# The number of composition factors of order $p$ in completely reducible groups of characteristic $p$

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Abstract. Let $q$ be a power of a prime $p$ and let $G$ be a completely reducible subgroup of $\mathrm{GL}(d, q)$. We prove that the number of composition factors of $G$ that have prime order $p$ is at most $(\varepsilon_q d - 1)/(p - 1)$, where $\varepsilon_q$ is a function of $q$ satisfying $1 \leqslant \varepsilon_q \leqslant 3/2$. For every $q$, we give examples showing this bound is sharp infinitely often.

## 1. Introduction

All groups considered in this paper are finite. Given a group $G$ and a prime $p$, let $c_p(G)$ denote the number of composition factors of $G$ of order $p$. Our main theorem is the following.

Theorem 1. *Let $q$ be a power of a prime $p$, say $q = p^f$. If $G$ is a completely reducible subgroup of* $\mathrm{GL}(d, q)$ *with $r$ irreducible components, then*

$$c_p(G) \leqslant \frac{\varepsilon_q d - r}{p - 1}, \quad \text{where } \varepsilon_q = \begin{cases} \frac{4}{3} & \text{if } p = 2 \text{ and } f \text{ is even,} \\ \frac{p}{p-1} & \text{if } p \text{ is a Fermat prime,} \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Recall that a *Fermat prime* is a prime of the form $2^{2^n} + 1$ for some $n \geqslant 0$, and that a subgroup $G$ of $\mathrm{GL}(V)$ is called *completely reducible* if $V$ is a direct sum of irreducible $G$-modules.

Our motivation for Theorem 1 arose from studying transitive permutation groups admitting paired orbitals with non-isomorphic subconstituents. In the case when both subconstituents are quasiprimitive, Knapp proved that one must be an epimorphic image of the other [**13**, Theorem 3.3]. This naturally led us to investigate the question of when a quasiprimitive group can be a non-trivial epimorphic image of another quasiprimitive group of the same degree. In an upcoming paper [**7**], we show that this is very rare. Our proof relies on Theorem 1 in the case when both quasiprimitive groups are of affine type.

Let $G$ be a group and let $a(G)$ be the product of the orders of the abelian composition factors of $G$. Note that $c_p(G) \leqslant \log_p(a(G))$ so upper bounds on $a(G)$ yield upper bounds on $c_p(G)$. It is proved in [**6**, Theorem 6.5] that, if $G$ is a completely reducible subgroup

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of $\mathrm{GL}(d, p^f)$, then $a(G) \leqslant \beta^{-1}(p^{fd})^{\gamma}$ where $\beta = 24^{1/3}$ and $\gamma = \log_9(48\beta)$, and thus $c_p(G) \leqslant \gamma df - \log_p \beta < \gamma df$. Our bound improves on this because it is independent of $f$, it involves the denominator $p-1$, and $\varepsilon_q < \gamma \approx 2.244$. Similarly, if $G$ is a primitive group of degree $n$ and $p \mid n$, then it follows from [**6**, Corollary 6.7] that $a(G) \leqslant \beta^{-1} n^{\gamma+1}$ and hence $c_p(G) \leqslant (\gamma+1)\log_p(n)$, whereas our result implies $c_p(G) \leqslant d + \frac{\varepsilon_p d - 1}{p-1}$ if $G$ is primitive of affine type and degree $n = p^d$.

After some preliminary results in Section 2, we exhibit some examples in Section 3 which show that, for every prime power $q$, Theorem 1 is sharp infinitely often. In particular, $\varepsilon_q$ is best possible. The bound in Theorem 1 can be sharpened (if $q$ is not an odd power of 2) to $\varepsilon_p(G) \leqslant \frac{\varepsilon_q d - s}{p-1}$ where $s$ is the number of *absolutely* irreducible components of $G$ since $G$ remains completely reducible over the algebraic closure of $\mathbb{F}_q$ by [**10**, §VII.2].

The proof of Theorem 1 is given in Section 4. The main idea is to use induction on $d$ and then split into cases, according to Aschbacher's classification of the subgroups of $\mathrm{GL}(d, p^f)$. The hardest case is when $G$ is a projectively almost simple absolutely irreducible '$\mathcal{C}_9$ group' with a 'non-geometric' linear action. We conclude with Corollary 9, which bounds $c_p(G/O_p(G))$ for $G$ an arbitrary subgroup of $\mathrm{GL}(d, p^f)$.

## 2. Preliminaries

Throughout the paper, $p$ will always denote a prime. Given a positive integer $n$, let $n_p$ denote the highest power of $p$ that divides $n$ and let $\mathrm{C}_n$ denote a cyclic group of order $n$. By Clifford's theorem [**4**], a normal subgroup of a completely reducible group is also completely reducible. This fact will be used repeatedly. The following lemmas will also be used repeatedly, sometimes without comment.

Lemma 2. *If $r$ is a positive integer, then* $\log_p r_p \leqslant \log_p (r!)_p \leqslant (r-1)/(p-1)$.

Proof. The first inequality is obvious. Consider the $p$-adic expansion $r = \sum_{k \geqslant 0} d_k p^k$ of $r$, with 'digits' $d_k \in \{0, 1, \ldots, p-1\}$ for each $k \geqslant 0$. Legendre proved that $\log_p (r!)_p = \sum_{k \geqslant 1} \lfloor r/p^k \rfloor = (r - s_p(r))/(p-1)$, where $s_p(r) = \sum_{k \geqslant 0} d_k$. The second inequality follows since $s_p(r) \geqslant 1$. □

Lemma 3. *Let $G$ be a group.*

(a) *If $1 = G_m \trianglelefteqslant G_{m-1} \trianglelefteqslant \cdots \trianglelefteqslant G_0 = G$ is a subnormal series for $G$, then $c_p(G) = \sum_{i=1}^{m} c_p(G_{i-1}/G_i)$.*
(b) *$c_p(G) \leqslant \log_p |G|_p$. If $G$ is $p$-soluble, then $c_p(G) = \log_p |G|_p$.*
(c) *If $G$ is a subgroup of a group $\Gamma$, then $c_p(G) \leqslant \log_p |\Gamma|_p$. In particular, if $G \leqslant \mathrm{Sym}(r)$, then $c_p(G) \leqslant (r-1)/(p-1)$.*
(d) *If $G$ is a subgroup of a direct product $H_1 \times \cdots \times H_r$ where the projection maps $\pi_i \colon G \to H_i$ are surjective for $1 \leqslant i \leqslant r$, then $c_p(G) \leqslant c_p(H_1) + \cdots + c_p(H_r)$.*
(e) *If $G$ is a subgroup of a central product $H_1 \circ \cdots \circ H_r$ where the projection maps $G \to H_i$ are surjective for $1 \leqslant i \leqslant r$, then $c_p(G) \leqslant c_p(H_1) + \cdots + c_p(H_r)$.*

PROOF. We prove these in order.

