On a family of groups defined by Said Sidki

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For Said Sidki on the occasion of his 75th birthday

Abstract

In a paper in 1982, Said Sidki defined a 2-parameter family of finitely-presented groups $Y(m,n)$ that generalise the Carmichael presentation for a finite alternating group satisfied by its generating 3-cycles $(1,2,t)$ for $t \geq 3$. For $m \geq 2$ and $n \geq 2$, the group $Y(m,n)$ is the abstract group generated by elements $a_1, a_2, \ldots, a_m$ subject to the defining relations $a_i^n = 1$ for $1 \leq i \leq m$ and $(a_i^k a_j^k)^2 = 1$ for $1 \leq i < j \leq m$ and $1 \leq k \leq [\frac{n}{2}]$. Sidki investigated the structure of various sub-families of these groups, for small values of m or n, and has conjectured that they are all finite. Sidki's conjecture remains open. In this paper it is shown that for all $m \geq 3$, the group $Y(m,6)$ is finite, and is isomorphic to a semi-direct product of an elementary abelian 2-group of order $2^{\frac{m(m+3)}{2}}$ by $Y(m,3) \cong A_{m+2}$. Also we exploit a computation for the group $Y(3,8)$ to prove that $Y(m,8)$ is a finite 2-group, for all m.

Keywords: Finitely-presented group; generators and relations; finiteness.

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1 Introduction

At the August 2016 Escola de Álgebra in Brazil, which celebrated the 75th birthday of Said Sidki (one of the founders of this biennial meeting), Said Sidki gave a lecture on a 2-parameter family of groups denoted by $Y(m,n)$, which he defined in a paper [6] in 1982.

For $m \geq 2$ and $n \geq 2$, the group $Y(m,n)$ is the abstract group with presentation

$$\langle\, a_1, a_2, \ldots, a_m \mid a_i^n = 1 \text{ for } 1 \leq i \leq m,\ (a_i^k a_j^k)^2 = 1 \text{ for } 1 \leq i < j \leq m \text{ and } 1 \leq k \leq [\tfrac{n}{2}]\,\rangle.$$

As noted by Sidki, for $n = 3$ this presentation generalises the one given by Carmichael [2] for the alternating group of given finite degree, satisfied by its generating 3-cycles $(1,2,t)$ for $t \geq 3$; see also [3, §6.3]. Sidki observed that $Y(m,2)$ is elementary abelian of order 2^m for all m, and that $Y(2,n)$ is metabelian of order $2^{n-1}n$, having an elementary abelian normal 2-subgroup of order 2^{n-1} with cyclic quotient of order n.

Sidki further investigated the structure of other sub-families of the groups $Y(m, n)$, for small values of m or n, as well as the general case where n is odd. In particular, he proved in [6] that $Y(m, 4)$ is a finite 2-group of order $2^{\frac{m(m+3)}{2}}$ and nilpotency class 3 for all $m \geq 2$. He also claimed in [6] that $Y(m, 6)$ is infinite for all $m \geq 3$, but then retracted this in a subsequent paper [7], after proving that $Y(3, n)$ is finite for all n.

Neubüser, Felsch and O'Brien used computational techniques to show that $Y(m, 5)$ is finite for $3 \leq m \leq 10$, and $Y(m, 7)$ is finite for $3 \leq m \leq 6$, and $Y(m, 11)$ is finite for $3 \leq m \leq 5$, and indeed that in each of these cases, $Y(m, n)$ is a simple orthogonal group of characteristic 2, or has such a group as a quotient by a normal 2-subgroup. The latter (unpublished) work was taken further recently by McInroy and Shpectorov [5] to show a definite connection with the orthogonal groups. Also the work by Sidki in [7] was taken further by Krstić and McCool to prove the non-finite presentability of the automorphism group $\Phi_2(\mathbb{Z})$ of the free $\mathbb{Z}$-group of rank two; see [4].

Based on these and other discoveries, Sidki has conjectured that the groups $Y(m, n)$ are all finite, and that they are 2-groups when n is a power of 2. As far as we are aware, and as reported in [5], these conjectures have not been resolved.

In this paper, we prove the following:

Theorem 1 *For all $m \geq 2$, the group $Y(m, 6)$ is finite, and is isomorphic to a semi-direct product of an elementary abelian 2-group of order $2^{\frac{m(m+3)}{2}}$ by $Y(m, 3) \cong A_{m+2}$.*

Theorem 2 *For all $m \geq 2$, the group $Y(m, 8)$ is a finite 2-group.*

In fact, computations using the MAGMA system [1] show that Theorem 1 is true in the cases $m = 2$, 3 and 4, with the elementary abelian normal subgroup having order 2^4, 2^9 and 2^{14} respectively. Also two different computations with MAGMA show that the group $Y(m, 6)$ has a quotient that is an extension by $Y(m, 3) \cong A_{m+2}$ of an elementary abelian group of order $2^{\frac{m(m+3)}{2}}$ when $m = 5$ or 6, and that $Y(3, 8)$ has order 2^{21}. A proof of much of Theorem 1 follows almost immediately from the fact that $Y(3, 6)$ is finite and has the required structure, but we give a computer-free proof in Sections 2 and 3. In both cases a key step involves consideration of the structure of 3-generator subgroups of the kernel N of the natural epimorphism from $Y(m, 6)$ to $Y(m, 3)$. Also we give a computer-assisted proof of Theorem 2 in Section 4, using Sidki's theorem on the groups $Y(m, 4)$ together with observed properties of the group $Y(3, 8)$.

