The Complexity of Multiwinner Voting Rules with Variable Number of Winners

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Abstract

We consider the approval-based model of elections, and undertake a computational study of voting rules which select committees whose size is not predetermined. While voting rules that output committees with a predetermined number of winning candidates are quite well studied, the study of elections with variable number of winners has only recently been initiated by Kilgour [18]. This paper aims at achieving a better understanding of these rules, their computational complexity, and on scenarios for which they might be applicable.

1 Introduction

We study the setting where a group of agents (the voters) want to select a set of candidates (a committee) based on these agents' preferences. Agents are asked which candidates they approve of for the inclusion into the committee and this input data needs to be aggregated. However, as opposed to the quickly growing body of work on electing committees of a fixed size [14, 15, 3, 1, 17], here we are interested in rules that derive both the size of the winning committee and its members from the voters' preferences. Recently, Kilgour [18] and Duddy et al. [13] initiated a systematic study of such voting rules; here we are interested in the complexity of computing their winners and in experimentally analyzing the sizes of the elected committees (for some early axiomatic results, we also point the reader to the work of Brandl and Peters [9]).

1.1 When Not To Fix the Size of the Committee?

There is a number of settings where it is not natural to fix the size of the committee to be elected and it is better to deduce it from the votes. Since so far committee elections with variable number of winners did not receive much attention in the AI literature, below we provide a number of examples of such settings. We do not mention this repeatedly, but one may wish to automate the processes in the examples below using AI techniques.

Initial Screening. Consider a situation where we need to select one item—among many possible ones—that has some desirable features. The final decision can only be done by a qualified expert, but we have a number of easy to evaluate (but imperfect) criteria that the selected items should satisfy (these criteria are soft and it may be that the best item actually fails some of them). We view each criterion as a voter (who "approves" the items that satisfy it) and we seek a committee, hopefully of a small size, of candidates from which the qualified expert will choose the final item.[1]

Initial screening is closely related to shortlisting [4, 14]. We use a different name for it to emphasize that we do not fix the number of candidates to choose, as is the case with shortlisting.

Finding a Set of Qualifying Candidates. Finding a set of candidates that satisfy all or almost all criteria is a common problem. Real-life examples include selecting baseball players for inclusion into the Hall of Fame and selecting students to receive an honors degree. In the former case, eligible voters (baseball writers) approve up to ten players and those approved by at least 75% of the voters are chosen to the Hall of Fame. In the latter case, the voting process is typically implicit; the university announces a set of criteria of excellency—which act as voters, "approving" the students that satisfy them[2]—and set rules such as "a student receives an honors degree if he or she meets at least five out of six criteria". It is often desirable that the selected committee is small (say, at most a few people for the Hall of Fame and some not-too-large percentage of the students for the honors degree), but this is not always the case. E.g, consider the task of selecting people for an in-depth medical check based on a number of simple criteria that jointly indicate elevated risk of a certain disease; everyone who is at risk should be checked regardless of the number of those patients.

One of the first procedures formally proposed for the task of selecting a group of qualifying candidates was the majority rule (MV), suggested by Brams, Kilgour, and Sanver [8]. The majority rule outputs the committee that includes all the candidates that are approved by at least half of the voters (satisfy at least half of the criteria). It is, of course, natural to consider MV with other thresholds, as, for example, in the Hall of Fame example.

Partitioning into Homogeneous Groups. For the case of partitioning candidates into *homogeneous* groups, we can no longer focus only on one of the groups (the committee), but rather we care about a partition into two groups so that each of them contains candidates that are as similar as possible. A prime example here is partitioning students in some class into two groups, e.g., a group of beginners and a group of advanced ones (say, regarding, their knowledge of a foreign language; depending on the setting, it may or may not be important to keep the sizes of these two groups close). The students are partitioned in this way to facilitate a better learning environment for everyone; in the context of voting, the issue of partitioning students was raised by Duddy et al. [13].

Finding a Representative Committee. An elected committee is representative if each voter approves at least one committee member (who then can represent this voter). The idea of choosing a representative committee of a fixed size received significant attention in the literature (see the works of Chamberlin and Courant [10], Monroe [21], Elkind et al. [14], and Aziz et al. [2] as some examples). However, as pointed out by Brams and Kilgour [7], committees of fixed size simply cannot always provide adequate representation. Thus, in some applications, it is natural to elect committees without prespecified sizes (while in others, fixing the committee size may be necessary).

A representative committee may be desired when some authorities are revising existing regulations and need to consult citizens, for which purpose they would like to select a representative focus groups in various cities. The people in a focus group do not have to represent the society proportionally (their role is to voice opinions and concerns and not to make final decisions), but

[1] One of the authors of this paper was once tasked with the problem of classifying a collection of daggers for a museum. The solution was to compute a number of partitions of the set of daggers into clusters, evaluate their qualities without reference to ethnographical knowledge on the daggers, and to present the best ones to the museum's experts, who chose one partition that led to an ethnographically meaningful classification.

[2] The criteria may include, e.g., never receiving a low grade from a course, taking some advanced classes, never being suspended, etc.

should cover all the spectrum of opinions in the society. Usually, small representative committees are more desirable than larger ones.

1.2 Our Contribution

Our main goal is to study computational aspects of voting rules tailored for elections with a variable number of winners. This direction was pioneered by Fishburn and Pekeč [16], who introduced the class of threshold rules and studied their computational complexity (somewhat surprisingly, only for the case when the committee size is fixed). We are not aware of other computational studies that followed their work.

