A weighted extremal function and equilibrium measure

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Abstract

We find an explicit formula for the weighted extremal function of $\mathbb{R}^n \subset \mathbb{C}^n$ with weight $(1+x_1^2+\cdots+x_n^2)^{-1/2}$ as well as its Monge-Ampère measure. As a corollary, we compute the Alexander capacity of $\mathbb{RP}^n$.

1 Introduction

For $K \subset \mathbb{C}^n$ compact, define the usual Siciak-Zaharjuta *extremal function*

$$V_K(z) = \sup\{u(z) : \ u \in L(\mathbb{C}^n), \ u \leq 0 \text{ on } K\}$$

where $L(\mathbb{C}^n)$ is the Lelong class of all plurisubharmonic (psh) functions u on $\mathbb{C}^n$ with the property that $u(z) - \log|z| = 0(1), \ |z| \to \infty$. Define

$$L^+(\mathbb{C}^n) := \{u \in L(\mathbb{C}^n) : u(z) \geq \log^+|z| + C\}$$

where C is a constant depending on u. We have

$$V_K(z) := \max\left\{0, \sup_p \left\{\frac{1}{\deg(p)} \log|p(z)| : p \text{ poly.}, \ ||p||_K := \max_{z\in K} |p(z)| \leq 1\right\}\right\}, \qquad (1.1)$$

where the supremum is taken over (non-constant) holomorphic polynomials p. Letting $V_K^*(z) := \limsup_{\zeta\to z} V_K(\zeta)$ be the uppersemicontinuous (usc) regularization, either $V_K^* \in L^+(\mathbb{C}^n)$ or $V_K^* \equiv \infty$, this latter case occurring when K is pluripolar; i.e., there exists $u \not\equiv -\infty$ psh on a neighborhood of K with $K \subset \{u = -\infty\}$.

If $K \subset \mathbb{C}^n$ is closed, a nonnegative usc function $w : K \to [0,\infty)$ with $\{z \in K : w(z) > 0\}$ not pluripolar is called a weight function on K and $Q(z) := -\log w(z)$ is the *potential* of w. The associated *weighted extremal function* is

$$V_{K,Q}(z) := \sup\{u(z) : \ u \in L(\mathbb{C}^n), \ u \leq Q \text{ on } K\}.$$

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Note $V_{K,0} = V_K$ for compact K. For unbounded K, the potential Q is required to grow at least like $\log|z|$. If $\liminf_{z\in K,\ |z|\to+\infty}\big(Q(z) - \log|z|\big) > -\infty$, Q is *weakly admissible*; if $|z|w(z) \to \infty$ as $|z| \to \infty,\ z \in K$, Q is *admissible*. In the former case, the Monge-Ampère measure $(dd^cV^*_{K,Q})^n$ might not have compact support. For bounded K, and for unbounded K with admissible Q (or even weakly admissble Q if $V_{K,Q}$ is continuous),

$$V_{K,Q}(z) = \sup\{\frac{1}{\deg(p)}\log|p(z)| : p \text{ poly.},\ ||pe^{-\deg(p)Q}||_K \leq 1\}.$$

If we let $X = \mathbb{P}^n$ with the usual Kähler form ω normalized so that $\int_{\mathbb{P}^n}\omega^n = 1$, we can define the class of $\omega-$*psh functions* (cf., [9])

$$PSH(X,\omega) := \{\phi \in L^1(X) : \phi \text{ usc},\ dd^c\phi + \omega \geq 0\}.$$

Let $\mathbf{z} := [z_0 : z_1 : \cdots : z_n]$ be homogeneous coordinates on $X = \mathbb{P}^n$. Identifying $\mathbb{C}^n$ with the affine subset of $\mathbb{P}^n$ given by $\{[1 : z_1 : \cdots : z_n]\}$, we can identify the $\omega-$psh functions with the Lelong class $L(\mathbb{C}^n)$, i.e., $PSH(X,\omega) \approx L(\mathbb{C}^n)$, and the bounded (from below) $\omega-$psh functions coincide with the subclass $L^+(\mathbb{C}^n)$. For example, if $\phi \in PSH(X,\omega)$, then

$$u(z) = u(z_1,...,z_n) := \phi([1 : z_1 : \cdots : z_n]) + \frac{1}{2}\log(1 + |z|^2) \in L(\mathbb{C}^n).$$

Abusing notation, we write $u = \phi + u_0$ where $u_0(z) := \frac{1}{2}\log(1+|z|^2)$. Given a closed subset $K \subset \mathbb{P}^n$ and a function q on K, we can define a *weighted $\omega-$psh extremal function*

$$v_{K,q}(\mathbf{z}) := \sup\{\phi(\mathbf{z}) : \phi \in PSH(X,\omega),\ \phi \leq q \text{ on } K\}.$$

Thus if $K \subset \mathbb{C}^n \subset \mathbb{P}^n$, for $[1 : z_1 : \cdots : z_n] = [1 : z] \in \mathbb{C}^n$ we have

$$v_{K,q}([1 : z]) = \sup\{u(z) : u \in L(\mathbb{C}^n),\ u \leq u_0+q \text{ on } K\} - u_0(z) = V_{K,u_0+q}(z) - u_0(z). \quad (1.2)$$

If $q = 0$, the *Alexander capacity* $T_\omega(K)$ of $K \subset \mathbb{P}^n$ was defined in [9] as

$$T_\omega(K) := \exp\,[-\sup_{\mathbb{P}^n} v_{K,0}].$$

This notion has applications in complex dynamics; cf., [8].

These extremal psh and $\omega-$psh functions $V_K, V_{K,Q}$ and $v_{K,0}, v_{K,q}$, as well as the homogeneous extremal psh function H_E of $E \subset \mathbb{C}^n$ (section 4), are very difficult to compute explicitly. Even when an explicit formula exists, computation of the associated Monge-Ampère measure is problematic. Our main goal in this paper is to utilize a novel approach to explicitly compute $V_{K,Q}$ and $(dd^cV_{K,Q})^n$ for the closed set $K = \mathbb{R}^n \subset \mathbb{C}^n$ with weakly admissible weight $w(z) = |f(z)| = |\frac{1}{(1+z^2)^{1/2}}|$ where $z^2 = z_1^2 + \cdots + z_n^2$.

Theorem 1.1. *For $K = \mathbb{R}^n \subset \mathbb{C}^n$ and weight $w(z) = |f(z)| = |\frac{1}{(1+z^2)^{1/2}}|$,*

$$V_{\mathbb{R}^n,Q}(z) = \frac{1}{2}\log\big([1 + |z|^2] + \{[1 + |z|^2]^2 - |1 + z^2|^2\}^{1/2}\big),\ z \in \mathbb{C}^n \text{ and} \quad (1.3)$$

$$(dd^cV_{\mathbb{R}^n,Q})^n = n!\frac{\omega_n}{(1 + x^2)^{\frac{n+1}{2}}}dx. \quad (1.4)$$

Here dx is Lebesgue measure on $\mathbb{R}^n$ and ω_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$. Note that for $n = 1$, it is easy to see that

$$V_{\mathbb{R}^n,Q}(z) = \max[\log|z-i|, \log|z+i|] \tag{1.5}$$

which agrees with formula (1.3). We remark that $V_{\mathbb{R}^n,Q}(z) = V_{L_{n+1}}(1,z)$ where L_{n+1} is the *Lie ball* in $\mathbb{C}^{n+1}$ (see (2.1)).

