A cone programming approach for solving the hyperbolic distance function

M. Hassanasab, D. Margaritis, I. Roshdi, P. Rouse

Annual CEPA International Workshop, 2nd-3rd November 2017, The University of Queensland
Färe, Grosskopf, and Lovell (1985) merged Farrell’s input and output oriented technical efficiency measures into a new graph-type approach known as hyperbolic distance function (HDF); it simultaneously contracts inputs and expands outputs using a single scaling factor on a path rather than a ray.

HDF is in general a nonlinear optimization approach.
Choice of input- or output-orientation is largely arbitrary, hence attractive features of HDF are

- greater flexibility for firms to modify inputs and outputs to increase efficiency;
- a more robust measure of efficiency that does not depend on the slope of the frontier at the projection point, particularly important for measuring efficiency in a dynamic setting;
- ability to better model undesirable outputs;

Its major drawback is that in general it is a nonlinear optimization problem Which is what we address here.
• Linearise the problem via a linear approximation of input scaling under VRS is found in Färe, Grosskopf, and Lovell (1985).

• Zofío and Prieto (2001) showed that this approximation is more acceptable closer to the efficient frontier while the gap between the true and approximate efficiency scores increases the further the unit (DMU) is from the efficient frontier.

• Recently, Färe et al. (2016), linked HDF to the directional distance function of Chambers et al. (1996, 1998) and proposed an efficient algorithm that generates the value of VRS-based HDF through solving a finite number of linear programs.
We transform the nonlinear HDF under VRS into an equivalent conic program with linear constraints plus a single extra cone constraint.

- This transformation not only provides a direct method for computing the exact value of VRS-based HDF
- It offers a dual counterpart for HDF as conic optimization.
The formulation of HDF with reference to $\mathcal{T}_V$ can be represented as (1):

$$\min \theta$$

s.t.  

$$\sum_{j \in J} \lambda_j x_j \leq \theta x_o,$$

$$\sum_{j \in J} \lambda_j y_j \geq \frac{1}{\theta} y_o,$$

$$\sum_{j \in J} \lambda_j = 1,$$

$$\lambda_j \geq 0, j \in J.$$  (1)
The model attempts to find a point on the production possibility set by maximally reducing inputs and expanding outputs of unit $D$ simultaneously using a single scaling factor along a path to the frontier.

**Figure: Geometrical example**
The formulation of the HDF can be represented as model (2) by adding the constraint $\varphi \theta \geq 1$,

$$\begin{align*}
\text{min} & \quad \theta \\
\text{s.t.} & \quad \sum_{j \in J} \lambda_j x_j \leq \theta x_o, \\
& \quad \sum_{j \in J} \lambda_j y_j \geq \varphi y_o, \\
& \quad \sum_{j \in J} \lambda_j = 1, \\
& \quad \varphi \theta \geq 1, \\
& \quad \lambda_i \geq 0, j \in J.
\end{align*}$$

(2)

Deliberately using inequality to obtain a conic constraint structure.

Optimal objective value of (1) and (2) are equal.
Toppled Ice Cream Cone

In model (2), $\theta$ and $\phi$ can obtain their value from a special cone, called a “toppled ice cream”:

$$\varphi\theta \geq \eta^2 \iff (\varphi, \theta, \eta) \in C_{\infty} \text{ and } \eta = 1 \iff (\varphi, \theta, \eta) \succeq C_{\infty} 0.$$  \hspace{1cm} (3)

Figure: $\mathbb{R}^3$ versa Ice Cream Cone.
We formulate the nonlinear programming model (2) as a cone based programming model (4).

$$\theta^H = \min \theta$$

s.t.  $$\sum_{j \in J} \lambda_j x_j \leq \theta x_o,$$

$$\sum_{j \in J} \lambda_j y_j \geq \varphi y_o,$$

$$\sum_{j \in J} \lambda_j = 1,$$

$$\eta = 1,$$

$$\lambda_j \geq 0, j \in J,$$

$$\left( \varphi, \theta, \eta \right) \succeq_c 0.$$

(4)
Figure: Efficiency measure and primal interpretation
Exploiting conic dual programming, we formulate the conic dual of Model (4) as

\[
\theta^H = \max \quad u_0 - \gamma \\
\text{s.t.} \quad uy_j - vx_j + u_0 \leq 0, j \in J, \quad \forall j \in J, \\
vx_o + \alpha = 1, \\
uy_o = \beta, \\
u, v \geq 0, \\
(\alpha, \beta, \gamma) \succeq c^* \quad 0.
\]
The dual of the toppled ice cream cone is

\[ C^*_\triangleleft = \{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha\beta \geq \frac{1}{4}\gamma^2 \text{ and } \alpha, \beta \geq 0 \}. \] (6)
For any optimal solution of (5), \((u^*, v^*, u_0^*, \alpha^*, \beta^*, \gamma^*)\), the following statements are satisfied:

i) \((\alpha^*, \beta^*, \gamma^*)\) lies on the boundary of the dual cone \(C^*_\odot\), i.e.,
\[
\alpha^* \beta^* = \frac{1}{4} \gamma^*^2;
\]

ii) \(\frac{\beta^*}{\alpha^*} = (\theta^H)^2\);

iii) \(\alpha^* = \beta^*\) for all DMUs on the frontier of the VRS technology;