In this paper we study the threshold rules of Fishburn and Pekeč [16], as well as a number of other rules, including those discussed by Kilgour [18]. We obtain the following results:

1. For each of the rules, we establish whether finding a winning committee under this rule is in P or is NP-hard (in which case we seek FPT algorithms parameterized by the numbers of candidates and by the number of voters).

2. We evaluate experimentally the average sizes of committees elected by our rules. We consider a basic model of preferences, where each voter approves each candidate independently, with some probability p (we focus on $p = 1/2$ but for some representative rules we consider a larger spectrum of probability values).

We only make preliminary comments regarding suitability of our rules for the tasks outlined above. While specifying the applications is important to facilitate future research, we believe that we are still at the level of identifying voting rules and gathering basic knowledge about them.

2 Preliminaries

An approval-based election $E = (C, V)$ consists of a set $C = \{c_1, \dots, c_m\}$ of candidates and a collection $V = (v_1, \dots, v_n)$ of voters. Voters express their preferences by filling approval ballots. The approval ballot of a voter specifies the subset of candidates that this voter approves. To simplify notation, we denote voter v_i's approval ballot also as v_i (whether we mean the voter or the subset of approved candidates will always be clear from the context). A collection V of voters, interpreted as a collection of approval ballots in a certain election, is called the *preference profile*. For a given subset S of the candidates from set C, by $\overline{S}$ we mean the candidates not in S, i.e., $\overline{S} = C \setminus S$. By the *approval score* of a candidate in an election, we mean the number of voters that approve of this candidate.

A voting rule for elections with a variable number of winners is a function $\mathcal{R}$ that, given an election $E = (C, V)$, returns a family of subsets of C (the set of committees which tie as winners). The main point of difference between the type of voting rules that we study here and the voting rules typically studied in the context of multiwinner elections is that we do not fix the size of the committee to be elected and we let it be deduced by the rule.

For an overview of multiwinner election procedures using approval balloting, we point to the works of Kilgour [17, 18], both for the discussions of rules with fixed and variable number of winners; Duddy et al. [9] and Brandl and Peters [13] discuss the polynomial-time computable Borda mean rule (not included in our discussion).

Our hardness results follow by reductions from the NP-complete problem Set Cover. An instance of Set Cover consists of a set $U = \{u_1, \dots, u_n\}$ of elements, a family $\mathcal{S} = \{S_1, \dots, S_m\}$ of subsets of U, and an integer k; we ask whether exist at most k sets from $\mathcal{S}$ whose union is U.

3 Simple Approval Rule

We start our discussion by considering the Approval rule (AV), one of the arguably simple rules for the setting with variable number of winners.

Approval Voting (AV). Under AV we output the (unique) committee of candidates with the highest approval score.

In essence, AV is the single-winner Approval rule which instead of breaking ties (among winning candidates) outputs all the candidates with the highest score. The rule is, of course, polynomial-time computable.

Proposition 1. *There is a polynomial-time algorithm that computes the unique winning committee under the AV rule.*

We should expect the winning committees for this rule to be very small and, indeed, the following experiment confirms this intuition (the AV rule is so simple that the experiment is not really necessary; we include it for the sake of completeness and to provide the setup for experiments regarding less intuitive rules).

Experiment 1. *We consider elections with* $m = 20$ *candidates and* $n = 20$ *voters, where for each candidate* c*, each voter approves* c *with probability* $p = 1/2$*. We have generated* $10,000$ *elections and the average committee size was* 1.52 *with standard deviation* 0.89*. (See Table 1 for the list of average committee sizes in this setting for our rules; to see how the number of voters affects the average committee sizes, we also included experiments for* 20 *candidates and* 100 *voters). We repeat this experiment for all the rules in this paper.*

Remark 1. *We have chosen a fairly small number of candidates and voters for our experiments as the results for such elections are fast to compute (even for* NP*-hard voting rules) and, yet, they appear to be sufficient to show the effects that we are interested in. We chose the model where each candidate is approved or disapproved independently by each voter because it is the most basic scenario which, we believe, one should start with (other scenarios should be considered in future works).*

4 (Generalized) Net-Approval Voting

In the framework where the size of the target committee is fixed, one may ask for a committee of candidates whose sum of approval scores is the highest. To adapt this idea to the variable number of winners, Brams and Kilgour [7] suggested the Net-Approval Voting (NAV) rule. This rule pays attention not only to approvals but also to disapprovals.

Net Approval Voting (NAV). The score of a committee S in election $E = (C, V)$ under NAV is defined to be $\sum_{v_i \in V} \left(|S \cap v_i| - |S \cap \overline{v_i}|\right)$; the committees with the highest score tie as co-winners.

Note that this rule is very close to the MV rule [8] mentioned in the introduction; the winning committees under NAV consist of all candidates approved by a strict majority of the voters and any subset of those approved by exactly half of the voters (MV includes all candidates approved by at least half of the voters).

Corollary 2. *There is a polynomial-time algorithm that computes the unique smallest*[3] *winning committee under the NAV rule.*

[3] In fact, there is a polynomial-time algorithm that computes a winning committee of any size, if exists.