The potential $Q(z)$ in this case is the standard Kähler potential $u_0(z)$ restricted to $\mathbb{R}^n$. Using (1.2) and the fact that $\mathbb{RP}^n \setminus \mathbb{R}^n$ is (locally) pluripolar in $\mathbb{P}^n$, for $z \in \mathbb{C}^n$ we have

$$V_{\mathbb{R}^n,Q}(z) = u_0(z) + v_{\mathbb{R}^n,0}([1:z]) = u_0(z) + v_{\mathbb{RP}^n,0}([1:z]).$$

As an application of (1.3) we can calculate the Alexander capacity $T_\omega(\mathbb{RP}^n)$ of $\mathbb{RP}^n$.

Corollary 1.2. *The unweighted $\omega-$psh extremal function of $\mathbb{RP}^n$ is given by*

$$v_{\mathbb{RP}^n,0}([1:z]) = \frac{1}{2}\log\big([1+|z|^2] + \{[1+|z|^2]^2 - |1+z^2|^2\}^{1/2}\big) - u_0(z)$$

$$= \frac{1}{2}\log\big(1 + [1 - \frac{|1+z^2|^2}{(1+|z|^2)^2}]^{1/2}\big) \tag{1.6}$$

for $[1:z] \in \mathbb{C}^n$ and

$$v_{\mathbb{RP}^n,0}([0:z]) = \limsup_{|t|\to\infty}\big[\frac{1}{2}\log\big(1+[1-\frac{|1+(tz)^2|^2}{(1+|tz|^2)^2}]^{1/2}\big)\big] = \frac{1}{2}\log\big(1+[1-\frac{|z^2|^2}{(|z|^2)^2}]^{1/2}\big). \tag{1.7}$$

Thus the exact value of the Alexander capacity $T_\omega(\mathbb{RP}^n)$ is $1/\sqrt{2}$.

The proofs of (1.3) and (1.4) are in sections 2 and 3. The proof of (1.3) in section 2 is by nature a verification of a formula found by other means. It is the purpose of section 6, based on the results in section 5, to provide readers interested in deriving formulas for other examples of K and Q an alternative, *deductive* proof of (1.3) from which this formula was originally discovered. It is our hope (and indeed expectation) that these techniques can be used in other cases. We would like to thank Ragnar Sigurdsson for many helpful suggestions, in particular, for the main calculation in the next section.

2 Relation with Lie ball and maximality of $V_{\mathbb{R}^n,Q}$

In this section we prove (1.3) of Theorem 1.1 as well as Corollary 1.2. Writing $Z := (z_0, z) = (z_0, z_1, ..., z_n) \in \mathbb{C}^{n+1}$, define the Lie ball

$$L_{n+1} = \{Z \in \mathbb{C}^{n+1} : |Z|^2 + \{|Z|^4 - |Z^2|^2\}^{1/2} \leq 1\}. \tag{2.1}$$

The extremal function of this circled set ($Z \in L_{n+1} \iff e^{i\theta}Z \in L_{n+1}$) is

$$V_{L_{n+1}}(Z) = \frac{1}{2}\log^+\big(|Z|^2 + \{|Z|^4 - |Z^2|^2\}^{1/2}\big); \text{ thus}$$

$$V(z) := V_{L_{n+1}}(1,z) = \frac{1}{2}\log\big([1+|z|^2] + \{[1+|z|^2]^2 - |1+z^2|^2\}^{1/2}\big) \qquad (2.2)$$

agrees with formula (1.3). The function $V_{L_{n+1}}(Z)$ in $\mathbb{C}^{n+1}$ is maximal outside L_{n+1}, i.e., $(dd^c V_{L_{n+1}})^{n+1} = 0$ there. We show: *our candidate function $V : \mathbb{C}^n \to \mathbb{R}$ in (2.2) for $V_{K,Q}$ in (1.3), where $K = \mathbb{R}^n \subset \mathbb{C}^n$ and $Q(x) = \frac{1}{2}\log(1+x^2)$, is maximal in $\mathbb{C}^n \setminus \mathbb{R}^n$.* Note for $x \in \mathbb{R}^n$, $|x|^2 = x^2$ and $V(x) = Q(x)$. Let $\|\cdot\|_c$ denote the Lie norm on $\mathbb{C}^{n+1}$: $\|z\|_c^2 = |z|^2 + \big(|z|^4 - |z^2|^2\big)^{\frac{1}{2}}$, $z \in \mathbb{C}^{n+1}$. This is a norm on $\mathbb{C}^{n+1}$ and $V(z) = \log\|(1,z)\|_c$.

For a C^2 function u on a domain D in $\mathbb{C}^{n+1}$ of the form $u = \log v$ we write the Levi form of u at $Z \in D$ applied to $w \in \mathbb{C}^{n+1}$ as

$$\mathcal{L}_u(Z;w) = \frac{1}{v(Z)}\bigg(\mathcal{L}_v(Z;w) - \frac{1}{v(Z)}\big|\langle \nabla v(Z), w\rangle\big|^2\bigg) \qquad (2.3)$$

where $\nabla v = (\partial v/\partial Z_1, \ldots, \partial v/\partial Z_{n+1})$ and $\langle a,b\rangle = \sum_{j=1}^{n+1} a_j b_j$.

Consider $v(Z) = \|Z\|_c^2 = |Z|^2 + \varphi(Z)^{\frac{1}{2}}$ where $\varphi(Z) = |Z|^4 - |Z^2|^2$. Note $|Z|^2 = |Z^2|$ if and only if $Z \in \mathbb{C}\cdot\mathbb{R}^{n+1}$. This occurs precisely when ReZ is a real multiple of ImZ. Hence $v \in C^\infty(\mathbb{C}^{n+1} \setminus \mathbb{C}\cdot\mathbb{R}^{n+1})$. Working on $\mathbb{C}^{n+1} \setminus \mathbb{C}\cdot\mathbb{R}^{n+1}$, we have

$$\frac{\partial v}{\partial Z_j} = \bar{Z}_j + \tfrac{1}{2}\varphi(Z)^{-\frac{1}{2}}\frac{\partial \varphi}{\partial Z_j} \text{ so that}$$

$$\frac{\partial^2 v}{\partial Z_j \partial \bar{Z}_k} = \delta_{jk} + \tfrac{1}{2}\varphi(Z)^{-\frac{1}{2}}\frac{\partial^2 \varphi}{\partial Z_j \partial \bar{Z}_k} - \tfrac{1}{4}\varphi(Z)^{-\frac{3}{2}}\frac{\partial \varphi}{\partial Z_j}\frac{\partial \varphi}{\partial \bar{Z}_k}.$$

The formula (2.3) for the Levi form of u becomes

$$\mathcal{L}_u(Z;w) = \frac{1}{v(Z)}\bigg(|w|^2 + \tfrac{1}{2}\varphi(Z)^{-\frac{1}{2}}\mathcal{L}_\varphi(Z;w) - \tfrac{1}{4}\varphi(Z)^{-\frac{3}{2}}\big|\langle\nabla\varphi(Z),w\rangle\big|^2$$
$$-\frac{1}{v(Z)}\big|\langle\bar{Z},w\rangle + \tfrac{1}{2}\varphi(Z)^{-\frac{1}{2}}\langle\nabla\varphi(Z),w\rangle\big|^2\bigg).$$