This offers an interpretation of the optimal values \((\alpha, \beta, \gamma)\) in terms of their association with the hyperbolic efficiency measure or its square (e.g., (ii) shows the conic ratio corresponds to the concept of the returns to the dollar).
Dual model offers a modified version of the shadow prices for the interior point \((x_o, y_o)\) scaling them by optimal dual conic variables \((\alpha^*, \beta^*)\).

**Figure:** Dual interpretation
A counterpart of the scale elasticity and returns to scale

\[ \alpha^+(\alpha^-) := \max(\min \alpha) \]

\[ \text{s.t. } uy_j - vx_j + u_0 \leq 0, j \in J, \]
\[ vx_o + \alpha = 1, \]
\[ uy_o - \alpha(\theta^H)^2 = 0, \]
\[ u_o + 2\alpha\theta^H = \theta^H \]
\[ u, v \geq 0, \alpha \geq 0. \]

\( \epsilon_{\text{min}} \) can be obtained by minimizing the same model.
Then applying the same procedure of Cooper et al. (2007) we have

1. $\alpha^- \geq 0.5$ if and only if DRS prevail at $(\theta^H x_o, \frac{1}{\theta^H y_o})$.

2. $\alpha^+ \leq 0.5$ if and only if IRS prevail at $(\theta^H x_o, \frac{1}{\theta^H y_o})$.

3. $\alpha^- \leq 0.5 \leq \alpha^+$ if and only if CRS prevail at $(\theta^H x_o, \frac{1}{\theta^H y_o})$. 
For an illustrative real-world example, we use FDIC data for a sample of 280 US banks from 2016 to measure hyperbolic technical efficiency.

We follow the intermediation approach defining bank inputs as: (\(x_1\)) total employees, (\(x_2\)) bank premises and fixed assets, and (\(x_3\)) total deposits; and outputs as: (\(y_1\)) net loans and leases, (\(y_2\)) total securities and (\(y_2\)) other earning assets. All variables are measured in thousands of their respective units.
The larger efficiency scores under HDF is because of its capability in permitting for simultaneous adjustment in both inputs and outputs rather than the BCC oriented measures for which technical efficiency is computed via either input or output adjustment but not both.

<table>
<thead>
<tr>
<th></th>
<th>x1</th>
<th>x2</th>
<th>x3</th>
<th>y1</th>
<th>y2</th>
<th>y3</th>
<th>(\theta^H)</th>
<th>(\theta^I)</th>
<th>(\theta^O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.10</td>
<td>89.79</td>
<td>5,585.44</td>
<td>4,994.16</td>
<td>1,276.28</td>
<td>253.71</td>
<td>0.82</td>
<td>0.71</td>
<td>0.68</td>
</tr>
<tr>
<td>STD</td>
<td>1.17</td>
<td>93.25</td>
<td>5,951.02</td>
<td>5,637.23</td>
<td>1,517.96</td>
<td>687.76</td>
<td>0.10</td>
<td>0.16</td>
<td>0.17</td>
</tr>
<tr>
<td>Min</td>
<td>0.10</td>
<td>2.24</td>
<td>1,177.86</td>
<td>495.74</td>
<td>3.36</td>
<td>0.66</td>
<td>0.58</td>
<td>0.35</td>
<td>0.38</td>
</tr>
<tr>
<td>Max</td>
<td>9.53</td>
<td>521.60</td>
<td>54,743.59</td>
<td>57,080.08</td>
<td>9,726.09</td>
<td>8,634.26</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
## Results for FDIC data

