Bounding the composition length of primitive permutation groups and completely reducible linear groups

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ABSTRACT. We obtain upper bounds on the composition length of a finite permutation group in terms of the degree and the number of orbits, and analogous bounds for primitive, quasiprimitive and semiprimitive groups. Similarly, we obtain upper bounds on the composition length of a finite completely reducible linear group in terms of some of its parameters. In almost all cases we show that the bounds are sharp, and describe the extremal examples.

MSC 2000 CLASSIFICATION: 20B15, 20H30, 20B05

1. Introduction

The composition length of a finite group is the length of any composition series of the group. It is sometimes viewed as a measure of its size or complexity. Often it is useful to have bounds in terms of parameters relevant to the way the group is represented, rather than the abstract group structure. In Subsection 1.1 we comment on the research questions which motivated our investigation, we describe how our results relate to other work, and mention some open questions.

We obtain upper bounds on the composition length of a finite permutation group in terms of the degree and the number of orbits (Theorem 1.2), and analogous bounds for primitive (Theorem 1.3), quasiprimitive and semiprimitive groups (Theorem 1.7). Similarly, we obtain upper bounds on the composition length of a finite completely reducible linear group in terms of some of its parameters (Theorem 1.4). We also show in almost all

Date: December 18, 2017.

cases that our bounds are sharp, and describe all extremal examples. For this purpose, we define the following concepts. A permutation group is *primitive* if it is transitive and preserves no nontrivial partition of the set on which it acts; transitive permutation groups that preserve some nontrivial point partition are said to be *imprimitive*.

DEFINITION 1.1. Let S_4 denote the symmetric group of degree 4 in its natural action and let k be a non-negative integer.

- Let $T_k = \mathsf{S}_4 \wr \cdots \wr \mathsf{S}_4$, the iterated imprimitive wreath product of k copies of S_4.
- Let $P_k = \mathsf{S}_4 \wr T_k$, in its primitive wreath product action.
- Let $L_k = \mathrm{GL}(2,2) \wr T_k$, viewed as an imprimitive linear subgroup of $\mathrm{GL}(2^{2k+1}, 2)$.

Note that T_k is a transitive group of degree 4^k; in particular $T_0 = 1$ has degree 1. Therefore P_k is a primitive group of degree 4^{4^k} which is abstractly isomorphic to T_{k+1}.

For a finite group G, let $c(G)$ denote its composition length.

THEOREM 1.2. *If G is a permutation group of degree n with r orbits, then*

$$c(G) \leqslant \frac{4}{3}(n-r).$$

Moreover, equality holds if and only if there exist nonnegative integers $k_1, \ldots, k_r$ such that the orbits of G have sizes $4^{k_1}, \ldots, 4^{k_r}$ and G is permutationally isomorphic to $T_{k_1} \times \cdots \times T_{k_r}$ in its natural action.

THEOREM 1.3. *If G is a primitive permutation group of degree n, then*

$$c(G) \leqslant \frac{8}{3}\log_2 n - \frac{4}{3}.$$

Moreover, equality holds if and only if $n = 4^{4^k}$ for some $k \geqslant 0$ and G is permutationally isomorphic to P_k.

These theorems depend on the finite simple group classification since the proof of Theorem 1.3 uses Theorem 1.2, and the proof of Theorem 1.2 uses an order bound for primitive groups from [**20**] which depends on the classification.

A group H of linear transformations of a vector space V is *completely reducible* if there is a direct decomposition $V = V_1 \oplus \cdots \oplus V_r$, with $r \geqslant 1$, such that each V_i is H-invariant and the restriction $H|_{V_i}$ is irreducible. The V_i are the *irreducible constituents* of H.

THEOREM 1.4. *If H is a completely reducible subgroup of $\mathrm{GL}(d,p^f)$ with r irreducible constituents $V_1,\dots,V_r$, then*

$$c(H) \leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - r\left(\log_2 f + \frac{4}{3}\right). \tag{1}$$

Moreover, equality holds if and only if one of the following occurs:

(a) $p^f = 2$ *and there exist positive integers* $k_1,\dots,k_r$ *such that* $\dim(V_i) = 2^{2k_i+1}$ *and* H *is linearly isomorphic to* $L_{k_1}\times\cdots\times L_{k_r}$, *or*
(b) $p^f = 2^2$, $d = r$ *and* H *is linearly isomorphic to* $\mathrm{GL}(1,4)^d \cong (\mathsf{C}_3)^d$.

1.1. Context, discussion, and more results. For a finite group G of order m, $c(G) \leqslant \log_2(m)$, with equality if and only if G is a 2-group (with each composition factor cyclic of order 2). Similarly each of the upper bounds in [**3**, **4**, **7**, **20**, **22**, **25**] on the orders of finite primitive permutation groups G of degree n yields an upper bound for $c(G)$ as a function of n. The best of these order bounds [**7**, Theorem 6.1(S)], due to Cameron in 1981, depends on the finite simple group classification: namely a primitive group G of degree n is of affine type, or is in a well understood family of primitive groups of product action type, or satisfies $|G| \leqslant n^{c\log_2\log_2 n}$ for a "computable constant c".

