Hyperbolic Discounting of the Far-Distant Future

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Abstract. We prove an analogue of Weitzman's [13] famous result that an exponential discounter who is uncertain of the appropriate exponential discount rate should discount the far-distant future using the lowest (i.e., most patient) of the possible discount rates. Our analogous result applies to a hyperbolic discounter who is uncertain about the appropriate hyperbolic discount rate. In this case, the far-distant future should be discounted using the probability-weighted *harmonic mean* of the possible hyperbolic discount rates.

Keywords: Hyperbolic discounting, Uncertainty.
JEL Classification: D71, D90.

1 Introduction

Consider an individiual – or Social Planner – who ranks streams of outcomes over a continuous and unbounded time horizon $T = [0, \infty)$ using a discounted utility criterion with discount function $D \colon T \to (0, 1]$. We assume throughout that D is differentiable, strictly decreasing and satisfies $D(0) = 1$. For example, D might have the *exponential* form

$$D(t) = e^{-rt}$$

for some constant *discount (or time preference) rate*, $r > 0$. Such discounting may be motivated by suitable preference axioms ([5]) or as a survival function with constant hazard rate, r ([12]). For an arbitrary (i.e., not necessarily exponential) discount function, Weitzman ([13]) defines the *local (or instantaneous) discount rate*, $r(t)$, using the relationship:

$$D(t) = \exp\left(-\int_0^t r(\tau)d\tau\right) \quad \Leftrightarrow \quad r(t) = -\frac{D'(t)}{D(t)} \tag{1}$$

Note that $r(t)$ is constant if (and only if) D has the exponential form.

Weitzman ([13]) considers a scenario in which the decision-maker is uncertain about the appropriate discount function to use. She may, for example, be uncertain about the true (constant) hazard rate of her survival function, as in [12]. The decision-maker entertains n possible scenarios corresponding to n possible discount functions D_i, $i = 1, 2, ..., n$, with associated local discount rate functions r_i. Suppose that scenario i has probability $p_i > 0$, with $\sum_{i=1}^n p_i = 1$, and that the decision-maker discounts according to the *expected (or certainty equivalent)* discount function

$$D = \sum_{i=1}^n p_i D_i \tag{2}$$

Such a discount function may also arise if the decision-maker is a utilitarian Social Planner for a population of n heterogeneous individuals, as in [6].

Let r be the local discount rate function associated with certainty equivalent discount function (2). Weitzman [13] studies the limit behaviour of $r(t)$ as $t \to \infty$. He proves that if each $r_i(t)$ converges to a limit, then $r(t)$ converges to the lowest of these limits. In other words, if

$$\lim_{t\to\infty} r_i(t) = r_i^*$$

for each i, then

$$\lim_{t\to\infty} r(t) = \min\{r_1^*, \ldots, r_n^*\}. \tag{3}$$

Moreover, if each r_i is constant (i.e., each D_i is exponential), then $r(t)$ declines *monotonically* to this limit ([13]).[1]

Example 1. *Suppose each D_i is exponential, so that $r_i(t) = r_i$ is constant. Then the results in [13] imply that $r(t)$ declines monotonically with $lim_{t\to\infty} r(t) = \min_i r_i$. Figure 1 illustrates for the case $n = 3$, $r_1 = 0.01$, $r_2 = 0.02$, $r_3 = 0.03$ and $p_1 = p_2 = p_3 = 1/3$.*

[1] In fact, this is true more generally – see [13, footnote 6].

