcal{Z}(V_i) \cap \mathcal{Z}(V_j) = \mathcal{F}(V_i) \cap \mathcal{F}(V_j) = \mathcal{E}(V_i) \cap \mathcal{E}(V_j) = \mathcal{S}(V_i) \cap \mathcal{S}(V_j) = \emptyset$ *for all i, j with $i \neq j$.*
(ii) $\bigcup_i \mathcal{Z}(V_i) \subset \mathcal{FP}_w$ *and* $\bigcup_i \mathcal{F}(V_i) \subset \mathcal{FP}_w$.
(iii) $\bigcup_i \mathcal{E}(V_i) \subset \mathcal{FP}$ *and* $\bigcup_i \mathcal{S}(V_i) \subset \mathcal{FP}$. ■

In the subsequent sections we prove the above theorems by introducing the notion of composition of computers to AIT, which corresponds to the notion of composition of systems in normal statistical mechanics.

V. Composition of systems

We first introduce the notion of *composition* of computers.

Definition 10 (composition of computers)**.**
Let $C_1, C_2, \dots, C_N$ be computers. The composition $C_1 \oslash C_2 \oslash \cdots \oslash C_N$ of C_1, C_2, $\dots$, and C_N is defined as the computer D such that (i) $\operatorname{dom} D = \{p_1 p_2 \dots p_N \mid p_1 \in \operatorname{dom} C_1 \ \&\ p_2 \in \operatorname{dom} C_2 \ \&\ \cdots \ \&\ p_N \in \operatorname{dom} C_N\}$, and (ii) $D(p_1 p_2 \dots p_N) = C_1(p_1)$ for every $p_1 \in \operatorname{dom} C_1$, $p_2 \in \operatorname{dom} C_2$, $\dots$, and $p_N \in \operatorname{dom} C_N$. ■

Theorem 11. *Let $C_1, C_2, \dots, C_N$ be computers. If C_1 is optimal then $C_1 \oslash C_2 \oslash \cdots \oslash C_N$ is also optimal.*

Proof: We first choose particular strings r_2, r_3, $\dots$, r_N with $r_2 \in \operatorname{dom} C_2$, $r_3 \in \operatorname{dom} C_3$, $\dots$, and $r_N \in \operatorname{dom} C_N$. Let C be an arbitrary computer. Then, by the definition of the optimality of C_1, there exists $d \in \mathbb{N}$ with the following property; for every $p \in \operatorname{dom} C$ there exists $q \in \{0,1\}^*$ for which $C_1(q) = C(p)$ and $|q| \le |p| + d$. It follows from the definition of the composition $C_1 \oslash C_2 \oslash \cdots \oslash C_N$ that for every $p \in \operatorname{dom} C$ there exists $q \in \{0,1\}^*$ for which $(C_1 \oslash C_2 \oslash \cdots \oslash C_N)(q r_2 r_3 \dots r_N) = C(p)$ and $|q r_2 r_3 \dots r_N| \le |p| + |r_2 r_3 \dots r_N| + d$. Thus $C_1 \oslash C_2 \oslash \cdots \oslash C_N$ is an optimal computer. ■

In the same manner as in normal statistical mechanics, we can prove Theorem 12 below for the thermodynamic quantities in AIT. In particular, the equations (7), (8), and (9) correspond to the fact that free energy, energy, and entropy are extensive parameters in thermodynamics, respectively.

Theorem 12. *Let $C_1, C_2, \dots, C_N$ be computers. Then the following hold for every $T \in (0,1)$.*

$$
\begin{aligned}
Z_{C_1 \oslash \cdots \oslash C_N}(T) &= Z_{C_1}(T) \cdots Z_{C_N}(T), \\
F_{C_1 \oslash \cdots \oslash C_N}(T) &= F_{C_1}(T) + \cdots + F_{C_N}(T), && (7) \\
E_{C_1 \oslash \cdots \oslash C_N}(T) &= E_{C_1}(T) + \cdots + E_{C_N}(T), && (8) \\
S_{C_1 \oslash \cdots \oslash C_N}(T) &= S_{C_1}(T) + \cdots + S_{C_N}(T). && (9)
\end{aligned}
$$

■

For any computer C and any $n \in \mathbb{N}^+$, the computer $\underbrace{C \oslash \cdots \oslash C}_{n}$ is denoted by $C^{\oslash n}$.