In 1993, Pyber [**24**, Theorem 2.10] states that, for a primitive permutation group G of degree n, $c(G) \leqslant (2+c)\log_2 n$ with c the constant in Cameron's result. A proof of this result appeared recently in [**14**, Corollary 6.7].[1] It has been used in several investigations. For example, it is used for the irreducible case of [**18**, Theorem C], which bounds the composition length of finite completely reducible linear groups, and it is used in [**9**, p. 305] to bound the invariable generation number for permutation groups. For the application in [**9**] the result [**18**, Theorem C] is applied with the constant $c = 2.25$. The paper [**14**] derives many bounds for permutation groups and linear groups G focussing on bounds for $|\mathrm{Out}(G)|$. In particular [**14**, Corollary 6.7] yields the bound $c(G) \leqslant (2+c)\log_2 n$ with the constant $c = \log_9(48\cdot 24^{1/3}) = 2.24\cdots$, that is to say, $c(G) \leqslant c'\log_2 n$ with $c' = 4.24\cdots$.

Our investigations began before [**14**] was published. Because we had been unable to find a proof of Pyber's result in the literature, and because of its diverse applications, we decided to seek the best value for a constant c' such that $c(G) \leqslant c'\log_2 n$ whenever G is a primitive permutation group of degree n. Further, we wondered if we could find sharp upper bounds and classify all groups attaining them. Our Theorem 1.3 achieves this, and in particular shows that the best value for such a constant c' is $8/3 = 2.66\cdots$.

[1] The statement in [**24**, Theorem 2.10] refers to a paper "in preparation" (reference [Py5] in [**24**]).

Whereas all the primitive permutation groups G achieving the bounds of Theorem 1.3 are of affine type, the primitive groups of degree n covered by Cameron's "order upper bound" $n^{c\log_2\log_2 n}$, are in particular not of affine type. The following companion result to Theorem 1.3 gives a sharp upper bound on the composition length of *non-affine primitive groups*, by which we mean primitive permutation groups with no nontrivial abelian normal subgroups.

THEOREM 1.5. *If G is a non-affine primitive permutation group of degree n, then*

$$c(G) \leqslant c_{\rm na}\log_2 n - \frac{4}{3}, \quad \text{where } c_{\rm na} = \frac{10}{3\log_2 5} = 1.43\cdots$$

with equality if and only if $n = 5^{4^k}$ and $G = \mathsf{S}_5 \wr T_k$ in product action, for some $k \geqslant 0$.

We note the striking difference between the logarithmic upper bounds on $c(G)$ for primitive groups G in Theorems 1.3 and 1.5, and the linear bound for general permutation groups in Theorem 1.2.

PROBLEM 1.6. *Which other infinite families of permutation groups have composition lengths bounded above by a logarithmic function of the degree?*

Our final main result gives examples of two such families. A permutation group is *quasiprimitive* if each of its nontrivial normal subgroups is transitive. It is *semiprimitive* if each of its normal subgroups is either semiregular or transitive. (A permutation group is *semiregular* if the only element fixing a point is the identity.)

THEOREM 1.7. *Let G be a permutation group of degree n.*

(a) *If G is quasiprimitive but not primitive, then*

$$c(G) \leqslant c_{\rm na}(\log_2 n - 1) - \frac{4}{3} = c_{\rm na}\log_2 n - 2.76\cdots \quad \text{where } c_{\rm na} = \frac{10}{3\log_2 5} = 1.43\cdots.$$

(b) *If G is semiprimitive but not quasiprimitive, then $c(G) \leqslant \frac{8}{3}\log_2 n - 3$.*

We give infinitely many examples to show that the bound in Theorem 1.7(b) is best possible (see Example 6.2). For a semiprimitive group G, a normal subgroup of G which is minimal subject to being transitive, is called a *plinth*. If G is a semiprimitive group which achieves the $\frac{8}{3}\log_2 n - 3$ bound in Theorem 1.7(b), then n is a power of 2 and each plinth of G is a 2-group (Remark 6.1). Unfortunately the bound for quasiprimitive groups is not sharp (Remark 6.3), and we do not even know the best constant c such that $c(G) \leqslant c \log_2 n$ for a quasiprimitive group G of degree n which is not primitive. By Theorem 1.7, $c \leqslant c_{\mathrm{na}} = 1.43\cdots$, and we give examples in Section 6 which show that $c \geqslant \frac{31}{12\log_2 5 + 9\log_2 3} = 0.73\cdots$.