(a) The given subnormal series for $G$ can be refined to a composition series for $G$. The result now follows from the definition of $c_p(G)$.

(b) The first claim is obvious. If $G$ is $p$-soluble, then, by definition, each composition factor has order $p$, or coprime to $p$. The result now follows from the definition of $c_p(G)$.

(c) Since $|G|_p \leqslant |\Gamma|_p$, we have $c_p(G) \leqslant \log_p |\Gamma|_p$. The second sentence follows from Lemma 2.

(d) Let $G_0 = G$. For $1 \leqslant i \leqslant r$, let $\pi_i \colon G \to H_i$ be the projection map, let $K_i = \ker(\pi_i)$ and let $G_i = G \cap K_1 \cap \cdots \cap K_i$. Note that $G_r = 1$. Hence $c_p(G) = \sum_{i=1}^{r} c_p(G_{i-1}/G_i)$ by (a). However,

$$\frac{G_{i-1}}{G_i} = \frac{G_{i-1}}{G_{i-1} \cap K_i} \cong \frac{G_{i-1}K_i}{K_i} \trianglelefteqslant \frac{G}{K_i} \cong H_i.$$

Thus $c_p(G_{i-1}/G_i) \leqslant c_p(H_i)$ by (a), and hence $c_p(G) \leqslant c_p(H_1) + \cdots + c_p(H_r)$.

(e) Let $H = H_1 \times \cdots \times H_r$ and let $N$ be a normal subgroup of $H$ such that $H/N = H_1 \circ \cdots \circ H_r$. Let $\Gamma$ be the preimage of $G$ in $H$. The projection maps $\Gamma \to H_i$ are surjective, hence $c_p(\Gamma) \leqslant \sum_{i=1}^{r} c_p(H_i)$ by (d). The result follows since $c_p(G) \leqslant c_p(\Gamma)$. □

## 3. Examples

LEMMA 4. *Let $p$ be a prime, let $r \geqslant 1$, let $q$ be a prime-power and let $\Gamma_1$ be an irreducible subgroup of $\mathrm{GL}(r,q)$. For every $n \geqslant 2$, let $\Gamma_n = \Gamma_{n-1} \wr \mathrm{C}_p$. Then, for every $n \geqslant 2$, $\Gamma_n$ is an imprimitive subgroup of $\mathrm{GL}(d_n,q)$ where $d_n = rp^{n-1}$. Furthermore,*

$$c_p(\Gamma_n) = \frac{\varepsilon d_n - 1}{p-1} \quad \textit{where} \quad \varepsilon = \frac{c_p(\Gamma_1)(p-1)+1}{r}.$$

PROOF. We first prove by induction that $\Gamma_n$ is an irreducible subgroup of $\mathrm{GL}(d_n,q)$. This is true for $n = 1$. Assume now that $n \geqslant 2$ and $\Gamma_{n-1}$ is an irreducible subgroup of $\mathrm{GL}(d_{n-1},q)$. Let $V = (\mathbb{F}_q)^{d_n}$ be the natural $\Gamma_n$-module. Restricting to the base group $N = (\Gamma_{n-1})^p$ of $\Gamma_n$, $V$ is a direct sum $V_1 \oplus \cdots \oplus V_p$ of pairwise nonisomorphic irreducible $N$-modules each of dimension $d_{n-1}$. Hence $\Gamma_n$ is an irreducible subgroup of $\mathrm{GL}(d_n,q)$ by Clifford's Theorem [**4**]. In particular, $\Gamma_n$ is imprimitive for $n \geqslant 2$. The formula for $c_p(\Gamma_n)$ is true when $n = 1$ as $d_1 = r$ and $c_p(\Gamma_1) = (\varepsilon r - 1)/(p-1)$. By Lemma 3(a), $c_p(\Gamma_n) = pc_p(\Gamma_{n-1}) + 1$. Hence the the formula for $c_p(\Gamma_n)$ also follows by induction. □

Using Lemma 4, we now give three families of examples that show that the bound in Theorem 1 is best possible.

EXAMPLE 5. Let $q$ be a power of a prime $p$ and let $\Gamma_1 = \Gamma\mathrm{L}(1,p^p) \cong \mathrm{GL}(1,p^p) \rtimes \mathrm{C}_p$. Note that $\Gamma_1$ is an absolutely irreducible subgroup of $\mathrm{GL}(p,p)$. Consequently, $\Gamma_1$ is an irreducible subgroup of $\mathrm{GL}(p,q)$. Note also that $c_p(\Gamma_1) = 1$. Applying Lemma 4 with $r = p$ yields, for every $n \geqslant 1$, an irreducible subgroup $\Gamma_n$ of $\mathrm{GL}(d_n,q)$ with $c_p(\Gamma_n) = (d_n - 1)/(p-1)$, where $d_n = p^n$.

Example 6. Let $q$ be an even power of 2 and let $\Gamma_1 = \mathrm{GU}(3,2) \cong 3^{1+2} \rtimes \mathrm{SL}(2,3)$. Note that $\Gamma_1$ is an absolutely irreducible subgroup of $\mathrm{GL}(3,2^2)$. Thus, $\Gamma_1$ is an irreducible subgroup of $\mathrm{GL}(3,q)$. Note also that $c_2(\Gamma_1) = 3$. Applying Lemma 4 with $(p,r) = (2,3)$ yields, for every $n \geqslant 1$, an irreducible subgroup $\Gamma_n$ of $\mathrm{GL}(d_n, q)$ with $c_2(\Gamma_n) = (4/3)d_n - 1$, where $d_n = 3 \cdot 2^{n-1}$.

Example 7. Let $p = 2^m + 1$ be a Fermat prime, let $q$ be a power of $p$, let $E$ denote an extraspecial group of order $2^{1+2m}$ and type $-$, let $P$ be a Sylow $p$-subgroup of the orthogonal group $\mathrm{GO}^-(2m,2)$, and let $\Gamma_1 = E \rtimes P$. Note that $\Gamma_1$ is an absolutely irreducible subgroup of $\mathrm{GL}(2^m, p) = \mathrm{GL}(p-1,p)$. Consequently, $\Gamma_1$ is an irreducible subgroup of $\mathrm{GL}(p-1,q)$. Note also that $|P| = p$ and $c_p(\Gamma_1) = 1$. Applying Lemma 4 with $r = p-1$ yields, for every $n \geqslant 1$, an irreducible subgroup $\Gamma_n$ of $\mathrm{GL}(d_n, q)$ with $c_p(\Gamma_n) = (\varepsilon d_n - 1)/(p-1)$, where $\varepsilon = p/(p-1)$ and $d_n = (p-1)p^{n-1}$.