2 Some properties of the groups $Y(m, 6)$

Let $Y = Y(m, 6)$ be the group defined as in the Introduction, with $m \geq 2$ and $n = 6$, and in this group, define $b_i = a_i^3$ and $c_{ij} = a_i^{-1} b_j a_i = a_i^{-1} a_j^3 a_i$ for all i and j in $\{1, 2, \ldots, m\}$. Also denote by R_{ij}^k the relation $(a_i^k a_j^k)^2 = 1$, which holds for all distinct $i, j \in \{1, 2, \ldots, m\}$ and all $k \in \mathbb{Z}$, and not just those i, j and k given in the defining presentation for $Y(m, n)$.

Before proceeding, we note that if $m \geq 3$ and $S = \{a_i, a_j, a_k\}$ is any subset of three of the given generators of Y, then those three elements satisfy the defining relations for $Y(3,6)$, and hence the subgroup generated by S is isomorphic to a quotient of $Y(3,6)$. In particular, many of the properties of the elements a_i, a_j and a_k follow immediately from the properties of the three given generators for $Y(3,6)$. Nevertheless we can prove the properties we need directly from the presentation for $Y(m,6)$, thereby avoiding reliance on the results of computer calculations for $Y(3,6)$. The first properties we need are easy.

Lemma 1 *In the group $Y(m,6)$, the following relations hold:*

(a) $b_j^2 = c_{ij}^{\,2} = 1$ *for all i and j ;*
(b) $[b_i, b_j] = (b_i b_j)^2 = 1$ *for all i and j ;*
(c) $a_i b_j a_i^{-1} = b_i b_j c_{ji}$ *for all i and j with $i \neq j$;*
(d) $[b_i, c_{ij}] = [b_j, c_{ij}] = 1$ *for all i and j with $i \neq j$;*
(e) $[c_{ij}, c_{ik}] = 1$ *whenever i, j and k are distinct.*

Proof. First, part (a) follows immediately from the relation $a_j^6 = 1$ and conjugation by a_i, and then (b) from the relation $(a_i^3 a_j^3)^2 = 1$. Also conjugation of the relation $[b_i, b_j] = 1$ by a_i gives $[b_i, c_{ij}] = 1$, which is the first part of (d). Similarly, conjugation of the relation $[b_j, b_k] = 1$ by a_i gives $[c_{ij}, c_{ik}] = 1$, which is (e). Next, using R_{ji}^2 and R_{ij}^1 we find $a_i b_j a_i^{-1} = a_i a_j^3 a_i^{-1} = a_i a_j^{-3} a_i^{-1} = a_i^3 a_i^{-2} a_j^{-2} a_j^{-1} a_i^{-1} = a_i^3 a_j^2 a_i^2 a_i a_j = a_i^3 a_j^3 a_j^{-1} a_i^3 a_j = b_i b_j c_{ji}$ for $i \neq j$, and so (c) holds. Finally if $i \neq j$ then part (c) gives $1 = a_j b_i^2 a_j^{-1} = (a_j b_i a_j^{-1})^2 = (b_j b_i c_{ij})^2 = [b_j b_i, c_{ij}]$, therefore c_{ij} commutes with $b_j b_i$ (since $(b_j b_i)^2 = c_{ij}^{\,2} = 1$), and then because $[b_i, c_{ij}] = 1$ we find that c_{ij} also commutes with b_j, giving the second part of (d). □

The next observations are more substantial.

Lemma 2 *In the group $Y(m,6)$, the following relations hold:*

(a) $a_i^{-1} c_{ij} a_i = b_i b_j c_{ji}$ *for all i and j with $i \neq j$;*
(b) $a_i^{-1} c_{ji} a_i = b_j b_i c_{ij}$ *for all i and j with $i \neq j$;*
(c) $a_i^{-1} c_{jk} a_i = c_{ij} b_i c_{ji} c_{jk} b_k b_i c_{ik} c_{ki} b_k c_{ik} c_{ij}$ *whenever i, j and k are distinct.*