Problem 1.8. (a) *Find a sharp upper bound on the composition length in terms of the degree, for quasiprimitive permutation groups which are not primitive.*

(b) *Determine all semiprimitive groups which achieve the bound in Theorem 1.7(a).*

(c) *For $G \leqslant \mathsf{S}_n$, with G semiprimitive and not quasiprimitive and with an insoluble plinth, find a sharp upper bound for $c(G)$ as a function of n.*

The proof of Theorem 1.3 proceeds by considering various types of finite primitive permutation groups. In particular a primitive subgroup $G \leqslant \mathrm{Sym}(\Omega) = \mathsf{S}_n$ may leave invariant a cartesian decomposition $\Omega = \Delta^r$ for some smaller set Δ and integer $r \geqslant 2$. In this case $n = m^r$ where $m = |\Delta|$, and the group G is permutationally isomorphic to a subgroup of the wreath product $\mathrm{Sym}(\Delta) \wr \mathsf{S}_r$ in product action. Moreover G must project to a transitive subgroup of S_r, and for $c(G)$ to be maximised we require the composition length of this transitive subgroup of S_r to be as large as possible. In other words, in order to prove Theorem 1.3 for these product action primitive groups we need the bound (and extreme examples) from Theorem 1.2 for transitive groups. We note that our result Theorem 1.2 extends early work by Fisher dating from 1974. Namely we improve [**10**, Lemma 2] by proving that permutation groups of the form $T_{k_1} \times \cdots \times T_{k_r}$ are the only examples, with r orbits, for which equality occurs in the upper bound in Theorem 1.2. (One reason for giving an independent proof is that there appears to be a small error in the proof of [**10**, Lemma 2]: the sentence beginning "If G is transitive and imprimitive" is incorrect.)

Another class of primitive groups which must be considered when proving Theorem 1.3 are those of affine type. These are groups of affine transformations of a finite vector space and have the form $N \rtimes G_0$, where N is the group of translations, and G_0 is an irreducible subgroup of linear transformations. Thus in order to prove Theorem 1.3 for affine primitive groups we need the bound (and extreme examples) from Theorem 1.4 for irreducible groups.

Our work on completely reducible groups also strengthens various results in the literature. As early as 1974, Fisher [**10**, **11**] obtained estimates for the polycyclic chief lengths of linear groups (over arbitrary fields). More recent work by Lucchini et al. [**18**, Theorem C], relying on the finite simple group classification, shows that, for a completely reducible subgroup $G \leqslant \mathrm{GL}(d, p^f)$ (with p prime), $c(G) \leqslant c_{\mathrm{cr}}(\log_2 p)dnf$ for some constant c_{cr}. Theorem 1.4 shows that the best possible constant c_{cr} is 8/3. The immediate motivation for our work was [12, Theorem 1] (on the number of composition factors C_p) which suggested that it might be possible to find sharp upper bounds on $c(G)$ for all finite completely reducible groups.

2. Preliminaries

We say that H is a *subdirect subgroup* of $G_1 \times \cdots \times G_r$ if H projects onto each direct factor. Given a group G_1 and a transitive permutation group G_2 of degree r, the *wreath product* $G_1 \wr G_2$ is $B \rtimes G_2$ where $B = B_1 \times \cdots \times B_r \cong G_1^r$, with G_2 acting naturally by conjugation on the B_i. We say that H is a *subwreath subgroup* of $G_1 \wr G_2$ if H projects onto G_2, and the normaliser in H of B_1 projects onto B_1.

Lemma 2.1. *Let G be a finite group.*

(a) *If $N \trianglelefteq G$, then $c(N) \leqslant c(G)$ with equality if and only if $N = G$.*
(b) *If H is a subdirect subgroup of $G_1 \times \cdots \times G_r$, then $c(H) \leqslant \sum_{i=1}^r c(G_i)$, with equality if and only if $H = G_1 \times \cdots \times G_r$.*
(c) *If G_2 is transitive permutation group of degree r and H is a subwreath subgroup of $G_1 \wr G_2$, then $c(H) \leqslant r \cdot c(G_1) + c(G_2)$, with equality if and only if $H = G_1 \wr G_2$.*

Proof. (a) Clearly $c(G) = c(N) + c(G/N)$ and $c(G/N) = 0$ if and only if $N = G$.

(b) Let $H_0 = H$ and, for $1 \leqslant i \leqslant r$, let K_i be the kernel of the projection map $\pi_i \colon H \to G_i$, and $H_i = H \cap K_1 \cap \cdots \cap K_i$. The normal series $H = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_r = 1$ has factor groups

$$\frac{H_{i-1}}{H_i} = \frac{H_{i-1}}{H_{i-1} \cap K_i} \cong \frac{H_{i-1}K_i}{K_i} \trianglelefteq \frac{H}{K_i} \cong G_i.$$

Therefore $c(H_{i-1}/H_i) \leqslant c(G_i)$ by part (a), and so

$$c(H) = \sum_{i=1}^r c(H_{i-1}/H_i) \leqslant \sum_{i=1}^r c(G_i) = c(G_1 \times \cdots \times G_r).$$

If equality holds, then for each i, $c(H_{i-1}/H_i) = c(G_i)$ which implies that $H_{i-1}/H_i \cong G_i$ by part (a). In particular, $|H_{i-1}/H_i| = |G_i|$ and so $|H| = \prod_{i=1}^r |H_{i-1}/H_i| = \prod_{i=1}^r |G_i| = |G|$ and thus $H = G$, as claimed.