The three examples above together show that, for every prime power $q$, Theorem 1 is sharp infinitely often. In Theorem 1 and these examples, the prime $p$ divides the field size. If $p$ does not divide $q$, then $c_p(G)$ cannot be bounded by a function of only $d$ and $p$, as the following example shows.

Example 8. Let $p \neq 2$ and $r$ be primes such that $r \equiv 1 \pmod p$, let $f$ be a positive power of $p$, let $q = r^f$ and let $G = \mathrm{GL}(d,q)$. Note that $G/\mathrm{SL}(d,q)$ is cyclic of order $q-1$ hence $c_p(G) \geqslant (r^f - 1)_p = (r-1)_p f_p = (r-1)_p f$.

## 4. Proof of Theorem 1

Let $p$ be a prime, let $f$ be a positive integer and let $q = p^f$. Let $V = (\mathbb{F}_q)^d$, viewed as a vector space over $\mathbb{F}_q$, and let $G$ be a completely reducible subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}(d,q)$. It is also useful to note that $\varepsilon_q \geqslant 1$.

Our proof now proceeds by induction on pairs $(d,f)$ where we use the lexicographic ordering $(d_1,f_1) < (d_2,f_2)$ if $d_1 < d_2$, or $d_1 = d_2$ and $f_1 < f_2$. The base case when $d = f = 1$ is trivial.

Since $\mathrm{GL}(d,q)/\mathrm{SL}(d,q)$ has order $q-1$ and thus coprime to $p$, it follows by Lemma 3(a) that $c_p(G) = c_p(G \cap \mathrm{SL}(d,q))$. We henceforth assume that $G \leqslant \mathrm{SL}(d,q)$. Let $Z = \mathsf{Z}(\mathrm{SL}(d,q))$. Note that $Z$ has order $\gcd(d,q-1)$ which is coprime to $p$ hence $c_p(G) = c_p(GZ)$. We thus assume henceforth that $Z \leqslant G$.

In fact, $c_p(\mathrm{SL}(d,q)) = 0$ unless $d = 2$ and $q \in \{2,3\}$, in which case $c_p(\mathrm{SL}(d,q)) = 1$. In both cases, (1) holds hence we assume $G < \mathrm{SL}(d,q)$.

Our proof relies heavily on Aschbacher's Theorem characterising the subgroups of $\mathrm{GL}(d,q)$ that do not contain $\mathrm{SL}(d,q)$, which asserts that $G$ lies in at least one of the following nine classes [**2**].

- $\mathcal{C}_1$ **(reducible subgroups):** In this case, $G$ fixes some proper nonzero subspace of $V$.

- $\mathcal{C}_2$ **(imprimitive subgroups):** In this case, $G$ fixes some decomposition $V = V_1 \oplus \cdots \oplus V_r$, where $r \geqslant 2$ and each $V_i$ has dimension $d/r$. In particular, $G \leqslant \mathrm{GL}(d/r, q) \wr \mathrm{Sym}(r)$.
- $\mathcal{C}_3$ **(extension field subgroups):** In this case, $G$ preserves the structure of $V$ as a $(d/r)$-dimensional vector space over $\mathbb{F}_{q^r}$ for some $r \geqslant 2$. In this case, $G \leqslant \mathrm{GL}(d/r, q^r) \rtimes \mathrm{C}_r$.
- $\mathcal{C}_4$ **(tensor product subgroups):** In this case, $G$ preserves a tensor product decomposition $V = U \otimes W$ with $d = \dim(U)\dim(W)$ and $\dim(U) \neq \dim(W)$. In particular, $G \leqslant \mathrm{GL}(U) \circ \mathrm{GL}(W)$.
- $\mathcal{C}_5$ **(subfield subgroups):** In this case, $q = q_0^r$ for some $r \geqslant 2$ and $G \leqslant \mathrm{GL}(d, q_0) \cdot \mathsf{Z}(\mathrm{GL}(d,q))$.
- $\mathcal{C}_6$ **(symplectic type $r$-groups):** In this case, there is a prime $r$ such that $d = r^m$ and an absolutely irreducible normal $r$-subgroup $R$ of $G$ such that $R/\mathsf{Z}(R)$ is elementary abelian of rank $2m$.
- $\mathcal{C}_7$ **(tensor-imprimitive subgroups):** In this case, $G$ preserves the tensor product decomposition $V = V_1 \otimes \cdots \otimes V_r$, where each $V_i$ has dimension $n$ and $d = n^r$. In particular, $G \leqslant (\mathrm{GL}(n,q) \circ \cdots \circ \mathrm{GL}(n,q)) \rtimes \mathrm{Sym}(r)$.
- $\mathcal{C}_8$ **(classical groups):** In this case, $G$ preserves a nondegenerate alternating, hermitian or quadratic form on $V$. Moreover, $G$ contains one of $\mathrm{Sp}(d,q)'$, $\mathrm{SU}(d,\sqrt{q})$ or $\Omega^\varepsilon(d,q)$, where $\varepsilon \in \{\pm, \circ\}$. For more details, see §§4.7.
- $\mathcal{C}_9$ **(nearly simple groups):** In this case, $G/Z$ is an almost simple group with socle $N/Z$ such that $Z \leqslant N$ and $N$ is absolutely irreducible.

We now consider these classes one by one.

**4.1.** $G \in \mathcal{C}_1$**.** As $G \in \mathcal{C}_1$ is completely reducible, $G$ preserves a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_r$ with $r \geqslant 2$ where the restriction $G_i$ of $G$ to $V_i$ is irreducible. By induction, we have $c_p(G_i) \leqslant \frac{\varepsilon_q d_i - 1}{p-1}$ where $d_i = \dim(V_i)$ for each $i$. Since $G \leqslant G_1 \times \cdots \times G_r$ and $G$ projects onto each $G_i$, Lemma 3(d) implies

$$c_p(G) \leqslant c_p(\prod_{i=1}^r G_i) = \sum_{i=1}^r c_p(G_i) \leqslant \sum_{i=1}^r \frac{\varepsilon_q d_i - 1}{p-1} = \frac{\varepsilon_q(\sum_{i=1}^r d_i) - r}{p-1} = \frac{\varepsilon_q d - r}{p-1}.$$

**4.2.** $G \in \mathcal{C}_2$**.** In this case $G$ is irreducible and preserves a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_r$ with $r \geqslant 2$. Thus $G$ acts transitively on the set $\Omega = \{V_1, \ldots, V_r\}$. Let $N$ be the kernel of the action of $G$ on $\Omega$, and let $N_i$ denote the restriction of $N$ to $V_i$. The stabiliser $G_i$ in $G$ of the subspace $V_i$ is irreducible on $V_i$. Thus $c_p(G_i) \leqslant \frac{\varepsilon_q d_i - 1}{p-1}$ by induction where $d_i = \dim(V_i) = d/r$ for each $i$. By definition, the projection maps $N \to N_i$ are surjective for all $i$ and, moreover, $N \leqslant N_1 \times \cdots \times N_r$. It follows by Lemma 3(d) that $c_p(N) \leqslant \sum_{i=1}^r c_p(N_i)$, and $N_i \trianglelefteqslant G_i$ implies $c_p(N_i) \leqslant c_p(G_i)$. On the other hand, $G/N$ is isomorphic to a subgroup of $\mathrm{Sym}(r)$ hence $c_p(G/N) \leqslant (r-1)/(p-1)$ by Lemma 3(c).