Proof. First $a_i^{-1} c_{ij} a_i = a_i^{-2} b_j a_i^2 = a_i a_i^3 b_j a_i^{-3} a_i^{-1} = a_i b_i b_j b_i^{-1} a_i^{-1} = a_i b_j a_i^{-1} = b_i b_j c_{ji}$ by Lemma 1(b) and 1(c), while $a_i^{-1} c_{ji} a_i = a_i^{-1} a_j^{-1} b_i a_j a_i = a_j a_i b_i a_i^{-1} a_j^{-1} = a_j b_i a_j^{-1} = b_j b_i c_{ij}$ by R_{ji}^1 and Lemma 1(c). The proof of part (c) is more tricky. We know from Magma computations that (c) holds in the group $Y(3,6)$, and hence it holds in $Y(m,6)$ for all $m \geq 3$, but here we give a proof that is free of (yet guided by) computer calculations. In fact we prove it backwards, by expanding the right-hand-side and then using known relations to reduce it, as follows:

$$\begin{aligned} & c_{ij} b_i c_{ji} c_{jk} b_k b_i c_{ik} c_{ki} b_k c_{ik} c_{ij} \\ = \; & a_i^{-1} a_j^3 a_i \, a_i^3 \, a_j^{-1} a_i^3 a_j \, a_j^{-1} a_k^3 a_j \, a_k^3 \, a_i^3 a_i^{-1} a_k^3 a_i \, a_k^{-1} a_i^3 a_k \, a_k^3 \, a_i^{-1} a_k^3 a_i \, a_i^{-1} a_j^3 a_i \\ = \; & a_i^{-1} a_j^{-1} (a_j^{-2} a_i^{-2} a_j^{-1} a_i^3 a_k^3 a_j a_k^3 a_i^2 a_k^2 a_k a_i a_k^{-1} a_i^3 a_k^{-2} a_i^{-1} a_k^3 a_j^2) \, a_j a_i \quad \text{by cancellation} \end{aligned}$$

$$= a_i^{-1}a_j^{-1}(a_i^2a_ja_i^3a_k^3a_ja_ka_i^{-2}a_i^{-1}a_k^{-2}a_i^3a_k^{-2}a_i^{-1}a_ka_j^{-2}a_k^{-2})\,a_ja_i \;\text{ by } R_{ij}^2,\, R_{ik}^2,\, R_{ik}^1 \text{ and } R_{jk}^2$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_i^2a_k^3a_ja_ka_i^3a_k^{-2}a_i^3a_k^{-2}a_i^{-2}a_ia_ka_j^{-2}a_k^{-2})\,a_ja_i \;\text{ by } R_{ij}^1$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_k^{-2}a_i^{-2}a_ka_ja_ka_i^3a_k^{-2}a_i^3a_i^2a_k^2a_k^{-1}a_i^{-1}a_j^{-2}a_k^{-2})\,a_ja_i \;\text{ by } R_{ik}^2,\, R_{ik}^2 \text{ and } R_{ik}^1$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_k^{-2}a_i^{-2}a_j^{-1}a_i^{-1}a_k^2a_ia_ka_i^{-1}a_j^{-2}a_k^{-2})\,a_ja_i \;\text{ by } R_{jk}^1 \text{ and } R_{ik}^2$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_k^{-2}a_i^{-1}a_ja_ka_i^{-2}a_j^{-2}a_k^{-2})\,a_ja_i \;\text{ by } R_{ij}^1 \text{ and } R_{ik}^1$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_k^{-2}a_i^{-1}a_k^{-1}a_j^{-1}a_i^{-2}a_j^{-2}a_k^{-2})\,a_ja_i \;\text{ by } R_{jk}^1$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_k^{-1}a_ia_ja_i^2a_k^{-2})\,a_ja_i \;\text{ by } R_{ik}^1 \text{ and } R_{ij}^2$$
$$= a_i^{-1}a_j^{-1}(a_ia_j^{-1}a_k^{-1}a_j^{-1}a_ia_k^{-2})\,a_ja_i \;\text{ by } R_{ij}^1$$
$$= a_i^{-1}a_j^{-1}(a_ia_ka_ia_k^{-2})\,a_ja_i \;\text{ by } R_{jk}^1$$
$$= a_i^{-1}a_j^{-1}a_k^3a_ja_i \;\text{ by } R_{ik}^1$$
$$= a_i^{-1}c_{jk}a_i. \qquad \square$$

Corollary 1 *In $Y(m,6)$, the relation $[c_{ij}, c_{ji}] = 1$ holds whenever $i \neq j$.*

Proof. By Lemmas 2(a) and 1(d) we have $1 = a_i^{-1}[b_j, c_{ij}]a_i = [a_i^{-1}b_ja_i, a_i^{-1}c_{ij}a_i] = [c_{ij}, b_ib_jc_{ji}] = [c_{ij}, c_{ji}]$. □

Corollary 2 *The relation in Lemma 2(c) can be simplified to $a_i^{-1}c_{jk}a_i = c_{ij}b_ic_{ji}c_{jk}b_ic_{ki}c_{ij}$.*

Proof. This follows from Corollary 1 and parts (b) and (d) of Lemma 1, with j replaced by k in each case. □

Corollary 3 *In $Y(m,6)$, the following relations hold whenever i, j and k are distinct:*
(a) $(b_ib_jc_{ik}c_{jk})^2 = 1$ (b) $(b_ic_{jk}c_{kj}c_{ki})^2 = 1$ (c) $(c_{ij}c_{jk}c_{ik})^2 = 1$ (d) $(c_{ij}c_{ik}c_{jk}c_{ki})^2 = 1$.