(c) Write $G_1 \wr G_2 = B \rtimes G_2$ where $B = B_1 \times \cdots \times B_r \cong G_1^r$ and G_2 permutes the B_i transitively by conjugation. Let $K = H \cap B$ and let N be the normaliser of B_1 in $G_1 \wr G_2$.

Since H is a subwreath subgroup of $G_1 \wr G_2$, we have $H/K \cong G_2$ and $H \cap N$ projects onto B_1. In particular,

$$c(H) = c(K) + c(G/K) = c(K) + c(G_2).$$

For each i, let $\pi_i \colon B \to B_i$ be the natural projection map and let $K_i = \pi_i(K)$. Since $B \trianglelefteqslant N$, we see $K = H \cap B \trianglelefteqslant H \cap N$ and therefore $\pi_1(K) \trianglelefteqslant \pi_1(H \cap N)$, that is $K_1 \trianglelefteqslant B_1$. Since G_2 is transitive on $\{B_1, \dots, B_r\}$, we have that $K_i \trianglelefteqslant B_i$ for each i. Hence, part (a) implies $c(K_i) \leqslant c(B_i) = c(G_1)$ for each i.

However, K is a subdirect subgroup of $K_1 \times \cdots \times K_r$ by the definition of K_i. Therefore by part (b), $c(K) \leqslant \sum_{i=1}^{r} c(K_i) = r \cdot c(K_1) \leqslant r \cdot c(G_1)$. Thus $c(H) \leqslant r \cdot c(G_1) + c(G_2)$.

We see that equality occurs only if $c(K_i) = c(B_i)$, and hence $K_i = B_i$, for each i. Thus K is a subdirect subgroup of $B = B_1 \times \cdots \times B_r$, and $c(K) = r \cdot c(G_1) = \sum_{i=1}^{r} c(B_i)$. This implies that $K = B$ by part (b), and hence that $H = G_1 \wr G_2$, as desired. □

Remark 2.2. Intransitive permutation groups give rise to subdirect subgroups, and imprimitive permutation groups give rise to subwreath subgroups. △

We use the following order bounds, from [2, 16], on the outer automorphism group $\mathrm{Out}(T)$ of a nonabelian simple group T.

Lemma 2.3. *Let T be a finite nonabelian simple group, and suppose that T has a proper subgroup of index m. Then*

(a) *either $|\mathrm{Out}(T)| < m/2$, or $(T, m, |\mathrm{Out}(T)|) = (\mathsf{A}_6, 6, 4)$, and*
(b) $|\mathrm{Out}(T)| \leqslant \log_2 |T|$.

Proof. If $T = \mathsf{A}_6$ then $|\mathrm{Out}(T)| = 4$, and either $m = 6$ or $m \geqslant 10$. Thus part (a) holds for A_6. If $T \neq \mathsf{A}_6$, then by [2, Lemma 2.7(i)], $|\mathrm{Out}(T)| < m/2$, so part (a) is proved. Part (b) is proved in [16]. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let G be a permutation group of degree n with r orbits. The proof is by induction on n. It is easy to check that the result holds for $n = 1$.

Suppose first that G is intransitive, that is $r \geqslant 2$. Let $\Omega_1, \dots, \Omega_r$ be the G-orbits and let $n_i = |\Omega_i|$ for each i. Let G_i be the permutation group induced by G on Ω_i. By induction, $c(G_i) \leqslant \frac{4}{3}(n_i - 1)$. Since G is a subdirect subgroup of $G_1 \times \cdots \times G_r$, it follows from Lemma 2.1(b) and induction that

$$c(G) \leqslant \sum_{i=1}^{r} c(G_i) \leqslant \sum_{i=1}^{r} \frac{4}{3}(n_i - 1) = \frac{4}{3}(n - r),$$

with equality only if $G = G_1 \times \cdots \times G_r$ and $c(G_i) = \frac{4}{3}(n_i - 1)$ for each i. By induction, $G_i = T_{k_i}$ for some k_i satisfying $n_i = 4^{k_i}$ and thus $G = T_{k_1} \times \cdots \times T_{k_r}$, as desired.

We may thus assume that G is transitive. Suppose that G is imprimitive and preserves a block system $\mathcal{B} := \{B_1, \dots, B_s\}$, where $1 < s < n$. Let G_2 be the (transitive) permutation group induced by G on $\mathcal{B}$, and let G_1 be the (transitive) permutation group induced on B_1 by the setwise stabiliser in G of B_1. Then G is a subwreath subgroup of $G_1 \wr G_2$ and hence, by Lemma 2.1(c), $c(G) \leqslant s \cdot c(G_1) + c(G_2)$. Since G_1, G_2 are transitive permutation groups of degree n/s and s, respectively, it follows by induction that

$$c(G) \leqslant s \cdot c(G_1) + c(G_2) \leqslant \frac{4s}{3}\left(\frac{n}{s} - 1\right) + \frac{4}{3}(s-1) = \frac{4}{3}(n-1),$$

with equality only if $G = G_1 \wr G_2$, $c(G_1) = \frac{4}{3}(\frac{n}{s} - 1)$ and $c(G_2) = \frac{4}{3}(s-1)$. By induction, this implies $G_1 = T_{k_1}$ and $G_2 = T_{k_2}$ for some integers k_1 and k_2 and thus $G = T_{k_1} \wr T_{k_2} = T_{k_1+k_2}$.