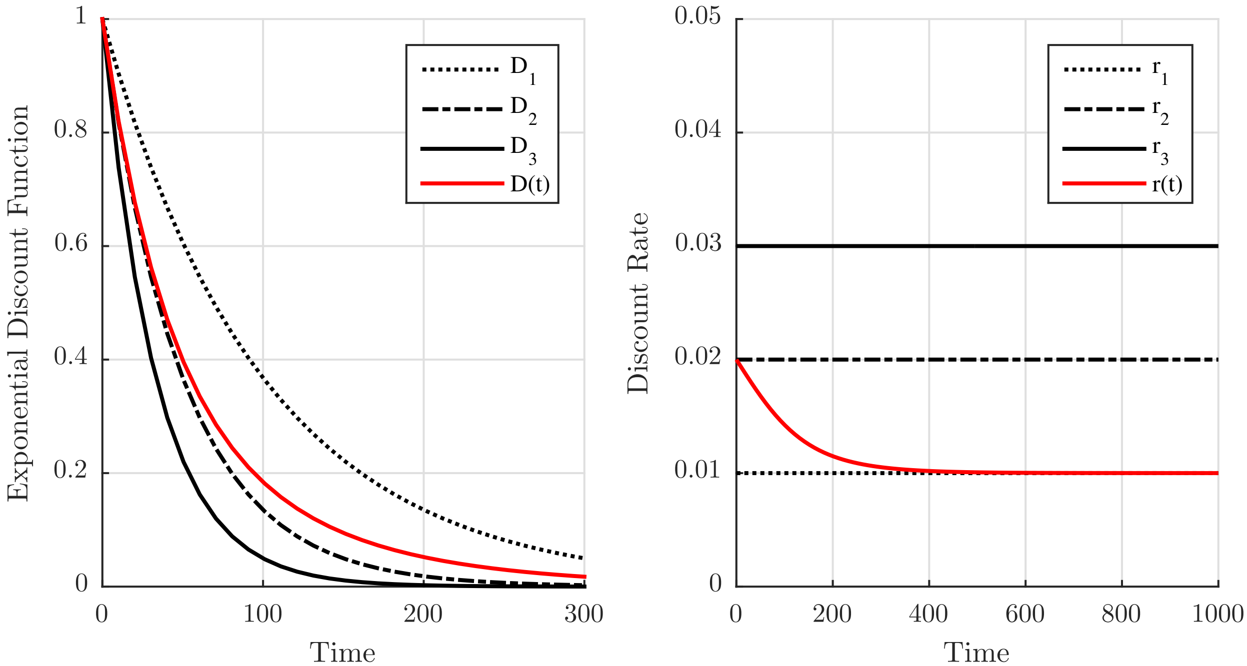

Figure 1: Exponential Discount Functions

Weitzman's result may be interpreted as saying that the certainty equivalent discount function (2) behaves locally as an exponential discount function with discount rate (3) when discounting outcomes in the far distant future. This result is most salient if the the individual discount functions are themselves exponential, as in Example 1. However, many individuals do *not* discount exponentially ([2]). If all the D_i functions are contained within some non-exponential class, it is natural to characterise the local asymptotic behaviour of (2) using a function from the same class. The next section considers the case of proportional hyperbolic functions.

2 Proportional hyperbolic discounting

It is well known (see, for example, [2] and [8]) that individuals typically have intertemporal preferences that are inconsistent with exponential discounting but compatible with time preference rates that *decrease* with t. Such preferences exhibit *decreasing impatience (DI)*.[2] The necessity of accommodating DI has made hyperbolic discounting a significant tool in behavioural economics. Several types of hyperbolic discount functions have been introduced, including quasi-hyperbolic [7, 10], proportional hyperbolic [4, 9], and generalized hyperbolic [1, 8].

[2]See Prelec [11] for a formal definition.

In this section we assume that each D_i has the proportional hyperbolic form

$$D_i(t) \quad = \frac{1}{1+h_i t}$$

where parameter $h_i > 0$ is the *hyperbolic discount rate.* Note that

$$r_i(t) \quad = \quad -\frac{D_i'(t)}{D_i(t)} \quad = \quad \frac{h_i}{1+h_i t}$$

so $r_i(t)$ decreases over time.