Experiment 2. *Repeating Experiment 1 for NAV, we obtain* 8.25 *as the average size of the smallest elected committee with standard deviation* 2.19. *This confirms the intuition that slightly fewer than half of the candidates would be elected in a typical election, where each candidate is approved independently with probability* $1/2$ *(since in the smallest winning committee it is necessary to be approved by* more than *half of the voters). We also computed the average size of a NAV committee for the case where voters approve candidates with other probabilities. Specifically, for each* $p \in \{0.05, 0.1, \ldots, 0.95\}$ *we generated* $10,000$ *elections with* 20 *candidates and* 20 *voters, where each voter approves each candidate with probability* p*, and we computed the average size of the NAV committee. We repeated the same experiment for* 20 *candidates and* 100 *voters. The results are presented in Figure 2. We see that when the number of voters becomes large, the graph becomes very close to the step function. This means that NAV should only be used in very specific settings (such as the baseball Hall of Fame example).*

It turns out that, using the main idea behind the NAV rule, it is possible to express many different voting rules. Below we suggest a language for describing such rules.

Generalized NAV. Let f and g be two non-decreasing, non-negative-valued functions, $f, g\colon \mathbb{N} \rightarrow \mathbb{N}$, such that $f(0) = g(0) = 0$. We define the (f, g)-NAV score of a committee S in election $E = (C, V)$ to be:

$$\textstyle\sum_{v_i \in V} \left(f(|S \cap v_i|) - g(|S \cap \overline{v_i}|)\right).$$

The committees with the highest score tie as co-winners. The intuition for this rule is that we would like to be able to count approvals and disapprovals differently. E.g., this can be explained as follows: at times, the lack of approval of a candidate is not really a disapproval but lack of information about him/her or simply no firm opinion.

Remark 2. *It would also be reasonable to include the terms* $f'(|\overline{S} \cap \overline{v_i}|)$ *and* $-g'(|\overline{S} \cap v_i|)$ *(for two additional functions* f' *and* g'*) in the definition of the score above. The first term, for example, would reflect the utility that voter* v_i *has from exclusion of candidates whom he/she did not approve.*

(f, g)-NAV rules are quite diverse. For example, if f and g are linear functions (e.g., $f(x) = x$ and $g(x) = 2x$) then (f, g)-NAV is a variant of the MV rule with a different threshold of approval (for the given example, a candidate would be included in the committee if it were approved by at least a $2/3$ fraction of the voters; thus, we refer to this rule also as $2/3$-NAV). Such rules seem quite appropriate for the task of choosing a set of qualifying candidates as, for each candidate c, the decision whether to include c in the committee or not is made based on approvals for c only (indeed, the decision if a patient should be sent for an in-depth medical check should not depend on the health of other patients).[4] Below we show that for nonlinear functions f and g, (f, g)-NAV rules might no longer have this independence property.

Let us consider the function $t_1(x)$, where $t_1(0) = 0$ and $t_1(k) = 1$ for each $k \geq 1$. Then, the rule $(t_1, 0)$-NAV, where we write 0 to mean the function that takes value 0 for all its inputs, seeks committees where each voter approves at least one committee member. In consequence, the committee that consists of all candidates is always winning under this rule (and, of course, also polynomial-time computable). However, it is far more interesting to seek the smallest $(t_1, 0)$-NAV winning committee and we refer to the rule that outputs such committees as the Minimum Representation Rule (MRC). A more intuitive description of this rule follows.

Minimal Representing Committee rule (MRC). Under the MRC rule, we output all the committees of smallest size such that each voter (with a nonempty approval ballot) approves at least one of the committee members.

[4] As a side comment, we mention that such rules are also typical in the lobbying scenarios [12, 6, 22].

Intuitively, MRC is very close to the approval variant of the Chamberlin–Courant rule [10, 23, 5]; we refer to the approval-based Chamberlin–Courant rule as CC. Under CC, we are given an approval election $E = (C, V)$, a committee size k, and our goal is to find a committee of size k such that as many voters as possible approve at least one of the committee members (for the case of CC, typically the fact that a voter approves a candidate is interpreted as saying that the voter would feel represented by this candidate). MRC is, in a sense, a variant of CC where we insist that each voter be represented, but we want to keep the committee as small as possible.

Since computing an MRC winning committee means, in essence, solving the minimization version of the Set Cover problem, we next proposition follows (missing hardness proofs are available in the supplementary material).

Proposition 3. *Given an election E and a positive integer k, it is* NP*-hard to decide if there is an MRC winning committee of size at most k.*

Proof. It suffices to note that our problem is equivalent to the Set Cover problem. To see this, consider a Set Cover instance with $I = (U, \mathcal{S}, k)$, where $U = \{u_1, \ldots, u_n\}$ is a set of elements, $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a family of subsets of U, and k is an integer. We form an election $E = (C, V)$ where for each set S_i we have a candidate s_i and for each element u_j we have a voter that approves exactly those candidates s_i for which $u_j \in S_i$. There is a winning MRC committee of size at most k if and only if there is a collection of at most k sets that cover U. □

Fortunately, computing MRC winning committees is fixed-parameter tractable (is in FPT) when parameterized by either the number of candidates or the number of voters (we omit the proof due to space restriction, but mention that the ideas are similar to those that we use for Theorem 9).