By Lemma 3(a), we have

$$c_p(G) = c_p(G/N) + c_p(N) \leqslant c_p(G/N) + \sum_{i=1}^{r} c_p(N_i) \leqslant c_p(G/N) + \sum_{i=1}^{r} c_p(G_i)$$
$$\leqslant \frac{r-1}{p-1} + \sum_{i=1}^{r} \frac{\varepsilon_q d_i - 1}{p-1} = \frac{r-1}{p-1} + \frac{\varepsilon_q d - r}{p-1} = \frac{\varepsilon_q d - 1}{p-1},$$

as desired. From now on, we assume that $G$ is irreducible and primitive. Therefore every normal subgroup $N$ of $G$ acts completely reducibly, indeed homogeneously.

**4.3.** $G \in \mathcal{C}_3$**.** In this case, there exists $r \geqslant 2$ such that $G \leqslant \mathrm{GL}(d/r, q^r) \rtimes \mathrm{C}_r$. Let $N = G \cap \mathrm{GL}(d/r, q^r)$. By the inductive hypothesis, we have $c_p(N) \leqslant \dfrac{\varepsilon_{q^r}(d/r) - 1}{p-1}$. Furthermore, by Lemma 2,

$$c_p(G/N) \leqslant c_p(\mathrm{C}_r) = \log_p r_p \leqslant \frac{r-1}{p-1}.$$

If $d = r$, then $N \leqslant \mathrm{GL}(1, q^d)$ and hence $|N|$ is coprime to $p$ and $c_p(N) = 0$. By Lemma 3(a), we have

$$c_p(G) = c_p(G/N) \leqslant \frac{r-1}{p-1} = \frac{d-1}{p-1} \leqslant \frac{\varepsilon_q d - 1}{p-1}.$$

We may thus assume that $d \geqslant 2r$. Note that

$$c_p(G) = c_p(N) + c_p(G/N) \leqslant \frac{\varepsilon_{q^r}(d/r) - 1}{p-1} + \frac{r-1}{p-1} = \frac{\varepsilon_{q^r}(d/r) + r - 2}{p-1}.$$

It thus suffices to show

$$\varepsilon_{q^r}(d/r) + r - 1 \leqslant \varepsilon_q d. \tag{2}$$

Suppose now that $\varepsilon_{q^r} \leqslant \varepsilon_q$. In this case, it suffices to show $\varepsilon_q d/r + r - 1 \leqslant \varepsilon_q d$ which is equivalent to $r - 1 \leqslant \varepsilon_q d(r-1)/r$, and hence equivalent to $r \leqslant \varepsilon_q d$. Since $\varepsilon_q \geqslant 1$, the latter holds and so we may thus assume that $\varepsilon_q < \varepsilon_{q^r}$. From the definition of $\varepsilon$, it follows that $\varepsilon_q = 1$ and $\varepsilon_{q^r} = 4/3$. Therefore, (2) becomes

$$\frac{4d}{3r} + r - 1 \leqslant d.$$

This is equivalent to $r^2 - r \leqslant d(r - 4/3)$ which holds since $d \geqslant 2r$ and $r \geqslant 2$.

**4.4.** $G \in \mathcal{C}_4$ **or** $\mathcal{C}_7$**.** In this case, $G$ preserves a non-trivial decomposition of $V$ as a tensor product, say $V = V_1 \otimes \cdots \otimes V_r$ where $r \geqslant 2$. If $G \in \mathcal{C}_4$, then $r = 2$ and $G$ fixes $V_1$ and $V_2$, otherwise $G$ permutes the factors $V_1, \ldots, V_r$. Note that $d = \prod_{i=1}^{r} d_i$, where $d_i = \dim(V_i) \geqslant 2$. Let $N$ be the kernel of the action of $G$ on the set $\{V_1, \ldots, V_r\}$ and let $N_i$ denote the restriction of $N$ to $V_i$. By definition, the projection maps $N \to N_i$ are surjective for all $i$ and, moreover, $N \leqslant N_1 \circ \cdots \circ N_r$. It follows by Lemma 3(e) that $c_p(N) \leqslant \sum_{i=1}^{r} c_p(N_i)$. By Clifford's Theorem, a subnormal subgroup of a completely reducible group is completely reducible. Since $N_i \trianglelefteqslant N \trianglelefteqslant G$, $N_i$ is completely reducible on $V$ and on $V_i$. Hence, by the inductive hypothesis, we have $c_p(N_i) \leqslant (\varepsilon_q d_i - 1)/(p-1)$. On the other hand, $G/N$ is isomorphic to a subgroup of $\mathrm{Sym}(r)$ hence $c_p(G/N) \leqslant (r-1)/(p-1)$ by Lemma 3(c). By Lemma 3(a), we have

$$\begin{aligned} c_p(G) = c_p(G/N) + c_p(N) \leqslant c_p(G/N) + \sum_{i=1}^{r} c_p(N_i) &\leqslant \frac{r-1}{p-1} + \sum_{i=1}^{r} \frac{\varepsilon_q d_i - 1}{p-1} \\ = \frac{\varepsilon_q(\sum_{i=1}^{r} d_i) - 1}{p-1} \leqslant \frac{\varepsilon_q(\prod_{i=1}^{r} d_i) - 1}{p-1} &= \frac{\varepsilon_q d - 1}{p-1}, \end{aligned}$$

where the last inequality follows from the fact that $d_i \geqslant 2$ for all $i$. This completes the proof of this case.

**4.5.** $G \in \mathcal{C}_5$**.** In this case, $G \leqslant \mathrm{GL}(d, q_0)\mathsf{Z}(\mathrm{GL}(d,q))$ where $q = q_0^r$ for some divisor $r$ of $f$ with $r \geqslant 2$. Let $G_0 = G \cap \mathrm{GL}(d, q_0)$. Note that $c_p(\mathsf{Z}(\mathrm{GL}(d,q))) = 0$ hence $c_p(G) = c_p(G_0)$. Since $q_0 = p^{f/r}$ and $(d, f/r) < (d, f)$ in our lexicographic ordering, the inductive hypothesis yields $c_p(G_0) \leqslant (\varepsilon_{q_0} d - 1)/(p-1)$. Since $q$ is a power of $q_0$, it follows from the definition of $\varepsilon$ that $\varepsilon_{q_0} \leqslant \varepsilon_q$ and the result follows.