Prelec [11] defines a local index of DI, analogous to the Arrow-Pratt index of absolute risk aversion for preferences over lotteries. For the proportional hyperbolic discount function D_i this index is

$$I_i(t) \quad = \quad -\frac{r_i'(t)}{r_i(t)} \quad = \quad \frac{h_i}{1+h_i t}$$

Prelec [11] also introduces a comparative notion of DI, analogous to the notion of "more risk averse than" for lottery preferences, and shows that D_i is "more decreasingly impatient than" D_j if $I_i(t) \geq I_j(t)$ for all t. For proportional hyperbolic discount functions, we observe that $I_i(t) \geq I_j(t)$ iff $h_i \geq h_j$. The hyperbolic discount rate therefore determines how rapidly impatience diminishes.

We henceforth assume that the discount functions have been indexed such that $h_1 > h_2 > \ldots > h_n$, so D_1 exhibits the most rapidly diminishing impatience and D_n the least. Nevertheless, the limiting behaviour of these discount functions is indistinguishable through Weitzman's lens, since

$$r_i^* \quad = \quad \lim_{t\to\infty} \frac{h_i}{1+h_i t} \quad = \quad 0$$

for each i. In other words, the limit of the local *exponential* discount rate is the same for each discount function, reflecting the fact that hyperbolic functions decline more slowly than exponentials for large t. Weitzman's result is therefore not very informative for this scenario.

Instead, we should like to have a local *hyperbolic* approximation to the certainty equivalent discount function (2). We follow Weitzman's example and define the *local (or instantaneous) hyperbolic discount rate*, $h(t)$, as follows:

$$D(t) \quad = \quad \frac{1}{1+h(t)t} \quad \Leftrightarrow \quad h(t) = \left(\frac{1}{D(t)} - 1\right)\frac{1}{t} \tag{4}$$

Note that $h(t)$ is constant if (and only if) D has the proportional hyperbolic form.

The question we wish to address is the following: *How does $h(t)$ behave as $t \to \infty$?* Theorems 1 and 2, which are proved in the Appendix, provide the answer. In order to state the second of these results, we remind the reader that the *weighted harmonic mean* of non-negative values $x_1, x_2, \ldots, x_n$ with non-negative weights $a_1, a_2, \ldots, a_n$ satisfying $a_1 + \ldots + a_n = 1$ is

$$H(x_1, a_1; \ldots; x_n, a_n) = \left(\sum_{i=1}^{n} \frac{a_i}{x_i}\right)^{-1}.$$

It is well-known that the weighted harmonic mean is smaller than the corresponding weighted arithmetic mean (i.e., expected value).

Theorem 1. *The local hyperbolic discount rate, $h(t)$, is strictly decreasing.*

Theorem 2. *The local hyperbolic discount rate, $h(t)$, converges to the probability-weighted harmonic mean of the individual hyperbolic discount rates. That is*

$$h(t) \to H(h_1, p_1; \ldots; h_n, p_n)$$

as $t \to \infty$.

The following example illustrates both results.

Example 2. *Suppose $n = 3$, $h_1 = 0.01$, $h_2 = 0.02$, $h_3 = 0.03$ and $p_1 = p_2 = p_3 = \frac{1}{3}$. Note that $h_2 = 0.02$ corresponds to the arithmetic mean of h_1, h_2 and h_3. Figure 2 displays the monotonic decline of $h(t)$ towards the weighted harmonic mean $H(h_1, p_1; h_2, p_; h_3, p_3) \approx$ 0.0164.*