Proposition 4. *The problem of deciding if there is an MRC winning committee of size at most k (in a given election E) is in* FPT*, when parameterized either by the number of candidates or the number of voters.*

Proof. The result for the number of candidates follows via a straightforward brute-force algorithm. For parameterization by the number of voters, we invoke the “candidate types” idea of Chen et al. [11]: There are at most 2^n “candidate types” (where the type of a candidate is simply the set of voters that approve of him or her). Then, we observe that it suffices to consider at most one candidate of each type, since a winning committee certainly never contains two candidates of the same type because we could remove one). In FPT-time, we try all possible committees of at most 2^n candidates (of different types). □

Experiment 3. *By applying Experiment 1 to MRC, we obtain that the average committee size is* 2.68 *with standard deviation* 0.46*. Since the rule is* NP*-hard, we have used the brute-force algorithm to try all possible committees. We also present results for other probabilities of approving each candidate (see Figure 2). A positive feature of this rule is that the size of a winning committee does not depend much on the number of voters.*

We can also use the standard greedy algorithm for Set Cover to find approximate MRC committees; indeed, we view this algorithm as a voting rule in its own right.

GreedyMRC. Under GreedyMRC, we start with an empty committee and perform a sequence of iterations. In each iteration we (a) add to the current committee a candidate c that is approved by the largest number of voters, and (b) we remove the voters that approve c from consideration. After we have removed all voters with nonempty approval ballots, we output the resulting committee (formally, the rule outputs all the committees that can be obtained by breaking the internal ties in some way).

Experiment 4. *By connection to* Set Cover*, GreedyMRC is guaranteed to find a committee that is at most a factor $O(\log m)$ larger than the one given by the exact MRC (where m is the number of candidates). In our experiment, with approval probabilities in $\{0.05, 0.1, \dots, 0.95\}$, the average sizes of the GreedyMRC committees where no more than 8% larger than the average sizes of the MRC ones (for the case of 20 voters) or no more than 11% larger (for the case of 100 voters).*

MRC and GreedyMRC appear to be well suited for choosing small committees of representative; our experiments confirm this intuition.

Recall that the function $t_1(x)$ is such that $t_1(0) = 0$ and $t_1(k) = 1$ for each $k \geq 1$. Then, consider the $(0, t_1)$-NAV rule, which elects all the committees that contain candidates approved by all the voters. While the empty set is trivially a winning committee under this rule, it is more interesting to ask about the largest winning committee; we refer to the rule that outputs the largest winning $(0, t_1)$-NAV rule as the unanimity rule:

Unanimity Voting (UV). Under the unanimity rule, we output the committee of all the candidates approved by all the voters.

While computing the smallest $(t_1, 0)$-NAV winning committee is hard (Proposition 4), it is easy to compute the (unique) largest $(0, t_1)$-NAV winning committee in polynomial time (i.e., there is a polynomial-time algorithm for UV).

Experiment 5. *As expected, in our experiment it never happened that some candidate was approved by all the voters (the probability of some candidate being approved by all* 20 *voters is* $20 \cdot 2^{-20}$ *and we considered only 10,000 elections).*

Both for $(t_1, 0)$-NAV and for $(0, t_1)$-NAV, it is trivial to compute *some* winning committee (the set of all candidates in the former case and the empty set in the latter). In general, however, this is not the case.

Theorem 5. *There exists an (f, g)-NAV rule for which deciding if there exists a committee with at least a given score is* NP*-hard.*

Proof. We consider specific functions f and g and show that for the corresponding (f, g)-NAV rule it is NP-hard to decide if there exists a committee with at least a given score. The specific functions f and g we consider are as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ 4, & x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 0, & x = 0 \\ 1, & x = 1 \\ 2, & x \geq 2 \end{cases}$$

To show NP-hardness, we reduce from the NP-hard X3C problem. In it, we are given sets $\mathcal{S} = \{S_1, \dots, S_n\}$ over elements $b_1, \dots, b_n$. Each set contains exactly three integers and each elements is contained in exactly three sets. The task is to decide whether there is a set of sets $S' \subseteq \mathcal{S}$ such that each element b_i is covered exactly once. We assume, without loss of generality, that $n > 39$.

Given an instance of X3C we create an election as follows. For each set S_j we create a candidate S_j. For each element b_i we create three voters: v_i^1 v_i^2, and v_i^3; v_i^1 and v_i^2 are referred to as *set voters* while v_i^3 is referred to as *antiset voter*. Both voters v_i^1 and v_i^2 approve exactly the candidates corresponding to the sets which contain b_i, while the voter v_i^3 approves exactly the candidates corresponding to the sets which do not contain b_i. With respect to the reduced election, we ask whether a committee with score at least $7n$ exists. This finishes the description of the reduction. Next we prove its correctness.

Let C be a committee for the reduced election and let b_i be some element of the X3C instance. First we show that C has at least six candidates in it. If it is not the case, then, since each set S_j covers exactly three elements, it follows that there are at least $n - 18$ elements not covered by C.

Let b_i be an element not covered by C. Then, the voters v_i^1, v_i^2, and v_i^3 corresponding to b_i give at most 2 points to C. If $C = \emptyset$ then each voter corresponding to b_i gives 0 points to C. Otherwise, if $C \neq \emptyset$, then each set voter (each of v_i^1 or v_i^2) gives at most -1 points to C, while the antiset voter gives at most 4 points. Thus, the voters corresponding to b_i give at most 2 points to C.

There are at most 18 elements which are covered by C. For each b_i which is covered by C, each of the voters v_i^1, v_i^2, and v_i^3 corresponding to b_i give at most 4 points to b_i (since this is the maximum number of points any voter gives to any committee). Thus, the voters corresponding to b_i give at most 12 points to C.