Proof. First, conjugating $[c_{kj}, c_{ki}] = 1$ (from Corollary 1) by a_k gives $[b_kb_jc_{jk}, b_kb_ic_{ik}] = 1$, and then since b_k commutes with b_j, c_{jk}, b_i and c_{ik}, it follows that

$$1 = [b_jc_{jk}, b_ic_{ik}] = c_{jk}b_jc_{ik}b_ib_jc_{jk}b_ic_{ik} = c_{jk}b_jb_ic_{ik}c_{jk}b_jb_ic_{ik} = c_{jk}(b_jb_ic_{ik}c_{jk})^2c_{jk}$$

and therefore $(b_ib_jc_{ik}c_{jk})^2 = (b_jb_ic_{jk}c_{ik})^2 = 1$, which is (a).

Now conjugating $b_ib_jc_{ik}c_{jk}$ by a_i and using Lemma 2 and Corollary 2 gives

$$a_i^{-1}(b_ib_jc_{ik}c_{jk})a_i = b_ic_{ij}(b_ib_kc_{ki})(c_{ij}b_ic_{ji}c_{jk}b_ic_{ki}c_{ij}) = c_{ij}b_kc_{ki}c_{ij}b_ic_{ji}c_{jk}b_ic_{ki}c_{ij}$$
$$= c_{ij}c_{ki}b_kb_ic_{ij}c_{ji}c_{jk}b_ic_{ki}c_{ij} = c_{ij}c_{ki}b_i(b_kc_{ij}c_{ji}c_{jk})b_ic_{ki}c_{ij}.$$

Thus $b_kc_{ij}c_{ji}c_{jk}$ is conjugate to $b_ib_jc_{ik}c_{jk}$, and so from (a) we obtain $(b_kc_{ij}c_{ji}c_{jk})^2 = 1$, and clearly (b) follows from this by a cyclic permutation of the subscripts.

Next, Corollary 2 gives $1 = a_i^{-1}c_{jk}^2a_i = (c_{ij}b_ic_{ji}c_{jk}b_ic_{ki}c_{ij})^2$, and then by conjugation and Lemma 1 we obtain $1 = (b_ic_{ki}c_{ij}c_{ij}b_ic_{ji}c_{jk})^2 = (b_ic_{ki}b_ic_{ji}c_{jk})^2 = (c_{ki}c_{ji}c_{jk})^2$. Further conjugation and a permutation of the subscripts gives (c).

Finally, for (d), we have $(b_kc_{ij}c_{ji}c_{jk})^2 = 1$ from (b), and then conjugation of this relation by a_i gives

$$1 = a_i^{-1}(b_k c_{ij} c_{ji} c_{jk})^2 a_i \;=\; (c_{ik}(b_i b_j c_{ji})(b_j b_i c_{ij}) c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij})^2 \;=\; (b_i c_{ik} c_{jk} c_{ki} b_i c_{ij})^2,$$

and further conjugation gives $1 = (b_i c_{ij} b_i c_{ik} c_{jk} c_{ki})^2 = (c_{ij} c_{ik} c_{jk} c_{ki})^2$. □

Lemma 3 *In $Y(m,6)$, we have $b_i c_{jk} b_i = c_{ij} c_{jk} c_{ij}$ whenever i, j and k are distinct.*

Proof. First, an easy application of Lemma 2, Corollary 2 and parts of Lemma 1 gives

$$\begin{aligned}
b_i^{-1} c_{jk} b_i \;&=\; a_i^{-3} c_{jk} a_i^3 \;=\; a_i^{-2}(c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij}) a_i^2 \\
&=\; a_i^{-1}((b_i b_j c_{ji}) b_i (b_i b_j c_{ij})(c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij}) b_i (b_i b_k c_{ik})(b_i b_j c_{ji}) a_i \\
&=\; a_i^{-1}(b_i b_j c_{ji} b_j b_i c_{ji} c_{jk} b_i c_{ki} c_{ij} b_k c_{ik} b_i b_j c_{ji}) a_i \\
&=\; a_i^{-1}(c_{jk} b_i c_{ki} c_{ij} b_k c_{ik} b_i b_j c_{ji}) a_i \\
&=\; a_i^{-1}(c_{jk} c_{ki} c_{ij} b_k c_{ik} b_j c_{ji}) a_i \\
&=\; (c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij})(b_i b_k c_{ik})(b_i b_j c_{ji}) c_{ik} (b_i b_k c_{ki}) c_{ij} (b_i b_j c_{ij}) \\
&=\; c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij} b_i b_k c_{ik} b_i b_j c_{ji} c_{ik} b_i b_k c_{ki} c_{ij} b_i b_j c_{ij} \\
&=\; c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij} b_k c_{ik} b_j c_{ji} c_{ik} b_k c_{ki} b_j.
\end{aligned}$$