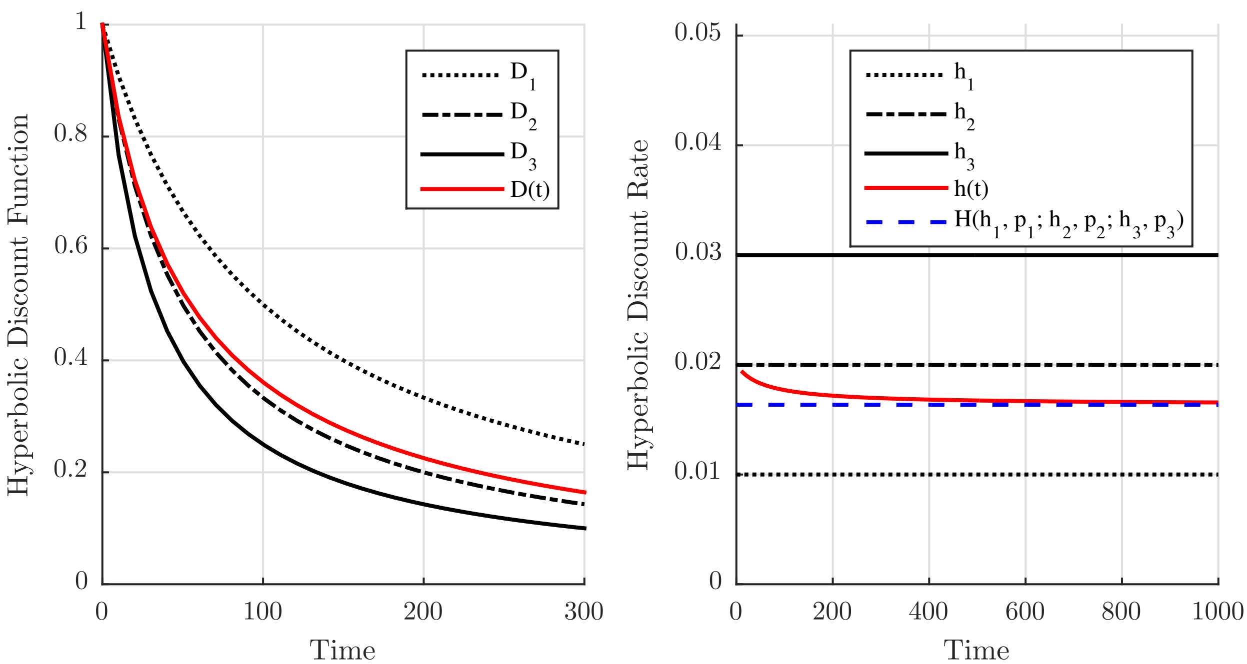

Figure 2: Hyperbolic Discount Functions

3 Discussion

With exponential discounting, uncertainty about the (exponential) discount rate implies that the far-distant future is discounted according to the most "patient" of the possible discount functions.[3] If discounting is hyperbolic, with uncertainty about the (hyperbolic) discount rate, *all* possible discount functions matter for the discounting of the far-distant future. The asymptotic local hyperbolic discount rate is, however, below the average (i.e., arithmetic mean) of the possible rates.

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A Appendix

A.1 Proof of Theorem 1

We prove this statement by induction on n. First we need to prove that the statement holds for $n = 2$. In this case:

$$h(t) = \left[\frac{1}{p_1(1+h_1t)^{-1} + p_2(1+h_2t)^{-1}} - 1\right]\frac{1}{t}$$

for each $t > 0$. Rearranging:

$$h(t) = \left[\frac{(1+h_1t)(1+h_2t)}{p_1(1+h_2t) + p_2(1+h_1t)} - 1\right]\frac{1}{t} \;=\; \left[\frac{1+(h_1+h_2)t + h_1h_2t^2}{p_1+p_2+(p_1h_2+p_2h_1)t} - 1\right]\frac{1}{t}.$$

Since $p_1 + p_2 = 1$ we obtain:

$$h(t) = \left[\frac{1+(h_1+h_2)\,t + h_1h_2t^2}{1+(p_1h_2+p_2h_1)\,t} - 1\right]\frac{1}{t} \;=\; \frac{p_1h_1+p_2h_2+h_1h_2t}{1+(p_1h_2+p_2h_1)\,t}.$$

By differentiating $h(t)$:

$$h'(t) = \frac{h_1h_2\,(1+(p_1h_2+p_2h_1)\,t) - (p_1h_1+p_2h_2+h_1h_2t)\,(p_1h_2+p_2h_1)}{[1+(p_1h_2+p_2h_1)t]^2} \tag{5}$$

We need to show that $h'(t) < 0$. Since the denominator of (5) is positive, the sign of $h'(t)$ depends on the sign of the numerator. Therefore, we denote the numerator of (5) by Q and analyse it separately:

$$\begin{aligned} Q(t) &= h_1h_2\,[1+(p_1h_2+p_2h_1)\,t] - (p_1h_1+p_2h_2+h_1h_2t)\,(p_1h_2+p_2h_1) \\ &= h_1h_2 - (p_1h_1+p_2h_2)\,(p_1h_2+p_2h_1)\,. \end{aligned}$$

[3] See, in particular, the important reformulation of Weitzman's result by Gollier and Weitzman ([3]), which resolves the so-called "Weitzman-Gollier puzzle".