We have the formulas

$$\frac{\partial\varphi}{\partial Z_j} = 2|Z|^2\bar{Z}_j - 2Z_j\big(\bar{Z}^2\big) \text{ and } \frac{\partial^2\varphi}{\partial Z_j\partial\bar{Z}_k} = 2|Z|^2\delta_{jk} + 2\bar{Z}_j Z_k - 4Z_j\bar{Z}_k$$

yielding

$$\langle\nabla\varphi(Z),w\rangle = 2\big(|Z|^2\langle\bar{Z},w\rangle - \big(\bar{Z}^2\big)\langle Z,w\rangle\big) \text{ and}$$
$$\mathcal{L}_\varphi(Z;w) = 2|Z|^2|w|^2 + 2|\langle\bar{Z},w\rangle|^2 - 4|\langle Z,w\rangle|^2.$$

In particular, $\langle\nabla\varphi(Z),\bar{Z}\rangle = 0$ and $\mathcal{L}_\varphi(Z;\bar{Z}) = -2\varphi(Z)$. Moreover, since $u(\lambda Z) = \log|\lambda| + u(Z)$ for $\lambda \in \mathbb{C}\setminus\{0\}$, $\mathcal{L}_u(Z;Z) = 0$. Hence

$$\mathcal{L}_u(Z;\bar{Z}) = \frac{1}{v(Z)}\bigg(|Z|^2 + \tfrac{1}{2}\varphi(Z)^{-\frac{1}{2}}(-2\varphi(Z)) - \frac{|Z^2|^2}{v(Z)}\bigg)$$

$$= \frac{1}{v(Z)^2}\bigg(\big(|Z|^2 + \varphi(Z)^{\frac{1}{2}}\big)\big(|Z|^2 - \varphi(Z)^{\frac{1}{2}}\big) - |Z^2|^2\bigg) = \frac{1}{v(Z)^2}\bigg(|Z|^4 - \varphi(Z) - |Z^2|^2\bigg) = 0.$$

Since $u = 2\log \|Z\|_c$ and $\|\cdot\|_c$ is a complex norm on $\mathbb{C}^{n+1}$ the function u is plurisubharmonic and $w \mapsto \mathcal{L}_u(Z;w)$ is positive semi-definite. The vectors Z and $\bar{Z}$ are linearly independent if and only if $Z \in \mathbb{C}^{n+1} \setminus \mathbb{C} \cdot \mathbb{R}^{n+1}$, so we have proved that the eigenspace at such Z corresponding to the eigenvalue 0 has dimension at least two and includes $\mathrm{Im} Z$.

For $V(z) = \log \|(1,z)\|_c$ in (2.2) the vector $(1,z)$ is in $\mathbb{C} \cdot \mathbb{R}^{n+1}$ if and only if $z \in \mathbb{R}^n$, so $V \in C^\infty(\mathbb{C}^n \setminus \mathbb{R}^n)$. We write $V = \frac{1}{2}u(1,z)$ where $z \in \mathbb{C}^n$. For $z \notin \mathbb{R}^n$,

$$\mathcal{L}_V(z; \mathrm{Im}\, z) = \tfrac{1}{2}\mathcal{L}_u((1,z);(0,\mathrm{Im}\, z)) = \tfrac{1}{2}\mathcal{L}_u((1,z);\mathrm{Im}\,(1,z)) = 0.$$

Thus V is maximal on $\mathbb{C}^n \setminus \mathbb{R}^n$, and since $V = Q$ on $\mathbb{R}^n$, we have $V = V_{\mathbb{R}^n,Q}$ in (1.3).

For the proof of Corollary 1.2, the formulas for $v_{\mathbb{RP}^n,0}$ follow immediately and it remains to get the value of $T_\omega(\mathbb{RP}^n)$ (see Example 5.12 of [9]). Since $|1+z^2| \le 1+|z|^2$ (and $|z^2| \le |z|^2$), upon taking $z = i(1/\sqrt{n}, ..., 1/\sqrt{n})$ in (1.6) or letting $z \to 0$ in (1.7),

$$\sup_{\mathbf{z}\in\mathbb{P}^n} v_{\mathbb{RP}^n,0}(\mathbf{z}) = \frac{1}{2}\log 2.$$

Thus $T_\omega(\mathbb{RP}^n) = \exp^{[-\sup_{\mathbb{P}^n} v_{\mathbb{RP}^n,0}]} = 1/\sqrt{2}$. We remark that Dinh and Sibony had observed that the value of $T_\omega(\mathbb{RP}^n)$ was independent of n (Proposition A.6 in [8]).

3 Calculation of $(dd^c V_{\mathbb{R}^n,Q})^n$ with $V_{\mathbb{R}^n,Q}$ in (1.3)

In this section we prove (1.4). First some background. Let $\delta(x;y)$ be a *Finsler metric* where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is a tangent vector at x; i.e., $y \to \delta(x;y)$ is a norm on $\mathbb{R}^n$ varying smoothly in x. We write $B_x := \{y : \delta(x;y) \le 1\}$ for the associated unit ball about x and

$$B_x^* := \{y : \delta(x;y) \le 1\}^* = \{a : a \cdot y = a^t y \le 1 \text{ for all } y \in B_x\}$$

for the dual unit ball (a^t denotes transpose of the vector a). Finsler metrics arise in pluripotential theory in the following setting: if $K = \bar{\Omega}$ where Ω is a bounded domain in $\mathbb{R}^n \subset \mathbb{C}^n$, the quantity

$$\delta_B(x;y) := \limsup_{t\to 0^+} \frac{V_K(x+ity)}{t} = \limsup_{t\to 0^+} \frac{V_K(x+ity) - V_K(x)}{t} \tag{3.1}$$

for $x \in K$ and $y \in \mathbb{R}^n$ defines a Finsler metric called the *Baran pseudometric* (cf., [5]). It is generally not Riemannian: such a situation yields more information on volumes of B_x and B_x^*. Recall ω_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$.

Proposition 3.1. *Suppose*

$$\delta(x;y)^2 = y^t G(x) y$$

is a Riemannian metric; i.e., $G(x)$ is a symmetric, positive definite matrix. Then

$$vol(B_x^*) \cdot vol(B_x) = \omega_n^2 \text{ and } vol(B_x^*) = \omega_n \sqrt{\det G(x)}.$$

Proof. Writing $G(x) = H^t(x)H(x)$, we have

$$\delta(x;y)^2 = y^t G(x) y = y^t H^t(x) H(x) y.$$

Letting $||\cdot||_2$ denote the standard Euclidean (l^2) norm, we then have

$$B_x = \{y \in \mathbb{R}^N : ||H(x)y||_2 \leq 1\} = H^{-1}(x)\big(\text{unit ball in } l^2\text{−norm}\big)$$

and

$$B_x^* = H(x)^t\big(\text{unit ball in } l^2\text{−norm}\big).$$

Hence $vol(B_x^*) \cdot vol(B_x) = \omega_n^2$ and

$$vol\big(\{y : \delta(x;y) \leq 1\}^*\big) = vol(B_x^*) = \omega_n \det H(x) = \omega_n \sqrt{\det G(x)}.$$