This can be taken further using Corollary 3, as follows:

$$\begin{aligned}
b_i^{-1} c_{jk} b_i \;&=\; c_{ij} b_i c_{ji} c_{jk} b_i c_{ki} c_{ij} b_k c_{ik} b_j c_{ji} c_{ik} b_k c_{ki} b_j \;=\; b_i(c_{ij} c_{ji} c_{jk} c_{ki} c_{ij} b_k c_{ik} b_j c_{ji} c_{ik} b_k c_{ki} b_j) b_i \\
&=\; b_i(c_{ki} c_{jk} c_{ji} b_k c_{ik} b_j c_{ji} c_{ik} b_k c_{ki} b_j) b_i \quad \text{since } (c_{ji} c_{jk} c_{ki} c_{ij})^2 = (c_{jk} c_{ji} c_{ki} c_{ij})^2 = 1 \text{ by part (d)} \\
&=\; b_i(c_{ki} c_{jk} c_{ji} c_{ik} b_k b_j c_{ji} c_{ki} b_k c_{ik} b_j) b_i \\
&=\; b_i(c_{ki} c_{jk} c_{ji} c_{ik} c_{ki} c_{ji} b_j c_{ik} b_j) b_i \quad \text{since } (b_k b_j c_{ji} c_{ki})^2 = (b_j b_k c_{ji} c_{ki})^2 = 1 \text{ by part (a)} \\
&=\; b_i(c_{ki} c_{ji} c_{jk} c_{ik} c_{ki} c_{ji} b_j c_{ik} b_j) b_i \\
&=\; b_i(c_{jk} c_{ji} c_{ki} c_{ik} c_{ki} c_{ji} b_j c_{ik} b_j) b_i \quad \text{since } (c_{jk} c_{ki} c_{ji})^2 = 1 \text{ by part (c)} \\
&=\; b_i(c_{jk} c_{ji} c_{ik} c_{ji} b_j c_{ik} b_j) b_i \;=\; (b_i c_{jk} b_i) c_{ji} c_{ik} c_{ji} b_j c_{ik} b_j.
\end{aligned}$$

Now from this we find that $1 = c_{ji} c_{ik} c_{ji} b_j c_{ik} b_j$, and hence that $b_j c_{ik} b_j = c_{ji} c_{ik} c_{ji}$, and then the result follows by swapping the subscripts i and j. □

Corollary 4 *In $Y(m,6)$, the relation $[b_i, c_{jk}] = 1$ holds whenever i, j and k are distinct.*

Proof. This is an easy consequence of earlier observations:

$$\begin{aligned}
1 &= (c_{ij} c_{jk} c_{ik})^2 = c_{ij} c_{jk} c_{ij} c_{ik} c_{jk} c_{ik} \quad \text{by Corollary 3(c) and Lemma 1(e)} \\
&= b_i c_{jk} b_i c_{ik} c_{jk} c_{ik} \quad \text{by Lemma 3} \\
&= b_i c_{jk} b_j c_{jk} c_{ik} b_i b_j c_{ik} = b_i b_j c_{ik} b_i b_j c_{ik} = b_j c_{ik} b_j c_{ik} \quad \text{by Corollary 3(a) and Lemma 1} \\
&= [b_j, c_{ik}],
\end{aligned}$$

and then the result follows by swapping the subscripts i and j. □

3 Structure and finiteness of the groups $Y(m,6)$

We can now prove our main theorem, namely that the group $Y = Y(m,6)$ is isomorphic to an extension by $Y(m,3) \cong A_{m+2}$ of an elementary abelian 2-group of rank $\frac{m(m+3)}{2}$, and hence finite, for all $m \geq 2$. We do this in steps.

Step 1 *The subgroup N generated by the elements b_i and c_{jk} (for $j \neq k$) is normal, with quotient Y/N isomorphic to the alternating group A_{m+2}.*

Proof. Note that $a_i^{-1} b_j a_i = c_{ij} \in N$ for all i and j (with $c_{ij} = c_{ii} = b_i$ when $i = j$), and that $a_i^{-1} c_{jk} a_i \in N$ for all i, j and k with $j \neq k$ by Lemma 2. Hence N is normal in Y. Moreover, it follows that N is generated by all conjugates of the elements $b_i = a_i^3$, and so the quotient Y/N is isomorphic to the group obtained from Y by adjoining the relations $a_i^3 = 1$ for $1 \leq i \leq m$. In particular, Y/N is isomorphic to $Y(m, 3)$, and hence to A_{m+2}. □

Step 2 *The subgroup N is abelian.*

This actually follows from the properties of the group $Y(3, 6)$ found by computation with MAGMA, but we can prove it directly:

Proof. By Lemma 1 and Corollaries 1 and 4 we have $[b_i, b_j] = [b_i, c_{ij}] = [b_i, c_{ji}] = 1$ for all distinct i and j, and $[b_i, c_{jk}] = 1$ whenever i, j and k are distinct. Thus every b_i is central in N. Moreover, by conjugation it follows that each c_{jk} $(= a_j^{-1} b_k a_j)$ is central in N as well, and therefore N is abelian. □

In particular, as N is generated by the m involutions b_i for $1 \leq i \leq m$ and the $m(m-1)$ involutions c_{jk} for distinct j and k in $\{1, 2, \ldots m\}$, it follows that N is an elementary abelian 2-group of rank at most $m + m(m-1) = m^2$. The next two steps reduce this upper bound on the rank of N to $\frac{m(m+3)}{2}$.