By expanding the brackets and using the fact that $p_1+p_2=1$ implies $1-p_1^2-p_2^2=2p_1p_2$ expression Q can be simplified further:

$$\begin{aligned} Q(t) &= h_1h_2 - p_1^2h_1h_2 - p_1p_2h_1^2 - p_1p_2h_2^2 - p_2^2h_1h_2 \\ &= h_1h_2(1-p_1^2-p_2^2) - p_1p_2(h_1^2+h_2^2) \\ &= -p_1p_2(h_1-h_2)^2. \end{aligned}$$

Therefore, since $h_1 \neq h_2$ we have $Q < 0$. Hence it follows that $h'(t) < 0$ and $h(t)$ is strictly decreasing.

Suppose that the proposition holds for $n = k$. We need to show that it also holds for $n = k+1$. When $n = k+1$ we have:

$$D \;=\; \sum_{i=1}^{k+1} p_i D_i \;=\; (1-p_{k+1}) \left(\sum_{i=1}^{k} \frac{p_i}{1-p_{k+1}} D_i \right) + p_{k+1} D_{k+1}.$$

Since

$$\sum_{i=1}^{k} \frac{p_i}{1-p_{k+1}} = 1,$$

we may write

$$D \;=\; (1-p_{k+1})\, D^{(k)} + p_{k+1} D_{k+1}.$$

where

$$D^{(k)} \;=\; \sum_{i=1}^{k} \frac{p_i}{1-p_{k+1}} D_i.$$

By the induction hypothesis it follows that

$$D^{(k)} = \frac{1}{1+h^{(k)}(t)t},$$

where $h^{(k)}$ is strictly decreasing. Therefore,

$$\begin{aligned} h(t) &= \left[\frac{1}{(1-p_{k+1})D^{(k)} + p_{k+1}D_{k+1}} - 1 \right] \frac{1}{t} \\ &= \left[\frac{1}{(1-p_{k+1})\left(1+h^{(k)}(t)t\right)^{-1} + p_{k+1}\left(1+h_{k+1}t\right)^{-1}} - 1 \right] \frac{1}{t}. \end{aligned}$$

Let $\hat{p_1} = 1-p_{k+1}$, $\hat{p_2} = p_{k+1}$, $\hat{h_1}(t) = h^{(k)}(t)$ and $\hat{h_2} = h_{k+1}$. Then we have

$$h(t) = \left[\frac{1}{\hat{p_1}(1+\hat{h_1}(t)t)^{-1} + \hat{p_2}(1+\hat{h_2}t)^{-1}} - 1 \right] \frac{1}{t}.$$

Analogously to the case $n = 2$, this expression can be rearranged to give:

$$h(t) = \frac{\hat{p_1}\hat{h_1}(t) + \hat{p_2}\hat{h_2} + \hat{h_1}(t)\hat{h_2}t}{1 + \hat{p_1}\hat{h_2}t + \hat{p_2}\hat{h_1}(t)t}.$$

However, in contrast to the case $n = 2$, $\hat{h_1}$ is now a function of t. Thus:

$$h'(t) \;=\; \frac{N(t)}{\left[1 + \hat{p_1}\hat{h_2}t + \hat{p_2}\hat{h_1}(t)t\right]^2}. \tag{6}$$

where

$$\begin{aligned} N(t) = & \left(\hat{p_1}\hat{h}_1'(t) + \hat{h_1}(t)\hat{h_2} + \hat{h}_1'(t)\hat{h_2}t\right)\left(1 + \hat{p_1}\hat{h_2}t + \hat{p_2}\hat{h_1}(t)t\right) \\ & - \left(\hat{p_1}\hat{h_1}(t) + \hat{p_2}\hat{h_2} + \hat{h_1}(t)\hat{h_2}t\right)\left(\hat{p_1}\hat{h_2} + \hat{p_2}\hat{h_1}(t) + \hat{p_2}\hat{h}_1'(t)t\right). \end{aligned}$$