**4.6.** $G \in \mathcal{C}_6$**.** In this case, $d = r^m$ for some prime $r$ with $r \mid (q-1)$, and $G$ normalises an absolutely irreducible $r$-subgroup $R$ where $R/\mathsf{Z}(R)$ is elementary abelian of rank $2m$. By [**12**, Proposition 4.6.5], the normaliser of $R$ in $\mathrm{SL}(d,q)$ is

$$\mathsf{Z}(\mathrm{SL}(d,q)) \circ (R \cdot \mathrm{Sp}(2m, r)).$$

Since $r \mid (q-1)$, we have $r \neq p$ and thus $c_p(\mathsf{Z}(\mathrm{SL}(d,q))) = c_p(R) = 0$. It follows by Lemma 3 that

$$c_p(G) \leqslant \log_p |\mathrm{Sp}(2m,r)|_p = \log_p \prod_{i=1}^{m} (r^{2i} - 1)_p.$$

It thus suffices to show that

$$\log_p \prod_{i=1}^{m} (r^{2i} - 1)_p \leqslant \frac{\varepsilon_q r^m - 1}{p-1}. \tag{3}$$

Let $\Delta = \prod_{i=1}^{m} (r^{2i} - 1)$. Suppose first that $p = 2$ and thus $r \geqslant 3$. Note that

$$\log_2 \Delta_2 = \log_2 \prod_{i=1}^{m} (r^{2i} - 1)_2 < \log_2 \prod_{i=1}^{m} r^{2i} = (m^2 + m) \log_2 r.$$

If $(m^2 + m)\log_2 r \leqslant r^m - 1$, then, clearly, (3) holds. We may thus assume that $(m^2 + m)\log_2 r > r^m - 1$ and it is not hard to see that this implies that $(m, r)$ is one of $(1,3)$, $(1,5)$ or $(2,3)$. If $(m,r) = (1,5)$, then $\log_2 \Delta_2 = 3$ and (3) follows by noting that $\varepsilon_q \geqslant 1$. Finally, if $r = 3$ then $q$ must be an even power of 2 and thus $\varepsilon_q = 4/3$ and again (3) can be verified directly for $m \in \{1,2\}$.

From now on, we assume that $p \geqslant 3$. Let $\ell$ be the order of $r^2$ modulo $p$, that is, the smallest integer $\ell \geqslant 1$ for which $(r^2)^\ell \equiv 1 \pmod p$. The key observation which follows from [**1**, Lemma 2.2(i)] is that

$$(r^{2i}-1)_p = \begin{cases} 1 & \text{if } \ell \nmid i, \\ (r^{2\ell}-1)_p \left(\frac{i}{\ell}\right)_p & \text{if } \ell \mid i. \end{cases}$$

Let $(r^{2\ell}-1)_p = p^e$ and note that $e \geqslant 1$. Hence

$$\Delta_p = \prod_{i=1}^{m}(r^{2i}-1)_p = \prod_{j=1}^{\lfloor m/\ell \rfloor}(r^{2j\ell}-1)_p = \prod_{j=1}^{\lfloor m/\ell \rfloor}(r^{2\ell}-1)_p j_p = p^{\lfloor m/\ell \rfloor e}\left(\left\lfloor \frac{m}{\ell}\right\rfloor !\right)_p.$$

Thus, by Lemma 2, $\log_p \Delta_p \leqslant \lfloor m/\ell \rfloor e + (\lfloor m/\ell \rfloor - 1)/(p-1)$. To prove (3), it thus suffices to prove

$$\lfloor m/\ell \rfloor e + \frac{\lfloor m/\ell \rfloor - 1}{p-1} \leqslant \frac{\varepsilon_q r^m - 1}{p-1}.$$

For this, it is sufficient to show that

$$\lfloor m/\ell \rfloor ep \leqslant \varepsilon_q r^m. \tag{4}$$

Suppose first that $p = 2^n + 1$ is a Fermat prime and $r = 2$. In this case $\ell = n$, $e = 1$ and $\varepsilon_q = \frac{p}{p-1}$. Hence (4) becomes $\lfloor m/n \rfloor \leqslant 2^{m-n}$. Writing $\alpha = m/n$, this inequality becomes $\lfloor \alpha \rfloor \leqslant 2^{n(\alpha-1)}$, which holds for all values of $\alpha$ since $n \geqslant 1$.

We now assume that $p \geqslant 3$ is not a Fermat prime or $r \geqslant 3$, and hence that $\varepsilon_q = 1$. Since $p^e \mid (r^{2\ell}-1)$, we see that $p^e \mid (r^\ell \pm 1)$ and hence $p^e \leqslant r^\ell + 1$. Suppose that equality holds. Since $p \geqslant 3$, we have $r = 2$ hence $p^e - 1$ is a power of 2 and thus so is $p-1$. In other words, $p$ is a Fermat prime, contradicting our assumption. We may thus assume that $p^e \leqslant r^\ell$ and hence $e \leqslant \ell \log_p r$. Using (4), it suffices to prove $mp \log_p r \leqslant r^m$.

We first consider the subcase when $p \leqslant r^{m/2}$. Under this hypothesis, it suffices to prove $m \log_p r \leqslant r^{m/2}$ and, since $p \geqslant 3$, even $m \log_3 r \leqslant r^{m/2}$ is sufficient. It is not hard to show that this always holds.

We now assume that $p \geqslant r^{m/2} + 1$ and hence $\ell \geqslant m/2$. If $\ell = m/2$, then $p = r^{m/2} + 1$ and hence $r = 2$ and $p$ is a Fermat prime, contrary to our hypothesis. Thus $\ell > m/2$. If $\ell > m$, then (4) clearly holds. We may thus assume that $m/2 < \ell \leqslant m$ and hence $\Delta_p = (r^{2\ell}-1)_p$.

Suppose that $p^2 \mid (r^{2\ell}-1)$. Since $p \geqslant 3$, this implies that $p^2 \mid r^\ell \pm 1$ and thus $p^2 \leqslant r^\ell + 1$. If $p^2 = r^\ell + 1$, then $r = 2$ and $p = 3$. We may thus assume that $p^2 \leqslant r^\ell$ and hence $p \leqslant r^{\ell/2} \leqslant r^{m/2}$, contrary to our hypothesis. Therefore $p^2 \nmid (r^{2\ell}-1)$, and it follows that $\Delta_p = p$. In particular, (3) holds since $p \leqslant r^m$. This concludes the proof of this case.