□

Motivated by (3.1) and Theorem 3.2 below, for $u(z) = V_{\mathbb{R}^n,Q}(z)$ in (1.3), we show

$$\delta_u(x;y) := \lim_{t\to 0^+} \frac{u(x+ity) - u(x)}{t}$$

exists. Fixing $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, let

$$F(t) := u(x+ity) = \frac{1}{2}\log\{(1+x^2+t^2y^2) + 2[t^2y^2 + t^2x^2y^2 - (x\cdot ty)^2]^{1/2}\}$$

$$= \frac{1}{2}\log\{(1+x^2+t^2y^2) + 2t[y^2 + x^2y^2 - (x\cdot y)^2]^{1/2}\}.$$

It follows that

$$\delta_u(x;y) = F'(0) = \frac{1}{2}\frac{2[y^2 + x^2y^2 - (x\cdot y)^2]^{1/2}}{1+x^2} = \frac{[y^2 + x^2y^2 - (x\cdot y)^2]^{1/2}}{1+x^2}.$$

We write

$$\delta_u^2(x;y) = \frac{y^2 + x^2y^2 - (x\cdot y)^2}{(1+x^2)^2} = y^t G(x) y \text{ where } G(x) := \frac{(1+x^2)I - xx^t}{(1+x^2)^2}.$$

Since this matrix is symmetric and positive definite, $\delta_u(x;y)$ defines a Riemannian metric.

The eigenvalues of the rank one matrix $xx^t \in \mathbb{R}^{n\times n}$ are $x^2, 0, \ldots, 0$ for $(xx^t)x = x(x^tx) = x^2 \cdot x$; and clearly $v \perp x$ implies $(xx^t)v = x(x^tv) = 0$. The eigenvalues of $(1+x^2)I - xx^t$ are then

$$(1+x^2) - x^2,\ (1+x^2) - 0, \ldots,\ (1+x^2) - 0 \ = \ 1, 1+x^2, \ldots, 1+x^2$$

and the eigenvalues of $G(x)$ are $\frac{1}{(1+x^2)^2}, \frac{1}{1+x^2}, \ldots, \frac{1}{1+x^2}$. This shows $\det G(x) = \frac{1}{(1+x^2)^{n+1}}$. From Proposition 3.1, for $\delta_u(x;y)$,

$$vol(B_x^*) = \omega_n \sqrt{\det G(x)} = \frac{\omega_n}{(1+x^2)^{\frac{n+1}{2}}} = \frac{\omega_n^2}{vol(B_x)}. \tag{3.2}$$

Note from (1.5) this agrees with the density of $\Delta V_{\mathbb{R}^n,Q}$ with respect to Lebesgue measure dx on $\mathbb{R}$ if $n = 1$ and this will be the case for the density of $(dd^c V_{\mathbb{R}^n,Q})^n$ with respect to Lebesgue measure dx on $\mathbb{R}^n$ for $n > 1$ as well. For motivation, we recall the main result of [7] (see [2] for the symmetric case $K = -K$):

Theorem 3.2. *Let $K \subset \mathbb{R}^n$ be a convex body; i.e., K is compact, convex and $int_{\mathbb{R}^n} K \neq \emptyset$. Let V_K be its Siciak-Zaharjuta extremal function. The limit*

$$\delta(x; y) := \lim_{t \to 0^+} \frac{V_K(x + ity)}{t} \tag{3.3}$$

exists for each $x \in \mathrm{int}_{\mathbb{R}^n} K$ and $y \in \mathbb{R}^n$ and

$$(dd^c V_K)^n = \lambda(x)dx \text{ where } \lambda(x) = n!vol(\{y : \delta(x; y) \leq 1\}^*) = n!vol(B_x^*). \tag{3.4}$$

The conclusion of Theorem 3.2 required Proposition 4.4 of [7]:

Proposition 3.3. *Let $D \subset \mathbb{C}^n$ and let $\Omega := D \cap \mathbb{R}^n$. Let v be a nonnegative locally bounded psh function on D which satisfies $\Omega = \{v = 0\}$; $(dd^c v)^n = 0$ on $D \setminus \Omega$; $(dd^c v)^n = \lambda(x)dx$ on Ω; for all $x \in \Omega$, $y \in \mathbb{R}^n$, the limit*

$$h(x, y) := \lim_{t \to 0^+} \frac{v(x + ity)}{t} \text{ exists and is continuous on } \Omega \times i\mathbb{R}^n;$$

and for all $x \in \Omega$, $y \to h(x, y)$ is a norm. Then $\lambda(x) = n!\mathrm{vol}\{y : h(x, y) \leq 1\}^*$.

We now give the proof of (1.4):

Proof. It will be useful to extend $Q(x) = \frac{1}{2}\log(1 + x^2)$ on $\mathbb{R}^n$ to all of $\mathbb{C}^n$ as

$$Q(z) = \frac{1}{2}\log|1 + z^2| \in L(\mathbb{C}^n).$$

With this extension of Q, and writing $u := V_{\mathbb{R}^n,Q}$, we claim

1. Q is pluriharmonic on $\mathbb{C}^n \setminus \mathcal{S}$ where $\mathcal{S} = \{z \in \mathbb{C}^n : 1 + z^2 = 0\}$;
2. $u - Q \geq 0$ in $\mathbb{C}^n$; and $\mathbb{R}^n = \{z \in \mathbb{C}^n : u(z) - Q(z) = 0\}$;
3. for each $x, y \in \mathbb{R}^n$
$$\lim_{t \to 0^+} \frac{Q(x + ity) - Q(x)}{t} = 0.$$

Item 1. is clear; 2. may be verified by direct calculation (the inequality also follows from the observation that $Q \in L(\mathbb{C}^n)$ and Q equals u on $\mathbb{R}^n$); and for 3., observe that

$$|1 + (x + ity)^2|^2 = (1 + x^2 - t^2y^2)^2 + 4t^2(x \cdot y)^2 = (1 + x^2)^2 + 0(t^2)$$

so that

$$Q(x+ity)-Q(x)=\frac{1}{2}\log|1+(x+ity)^2|-\frac{1}{2}\log(1+x^2)$$
$$=\frac{1}{4}\log\frac{(1+x^2)^2+0(t^2)}{(1+x^2)^2}\approx\frac{1}{4}\frac{0(t^2)}{(1+x^2)^2}\text{ as } t\to 0.$$

Thus 1. and 2. imply that $v:=u-Q$ defines a nonnegative plurisubharmonic function in $\mathbb{C}^n\setminus\mathcal{S}$, in particular, on a neighborhood $D\subset\mathbb{C}^n$ of $\mathbb{R}^n$; from 1.,

$$(dd^c v)^n=(dd^c u)^n \text{ on } D; \tag{3.5}$$

and from 3., for each $x,y\in\mathbb{R}^n$

$$\lim_{t\to0^+}\frac{v(x+ity)-v(x)}{t}=\lim_{t\to0^+}\frac{u(x+ity)-Q(x+ity)-u(x)+Q(x)}{t}$$
$$=\lim_{t\to0^+}\frac{u(x+ity)-u(x)}{t}-\lim_{t\to0^+}\frac{Q(x+ity)-Q(x)}{t}=\delta_u(x;y).$$

Then (3.5), (3.2) and Proposition 3.3 give (1.4). □

This completes the proof of Theorem 1.1.