The denominator of (6) is strictly positive, so the sign of the derivative is the same as that of $N(t)$. Note that

$$N(t) \;=\; \hat{Q}\,(t) + \hat{h}_1'(t)\left[\left(\hat{p}_1 + \hat{h}_2 t\right)\left(1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t)t\right) - \hat{p}_2 t\left(\hat{p}_1\hat{h}_1(t) + \hat{p}_2\hat{h}_2 + \hat{h}_1(t)\hat{h}_2 t\right)\right]$$

where $\hat{Q}\,(t)$ is defined as above, but with $h_1 = \hat{h}_1\,(t)$ and $h_2 = \hat{h}_2$. Since $\hat{Q}\,(t) \le 0$ (with equality if and only if $\hat{h_1}\,(t) = h_2$) and $\hat{h}_1' < 0$, it suffices to show

$$\left(\hat{p}_1 + \hat{h}_2 t\right)\left(1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t)t\right) - \hat{p}_2 t\left(\hat{p}_1\hat{h}_1(t) + \hat{p}_2\hat{h}_2 + \hat{h}_1(t)\hat{h}_2 t\right) > 0 \tag{7}$$

Cancelling terms on the left-hand side of (7) leaves us with:

$$\hat{p}_1\left(1 + \hat{p}_1\hat{h}_2 t\right) + \hat{h}_2 t\left(1 + \hat{p}_1\hat{h}_2 t\right) - (\hat{p}_2)^2\,\hat{h}_2 t.$$

We now use the fact that $(\hat{p}_2)^2 = (1 - \hat{p}_1)^2 = 1 - 2\hat{p}_1 + (\hat{p}_1)^2$ to get

$$\hat{p}_1\left(1 + \hat{p}_1\hat{h}_2 t\right) + \hat{h}_2 t\left(1 + \hat{p}_1\hat{h}_2 t\right) - \left[1 - 2\hat{p}_1 + (\hat{p}_1)^2\right]\hat{h}_2 t \;=\; \hat{p}_1 + \left(t\hat{h}_2\right)^2\hat{p}_1 + 2\hat{p}_1\hat{h}_2 t,$$

which is strictly positive. This establishes the required inequality (7) and completes the proof.

A.2 Proof of Theorem 2

We note that

$$\frac{p_i}{1 + h_i t} \;=\; \frac{p_i}{h_i t} + \epsilon_i(t),$$

where $\epsilon_i(t)/t^2 \to 0$ when $t \to \infty$. Let $\epsilon(t) = \epsilon_1(t) + \ldots + \epsilon_n(t)$. Hence it follows that:

$$\begin{aligned} \frac{1}{1 + h(t)t} \;=\; \sum_{i=1}^{n} p_i D_i(t) \;&=\; \frac{p_1}{1 + h_1 t} + \ldots + \frac{p_n}{1 + h_n t} \\ &=\; \frac{p_1}{h_1 t} + \ldots + \frac{p_n}{h_n t} + \epsilon(t) \\ &=\; \left(\frac{p_1}{h_1} + \ldots + \frac{p_n}{h_n}\right)\frac{1}{t} + \epsilon(t) \\ &=\; \frac{1}{H(h_1, p_1; \ldots; h_n, p_n)t} + \epsilon(t) \\ &=\; \frac{1}{1 + H(h_1, p_1; \ldots; h_n, p_n)t} + \hat{\epsilon}(t), \end{aligned}$$

where $\hat{\epsilon}(t)/t^2 \to 0$ as $t \to \infty$. This implies that $h(t) \to H(h_1, p_1; \ldots; h_n, p_n)$ as $t \to \infty$.

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