Summarizing the above two paragraphs, we have that the total score of C which has at most six candidates is at most $(3k - 18) \cdot 2 + 18 \cdot 12$. Since we assume, without loss of generality, that $n > 39$, we have that this quantity is strictly less than $7n$. Therefore, from now on we assume that C has at least six candidates in it.

Thus, let C be a committee with at least six candidates in it and let b_i be an element. Let $V_i = \{v_i^1, v_i^2, v_i^3\}$ and consider the following four cases depending on the number of times b_i is covered by the sets S_j corresponding to the candidates in C.

- **b_i is not covered by C:** In this case, the score given to C by V_i is at most $(-2)+(-2)+4 = 0$. To see this, observe that each of the set voters (v_i^1, v_i^2) gives to C exactly -2 points, since they do not approve any candidate from C but disapprove all candidates in C; further, observe that the antiset voter (v_i^3) gives to C at most 4 points, as this is the maximum number of points any voter can give to any committee.

- **b_i is covered exactly once by C:** In this case, the score given to C by V_i is $2+2+4-1 = 7$. To see this, observe that each of the set voters (v_i^1, v_i^2) gives to C exactly 2 points, since they approve one candidate from C (the one candidate corresponding to the one set covering b_i) and disapprove all other candidates in C; further, observe that the antiset voter (v_i^3) gives to C exactly 3 points, since it approves at least one candidate in C and disapprove exactly one candidate in C (the one candidate corresponding to the one set covering b_i).

- **b_i is covered more than once by C:** In this case, the score given to C by V_i is $2+2+4-2 = 6$. To see this, observe that each of the set voters (v_i^1, v_i^2) gives to C exactly 2 points, since they approve more than one candidate from C (the two or three candidates corresponding to the two or three sets covering b_i) and disapprove all other candidates in C; further, observe that the antiset voter (v_i^3) gives to C exactly 2 points, since it approves at least one candidate in C and disapprove two or three candidates in C (the two or three candidates corresponding to the two or three sets covering b_i).

As there are exactly n elements, it follows from the case analysis above that a committee C with score at least $7n$ shall correspond to an exact cover. □

Naturally, one can come up with many other interesting variants of the generalized Net-Approval voting rules. We recommend analysis of this class of rules for future research.

5 (Net-)Capped Satisfaction and FirstMajority

Kilgour and Marshall [19] introduced the following rule in the context of electing committees of fixed size, and Kilgour [18] recalled it in the context of elections with a variable number of winners, suggesting its net version.

Capped Satisfaction Approval (CSA). The Capped Satisfaction Approval (CSA) score of a committee S is defined to be $\sum_{v_i \in V} \frac{|S \cap v_i|}{|S|}$. The committees with the highest score tie as co-winners.

Net Capped Satisfaction Approval (NCSA). The NCSA rule uses the "net" variant of CSA score; specifically, the score of a committee S is defined to be $\sum_{v_i \in V} \frac{|S \cap v_i|}{|S|} - \frac{|S \cap \overline{v_i}|}{|S|}$ and the committees with the highest score tie as co-winners.

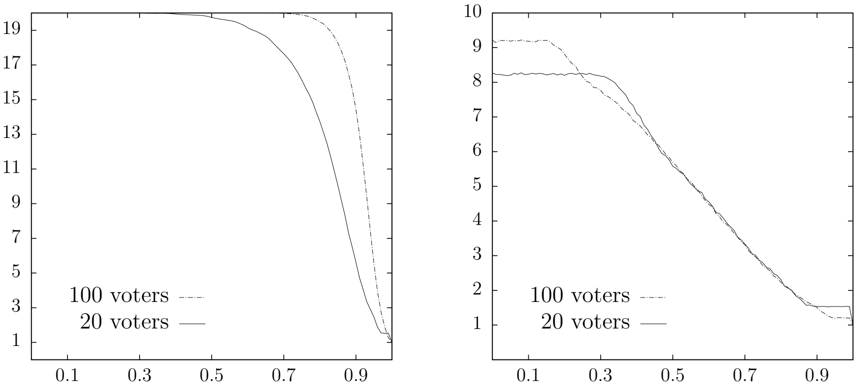
In the definitions above, the idea behind dividing the scores by the size of the committee is to ensure that committees which are too large will not be elected. Unfortunately, for the rules as defined by Kilgour [18], this effect is too strong, leading mostly to committees containing only the candidate(s) with the highest approval score. We explain why this is the case and suggest a modification.