VI. The proof of the main result

In order to prove the main result, Theorem 9, we next introduce the notion of physically reasonable computer.

Definition 13 (physically reasonable computer)**.** *For any computer C, we say that C is physically reasonable if there exist $p, q \in \operatorname{dom} C$ such that $|p| \neq |q|$.* ■

Example 14. *The following two computers are examples of physically reasonable computers.*

(i) Two level system: *Let B be a particular computer for which $\operatorname{dom} B = \{1, 01\}$. Then we see that, for every $T > 0$,*

$$
\begin{aligned}
Z_B(T) &= 2^{-1/T} + 2^{-2/T}, \\
F_B(T) &= -T \log_2 Z_B(T), \\
E_B(T) &= \frac{1}{Z_B(T)} \left(2^{-1/T} + 2 \cdot 2^{-2/T} \right), \\
S_B(T) &= (E_B(T) - F_B(T))/T.
\end{aligned}
$$

(ii) One dimensional harmonic oscillator: *Let O be a particular computer for which $\operatorname{dom} O = \{0^l 1 \mid l \in \mathbb{N}\}$. Then we* *see that, for every $T>0$,*

$$\begin{aligned}
Z_O(T) &= \frac{1}{2^{1/T}-1},\\
F_O(T) &= T\log_2\left(2^{1/T}-1\right),\\
E_O(T) &= \frac{2^{1/T}}{2^{1/T}-1},\\
S_O(T) &= (E_O(T)-F_O(T))/T.
\end{aligned}$$

■

We can prove Theorem 15 below in a similar manner to the proof of Theorem 7 of [15]. We can directly check that Theorem 15 holds for the above two examples B and O of physically reasonable computers.

Theorem 15. *Let C be a computer. Suppose that C is physically reasonable. Then each of the mapping $(0,1)\ni T\mapsto Z_C(T)$, the mapping $(0,1)\ni T\mapsto E_C(T)$, and the mapping $(0,1)\ni T\mapsto S_C(T)$ is an increasing real function. On the other hand, the mapping $(0,1)\ni T\mapsto F_C(T)$ is a decreasing real function.* ■

Based on Theorem 12 and the physically reasonable computer given in Example 14, the main result is proved as follows.

The proof of Theorem 9: Let O be the computer considered in Example 14 (ii). For each $n\in\mathbb{N}^+$, we denote the computer $U\oslash(O^{\oslash n})$ by V_n. Recall here that U is the optimal computer used to define $H(s)$. Then, by Theorem 11, we first see that V_n is optimal for every $n\in\mathbb{N}^+$. Furthermore, it is easy to see that the infinite sequence $V_1,V_2,V_3,\dots$ of computers is recursive. It follows from Theorem 12 that, for every $T\in(0,1)$,

$$\begin{aligned}
Z_{V_n}(T) &= Z_U(T)Z_O(T)^n,\\
F_{V_n}(T) &= F_U(T)+nF_O(T),\\
E_{V_n}(T) &= E_U(T)+nE_O(T),\\
S_{V_n}(T) &= S_U(T)+nS_O(T).
\end{aligned}\tag{10}$$

Let m and n be arbitrary two positive integers with $m>n$. Then it follows from the equations (10) that

$$Z_{V_m}(T) = Z_{V_n}(T)Z_O(T)^{m-n},\tag{11}$$
$$F_{V_m}(T) = F_{V_n}(T)+(m-n)F_O(T),\tag{12}$$
$$E_{V_m}(T) = E_{V_n}(T)+(m-n)E_O(T),\tag{13}$$
$$S_{V_m}(T) = S_{V_n}(T)+(m-n)S_O(T)\tag{14}$$

for every $T\in(0,1)$. In what follows, using (12) we show that $\mathcal{F}(V_m)\cap\mathcal{F}(V_n)=\emptyset$. In a similar manner, using (11), (13), and (14) we can show that $\mathcal{Z}(V_m)\cap\mathcal{Z}(V_n)=\mathcal{E}(V_m)\cap\mathcal{E}(V_n)=\mathcal{S}(V_m)\cap\mathcal{S}(V_n)=\emptyset$ as well.