Finally, we assume that G is primitive. We used a database of primitive groups of small degree (see [8]) to check that the bound is satisfied when $n \leqslant 24$ and equality holds only for $T_1 = \mathsf{S}_4$. We thus assume that $n \geqslant 25$. If G contains the alternating group of degree n, then $c(G) \leqslant 2$ and again the result holds. We may thus assume that this is not the case and, by [20, Corollary 1.4] we have $|G| \leqslant 2^{n-1}$. This implies that $c(G) \leqslant \log_2(2^{n-1}) = n - 1 < \frac{4}{3}(n-1)$. This completes the proof of Theorem 1.2. □

4. Proof of Theorem 1.4

Proof of Theorem 1.4. Let $H \leqslant \mathrm{GL}(d, p^f)$, such that H is completely reducible on $V = \mathbb{F}_{p^f}^d$. We fix the prime p and use induction on pairs (d, f) which are ordered lexicographically, where $(d_1, f_1) < (d_2, f_2)$ means $d_1 < d_2$, or $d_1 = d_2$ and $f_1 < f_2$. The case $d = 1$ below will serve as the base of our induction.

Case 0. $d = 1$. As $\mathrm{GL}(1, p^f) \cong \mathsf{C}_{p^f-1}$ is cyclic, we have $c(H) \leqslant c(\mathrm{GL}(1, p^f))$. Here $d = r = 1$ so it suffices to show that $c(\mathrm{GL}(1, p^f)) \leqslant \left(\frac{8}{3}\log_2 p - 1\right) f - (\log_2 f + \frac{4}{3})$ with equality if and only if $p = f = 2$. Suppose first that $p = 2$. The claim is easily verified for $f \leqslant 3$. For $f \geqslant 4$, as $2^f - 1$ is odd, we have

$$c(\mathsf{C}_{2^f-1}) \leqslant \log_3(2^f - 1) < f \log_3 2 < \frac{5}{3}f - \log_2 f - \frac{4}{3}.$$

We may thus assume that $p \geqslant 3$. One can check that, for all positive f, we have $\log_2 f + \frac{4}{3} \leqslant \left(\frac{5}{3}\log_2 p - 1\right) f$ and thus

$$c(\mathsf{C}_{p^f-1}) \leqslant \log_2(p^f - 1) < f \log_2 p \leqslant \left(\frac{8}{3}\log_2 p - 1\right) f - \left(\log_2 f + \frac{4}{3}\right).$$

This completes the proof of the case $d = 1$. From now on, we assume that $d \geqslant 2$. □

We divide the proof into cases mirroring Aschbacher's classification of finite linear groups [1] into nine classes $\mathcal{C}_1, \dots, \mathcal{C}_9$. The end of a case will be denoted by □.

Case 1. $H \in \mathcal{C}_1$. Here H is reducible. As H is completely reducible, it leaves invariant a direct decomposition $V = V_1 \oplus \cdots \oplus V_r$ with H acting irreducibly on each of the V_i, and $r \geqslant 2$. Let $d_i = \dim(V_i)$ and $H_i = H|_{V_i}$. Note that H_i is an irreducible subgroup of $\mathrm{GL}(V_i)$ and H is a subdirect subgroup of $H_1 \times H_2 \times \cdots \times H_r$. By induction,

$$c(H_i) \leqslant \left(\frac{8}{3}\log_2 p - 1\right) d_i f - \left(\log_2 f + \frac{4}{3}\right)$$

for each i. Since $\sum_{i=1}^r d_i = d$, Lemma 2.1(b) implies $c(H) \leqslant \sum_{i=1}^r c(H_i)$ and so

$$c(H) \leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - r\left(\log_2 f + \frac{4}{3}\right),$$

with equality if and only if $H = H_1 \times \cdots \times H_r$ and $c(H_i) = \left(\frac{8}{3}\log_2 p - 1\right) d_i f - \left(\log_2 f + \frac{4}{3}\right)$ for each i. By induction, this occurs if and only if either $p^f = 2$ and each H_i equals L_{k_i} for some k_i, or $p^f = 2^2$ and each H_i equals $\mathrm{GL}(1,4)$. Since the value of p^f is independent of i, equality holds if and only if H is as in Theorem 1.4. □

From now on, we assume that $r = 1$, or equivalently, that H is irreducible.