4 Known results on extremal functions

We list some results on extremal functions used in the sequel. We know much information about V_K is when K is a convex body in $\mathbb{R}^n$. Through every point $z\in\mathbb{C}^n\setminus K$ there is either a complex ellipse or a complex line L with $z\in L$ such that V_K restricted to L is harmonic on $L\setminus K$ (cf., [1], [6]). For $K=B_n$, the real unit ball in $\mathbb{R}^n\subset\mathbb{C}^n$, the real ellipses and lines $L\cap B_n$ are symmetric with respect to the origin and, other than great circles in the real boundary of B_n, each $L\cap B_n$ hits this boundary at exactly two antipodal points. Lundin proved [11], [1] that

$$V_{B_n}(z)=\frac{1}{2}\log h(|z|^2+|z^2-1|), \tag{4.1}$$

where h is the inverse Joukowski map $h(\frac{1}{2}(t+\frac{1}{t}))=t$ for $1\le t\in\mathbb{R}$. Moreover,

$$(dd^cV_{B_n})^n=n!\ vol(B_n)\ \frac{dx}{(1-|x|^2)^{\frac{1}{2}}}=n!\frac{\omega_n}{(1-|x|^2)^{\frac{1}{2}}}dx.$$

We may consider the class of *logarithmically homogeneous* psh functions

$$H:=\{u\in L(\mathbb{C}^n):\ u(tz)=\log|t|+u(z),\ t\in\mathbb{C},\ z\in\mathbb{C}^n\}$$

and, for $E\subset\mathbb{C}^n$, the *homogeneous extremal function of E,* denoted H_E^*, where

$$H_E(z):=\max[0,\sup\{u(z):u\in H,\ u\le 0 \text{ on } E\}].$$

Note that $H_E(z) \leq V_E(z)$. If E is compact, we have

$$H_E(z) = \max[0, \sup\{\frac{1}{\deg(h)} \log |h(z)| : h \text{ homogeneous polynomial, } ||h||_E \leq 1\}]. \quad (4.2)$$

Finally, we mention the following beautiful result of Sadullaev [12].

Theorem 4.1. *Let A be a pure $m-$dimensional, irreducible analytic subvariety of $\mathbb{C}^n$ where $1 \leq m \leq n-1$. Then A is algebraic if and only if for some (all) $K \subset A$ compact and nonpluripolar in A, V_K in (1.1) is locally bounded on A.*

Note that A and hence K is pluripolar in $\mathbb{C}^n$ so $V_K^* \equiv \infty$; moreover, $V_K = \infty$ on $\mathbb{C}^n \setminus A$. In this setting, $V_K|_A$ (precisely, its usc regularization in A) is maximal on the regular points A^{reg} of A outside of K; i.e., $(dd^c V_K|_A)^m = 0$ there, and $V_K|_A \in L(A)$. Here $L(A)$ is the set of psh functions u on A (u is psh on A^{reg} and locally bounded above on A) with the property that $u(z) - \log |z| = 0(1)$ as $|z| \to \infty$ through points in A, see [12].

5 Relating extremal functions

Let $K \subset \mathbb{C}^n$ be closed and let f be holomorphic on a neighborhood Ω of K. We define $F : \Omega \subset \mathbb{C}^n \to \mathbb{C}^{n+1}$ as

$$F(z) := (f(z), zf(z)) = W = (W_0, W') = (W_0, W_1, ..., W_n)$$

where $W' = (W_1, ..., W_n)$. Thus

$$W_0 = f(z), \ W_1 = z_1 f(z), ..., \ W_n = z_n f(z).$$

Moreover we assume there exists a polynomial $P = P(z_0, z)$ in $\mathbb{C}^{n+1}$ with $P(f(z), z) = 0$ for $z \in \Omega$; i.e., f is *algebraic*. Taking such a polynomial P of minimal degree, let

$$A := \{W \in \mathbb{C}^{n+1} : P(W_0, W'/W_0) = P(W_0, W_1/W_0, ..., W_n/W_0) = 0\}. \quad (5.1)$$

Note that writing $P(W_0, W'/W_0) = \widetilde{P}(W_0, W')/W_0^s$ where $\widetilde{P}$ is a polynomial in $\mathbb{C}^{n+1}$ and s is the degree of $P(z_0, z)$ in z we see that A differs from the algebraic variety

$$\widetilde{A} := \{W \in \mathbb{C}^{n+1} : \widetilde{P}(W_0, W') = 0\}$$

by at most the set of points in A where $W_0 = 0$, which is pluripolar in A. Thus we can apply Sadullaev's Theorem 4.1 to nonpluripolar subsets of A. Now $P(f(z), z) = 0$ for $z \in \Omega$ says that

$$F(\Omega) = \{(f(z), zf(z)) : z \in \Omega\} \subset A.$$

We can define a weight function $w(z) := |f(z)|$ which is well defined on all of Ω and in particular on K; as usual, we set

$$Q(z) := -\log w(z) = -\log |f(z)|. \quad (5.2)$$

We will need our potentials defined in (5.2) to satisfy

$$Q(z) := \max\{-\log|W_0| : W \in A,\ W'/W_0 = z\} \tag{5.3}$$

and we mention that (5.3) can give an a priori definition of a potential for those $z \in \mathbb{C}^n$ at which there exist $W \in A$ with $W'/W_0 = z$.

We observe that for $K \subset \Omega$, we have two natural associated subsets of A:

1. $\widetilde{K} := \{W \in A : W'/W_0 \in K\}$ and
2. $F(K) = \{W = F(z) \in A : z \in K\}$.

Note that $F(K) \subset \widetilde{K}$ and the inclusion can be strict.

Proposition 5.1. *Let $K \subset \mathbb{C}^n$ be closed with Q in (5.2) satisfying (5.3). If $F(K)$ is nonpluripolar in A,*

$$V_{K,Q}(z) - Q(z) \leq H_{F(K)}(W) \text{ for } z \in \Omega \text{ with } f(z) \neq 0$$

where the inequality is valid for $W = F(z) \in F(\Omega)$.

In general, Proposition 5.1 only gives estimates for $V_{K,Q}(z)$ if $z \in \Omega$ and $f(z) \neq 0$. We use this and Proposition 5.4 in the next section to get formula (1.3) for $V_{\mathbb{R}^n,Q}(z)$ with weight $w(z) = |f(z)| = |\frac{1}{(1+z^2)^{1/2}}|$ for z in a neighborhood Ω of $\mathbb{R}^n$.

Proof. First note that for $z \in K$ and $W = F(z) \in F(K)$, given a polynomial p in $\mathbb{C}^n$,

$$|w(z)^{\deg(p)} p(z)| = |f(z)|^{\deg(p)} |p(z)| = |W_0^{\deg(p)} p(W'/W_0)| = |\widetilde{p}(W)| \tag{5.4}$$

where $\widetilde{p}$ is the homogenization of p. Thus $||w^{\deg(p)} p||_K \leq 1$ implies $|\widetilde{p}| \leq 1$ on $F(K)$.