Consider an election $E = (C, V)$ with candidate set $C = \{c_1, \ldots, c_m\}$ and preference profile $V = (v_1, \ldots, v_n)$. Let $\mathrm{s}(c_1), \ldots, \mathrm{s}(c_m)$ be the approval scores of the candidates, and, without loss of generality, assume that $\mathrm{s}(c_1) \geq \mathrm{s}(c_2) \geq \ldots \geq \mathrm{s}(c_m)$. Note that, if there are no ties regarding the approval scores, then for each k, the highest-scoring CSA committee of size k is simply $S_k = \{c_1, \ldots, c_k\}$ and its score is $\sum_{v_i \in V} \frac{|S_k \cap v_i|}{|S_k|} = \frac{1}{k} \sum_{v_i \in V} |S_k \cap v_i| = \frac{\mathrm{s}(c_1) + \ldots + \mathrm{s}(c_k)}{k}$. This value, however, never increases with k and, so, typically CSA outputs very small committees (which contain only the candidates with the highest approval score; the same reasoning applies to NCSA). Thus, we introduce the q-CSA and the q-NCSA rules, where q is a real number, $0 \leq q \leq 1$, and (a) the q-CSA score of a committee S in election $E = (C, V)$ is $\sum_{v_i \in V} \frac{|S \cap v_i|}{|S|^q}$, and (b) the q-NCSA score of this committee is $\sum_{v_i \in V} \frac{|S \cap v_i|}{|S|^q} - \frac{|S \cap \overline{v_i}|}{|S|^q}$. We note that for $q = 1$ these rules are, simply, CSA and NCSA, whereas 0-NCSA is NAV and 0-CSA is a rule that outputs the committee that includes all the candidates that receive any approvals.

By the reasoning above, for each rational value of q, both q-CSA and q-NCSA are polynomial-time computable (using notation from previous paragraph, it suffices to consider the committees $S_1, S_2, \ldots, S_m$ and output the one with the highest q-CSA or q-NCSA score, respectively).

Proposition 6. *For each rational value of q, there is a polynomial-time algorithm that, given an election, computes a winning committee for q-CSA and for q-NCSA.*

Experiment 6. *To obtain some better understanding of the influence of the parameter q on the size of the committees elected according to q-CSA and q-NCSA, we have repeated Experiment 1 (with approval probability $p = 1/2$) for these rules for q values between* 0 *and* 1 *with step* 0.01*. The sizes of the average committee that we obtained are presented in Figure 1. The figures show results for the case of* 20 *candidates and either* 20 *or* 100 *voters. While the average committee size for q-NCSA does not depend very strongly on the number of voters (and its dependence on q is appealing), the results for q-CSA are worrying. Not only does the rule elect (nearly) all candidates for most values of q, but also for the values where it is more selective (e.g., $q = 0.9$), the average size of its committees depends very strongly on the number of voters. In Figure 2 we show average sizes of* 0.9*-CSA and of* 0.9*-NCSA committees, depending on the probability of candidate approval. These figures confirm our*



(a) q-CSA

(b) q-NCSA

Figure 1: Average committee sizes (y-axis) under q-CSA and q-NCSA rules for different values of q (x-axis); see Experiment 1 for information on how the elections were generated.

worries regarding q-CSA rules. While the dependence of the average committee size on the candidate approval probability for 0.9-*NCSA has the same nature irrespective of the number of voters (it is, roughly speaking, convex both for* 20 *and* 100 *voters), the same dependence for* 0.9-*CSA changes its nature (from roughly convex for the case of* 20 *voters to roughly concave for the case of* 100 *voters). In Table 1 we also show average committee sizes for* 0.5-*CSA and* 0.5-*NCSA for the candidate approval probability* $p = 1/2$.

Given the above experiments, we believe that for practical applications, where we may have limited control on the number of candidates, the number of voters, and the types of the votes cast, choosing an appropriate value of the parameter q for q-CSA rules (e.g., to promote committees close to a particular size) would be very difficult. On the other hand, q-NCSA might be robust enough as to be practical.

One could consider generalized variants of the q-CSA and q-NCSA rules in the same way as we have considered generalized NAV rules. We leave this as future work and we conclude the section by considering a different rule of Kilgour [18], which is not an (N)CSA rule, but which is somewhat similar since it also chooses a certain number of candidates with the highest approval scores.

FirstMajority. Consider an election with candidates $C = \{c_1, \ldots, c_m\}$. For each $c_i \in C$ denote by $s(c_i)$ the approval score of c_i. Reorder the candidates so that $\mathrm{s}(c_1) \geq \mathrm{s}(c_2) \geq \cdots \geq \mathrm{s}(c_m)$. The FirstMajority rule outputs the smallest committee of the form $\{c_1, c_2, \ldots, c_j\}$ such that $\sum_{t=1}^{j} \mathrm{s}(c_t) > \sum_{q=j+1}^{m} \mathrm{s}(c_q)$. Note that if some candidates are approved by the same number of voters then this rule may return more than one committee, corresponding to the various possible ways of reordering the candidates.

The very definition of FirstMajority gives a polynomial-time algorithm for computing its winning committees.

Proposition 7. *There is a polynomial-time algorithm that finds some winning committee under the FirstMajority rule.*

Experiment 7. *Under our experimental setup (see Experiment 1), on the average, the FirstMajority rule outputs committees of size* 9.51 *(with standard deviation* 0.43*). Further, the size of the committee is almost independent of the number of voters and the candidate approval probability (see Figure 2).*

6 Threshold Rules

We conclude our discussion by considering the threshold rules of Fishburn and Pekeč [16]. Let $t\colon \mathbb{N} \to \mathbb{N}$ be some function referred to as the *threshold function.* The t-Threshold rule is defined as follows.

***t*-Threshold.** Consider an election $E = (C, V)$. Under the t-Threshold rule, we say that a voter $v_i \in V$ approves a committee S if $|S \cap v_i| \geq t(|S|)$. The t-Threshold rule outputs those committees that are approved by the largest number of voters.