**4.7.** $G \in \mathcal{C}_8$**.** In this case $G$ has a normal subgroup $N$ such that $N = \Omega^\varepsilon(d,q)$ for $d$ even or $dq$ odd, $\mathrm{Sp}(d,q)'$ for $d$ even, or $\mathrm{SU}(d,\sqrt{q})$ for $q$ a square. Moreover, $G$ is contained in $\mathrm{GO}^\varepsilon(d,q)$, $\mathrm{GSp}(d,q)$ or $\mathrm{GU}(d,\sqrt{q})Z$, where $\mathrm{GO}^\varepsilon(d,q)$ and $\mathrm{GSp}(d,q)$ denote the groups of all similarities of the quadratic or alternating form respectively, while $\mathrm{GU}(d, \sqrt{q})$ denotes the group of all isometries of the hermitian form. By excluding previous cases, we may also assume that $N/(N \cap Z)$ is nonabelian and simple [**15**, §VI.1–2]. Thus $c_p(N) = 0$ and hence $c_p(G) = c_p(G/N)$.

Now, $|\mathrm{GU}(d, \sqrt{q})Z : \mathrm{SU}(d, \sqrt{q})| = q - 1$ which is coprime to $p$ hence $c_p(G) = 0$ when $N = \mathrm{SU}(d, \sqrt{q})$.

Similarly, $|\mathrm{GSp}(d, q) : \mathrm{Sp}(d, q)| = q-1$, while $\mathrm{Sp}(d, q)' = \mathrm{Sp}(d, q)$ unless $(d, q) = (4, 2)$. Thus, if $N = \mathrm{Sp}(d, q)'$, then $c_p(G) = 0$ unless $(d, q) = (4, 2)$, in which case $c_p(G) = 1$. In both cases, (1) holds.

Finally, $|\mathrm{GO}^{\varepsilon}(d, q) : \Omega^{\varepsilon}(d, q)| = 2(q - 1)\gcd(2, d, q - 1)$ which is coprime to $p$ unless $p = 2$. Thus, if $N = \Omega^{\varepsilon}(d, q)$, then $c_p(G) = 0$ unless $p = 2$, in which case $c_p(G) = 1$. Again, (1) holds in both cases.

**4.8.** $G \in \mathcal{C}_9$**.** In this case, $G$ has a normal series $G \trianglerighteq N \triangleright Z \trianglerighteq 1$ where $G/Z$ is almost simple with socle $N/Z$ and, moreover, $N$ is absolutely irreducible. Let $T = N/Z$. Note that $c_p(Z) = c_p(T) = 0$ and thus $c_p(G) = c_p(G/N)$. Note also that $G/N$ is isomorphic to a subgroup of $\mathrm{Out}(T)$. It follows by Lemma 3 that $c_p(G/N) \leqslant \log_p |\mathrm{Out}(T)|_p$.

If $|\mathrm{Out}(T)|_p \leqslant 2$, then $c_p(G) = c_p(G/N) \leqslant c_p(\mathrm{Out}(T)) \leqslant 1$ with equality if and only if $p = 2$. Certainly (1) is satisfied if $p > 2$, and it is satisfied when $p = 2$ provided $1 \leqslant \varepsilon_2 d - 1$. This is true as $d \geqslant 2$ and $\varepsilon_2 = 4/3$. We may thus assume that $|\mathrm{Out}(T)|_p \geqslant 3$. This already rules out the case when $T$ is a sporadic group or an alternating group $\mathrm{Alt}(n)$, with $n \neq 6$. In view of the exceptional isomorphism $\mathrm{Alt}(6) \cong \mathrm{PSL}(2, 9)$, we will therefore assume that $T$ is a nonabelian simple group of Lie type. We rule out the Tits group ${}^2\mathrm{F}_4(2)'$ as we view it as a sporadic group.

Suppose that $T$ is defined over a field $F'$ of characteristic $p'$ and order $(p')^{f'}$. Let $q' = |F'| = (p')^{f'}$ if $T$ is an untwisted group of Lie type, and $(q')^k = |F'| = (p')^{f'}$ if $T$ is twisted with respect to a graph symmetry of order $k$.

It is well known that $|\mathrm{Out}(T)| = \delta f' \gamma$ where $\delta$ and $\gamma$ are the number of "diagonal" and "graph" outer automorphisms, respectively (see [**5**, p. (xv)] and [**5**, p.(xvi) Table 5]). It follows that $c_p(G) \leqslant \log_p(\delta f' \gamma)_p$. We now split into two cases, according to whether or not $p = p'$.

4.8.1. $p = p'$. By [**5**, Table 5], $\delta$ is coprime to $p$ and thus $\delta_p = 1$. We first suppose that $p \leqslant 3$ and $\gamma_p = 1$. Recall that the field automorphisms yield a cyclic subgroup of $\mathrm{Out}(T)$ of order $f'$, while a Sylow $p$-subgroup of $\mathrm{GL}(d, p^f)$ has exponent $p^{\lceil \log_p d \rceil}$(see [**11**, §16.5], for example). It follows that $\log_p f'_p \leqslant \lceil \log_p d \rceil$ hence

$$c_p(G) \leqslant \log_p f'_p \leqslant \lceil \log_p d \rceil. \tag{5}$$

When $p = 2$, we have $\varepsilon_q \geqslant 1$ and $\lceil \log_2 d \rceil \leqslant d - 1$ always holds. When $p = 3$, we have $\varepsilon_q = 3/2$ and $\lceil \log_3 d \rceil \leqslant (\varepsilon_q d - 1)/2$ always holds. Thus (1) is true in this case.

We may now assume that either $p \geqslant 5$ or $\gamma_p \neq 1$. In particular, $T$ is neither a Suzuki group nor a Ree group (these have $p \leqslant 3$ and $\gamma = 1$). By [**5**, p. (xv)], $\mathrm{Out}(T)$ has the form $(O_D \rtimes O_F) \rtimes O_G$ where $O_D, O_F, O_G$ denote groups of diagonal, field, and graph outer automorphisms, respectively. Conjugation induces on $N/Z \cong T$ a homomorphism $\overline{\phantom{x}}\colon G \to \mathrm{Out}(T)$, with kernel containing $N$. We must bound $c_p(G) = c_p(\overline{G})$. Since $\delta_p = 1$, $O_D$ is a $p'$-group. Write $|\overline{G} \cap (O_D \rtimes O_F)|_p = p^\ell$. Then $c_p(G) \leqslant \ell + \log_p \gamma_p$ where $\log \gamma_p \leqslant 1$ for $p \leqslant 3$, and $\log \gamma_p = 0$ otherwise. We digress from bounding $c_p(G)$ (for three paragraphs) to show that $G$ contains an element of order $p^{\ell+1}$. This is trivially true if $\ell = 0$ so assume that $\ell \geqslant 1$.