Now fix $z \in \Omega$ at which $f(z) \neq 0$ (so $Q(z) < \infty$) and fix $\epsilon > 0$. Choose a polynomial $p = p(z)$ with $||w^{\deg(p)} p||_K \leq 1$ and

$$\frac{1}{\deg(p)} \log|p(z)| \geq V_{K,Q}(z) - \epsilon.$$

Thus for $W \in A$ with $W_0 \neq 0$ and $W'/W_0 = z$,

$$V_{K,Q}(z) - \epsilon - Q(z) \leq \frac{1}{\deg(p)} \log|p(W'/W_0)| - Q(W'/W_0) \leq \frac{1}{\deg(p)} \log|p(W'/W_0)| + \log|W_0|$$

with (5.3) used in the second inequality. By (5.4) and the fact that $\|\widetilde{p}\|_{F(K)} \leq 1$,

$$\frac{1}{\deg(p)} \log|p(W'/W_0)| + \log|W_0| = \frac{1}{\deg(\widetilde{p})} \log|\widetilde{p}(W)| \leq H_{F(K)}(W).$$

This shows that $V_{K,Q}(z) - \epsilon - Q(z) \leq H_{F(K)}(W)$. Finally, let $\epsilon \to 0$. $\square$

Next we prove a lower bound involving $\widetilde{K}$ which will be applicable in our special case.

Definition 5.2. Let $A \subset \mathbb{C}^{n+1}$ be an algebraic hypersurface. We say that A is *bounded on lines through the origin* if there exists a uniform constant $c \geq 1$ such that for all $W \in A$, if $\alpha W \in A$ also holds for some $\alpha \in \mathbb{C}$, then $|\alpha| \leq c$.

Example 5.3. A simple example of a hypersurface bounded on lines through the origin is one given by an equation of the form $p(W) = 1$, where p is a homogeneous polynomial. In this case, if $\alpha W \in A$ then

$$1 = p(\alpha W) = \alpha^{\deg(p)} p(W) = \alpha^{\deg(p)},$$

so α must be a root of unity. Hence we may take $c = 1$.

Proposition 5.4. *Let $K \subset \mathbb{C}^n$ and let $Q(z) = -\log|f(z)|$ with f defined and holomorphic on $\Omega \supset K$. Define A as in (5.1) and assume Q satisfies (5.3). We suppose A is bounded on lines through the origin, $\widetilde{K}$ is a nonpluripolar subset of A, and that Q has an extension to $\mathbb{C}^n$ (which we still call Q) satisfying (5.3) such that $Q \in L(\mathbb{C}^n)$. Then given $z \in \mathbb{C}^n$,*

$$H_{\widetilde{K}}(W) \leq V_{\widetilde{K}}(W) \leq V_{K,Q}(z) - Q(z)$$

for all $W = (W_0, W') \in A$ with $W'/W_0 = z$.

Proof. The left-hand inequality $H_{\widetilde{K}}(W) \leq V_{\widetilde{K}}(W)$ is immediate. For the right-hand inequality, we first note that $V_{\widetilde{K}}(W) \in L(A)$ if $\widetilde{K}$ is nonpluripolar in A. Hence there exists a constant $C \in \mathbb{R}$ such that

$$V_{\widetilde{K}}(W) \leq \log|W| + C = \log|W_0| + \frac{1}{2}\log(1 + |W'/W_0|^2) + C$$

for all $W \in A$ with $W_0 \neq 0$.

Define the function

$$U(z) := \max\{V_{\widetilde{K}}(W) : W \in A, W'/W_0 = z\} + Q(z).$$

Note that the right-hand side is a locally finite maximum since A is an algebraic hypersurface. Away from the singular points A^{sing} of A one can write $V_{\widetilde{K}}(W)$ as a psh function in z by composing it with a local inverse of the map $A \ni W \mapsto z = W'/W_0 \in \mathbb{C}^n$. Hence U is psh off the pluripolar set

$$\{z \in \mathbb{C}^n : z = W'/W_0 \ \text{ for some } \ W \in A^{sing}\},$$

and hence psh everywhere since it is clearly locally bounded above on $\mathbb{C}^n$. Also, since $V_{\widetilde{K}} = 0$ on $\widetilde{K}$ it follows that $U \leq Q$ on K.

We now verify that $U \in L(\mathbb{C}^n)$ by checking its growth. By the definitions of U and Q and (5.3), given $z \in \mathbb{C}^n$ there exist $W, V \in A$, with $z = W'/W_0 = V'/V_0$, such that

$$U(z) = V_{\widetilde{K}}(W) + Q(z) \ \text{ and } \ Q(z) = -\log|V_0|.$$

Note that $W = \alpha V$, and since A is uniformly bounded on lines through the origin, there is a uniform constant c (independent of W, V) such that $|\alpha| \leq c$. We then compute

$$\begin{aligned} U(z) = V_{\widetilde{K}}(W) - \log|V_0| &\leq V_{\widetilde{K}}(W) - \log|W_0| + \log c \\ &\leq \log|W| + C - \log|W_0| + \log c \\ &= \log|W/W_0| + C + \log c = \tfrac{1}{2}\log(1+|z|^2) + C + \log c \end{aligned}$$

where $C > 0$ exists since $V_{\widetilde{K}} \in L(A)$. Hence $U \in L(\mathbb{C}^n)$, and since $U \leq Q$ on K this means that $U(z) \leq V_{K,Q}(z)$. By the definition of U,

$$V_{\widetilde{K}}(W) + Q(z) \leq V_{K,Q}(z)$$

for all $W \in A$ such that $W'/W_0 = z$, which completes the proof. □

Remark 5.5. When $f \equiv 1$ we have $Q \equiv 0$, $F(z) = (1, z)$, and $F(K) = \widetilde{K} = \{1\} \times K$. Combining Propositions 5.1 and 5.4 yields $V_K(z) = H_{\{1\}\times K}(1, z)$, which is an instance of the H-principle of Siciak that relates functions in $L(\mathbb{C}^n)$ to functions in $H(\mathbb{C}^{n+1})$.

A weighted version of this equality also holds. Given $K \subset \mathbb{C}^n$ closed and w a weight function on K (with $Q = -\log w$), form the circled set

$$Z(K) := \{(t, tz) \in \mathbb{C}^{n+1} : z \in K,\ |t| = w(z)\}.$$

Then from Bloom (cf., [4] and [3]), $H_{Z(K)}(1, z) = V_{K,Q}(z)$.