Case 2. $H \in \mathcal{C}_2$. Here H is an imprimitive linear group. That is, H preserves a nontrivial direct decomposition $V = V_1 \oplus \cdots \oplus V_b$, where $d = ab$, $b \geqslant 2$, and $\dim(V_i) = a$ for each i. Let H_2 be the permutation group induced by the action of H on $\{V_1, \dots, V_b\}$ and let K be the kernel of this action. Note that H_2 is transitive. Since H is irreducible, the setwise stabiliser of V_1 in H induces on V_1 an irreducible subgroup H_1 of $\mathrm{GL}(a, p^f)$, and $K|_{V_1}$ is normal in H_1. Moreover H is conjugate to a subwreath subgroup of $H_1 \wr H_2$, and so by Lemma 2.1(c), $c(H) \leqslant b \cdot c(H_1) + c(H_2)$. Since $a < d$, it follows by induction that $c(H_1) \leqslant \left(\frac{8}{3}\log_2 p - 1\right) af - \left(\log_2 f + \frac{4}{3}\right)$. By Theorem 1.2, $c(H_2) \leqslant \frac{4}{3}(b-1)$ hence

$$\begin{aligned} c(H) \leqslant b \cdot c(H_1) + c(H_2) &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - b\left(\log_2 f + \frac{4}{3}\right) + \frac{4}{3}(b-1) \\ &= \left(\frac{8}{3}\log_2 p - 1\right) df - \left(b\log_2 f + \frac{4}{3}\right). \end{aligned}$$

As $r = 1$, this expression is less than or equal to the upper bound in (1). Suppose now that equality holds. This implies that $b\log_2 f = \log_2 f$ and thus $f = 1$. Equality holding also implies that $c(H_2) = \frac{4}{3}(b-1)$ which, by Theorem 1.2, implies $H_2 = T_{k_2}$ for some $k_2 \geqslant 1$. Similarly, we must have $c(H_1) = \left(\frac{8}{3}\log_2 p - 1\right) af - \left(\log_2 f + \frac{4}{3}\right)$. Since H_1 is irreducible, it follows by induction that $p^f = 2$ and $H_1 = L_{k_1}$ for some $k_1 \geqslant 0$. Finally, Lemma 2.1(c) implies that

$$H = H_1 \wr H_2 = (\mathrm{GL}(2,2) \wr T_{k_1}) \wr T_{k_2} = \mathrm{GL}(2,2) \wr (T_{k_1} \wr T_{k_2}) = \mathrm{GL}(2,2) \wr T_{k_1+k_2} = L_{k_1+k_2}. \square$$

From now on, we assume that H is a primitive linear group.

Case 3. $H \in \mathcal{C}_3$. In this case, H preserves on V the structure of a b-dimensional vector space V' over a field of order p^{fa}, where $d = ab$ and $a \geqslant 2$. Note that H is conjugate to a subgroup of $\Gamma\mathrm{L}(b, p^{fa}) = \mathrm{GL}(b, p^{fa}) \rtimes \mathsf{C}_a$. Let $K = H \cap \mathrm{GL}(b, p^{fa})$. Then $H/K \leqslant \mathsf{C}_a$ and $c(H) = c(K) + c(H/K) \leqslant c(K) + \log_2 a$. By Clifford's Theorem [**19**, Theorem 3.6.2], K acts completely reducibly on V, and by [**19**, Theorem 1.8.4], K acts completely reducibly on V', say with r' irreducible constituents. Since $b \leqslant d/2$, the inductive hypothesis yields

$$\begin{aligned} c(K) &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) b(fa) - r'\left(\log_2(fa) + \frac{4}{3}\right) \\ &= \left(\frac{8}{3}\log_2 p - 1\right) df - r'\left(\log_2(fa) + \frac{4}{3}\right) \end{aligned}$$

and thus

$$\begin{aligned} c(H) &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - r'\left(\log_2(fa) + \frac{4}{3}\right) + \log_2 a \\ &= \left(\frac{8}{3}\log_2 p - 1\right) df - r'\left(\log_2 f + \frac{4}{3}\right) - (r'-1)\log_2 a. \end{aligned}$$

As $r' \geqslant 1$, the required inequality (1) for $c(H)$ follows from this. Suppose now that equality holds. It follows that $r' = 1$ and $c(K) = \left(\frac{8}{3}\log_2 p - 1\right) b(fa) - r'\left(\log_2(fa) + \frac{4}{3}\right)$. Since $a \geqslant 2$ and $b < d$, induction yields that $K = \mathrm{GL}(1,4)$, so $b = 1$ and $p^{fa} = 2^2$, which implies that $(p, f, a) = (2, 1, 2)$. Thus $d = ab = 2$, $p^f = 2$, $H/K = \mathsf{C}_2$ and $H \cong \Gamma\mathrm{L}(1,4)$ so $H = \mathrm{GL}(2,2) = L_0$. This concludes the proof in the extension field case. $\square$

We subsequently assume that H preserves no extension field structure on V. Hence H is absolutely irreducible. When (1) holds strictly, as below, equality is impossible.