Now, let us assume contrarily that $\mathcal{F}(V_m)\cap\mathcal{F}(V_n)\neq\emptyset$. Then there exists $T_c\in(0,1)$ such that both $F_{V_m}(T_c)$ and $F_{V_m}(T_c)$ are computable. It follows from (12) that

$$F_O(T_c)=\frac{1}{m-n}\left(F_{V_m}(T_c)-F_{V_n}(T_c)\right).$$

Thus, $F_O(T_c)$ is also computable. Hence, from the definition of the computability of real, we can show that there exist total recursive functions $a\colon\mathbb{N}^+\to\mathbb{Q}$ and $b\colon\mathbb{N}^+\to\mathbb{Q}$ such that (i) $a(n)\le F_O(T_c)\le b(n)$ for all $n\in\mathbb{N}^+$ and (ii) $\lim_{n\to\infty}a(n)=\lim_{n\to\infty}b(n)=F_O(T_c)$.

On the other hand, since $F_O(r)=r\log_2\left(2^{1/r}-1\right)$ for every $r\in\mathbb{Q}^+$, it is shown that there exist total recursive functions $c\colon\mathbb{N}^+\times\mathbb{Q}^+\to\mathbb{Q}$ and $d\colon\mathbb{N}^+\times\mathbb{Q}^+\to\mathbb{Q}$ such that (i) $c(n,r)\le F_O(r)\le d(n,r)$ for all $n\in\mathbb{N}^+$ and all $r\in\mathbb{Q}^+$ and (ii) $\lim_{n\to\infty}c(n,r)=\lim_{n\to\infty}d(n,r)=F_O(r)$ for all $r\in\mathbb{Q}^+$. Since the mapping $(0,1)\ni T\mapsto F_O(T)$ is a decreasing real function by Theorem 15, it is then easy to see that, given $k\in\mathbb{N}^+$, one can find $r_1,r_2\in\mathbb{Q}^+$ and $n\in\mathbb{N}^+$ such that (i) $0<r_1<r_2<1$, (ii) $|r_2-r_1|<1/k$, (iii) $c(n,r_1)\ge b(n)$, and (iv) $a(n)\ge d(n,r_2)$ by searching such r_1,r_2 and n exhaustively. Since the mapping $(0,1)\ni T\mapsto F_O(T)$ is a decreasing real function by Theorem 15 again, we see that $r_1\le T_c\le r_2$ and therefore $|T_c-r_1|<1/k$. Thus, there exists a total recursive function $f\colon\mathbb{N}^+\to\mathbb{Q}$ such that $|T_c-f(k)|<1/k$ for all $k\in\mathbb{N}^+$. Hence, T_c is computable.

Since V_m is optimal and T_c is computable, it follows from Theorem 5 (i) that $F_{V_m}(T_c)$ is weakly Chaitin T_c-random. However, this contradicts the fact that $F_{V_m}(T_c)$ is computable. Thus we have $\mathcal{F}(V_m)\cap\mathcal{F}(V_n)=\emptyset$. This completes the proof of Theorem 9 (i).

Theorem 9 (ii) and (iii) follow immediately from Theorem 7 and the fact that V_i is optimal for all i. ■

VII. Conclusion

As a sequel to our former works, in this work we developed the statistical mechanical interpretation of AIT further and pursued its formal correspondence to normal statistical mechanics. In particular, we investigated the structure of the set of fixed points on partial randomness in greater detail by introducing the notion of composition of computers to AIT, which corresponds to the notion of composition of systems in normal statistical mechanics.

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