6 Theorem 1.1 revisited

Let $K = \mathbb{R}^n \subset \mathbb{C}^n$ and $w(z) = |f(z)| = |\frac{1}{(1+z^2)^{1/2}}|$. Here $f(z) \neq 0$ and we may extend $Q(z) = -\log|f(z)|$ to all of $\mathbb{C}^n$ as $Q(z) = \frac{1}{2}\log|1+z^2| \in L(\mathbb{C}^n)$. We use the results of the previous section to give our original proof of Theorem 1.1; this also shows where the formula (1.3) arose. Since $(1+z^2)\cdot f(z)^2 - 1 = 0$, we take

$$P(z_0, z) = (1+z^2)z_0^2 - 1.$$

Here,

$$A = \{W : P(W_0, W'/W_0) = (1 + W'^2/W_0^2)W_0^2 - 1 = W_0^2 + W'^2 - 1 = 0\}$$

is the complexified sphere in $\mathbb{C}^{n+1}$. From Definition 5.2 and Example 5.3, A is bounded on lines through the origin. Note that f is clearly holomorphic in a neighborhood of $\mathbb{R}^n$; thus we can take, e.g., $\Omega = \{z = x + iy \in \mathbb{C}^n : y^2 = y_1^2 + \cdots + y_n^2 < s < 1\}$ in Propositions 5.1 and 5.4 where $z_j = x_j + iy_j$. Condition (5.3) also holds for $Q(z) = \frac{1}{2}\log|1+z^2|$: given $z = W'/W_0$ for some $W = (W_0, W') \in A$, we have $W_0^2 = \frac{1}{1+z^2}$. Hence $-\log|W_0| = \frac{1}{2}\log|1+z^2|$ is the same value for all such W. We have

$$F(K) = \{(f(z), zf(z)) : z = (z_1, ..., z_n) \in K = \mathbb{R}^n\} = \{(\frac{1}{(1+x^2)^{1/2}}, \frac{x}{(1+x^2)^{1/2}}) : x \in \mathbb{R}^n\}.$$

Writing $u_j = \mathrm{Re} W_j$, we see that

$$F(K) = \{(u_0, ..., u_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} u_j^2 = 1,\ u_0 > 0\}.$$

On the other hand,

$$\widetilde{K} = \{W \in A : W'/W_0 \in K\} = \{(u_0, ..., u_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} u_j^2 = 1\}.$$

Clearly $\widetilde{K}$ is nonpluripolar in A which completes the verification that Proposition 5.4 is applicable. We also observe that since for any homogeneous polynomial $h = h(W_0, ..., W_n)$ we have

$$|h(-u_0, u_1, ..., u_n)| = |h(u_0, -u_1, ..., -u_n)|,$$

the homogeneous polynomial hulls of $\widetilde{K}$ and $\overline{F(K)}$ in $\mathbb{C}^{n+1}$ coincide so that $H_{\widetilde{K}} = H_{\overline{F(K)}}$ in A (see (4.2)). Since

$$\overline{F(K)} \setminus F(K) = \{(u_0, ..., u_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} u_j^2 = 1,\ u_0 = 0\} \subset A \cap \{W_0 = 0\}$$

is a pluripolar subset of A,

$$H_{\widetilde{K}} = H_{F(K)} \tag{6.1}$$

on $A \setminus P$ where $P \subset A$ is pluripolar in A. Combining (6.1) with Propositions 5.1 and 5.4, we have

$$H_{\widetilde{K}}(W) = V_{\widetilde{K}}(W) = V_{K,Q}(z) - Q(z) = H_{F(K)}(W) \tag{6.2}$$

for $z \in \widetilde{\Omega} := \Omega \setminus \widetilde{P}$ and $W = F(z)$ where $\widetilde{P}$ is pluripolar in $\mathbb{C}^n$.

To compute the extremal functions in this example, we first consider $V_{\widetilde{K}}$ in A. Let

$$B := B_{n+1} = \{(u_0, ..., u_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} u_j^2 \leq 1\}$$

be the real $(n+1)-$ball in $\mathbb{C}^{n+1}$.

Proposition 6.1. *We have $V_B(W) = V_{\widetilde{K}}(W)$ for $W \in A$.*

Proof. Clearly $V_B|_A \leq V_{\widetilde{K}}$. To show equality holds, the idea is that if we consider the complexified extremal ellipses L_α for B whose real points S_α are great circles on $\widetilde{K}$, the boundary of B in $\mathbb{R}^{n+1}$, then the union of these varieties fill out A: $\cup_\alpha L_\alpha = A$. Since $V_B|_{L_\alpha}$ is *harmonic*, we must have $V_B|_{L_\alpha} \geq V_{\widetilde{K}}|_{L_\alpha}$ so that $V_B|_A = V_{\widetilde{K}}$.

To see that $\cup_\alpha L_\alpha = A$, we first show $A \subset \cup_\alpha L_\alpha$. If $W \in A \setminus \widetilde{K}$, then W lies on *some* complexified extremal ellipse L whose real points $\mathcal{E}$ are an inscribed ellipse in B with boundary in $\widetilde{K}$ (and $V_B|_L$ is harmonic). If $L \neq L_\alpha$ for some α, then $\mathcal{E} \cap \widetilde{K}$ consists of two antipodal points $\pm p$. By rotating coordinates we may assume $\pm p = (\pm 1, 0, ..., 0)$ and

$$\mathcal{E} \subset \{(u_0, ..., u_n) : \ u_2 = \cdots = u_n = 0\}.$$

We have two cases:

1. $\mathcal{E} = \{(u_0, ..., u_n) : |u_0| \leq 1, \ u_1 = 0, \ u_2 = \cdots = u_n = 0\}$, a real interval:

   In this case
   $$L = \{(W_0, 0, ..., 0) : W_0 \in \mathbb{C}\}.$$
   But then $L \cap A = \{(W_0, 0, ..., 0) : W_0 = \pm 1\} = \{\pm p\} \subset \widetilde{K}$, contradicting $W \in A \setminus \widetilde{K}$.

2. $\mathcal{E} = \{(u_0, ..., u_n) : u_0^2 + u_1^2/r^2 = 1, \ u_2 = \cdots = u_n = 0\}$ where $0 < r < 1$, a nondegenerate ellipse:

   In this case,
   $$L := \{(W_0, ..., W_n) : W_0^2 + W_1^2/r^2 = 1, \ W_2 = \cdots = W_n = 0\}.$$
   But then if $W \in L \cap A$ we have
   $$W_0^2 + W_1^2/r^2 = 1 = W_0^2 + W_1^2$$
   so that $W_1 = \cdots = W_n = 0$ and $W_0^2 = 1$; i.e., $L \cap A = \{\pm p\} \subset \widetilde{K}$ which again contradicts $W \in A \setminus \widetilde{K}$.

For the reverse inclusion, recall that the variety A is defined by $\sum_{j=0}^n W_j^2 = 1$. If $W = u + iv$ with $u, v \in \mathbb{R}^{n+1}$, we have

$$\sum_{j=0}^n W_j^2 = \sum_{j=0}^n [u_j^2 - v_j^2] + 2i \sum_{j=0}^n u_j v_j.$$

Thus for $W = u + iv \in A$, we have $\sum_{j=0}^n [u_j^2 - v_j^2] = 1$ and $\sum_{j=0}^n u_j v_j = 0$.

If we take an orthogonal transformation T on $\mathbb{R}^{n+1}$, then, by definition, T preserves Euclidean lengths in $\mathbb{R}^{n+1}$; i.e.,

$$\sum_{j=0}^n u_j^2 = \sum_{j=0}^n (T(u)_j)^2 \text{ and } \sum_{j=0}^n v_j^2 = \sum_{j=0}^n (T(v)_j)^2.$$

Moreover, if u, v are orthogonal; i.e., $\sum_{j=0}^n u_j v_j = 0$, then $\sum_{j=0}^n (T(u))_j \cdot (T(v))_j = 0$. Extending T to a complex-linear map on $\mathbb{C}^{n+1}$ via

$$T(W) = T(u + iv) := T(u) + iT(v),$$

we see that if $W \in A$, then $\sum_{j=0}^{n}(T(u))_j \cdot (T(v))_j = 0$, so that

$$\sum_{j=0}^{n}(T(W)_j)^2 = \sum_{j=0}^{n}[(T(u)_j)^2 - (T(v)_j)^2] = \sum_{j=0}^{n}[u_j^2 - v_j^2] = 1.$$

Thus T preserves A.