Choose $H \leqslant G$ such that $N \leqslant H$, $\overline{H} \leqslant O_D \rtimes O_F$, and $|H : N| = p^\ell$. Since $O_F$ is cyclic and $|O_D|_p = \delta_p = 1$, Sylow's Theorem implies that $H/Z$ is unique up to isomorphism. Thus we may assume that $H = \langle N, \varphi \rangle$, where the automorphism $\widetilde{\varphi} \in \mathrm{Aut}(T)$ induced by $\varphi$ on $T = N/Z$ is a standard field automorphism of order $p^\ell$.

Suppose first that $T$ is an untwisted group of Lie type. Since $\widetilde{\varphi}$ is a standard field automorphism there is a root system $\Phi$ for $T$ such that $T$ is generated by the set of all root elements $x_r(\lambda)$ for $r \in \Phi$ and $\lambda \in F'$, and there is an automorphism $\psi$ of the field $F'$ of order $p^\ell$ such that $\widetilde{\varphi} \in \mathrm{Aut}(T)$ maps each $x_r(\lambda)$ to $x_r(\lambda^\psi)$ (see [**3**]). Let $(F')^\psi$ be the fixed subfield of $\psi$ and let $\mathrm{Tr}\colon F' \to (F')^\psi$ be the (surjective) trace map $\mathrm{Tr}(\lambda) = \sum_{i=0}^{p^\ell - 1} \lambda^{\psi^i}$. Calculating in $\mathrm{Aut}(T)$, with $T$ identified with $\mathrm{Inn}(T)$, we have

$$\begin{aligned}
(\widetilde{\varphi} x_r(\lambda))^{p^\ell} &= (x_r(\lambda))^{\psi^{p^\ell - 1}} (x_r(\lambda))^{\psi^{p^\ell - 2}} \cdots (x_r(\lambda))^{\psi} x_r(\lambda) \\
&= x_r(\lambda^{\psi^{p^\ell - 1}}) x_r(\lambda^{\psi^{p^\ell - 2}}) \cdots x_r(\lambda^\psi) x_r(\lambda) \\
&= x_r(\lambda^{\psi^{p^\ell - 1}} + \lambda^{\psi^{p^\ell - 2}} + \cdots + \lambda^\psi + \lambda) = x_r(\mathrm{Tr}(\lambda)).
\end{aligned}$$

Choosing $\lambda \in F'$ such that $\mathrm{Tr}(\lambda) \neq 0$ yields an element $\widetilde{\varphi} x_r(\lambda)$ of order $p^{\ell+1}$. Thus $H$, and hence $G$, has an element of order $p^{\ell+1}$, as desired.

Suppose now that $T$ is a twisted group of Lie type arising from an untwisted group $L$ with root system $\Phi$. Since $T$ is twisted, $\gamma = 1$ hence $p \geqslant 5$ and all roots in a fundamental system for $\Phi$ have the same length. Moreover, there is a graph automorphism $\rho$ of order $k$ arising from a symmetry of the Dynkin diagram of $L$ and a field automorphism $\sigma$ of order $k$ such that $T$ is the centraliser in $L$ of the automorphism $\rho\sigma$. By [**3**, Proposition 13.6.3], if $k = 2$ and $T \neq \mathrm{PSU}(3, q')$, then there is a root $r$ with image $\overline{r}$ under the symmetry of the Dynkin diagram such that, for all $\lambda \in F'$, the element $x_S(\lambda) := x_r(\lambda) x_{\overline{r}}(\lambda^\sigma)$ lies in $T$. Similarly, if $k = 3$, then there is a root $r$ with images $\overline{r}$ and $\overline{\overline{r}}$ such that, for all $\lambda \in F'$, the element $x_S(\lambda) = x_r(\lambda) x_{\overline{r}}(\lambda^\sigma) x_{\overline{\overline{r}}}(\lambda^{\sigma^2})$ lies in $T$. In both cases, a calculation similar to the earlier one shows that $(\widetilde{\varphi} x_S(\lambda))^{p^\ell} = x_S(\mathrm{Tr}(\lambda))$ and hence, by choosing $\lambda$ appropriately, we ensure that $\widetilde{\varphi} x_S(\lambda)$ has order $p^{\ell+1}$. Finally, if $T = \mathrm{PSU}(3, q')$, then, for a simple root $r$, we have $r + \overline{r} \in \Phi$ and hence $T$ contains elements $x_{r+\overline{r}}(\lambda)$ for all $\lambda$ in the index 2 subfield of $F'$ fixed by the field automorphism of order 2. Since $p$ is odd, such a subfield contains elements with nonzero trace and we again find an element $\widetilde{\varphi} x_{r+\overline{r}}(\lambda)$ of order $p^{\ell+1}$.

We have shown that, in all cases, $G$ contains an element of order $p^{\ell+1}$. Recall that a Sylow $p$-subgroup of $\mathrm{GL}(d, p^f)$ has exponent $p^m$ where $p^{m-1} < d \leqslant p^m$. Thus $\ell + 1 \leqslant m$

and

$$\ell \leqslant m-1 \leqslant \frac{p^{m-1}-1}{p-1} < \frac{d-1}{p-1}.$$

In particular, (1) holds if $c_p(G)=\ell$. We may thus assume that $c_p(G)>\ell$ which implies that $\gamma_p \neq 1$ and $p \leqslant 3$. By [**5**, Table 5], $\gamma$ divides 6 hence $\log_p \gamma_p = 1$. It follows that

$$c_p(G) = \ell + 1 \leqslant m = \lceil \log_p d \rceil$$

but, as we saw earlier in the sentences following (5), this implies (1) when $p \leqslant 3$.

4.8.2. $p \neq p'$. In this case, we have an absolutely irreducible cross-characteristic representation $N \to \mathrm{GL}(d,q)$. This gives rise to a projective representation $T \to \mathrm{PGL}(d,q)$ and Landazuri and Seitz [**14**, Theorem] give lower bounds on $d$ with respect to $q'$. Furthermore, possibilities for quasisimple groups $N$ and small dimensions $d$ are listed in [**8**,**9**].

We first assume that $d \leqslant 5$. Suppose that $T \cong \mathrm{PSL}(2,q')$. By [**8**, Table 2], we have $d \in \{q', q' \pm 1, (q' \pm 1)/2\}$. Since $d \leqslant 5$, this implies that $q' \leqslant 11$ and $q' \neq 8$ and, as $|F'| = q' = (p')^{f'}$, we see that $f' \leqslant 2$. Since $|\mathrm{Out}(T)|$ divides $2f'$ and $|\mathrm{Out}(T)|_p \geqslant 3$, it follows that $p = f' = 2$. As $p' \neq p$, this implies that $q' = 9$ and thus $d \geqslant (9-1)/2 = 4$ hence (1) holds. Suppose now that $T$ is a group of Lie type other than $\mathrm{PSL}(2,q')$. By [**9**, Table 2], the possible choices for $T$ with $d \leqslant 5$ are $\mathrm{PSL}(3,4)$ and $\mathrm{PSU}(4,2)$ with $|\mathrm{Out}(T)|$ being 12 and 2, respectively. As $p \neq p'$ and $|\mathrm{Out}(T)|_p \geqslant 3$, we have $|\mathrm{Out}(T)|_p = p = 3$ and $c_p(G) \leqslant 1$ hence (1) holds. We henceforth assume that $d \geqslant 6$.