CASE 4. $H \in \mathcal{C}_4$. Here H is tensor decomposable. That is, H preserves a decomposition $V = U \otimes W$, where $a := \dim(U) \geqslant 2$, $b := \dim(W) \geqslant 2$, and $d = ab$. We allow $a = b$. Thus $H \leqslant \mathrm{GL}(U) \circ \mathrm{GL}(W)$, and H projects onto irreducible subgroups of $H_1 \leqslant \mathrm{GL}(U)$ and $H_2 \leqslant \mathrm{GL}(W)$. Hence $H/\mathbb{Z}(H)$ is a a subdirect subgroup $H_1 \times H_2 \leqslant \mathrm{GL}(a, p^f) \times \mathrm{GL}(b, p^f)$. By Lemma 2.1(b) we have $c(H) \leqslant c(H_1 \times H_2)$. It follows by induction that

$$\begin{aligned} c(H) &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) af - \left(\log_2 f + \frac{4}{3}\right) + \left(\frac{8}{3}\log_2 p - 1\right) bf - \left(\log_2 f + \frac{4}{3}\right) \\ &= \left(\frac{8}{3}\log_2 p - 1\right)(a+b)f - 2\left(\log_2 f + \frac{4}{3}\right) \\ &< \left(\frac{8}{3}\log_2 p - 1\right)(ab)f - \left(\log_2 f + \frac{4}{3}\right). \end{aligned}$$

□

Assume now that Case 4 does not apply. As the $\mathcal{C}_7$ case is similar to $\mathcal{C}_4$ case, we treat it next, and out of order.

CASE 7. $H \in \mathcal{C}_7$. Here H is tensor imprimitive and tensor indecomposable. Therefore H preserves a decomposition $V = V_1 \otimes \cdots \otimes V_b$, where $d = a^b$, $a \geqslant 2$, $b \geqslant 2$, and $\dim(V_i) = a$ for each i. Then $H \leqslant K \rtimes \mathsf{S}_b$, where $K = \mathrm{GL}(V_1) \circ \cdots \circ \mathrm{GL}(V_b)$ contains the scalars $Z \cong \mathsf{C}_{p^f-1}$ and $K/Z = \mathrm{PGL}(a, p^f)^b$. Since H is not tensor decomposable, $H/(H \cap K) \cong HK/K$ is a transitive subgroup of S_b, and so by Theorem 1.2, $c(H/(H \cap K)) \leqslant \frac{4}{3}(b-1)$. The subgroups H_i of $\mathrm{PGL}(V_i)$ induced by $H \cap K$ are permuted transitively by H. Hence the H_i are irreducible and pairwise isomorphic. Induction and Lemma 2.1(c) imply

$$\begin{aligned} c(H) &\leqslant b\left(\left(\frac{8}{3}\log_2 p - 1\right) af - \log_2 f - \frac{4}{3}\right) + \frac{4}{3}(b-1) \\ &= \left(\frac{8}{3}\log_2 p - 1\right)(ab)f - \left(b\log_2 f + \frac{4}{3}\right) \\ &< \left(\frac{8}{3}\log_2 p - 1\right)(a^b)f - \left(\log_2 f + \frac{4}{3}\right). \end{aligned}$$

The final inequality uses the fact that $ab \leqslant a^b$ for $a, b \geqslant 2$. In summary, the desired bound holds strictly, when H is tensor imprimitive. □

CASE 5. $H \in \mathcal{C}_5$. Here H is realisable over a proper subfield, modulo scalars. That is, $f = ab$ with $b \geqslant 2$ and we may assume that $H \leqslant \mathsf{C}_{p^f-1} \circ \mathrm{GL}(d, p^a)$ where the subgroup

C_{p^f-1} of non-zero $\mathbb{F}_{p^f}$-scalars meets $\mathrm{GL}(d,p^a)$ in the subgroup of $\mathbb{F}_{p^a}$-scalars. Moreover, $N := H \cap \mathrm{GL}(d,p^a)$ is normal in H and

$$\frac{H}{N} = \frac{H}{H \cap \mathrm{GL}(d,p^a)} \cong \frac{H\mathrm{GL}(d,p^a)}{\mathrm{GL}(d,p^a)} \leqslant \frac{\mathsf{C}_{p^f-1} \circ \mathrm{GL}(d,p^a)}{\mathrm{GL}(d,p^a)} \cong \frac{\mathsf{C}_{p^f-1}}{\mathsf{C}_{p^a-1}}.$$

Therefore $|H/N| \leqslant (p^f-1)/(p^a-1) < p^f/(\frac{1}{2}p^a) = 2p^{f-a}$. Since N is an irreducible subgroup of $\mathrm{GL}(d,p^a)$ and $a < f$, the inductive hypothesis gives:

$$\begin{aligned} c(H) &= c(N) + c(H/N) < c(N) + \log_2(2p^{f-a}) \\ &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) da - \left(\log_2 a + \frac{4}{3}\right) + (f-a)\log_2 p + 1. \end{aligned}$$

In order to prove the desired bound (1) with strict inequality, it suffices to show

$$\log_2 f - \log_2 a + (f-a)\log_2 p + 1 \leqslant \left(\frac{8}{3}\log_2 p - 1\right) d(f-a). \tag{2}$$

Since $1 \leqslant a \leqslant f/2$ and $2 \leqslant d$ we have $f \leqslant d(f-a)$ and it suffices to show

$$\log_2 f + (f-1)\log_2 p + 1 \leqslant \left(\frac{8}{3}\log_2 p - 1\right) f. \tag{3}$$

This is true when $p = 2$ since $\log_2 f \leqslant 2f/3$ for $f \geqslant 1$. Suppose now that $p \geqslant 3$. Since $\log_2 f \leqslant f - 1$ and $2 \leqslant \frac{5}{3}\log_2 p$, we see that $c(H)$ is strictly less than the bound in (1). □

From now on assume Case 5 does not apply.