Clearly the ellipse

$$L_0 := \{(W_0, ..., W_n) : W_0^2 + W_1^2 = 1,\ W_2 = \cdots = W_n = 0\}$$

corresponding to the great circle $S_0 := \{(u_0, ..., u_n) : u_0^2 + u_1^2 = 1,\ u_2 = \cdots = u_n = 0\}$ lies in A and any other great circle S_α can be mapped to S_0 via an orthogonal transformation T_α. From the previous paragraph, we conclude that $\cup_\alpha L_\alpha \subset A$. □

We use the Lundin formula $V_B(W) = \frac{1}{2}\log h\big(|W|^2 + |W^2 - 1|\big)$ in (4.1) where $h(t) = t + \sqrt{t^2 - 1}$ for $t \in \mathbb{C} \setminus [-1, 1]$. Now the formula for $V_{\widetilde{K}}$ can only be valid on A; and indeed, since $W^2 = 1$ on A, by the previous proposition we obtain

$$V_{\widetilde{K}}(W) = \frac{1}{2}\log h(|W|^2),\ W \in A.$$

Remark 6.2. Note that since the real sphere $\widetilde{K}$ and the complexified sphere A are invariant under real rotations, the Monge-Ampère measure

$$(dd^c V_{\widetilde{K}}(W))^n = (dd^c \frac{1}{2}\log h(|W|^2))^n$$

must be invariant under real rotations as well and hence is normalized surface area measure on the real sphere $\widetilde{K}$. This can also be seen as a consequence of $V_{\widetilde{K}}$ being the *Grauert tube function* for $\widetilde{K}$ in A as $(dd^c V_{\widetilde{K}}(W))^n$ gives the volume form dV_g on $\widetilde{K}$ corresponding to the standard Riemannian metric g there (cf., [13]).

Getting back to the calculation of $V_{K,Q}$, note that since $W = (\frac{1}{(1+z^2)^{1/2}}, \frac{z}{(1+z^2)^{1/2}})$,

$$|W|^2 := |W_0|^2 + |W_1|^2 + \cdots + |W_n|^2 = \frac{1 + |z|^2}{|1 + z^2|}.$$

Plugging in to (6.2)

$$V_{\widetilde{K}}(W) = V_B(W) = V_{K,Q}(z) - Q(z) = V_{K,Q}(z) - \frac{1}{2}\log|1 + z^2|$$

gives

$$V_{K,Q}(z) = \frac{1}{2}\log\big([1 + |z|^2] + \{[1 + |z|^2]^2 - |1 + z^2|^2\}^{1/2}\big)$$

for $z \in \widetilde{\Omega}$, agreeing with (1.3).

A similar observation leads to another derivation of the above formula. Consider $\overline{F(K)}$ as the upper hemisphere

$$S := \{(u_0, ..., u_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} u_j^2 = 1,\ u_0 \geq 0\}$$

in $\mathbb{R}^{n+1}$ and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection $\pi(u_0, ..., u_n) = (u_1, ..., u_n)$ which we extend to $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ via $\pi(W_0, ..., W_n) = (W_1, ..., W_n)$. Then

$$\pi(S) = B_n := \{(u_1, ..., u_n) \in \mathbb{R}^n : \sum_{j=1}^{n} u_j^2 \leq 1\}$$

is the real $n-$ball in $\mathbb{C}^n$. Each great semicircle C_α in S – these are simply half of the L_α's from before – projects to half of an inscribed ellipse $\mathcal{E}_\alpha$ in B_n, while the other half of $\mathcal{E}_\alpha$ is the projection of the great semicircle given by the negative $u_1, ..., u_n$ coordinates of C_α (still in $F(K)$, i.e., with $u_0 > 0$). As before, the complexification $\mathcal{E}_\alpha^*$ of the ellipses $\mathcal{E}_\alpha$ correspond to complexifications of the great circles.

Proposition 6.3. *We have*

$$H_{F(K)}(W_0, ..., W_n) = V_{B_n}(\pi(W)) = V_{B_n}(W_1, ..., W_n) = V_{B_n}(W') \leq V_{\widetilde{K}}(W_0, ..., W_n)$$

for $W = (W_0, ..., W_n) = (W_0, W') \in A$.

Proof. Clearly $V_{B_n}(\pi(W)) \leq V_{\widetilde{K}}(W)$. For the inequality $H_{F(K)}(W) \leq V_{B_n}(\pi(W))$, note that for $W \in A$ with $W = (W_0, W')$, we have $\pi^{-1}(W') = (\pm W_0, W') \in A$ but the value of $H_{F(K)}$ is the same at both of these points. Thus $W' \to H_{F(K)}(\pi^{-1}(W'))$ is a well-defined function of W' for $W \in A$ which is clearly in $L(\mathbb{C}^n)$ (in the W' variables) and nonpositive if $W' \in B_n$; hence $H_{F(K)}(\pi^{-1}(W')) \leq V_{B_n}(W')$. □

From (6.2),

$$H_{\widetilde{K}}(W) = V_{\widetilde{K}}(W) = V_{K,Q}(z) - Q(z) = H_{F(K)}(W)$$

for $z \in \widetilde{\Omega}$ and $W = F(z)$ so that we have equality for such W in Proposition 6.3 and an alternate way of computing $V_{K,Q}$. From the Lundin formula, for $(W_0, W') \in A$ we have $W_0^2 + W'^2 = 1$ so

$$V_{B_n}(W') = \frac{1}{2} \log h\big(|W'|^2 + |W'^2 - 1|\big) = \frac{1}{2} \log h(|W|^2)$$

and we get the same formula (1.3)

$$V_{K,Q}(z) = \frac{1}{2} \log\big([1 + |z|^2] + \{[1 + |z|^2]^2 - |1 + z^2|^2\}^{1/2}\big) =: V(z)$$

for $z \in \widetilde{\Omega}$.

To show this formula holds on all of $\mathbb{C}^n$, we know $V \leq V_{K,Q}$ on $\mathbb{C}^n$ since $V \leq Q$ on $\mathbb{R}^n$. Now $V \in L^+(\mathbb{C}^n)$ since, e.g., $u(z) = V_{L_{n+1}}(1,z)$ and $V_{L_{n+1}} \in L^+(\mathbb{C}^{n+1})$. Thus $V_{K,Q} \in L^+(\mathbb{C}^n)$ as well. This implies that the total Monge-Ampere mass of V and $V_{K,Q}$ are the same (cf. [10], Corollary 5.5.3). But $V_{K,Q}$ is maximal outside of $\mathbb{R}^n$ and $(dd^cV)^n = (dd^cV_{K,Q})^n$ on $\Omega \supset \mathbb{R}^n$. Thus $(dd^cV)^n$ must vanish outside of Ω; i.e., V is maximal on $\mathbb{C}^n \setminus \mathbb{R}^n$ and $V = V_{K,Q}$ on $\mathbb{C}^n$.

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