Step 3 *If $m \geq 4$ then $c_{ij} c_{ji} c_{k\ell} c_{\ell k} c_{ik} c_{ki} c_{j\ell} c_{\ell j} = 1$ whenever i, j, k and ℓ are distinct.*

Proof. First, by the observations in Section 2 and the fact that N is abelian, we have

$$
\begin{array}{ll}
a_i^{-1} b_j a_i = c_{ij}, & a_i b_j a_i^{-1} = a_i^{-2} b_j a_i^2 = a_i^{-1} c_{ij} a_i = b_i b_j c_{ji}, \\
a_i^{-1} c_{ij} a_i = b_i b_j c_{ji}, & a_i c_{ij} a_i^{-1} = b_j, \\
a_i^{-1} c_{ji} a_i = b_j b_i c_{ij}, & a_i c_{ji} a_i^{-1} = a_i^{-2} c_{ji} a_i^2 = a_i^{-1} b_j b_i c_{ij} a_i = b_j c_{ij} c_{ji}, \\
a_i^{-1} c_{jk} a_i = c_{ji} c_{jk} c_{ki}, & a_i c_{jk} a_i^{-1} = a_i^{-2} c_{jk} a_i^2 = a_i^{-1} c_{ji} c_{jk} c_{ki} a_i = b_j b_k c_{ij} c_{ji} c_{jk} c_{ki} c_{ik},
\end{array}
$$

whenever i, j, k and ℓ are distinct. It follows that

$$a_\ell^{-1} a_i^{-1} c_{jk} a_i a_\ell = a_\ell^{-1} c_{ji} c_{jk} c_{ki} a_\ell = (c_{j\ell} c_{ji} c_{i\ell})(c_{j\ell} c_{k\ell} c_{k\ell})(c_{ki} c_{jk} c_{i\ell}) = c_{ji} c_{jk} c_{ki},$$

while on the other hand

$$
\begin{aligned}
a_\ell^{-1} a_i^{-1} c_{jk} a_i a_\ell &= a_i a_\ell c_{jk} a_\ell^{-1} a_i^{-1} = a_i (b_j b_k c_{\ell j} c_{j\ell} c_{jk} c_{k\ell} c_{\ell k}) a_i^{-1} \\
&= (b_i b_j c_{ji})(b_i b_k c_{ki})(b_\ell b_j c_{i\ell} c_{\ell i} c_{\ell j} c_{ji} c_{ij})(b_j b_\ell c_{ij} c_{ji} c_{j\ell} c_{\ell i} c_{i\ell})(b_j b_k c_{ij} c_{ji} c_{jk} c_{ki} c_{ik}) \\
&\quad (b_k b_\ell c_{ik} c_{ki} c_{k\ell} c_{\ell i} c_{i\ell})(b_\ell b_k c_{i\ell} c_{\ell i} c_{\ell k} c_{ki} c_{ik}) \\
&= c_{\ell j} c_{j\ell} c_{ij} c_{jk} c_{k\ell} c_{\ell k} c_{ik}.
\end{aligned}
$$

Comparing the two expressions found gives $c_{ij} c_{ji} c_{k\ell} c_{\ell k} c_{ik} c_{ki} c_{j\ell} c_{\ell j} = 1$, as required. □

Step 4 *The rank of the subgroup N is at most $\frac{m(m+3)}{2}$.*

Proof. By the previous step, $c_{\ell k} \in \langle\, c_{ij}, c_{ji}, c_{k\ell}, c_{ik}, c_{ki}, c_{j\ell}, c_{\ell j}\,\rangle$ whenever i, j, k and ℓ are distinct. This observation can be used to eliminate $c_{\ell k}$ as a generator for N whenever

(a) $3 \le k < \ell \le m$, by taking $(i, j) = (1, 2)$,

(b) $(k, \ell) = (2, m)$, $(2, m{-}1), \ldots, (2, 4)$, by taking $(i, j) = (1, \ell{-}1)$ in each case in turn.

The number of generators eliminated in this way is $\binom{m-2}{2} = \frac{(m-2)(m-3)}{2}$ in case (a), and $m{-}3$ in case (b), making a total of $\frac{(m-2)(m-3)}{2} + m{-}3 = \frac{m(m-3)}{2}$, and leaving a generating set of size $m^2 - \frac{m(m-3)}{2} = \frac{m(m+3)}{2}$. □

Step 5 *A semi-direct product G of an elementary abelian 2-group of order $2^{\frac{m(m+3)}{2}}$ by $Y(m, 3) \cong A_{m+2}$ can be constructed as a quotient of $Y(m, 6)$.*

Proof. Let $A = A_{m+2} \cong Y(m, 3)$, with generating set $\{x_1, x_2, \ldots, x_m\}$ satisfying the relations for $Y(m, 3)$, namely $x_i^3 = 1$ for $1 \le i \le m$ and $(x_j x_k)^2 = 1$ for $1 \le j < k \le m$, and let $B \cong C_2^{\frac{m(m+3)}{2}}$ be an elementary abelian 2-group of rank $\frac{m(m+3)}{2}$, with generating set $\{b_i : 1 \le i \le m\} \cup \{c_{jk} : (j, k) \in S\}$, where S consists of all pairs (j, k) for which c_{jk} was not eliminated in (a) or (b) of the proof of Step 4 above, namely all (j, k) with either $1 \le j < k \le m$, or $j > k = 1$, or $(j, k) = (3, 2)$. Here we may note that the relation proved in Step 3 cannot be used to eliminate any further c_{jk} with $(j, k) \in S$.