Suppose first that $\delta \geqslant 5$. This implies that $T = \mathrm{PSL}(n,q')$ or $\mathrm{PSU}(n,q')$ and $n \geqslant 4$. It follows by [**14**, Theorem] that

$$d \geqslant \frac{q'((q')^4-1)}{q'+1} = q'(q'-1)((q')^2+1). \tag{6}$$

(Note that the exceptions for $\mathrm{PSL}(n,q')$ and $\mathrm{PSU}(n,q')$ in [**14**, Theorem] do not arise because $n \geqslant 4$.) Since $T = \mathrm{PSL}(n,q')$ or $\mathrm{PSU}(n,q')$, it follows by [**5**, Table 5] that $\delta = \gcd(n+1, q' \pm 1) \leqslant q'+1$ and thus

$$d \geqslant q'(q'-1)((q')^2+1) \geqslant (\delta-1)(\delta-2)\delta \geqslant 12\delta. \tag{7}$$

Similarly, (6) implies $d \geqslant (q')^3$. As $(p')^{f'} = (q')^k$ for some $k \leqslant 2$, we have

$$f' = k \log_{p'} q' \leqslant 2 \log_{p'} q' \leqslant 2 \log_2 q' \leqslant 2 \log_2 d^{1/3} \leqslant 2(d-1)/3. \tag{8}$$

Combining (7) and (8) gives $\delta + f' \leqslant d-1$ and thus

$$\begin{aligned} c_p(G) \leqslant \log_p \gamma_p + \log_p \delta_p + \log_p f'_p &\leqslant \frac{(p-1)\log_p \gamma_p}{p-1} + \frac{\delta-1}{p-1} + \frac{f'-1}{p-1} \\ &\leqslant \frac{(p-1)\log_p \gamma_p + d - 3}{p-1}. \end{aligned}$$

If $p \geqslant 5$, then $\gamma_p = 1$ and thus (1) holds. If $p \leqslant 3$, then $\log_p \gamma_p \leqslant 1$ and $p-1 \leqslant 2$ and again (1) holds.

We may thus assume that $\delta \leqslant 4$. We will show that $f' \leqslant \log_{p'}(d+1)^2$. First, suppose that $k = 1$. It follows by [**14**, Theorem] that $d \geqslant (q'-1)/2$. (As $d \geqslant 6$, we can assume that $q' > 13$, which rules out the exceptional cases in [**14**, Theorem].) This implies that

$$f' = \log_{p'} q' \leqslant \log_{p'}(2d+1) < \log_{p'}(d+1)^2.$$

Next, if $k = 2$, then [**14**, Theorem] implies that $d \geqslant q' - 1$. This implies that

$$f' = 2\log_{p'} q' \leqslant 2\log_{p'}(d+1) = \log_{p'}(d+1)^2.$$

Finally, if $k = 3$, then $d \geqslant (q')^3$ by [**14**, Theorem] and

$$f' = 3\log_{p'} q' \leqslant 3\log_{p'} d^{1/3} < \log_{p'}(d+1)^2.$$

This completes our proof that $f' \leqslant \log_{p'}(d+1)^2$.

Suppose first that $p \geqslant 5$. By [**5**, Table 5], we have $\gamma_p = 1$ and $\delta \leqslant 4$ implies that $\delta_p = 1$. As $d \geqslant 6$, we have $f' \leqslant \log_{p'}(d+1)^2 \leqslant \log_2(d+1)^2 \leqslant d$. It follows that

$$c_p(G) \leqslant \log_p f'_p \leqslant \frac{f'-1}{p-1} \leqslant \frac{d-1}{p-1} \leqslant \frac{\varepsilon_q d - 1}{p-1},$$

as desired. For $p = 3$, we have

$$c_p(G) \leqslant \log_3 \gamma_3 + \log_3 \delta_3 + \log_3 f' \leqslant 1 + 1 + \log_3 \log_2(d+1)^2.$$

It is not hard to see that $2 + \log_3 \log_2(d+1)^2 \leqslant \frac{(3/2)d-1}{2} = \frac{\varepsilon_3 d - 1}{3-1}$ when $d \geqslant 6$. This completes the case $p = 3$. Finally, suppose that $p = 2$ and thus $p' \geqslant 3$. It follows that

$$c_p(G) \leqslant \log_2 \gamma_2 + \log_2 \delta_2 + \log_2 f' \leqslant 1 + 2 + \log_2 \log_3(d+1)^2.$$

Again, it is not hard to see that $3 + \log_2 \log_3(d+1)^2 \leqslant d - 1$ when $d \geqslant 6$, establishing the case $p = 2$. This completes the induction and thus the proof. $\square$

Corollary 9. *Let $V = (\mathbb{F}_q)^d$ be the natural module for $G \leqslant \mathrm{GL}(d,q)$ where $q = p^f$. If $V$ has a composition series with $r$ simple factors, and $\varepsilon_q$ is defined by (1), then*

$$c_p(G/O_p(G)) \leqslant \frac{\varepsilon_q d - r}{p-1}.$$

Proof. Fix a composition series $V > V_1 > \cdots > V_r = \{0\}$ for $V$ and consider the homomorphism $\phi\colon G \to \prod_{i=1}^r \mathrm{GL}(W_i)$ where $W_i := V_{i-1}/V_i$ for $1 \leqslant i \leqslant r$. Let $G_i$ be the subgroup of $\mathrm{GL}(W_i)$ induced by $G$. Then $G_i$ acts irreducibly on $W_i$. Hence the largest normal $p$-subgroup $O_p(G_i)$ of $G_i$ is trivial. (Note that $[O_p(G_i), W_i]$ is $G_i$-invariant and $[O_p(G_i), W_i] < W_i$, so $[O_p(G_i), W_i] = \{0\}$.) It follows that $\ker(\phi) = O_p(G)$.

We have $d = d_1 + \cdots + d_r$ where $d_i = \dim(W_i)$. Applying Theorem 1 gives

$$c_p(G/O_p(G)) = c_p(G/\ker(\phi)) = c_p(\mathrm{im}(\phi)) \leqslant \sum_{i=1}^r c_p(G_i) \leqslant \sum_{i=1}^r \frac{\varepsilon_q d_i - 1}{p-1} = \frac{\varepsilon_q d - r}{p-1}. \quad \square$$

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