<table>
<thead>
<tr>
<th>Bank</th>
<th>$\theta^H$</th>
<th>$\theta^I$</th>
<th>$\theta^O$</th>
<th>$\alpha^*$</th>
<th>$\beta^*$</th>
<th>$\gamma^*$</th>
<th>$\alpha^-$</th>
<th>$\alpha^+$</th>
<th>RTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>D2</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.68</td>
<td>0.68</td>
<td>-1.36</td>
<td>0.59</td>
<td>0.73</td>
<td>DRS</td>
</tr>
<tr>
<td>D3</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.72</td>
<td>0.72</td>
<td>-1.44</td>
<td>0.50</td>
<td>0.91</td>
<td>CRS</td>
</tr>
<tr>
<td>D4</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.88</td>
<td>0.88</td>
<td>-1.77</td>
<td>0.32</td>
<td>0.93</td>
<td>CRS</td>
</tr>
<tr>
<td>D5</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.67</td>
<td>0.67</td>
<td>-1.34</td>
<td>0.41</td>
<td>0.80</td>
<td>CRS</td>
</tr>
<tr>
<td>D6</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.55</td>
<td>0.55</td>
<td>-1.09</td>
<td>0.51</td>
<td>0.56</td>
<td>DRS</td>
</tr>
<tr>
<td>D7</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.86</td>
<td>0.86</td>
<td>-1.71</td>
<td>0.29</td>
<td>0.97</td>
<td>CRS</td>
</tr>
<tr>
<td>D8</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.74</td>
<td>0.74</td>
<td>-1.47</td>
<td>0.48</td>
<td>1.00</td>
<td>CRS</td>
</tr>
<tr>
<td>D9</td>
<td>0.94</td>
<td>0.87</td>
<td>0.89</td>
<td>0.52</td>
<td>0.46</td>
<td>-0.97</td>
<td>0.52</td>
<td>0.52</td>
<td>DRS</td>
</tr>
<tr>
<td>D10</td>
<td>0.97</td>
<td>0.93</td>
<td>0.94</td>
<td>0.54</td>
<td>0.50</td>
<td>-1.04</td>
<td>0.54</td>
<td>0.54</td>
<td>DRS</td>
</tr>
<tr>
<td>D11</td>
<td>0.87</td>
<td>0.72</td>
<td>0.78</td>
<td>0.57</td>
<td>0.43</td>
<td>-0.99</td>
<td>0.57</td>
<td>0.57</td>
<td>DRS</td>
</tr>
<tr>
<td>D12</td>
<td>0.82</td>
<td>0.61</td>
<td>0.70</td>
<td>0.58</td>
<td>0.39</td>
<td>-0.95</td>
<td>0.58</td>
<td>0.58</td>
<td>DRS</td>
</tr>
<tr>
<td>D13</td>
<td>0.89</td>
<td>0.79</td>
<td>0.80</td>
<td>0.52</td>
<td>0.41</td>
<td>-0.93</td>
<td>0.52</td>
<td>0.52</td>
<td>DRS</td>
</tr>
<tr>
<td>D14</td>
<td>0.74</td>
<td>0.56</td>
<td>0.54</td>
<td>0.49</td>
<td>0.27</td>
<td>-0.72</td>
<td>0.49</td>
<td>0.49</td>
<td>IRS</td>
</tr>
<tr>
<td>D15</td>
<td>0.93</td>
<td>0.86</td>
<td>0.87</td>
<td>0.52</td>
<td>0.45</td>
<td>-0.97</td>
<td>0.52</td>
<td>0.52</td>
<td>DRS</td>
</tr>
<tr>
<td>D16</td>
<td>0.87</td>
<td>0.75</td>
<td>0.78</td>
<td>0.53</td>
<td>0.41</td>
<td>-0.93</td>
<td>0.53</td>
<td>0.53</td>
<td>DRS</td>
</tr>
<tr>
<td>D17</td>
<td>0.87</td>
<td>0.74</td>
<td>0.77</td>
<td>0.53</td>
<td>0.40</td>
<td>-0.92</td>
<td>0.53</td>
<td>0.53</td>
<td>DRS</td>
</tr>
<tr>
<td>D18</td>
<td>0.84</td>
<td>0.67</td>
<td>0.72</td>
<td>0.54</td>
<td>0.38</td>
<td>-0.91</td>
<td>0.54</td>
<td>0.54</td>
<td>DRS</td>
</tr>
<tr>
<td>D19</td>
<td>0.98</td>
<td>0.95</td>
<td>0.96</td>
<td>0.59</td>
<td>0.56</td>
<td>-1.14</td>
<td>0.59</td>
<td>0.59</td>
<td>DRS</td>
</tr>
<tr>
<td>D20</td>
<td>0.87</td>
<td>0.74</td>
<td>0.78</td>
<td>0.54</td>
<td>0.41</td>
<td>-0.94</td>
<td>0.54</td>
<td>0.54</td>
<td>DRS</td>
</tr>
</tbody>
</table>
A direct method to compute the HDF within a conic optimization setting;

An exact multiplier version of the HDF model;

The conic primal-dual HDF formulations have several desirable properties and offer interesting insights;

We believe that this will open up new possibilities for research in both hyperbolic modelling and the application of conic programming to DEA problems since conic programming software is now widely available.
Thank you!