Next, define $c_{\ell k}$ for $2 \le k < \ell \le m$ with $(\ell, k) \notin S$ by the reverse of the process used in Step 4, via the instances of the relation $c_{ij}c_{ji}c_{k\ell}c_{\ell k}c_{ik}c_{ki}c_{j\ell}c_{\ell j} = 1$ proved in Step 3. Then it is not difficult to see that the latter relation holds generally in the group B. (In fact B can be viewed as a quotient of the perhaps more natural elementary abelian 2-group of rank $m + m(m-1) = m^2$, by the subgroup generated by the relators from Step 3.)

Now define G as the semi-direct product $B \rtimes A$, with conjugation of B by A given by

$$\begin{aligned}
&x_i^{-1} b_i x_i = b_i \quad \text{for all } i \text{ in } \{1, 2, \ldots, m\},\\
&x_i^{-1} b_j x_i = c_{ij} \quad \text{for distinct } i \text{ and } j \text{ in } \{1, 2, \ldots, m\},\\
&x_i^{-1} c_{ij} x_i = b_i b_j c_{ji} \quad \text{for distinct } i \text{ and } j \text{ in } \{1, 2, \ldots, m\},\\
&x_i^{-1} c_{ji} x_i = b_i b_j c_{ij} \quad \text{for distinct } i \text{ and } j \text{ in } \{1, 2, \ldots, m\},\\
&x_i^{-1} c_{jk} x_i = c_{ji} c_{jk} c_{ki} \quad \text{for distinct } i,\ j \text{ and } k \text{ in } \{1, 2, \ldots, m\}.
\end{aligned}$$

It is an easy exercise to verify that this definition gives valid action of A on B, consistent with the relation proved in Step 3. For example, if i and j are distinct then the involution $x_i x_j$ induces the automorphism of B that swaps

b_i with c_{ji}, and b_j with $b_i b_j c_{ji}$, and c_{ij} with $b_i c_{ij} c_{ji}$, and

b_k with $c_{ij} c_{ik} c_{kj}$, and c_{ik} with $c_{ij} c_{ji} c_{jk} c_{kj} c_{ki}$, and c_{jk} with $b_i b_k c_{ki}$,
and c_{ki} with $c_{ij} c_{ji} c_{jk} c_{kj} c_{ik}$, and c_{kj} with $b_i b_k c_{jk} c_{ki} c_{kj}$, whenever $k \notin \{i, j\}$, and

$c_{k\ell}$ with $c_{ki} c_{k\ell} c_{\ell i}$ whenever i, j, k and ℓ are distinct.

Here we note that the fact that conjugation by $x_i x_j$ takes $c_{ki} c_{k\ell} c_{\ell i}$ back to $c_{k\ell}$ is a consequence of the relation $c_{ik}c_{ki}c_{j\ell}c_{\ell j}c_{i\ell}c_{\ell i}c_{jk}c_{kj} = 1$. Also the latter relation is preserved under conjugation by x_t, for every t.

Finally, define $g_i = b_i x_i$ in G, for $1 \le i \le m$. Then $g_i^3 = b_i^3 x_i^3 = b_i$, so $g_i^6 = 1$, for all i. Also if $i \ne j$ then $(g_i g_j)^2 = (b_i x_i b_j x_j)^2 = (b_j c_{ji} x_i x_j)^2 = 1$ (since $x_i x_j$ centralises $b_j c_{ji}$), while $(g_i^2 g_j^2)^2 = (b_i^2 x_i^2 b_j^2 x_j^2)^2 = (x_i^2 x_j^2)^2 = 1$ and $(g_i^3 g_j^3)^2 = (b_i b_j)^2 = 1$, so the elements $g_1, g_2, \ldots, g_m$ satisfy the defining relations for $Y(m,6)$. The subgroup generated by these elements contains $g_i^3 = b_i$ for all i, and so also contains $g_i^{-1} b_j g_i = x_i^{-1} b_i b_j b_i x_i = x_i^{-1} b_j x_i = c_{ij}$ for all distinct i and j, and hence equals G. Thus G is a quotient of $Y(m,6)$, as required. □

By Steps 4 and 5, we deduce that $Y(m,6)$ has order exactly $2^{\frac{m(m+3)}{2}}$, and this completes the proof of Theorem 1.

4 Structure and finiteness of the groups $Y(m,8)$

In this final section, we use Sidki's theorem on the groups $Y(m,4)$ and some computational analysis of $Y(3,8)$ to prove Theorem 2, namely that $Y(m,8)$ is a finite 2-group for all m.