We consider the following three (in some sense extreme) examples of threshold functions: (a) the unit function $t_{\mathrm{unit}} = t_1$ (recall the discussion of generalized net-approval rules); (b) the majority function, $t_{\mathrm{maj}}(k) = k/2$; and (c) the full function, $t_{\mathrm{full}}(k) = k$.

The t_{unit}-Threshold rule is very similar to MRC because, under the t_{unit} threshold function, a voter approves a committee if it includes at least one candidate that this voter approves. Thus the rule outputs all committees S such that each voter with a nonempty approval ballot approves some member of S (MRC outputs the smallest of these committees). Thus finding the largest winning committee is easy (take all the candidates), but finding the smallest one is hard (as then we have the MRC rule).

On the other hand, the t_{full}-Threshold rule outputs exactly those committees S that (a) each candidate in S has the highest approval score and (b) all the candidates in S are approved by the same group of voters. It seems, however, that AV is a simpler and more natural rule than the t_{full}-Threshold rule.

Finally, we consider the t_{maj}-Threshold rule, introduced and studied by Fishburn and Pekeč [16]; t_{maj}-Threshold winning committees receive broad support from the voters, and—as suggested by Fishburn and Pekeč—should be "of moderate size". Computing t_{maj}-Threshold winning committees is NP-hard, but there are FPT algorithms.

Theorem 8. *The problem of deciding if there is a nonempty committee that satisfies all the voters under the t_{maj}-threshold rule is* NP*-hard.*

Proof. We give a reduction from the Set Cover problem. Let our input instance be $I = (U, \mathcal{S}, k)$, where $U = \{u_1, \dots, u_n\}$ is a set of elements, $\mathcal{S} = \{S_1, \dots, S_m\}$ is a family of subsets of U, and k is a positive integer. Without loss of generality, we can assume that $m > k$ (otherwise there would be a trivial solution for our input instance).

We form an election with the candidate set $C = F \cup \mathcal{S}$, where $F = \{f_1, \dots, f_k\}$ is a set of filler candidates and $\mathcal{S}$ is a set of candidates corresponding to the sets from the Set Cover instance (by a small abuse of notation, we use the same symbols for $\mathcal{S}$ and its contents irrespective if we interpret it as part of the Set Cover instance or as candidates in our elections). We introduce $kn + 2$ voters:

1. The first voter approves of all the filler candidates and the second voter approves all the set candidates. We refer to these voters as the balancing voters.

2. For each element $u_i \in U$, we have a group of k voters, so that the jth voter in this group ($j \in [k]$) approves of all the filler candidates except f_j, and also of exactly those set candidates that correspond to sets containing u_i.

We claim that there is a nonempty committee S such that every voter approves at least half of the members of S (i.e., every voter is satisfied) if and only if I is a *yes*-instance.

Let us assume that S is a committee that satisfies all the voters. We note that S must contain the same number of filler and set candidates. If it contained more set candidates than filler candidates then the first balancing voter would not be satisfied, and if it contained more filler candidates than set candidates, then the second balancing voter would not be satisfied. Thus, there is a number k' such that $|S| = 2k'$, $k' \leq k$, and S contains exactly k' filler and k' set candidates.

We claim that these k' set candidates correspond to a cover of U. Consider some arbitrary element u_i and some filler candidate f_j such that f_j does not belong to S (since $m > k \geq k'$ such candidates must exist). There is a voter that approves all the filler candidates except f_j and all the set candidates that contain u_i. Thus, the committee contains exactly $k' - 1$ filler candidates that this voter approves and—to satisfy this voter—must contain at least one set candidate that contains u_i. Since u_i was chosen arbitrarily, we conclude that the set candidates from S form a cover of U. There are at most k of them, so I is a *yes*-instance.

On the other hand, if there is a family of $k' \leq k$ sets that jointly cover U, then a committee that consists of arbitrarily chosen k' filler candidates and the set candidates corresponding to the cover satisfies all the voters. □

Theorem 9. *Let t be a linear function (i.e., $t(k) = \alpha k$ for some $\alpha \in [0, 1]$). There are* FPT *algorithms for computing the smallest and the largest winning committees under the t-Threshold rule in* FPT *time for parameterizations by the number of candidates and by the number of voters.*

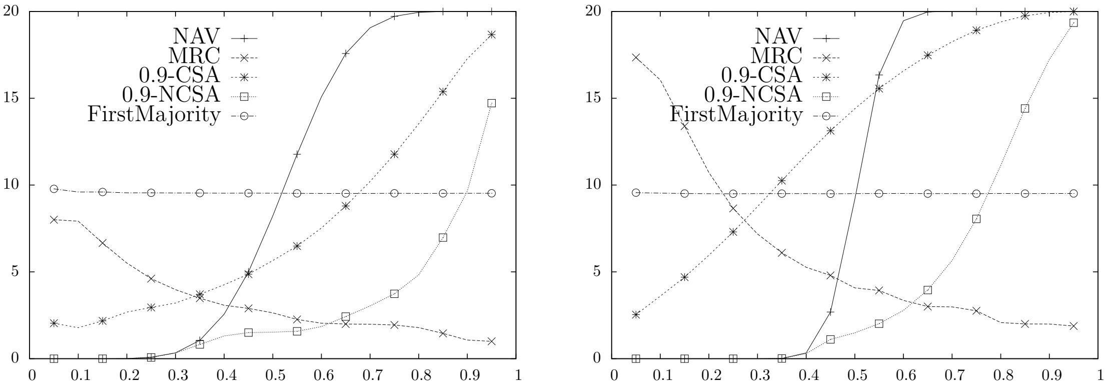
Proof. For parameterization by the number of candidates it suffices to try all possible committees. For parameterization by the number of voters, we combine the candidate-type technique of Chen et al. [11] and an integer linear programming (ILP) approach. The *type* of candidate c is the subset of voters that approve c. For an election with n voters, each candidate has one of at most 2^n types. We describe an algorithm for computing a committee approved by at least N voters (where N is part of the input; it suffices to try all values of $N \in [n]$ to find a committee with the highest score). We focus on computing the largest winning committee.