Computer-assisted proof of Theorem 2.

Let $Y = Y(m,8)$ be as defined as in the Introduction, and in this group, let N be the subgroup generated by the elements $u_i = a_i^4$ and $v_{jk} = a_j^{-1} u_k a_j = a_j^{-1} a_k^4 a_j$ and $w_{jk} = a_j^{-2} u_k a_j^2 = a_j^{-2} a_k^4 a_j^2$, for all i, j and k in $\{1, 2, \ldots, m\}$. Note that each of these generators for N has order at most 2, and that $Y(2,8)$ is metabelian of order $2^{8-1}8 = 1024$, so we may assume that $m \ge 3$. Also we note that $Y(m,4)$ is a 2-group of order $2^{\frac{m(m+3)}{2}}$, by Sidki's theorem in [6, §3.1].

A 45-minute computation with Magma [1] shows that the following hold when $m = 3$:

(a) N can be generated by $\{u_1, u_2, u_3, v_{12}, v_{21}, v_{13}, v_{31}, v_{23}, v_{32}, w_{12}, w_{23}, w_{31}\}$,

(b) N is a normal subgroup of Y, of index 512,

(c) the abelianisation N/N' of N is elementary abelian of order $2^{12} = 4096$, and

(d) N itself has order 4096, and hence is an elementary abelian 2-group.

For the interested reader, we give the Magma code and resulting output in an Appendix. Note that here the quotient Y/N is isomorphic to the 2-group $Y(3,4)$, of order $2^{\frac{3\cdot 6}{2}} = 512$.

From this computation we find that in the general case (for $m \ge 3$), the following hold:

(d) For any i, j, k in $\{1, 2, \ldots, m\}$, each of $a_i^{-1} u_j a_i$, $a_i^{-1} v_{jk} a_i$ and $a_i^{-1} w_{jk} a_i$ is expressible as a word in $\{u_i, u_j, u_k, v_{ij}, v_{ji}, v_{ki}, v_{ik}, v_{jk}, v_{kj}, w_{ij}, w_{jk}, w_{ki}\}$, and so lies in N, and

(e) Each element u_i commutes with every other u_j, and with v_{jk} and w_{jk} for every j and k in in $\{1, 2, \ldots, m\}$.

By (e) it follows that N is normal in Y, with the quotient Y/N being isomorphic to the finite 2-group $Y(m,4)$. Next, by (f) we deduce that each of the elements u_i is central in N, and then by conjugation, so are each of the elements v_{jk} and w_{jk}. Hence N is abelian.

Moreover, since it is generated by at most (indeed fewer than) $m + 2m(m-1) = 2m^2 - m$ involutions, N is a finite elementary abelian 2-group. Thus Y is a finite 2-group. □

A similar approach using $Y(3, 10)$ shows also that $Y(m, 10)$ is a finite group (with a normal 2-subgroup N such that $Y(m, 10)/N \cong Y(m, 5)$), for all m.

Finally, we believe that some of the arguments presented above can be adapted to prove the following, which may be the subject of a sequel:

Conjecture If $Y(m, n)$ is finite, then the group $Y(m, 2n)$ has a finite normal elementary abelian 2-subgroup N such that $Y(m, 2n)/N \cong Y(m, n)$.

Note that if this conjecture is true, it will follow that $Y(m, 2^s)$ is a finite 2-group for all m and all s.

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Appendix

Code for Y(3,8):

```
F:=FreeGroup(3);
Rels:=[ F.i^8 : i in [1..3] ]
 cat [ (F.i*F.j)^2 : i in [1..3], j in [1..3] | i ne j ]
 cat [ (F.i^2*F.j^2)^2 : i in [1..3], j in [1..3] | i ne j ]
 cat [ (F.i^3*F.j^3)^2 : i in [1..3], j in [1..3] | i ne j ]
 cat [ (F.i^4*F.j^4)^2 : i in [1..3], j in [1..3] | i ne j ];
Y:=quo<F|Rels>;
N:=sub<Y| Y.1^4, Y.2^4, Y.3^4,
 (Y.1^4)^Y.2, (Y.2^4)^Y.1, (Y.1^4)^Y.3, (Y.3^4)^Y.1, (Y.2^4)^Y.3, (Y.3^4)^Y.2,
 (Y.2^4)^(Y.1^2), (Y.3^4)^(Y.2^2), (Y.1^4)^(Y.3^2) >;
print "Other three generators in N?",
 (Y.1^4)^(Y.2^2) in N, (Y.2^4)^(Y.3^2) in N, (Y.3^4)^(Y.1^2) in N;
N:=Rewrite(Y,N); print "Is N normal?",IsNormal(Y,N);
print "Order of quotient Y/N is",Index(Y,N);
aqs:=AQInvariants(N); print "Abelian invariants for N are",aqs;
print "Abelianisation of N has rank",#aqs;
print "Order of N is",Order(N);
```

Output:

```
Other three generators in N? true true true
Is N normal? true
Order of quotient Y/N is 512
Abelian invariants for N are [ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 ]
Abelianisation of N has rank 12
Order of N is 4096
```