Let $E = (C, V)$ be the input election with n voters. We form an instance of the ILP problem as follows. For each candidate type i, $i \in [2^n]$, we introduce integer variable x_i (intuitively x_i is the number of candidates of type i that are included in the winning committee). For each $i \in [2^n]$, we form constraint $0 \leq x_i \leq n_i$, where n_i is the number of candidates of type i in election E. We also add constraint $\sum_{i \in [2^n]} x_i \geq 1$ as the winning committee must be nonempty.

For each voter $j \in [n]$, we define an integer variable v_j (the intention is that v_j is 1 if the jth voter approves of the committee specified by variables $x_0, \ldots, x_{2^n-1}$ and it is 0 otherwise; see also comments below). For each $j \in [n]$, we introduce constraints $0 \leq v_j \leq 1$, and:

$$\left(\textstyle\sum_{i \in \mathrm{types}(v_j)} x_i\right) - t\left(\textstyle\sum_{i \in [2^n]} x_i\right) \geq -(1 - v_j)n, \qquad (1)$$

where $\mathrm{types}(v_j)$ is the set of all candidate types approved by voter v_j. To understand these constraints, note that $\sum_{i \in [2^n]} x_i$ is the size of the selected committee, $\sum_{i \in \mathrm{types}(v_j)} x_i$ is the number of committee members approved by the jth voter, and, thus, Eq. (1) is satisfied either if $v_j = 0$ or $v_j = 1$ and there is an integer k such that the jth voter approves at least $t(k)$ members of the selected size-k committee. We add constraint $v_1 + \cdots + v_n \geq N$ (i.e., we require that at least N voters are satisfied with the selected committee; this also prevents satisfying Eq. (1) by setting $v_j = 0$ for all $j \in [n]$).



(a) 20 voters

(b) 100 voters

Figure 2: Average committee sizes for some of our rules (20 candidates and either 20 or 100 voters; approval probability is on the x-axis).

rule	avg. committee size ± its std. deviation		complexity
	(20 voters)	(100 voters)	
$^2/_3$-NAV	1.02 ± 1.01	0.01 ± 0.09	P
AV	1.52 ± 0.89	1.20 ± 0.50	P
0.9-NCSA	1.52 ± 0.89	1.50 ± 0.78	P
MRC	2.63 ± 0.48	4.08 ± 0.26	NP-hard
GreedyMRC	2.75 ± 0.46	4.55 ± 0.53	P
t_{maj}-Thr (min)	2.75 ± 1.33	$\mathbf{2.05 \pm 0.34}$	NP-hard
0.5-NCSA	5.57 ± 2.14	5.57 ± 2.18	P
0.9-CSA	5.63 ± 3.02	$\mathbf{14.67 \pm 2.75}$	P
t_{maj}-Thr (max)	7.68 ± 3.27	$\mathbf{2.20 \pm 0.78}$	NP-hard
NAV	8.25 ± 2.19	9.19 ± 2.23	P
FirstMajority	9.51 ± 0.43	9.50 ± 0.25	P
0.5-CSA	19.74 ± 0.52	20.00 ± 0.00	P

Table 1: Average committee sizes (see Experiment 1 for information on how the elections were generated). Rules are sorted with respect to the average committee size for 20 voters (results in bold are those that would change their position if we sorted for the average committee size with 100 voters). t_{maj}-Thr. (min) and t_{maj}-Thr (max) refer to the smallest and largest committees under the t_{maj}-Threshold rule.

To compute the largest committee approved by at least N voters, we find a feasible solution (for the above-described integer linear program) that maximizes $\sum_{i \in [2^n]} x_i$ (we use the famous FPT-time algorithm of Lenstra [20]). □

Experiment 8. *In our experiments, the average size of the smallest* t_{maj}*-Threshold committee was* 2.84*. On the other hand, the largest committee contained, on average,* 7.52 *candidates. Yet, for the* *case of* 100 *voters the difference between the sizes of the largest committee and the smallest committee are much more modest (see Table 1).*

Largest winning committees under t_{maj}-Threshold are typically of even size (if S is a winning committee of odd size then it still wins after adding an arbitrary candidate).

7 Conclusion and Further Research

We have argued that elections with variable number of winners are useful and we have analyzed a number of such rules already present in the literature and provided generalizations for some of them, finding polynomial algorithms in most cases, but also identifying interesting NP-hard rules. Further analysis (both axiomatic and computational) is the most pressing direction for future research. We also note that in many practical cases there is a societal preference on the size of the committee to be elected, which is usually single-peaked. Incorporating this preference into the voting rules is an interesting direction of research. Finally, as in principle multiwinner voting rules with variable number of winners cluster the candidates into two sets (those in the winning committee, and the rest), adapting ideas from data and cluster analysis might prove useful in designing other rules better tailored for this task.

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