Case 6. $H \in \mathcal{C}_6$. Here H is of symplectic type. Thus $d = s^a$ where s is a prime dividing $p^f - 1$, and $H \leqslant \mathsf{C}_{p^f-1} \circ S.\, H_0$, where S is an extraspecial group of order s^{1+2a} whose center C_s is amalgamated in $\mathsf{C}_{p^f-1} \circ S$, and $H_0 \leqslant \mathrm{Sp}(2a,s)$. By [**23**, Table 4], $|\mathrm{Sp}(2a,s)| \leqslant s^{2a^2+a}$ so $c(H_0) \leqslant (2a^2+a)\log_2 s$ and $c(H) < f\log_2 p + 2a + (2a^2+a)\log_2 s$. For convenience set $z = 2a + (2a^2+a)\log_2 s$. We want to show that

$$\left(\frac{8}{3}\log_2 p - 1\right) s^a f - \log_2 f - \frac{4}{3} \geqslant f\log_2 p + z.$$

Assume to the contrary this does not hold, that is to say,

$$\left(\frac{8}{3}s^a f - f\right)\log_2 p - s^a f - \log_2 f - \frac{4}{3} < z. \tag{4}$$

Since $\log_2 p \geqslant 1$ and $\frac{4f}{3} \geqslant \log_2 f + \frac{4}{3}$, equation (4) implies

$$\frac{5}{3}s^a f - \frac{7}{3}f < z. \tag{5}$$

As $f \geqslant 1$, (5) implies $\frac{5}{3}s^a - \frac{7}{3} < z$ which, in turn, implies $s^a \in \{2, 2^2, 2^3, 2^4, 2^5, 3, 3^2, 5, 7\}$. For fixed $s^a \geqslant 2$ (and thus fixed z), (5) is a linear inequality with only finitely many solutions in f. Similarly, for fixed s^a and f, (4) is a linear inequality with only finitely many solutions in $\log_2 p$. It is thus routine to find all the solutions to (4) with p and s primes, $f \geqslant 1$ and $s^a \geqslant 2$. (This can also easily be automated.) The solutions are:

$$(s^a, p^f) \in \{(2,2), (2,3), (2,2^2), (2,2^3), (3,2), (3,2^2), (3^2,2), (2^2,2), (2^2,3), (2^2,2^2), (2^3,2), (2^3,3), (2^3,2^2), (2^4,2), (2^5,2), (5,2)\}.$$

Since s divides $p^f - 1$, the possibilities reduce to $(s^a, p^f) \in \{(2,3), (3,2^2), (2^2,3), (2^3,3)\}$. Finally, using [6] we can determine the $\mathcal{C}_6$-subgroups of $\mathrm{GL}(2,3)$, $\mathrm{GL}(3,4)$, $\mathrm{GL}(4,3)$ and $\mathrm{GL}(8,3)$; in all these cases, the bound (1) holds strictly. This concludes the $\mathcal{C}_6$ case. □

From Cases 1–7 we may assume that $H \notin \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_7$. We now define groups X and Y satisfying $X \leqslant H \leqslant Y$ depending on the nature of the form preserved by H. In Case 8 we treat the case where $X \leqslant H \leqslant Y$.

(a) H preserves no non-degenerate Hermitian, alternating or quadratic form on V, up to scalar multiplication. Here $d \geqslant 2$, $X = \mathrm{SL}(d, p^f)$ and $Y = \mathrm{GL}(d, p^f)$.
(b) H preserves, a non-degenerate Hermitian form on V modulo scalars. Here $d > 2$, f is even[2], $X = \mathrm{SU}(d, p^f)$ and $Y = \mathsf{C}_{p^f-1} \circ \mathrm{GU}(d, p^f)$.
(c) H preserves, modulo scalars, a non-degenerate alternating form but no non-degenerate quadratic form on V. Here $d \geqslant 4$ is even, $X = \mathrm{Sp}(d, p^f)$ and $Y = \mathsf{C}_{p^f-1} \circ \mathrm{GSp}(d, p^f)$.
(d) H preserves, modulo scalars, a non-degenerate quadratic form on V. Here $d > 2$, $X = \Omega^\varepsilon(d, p^f)$, where $\varepsilon = \pm$ if d is even and $\varepsilon = \circ$ if d is odd and $Y = \mathsf{C}_{p^f-1} \circ \mathrm{GO}(d, p^f)$. If d is odd we additionally assume that q is odd, since X is irreducible on V.

Case 8. $X \leqslant H \leqslant Y$. First we consider the case where the derived group X' modulo scalars is not a nonabelian simple group. Then X is one of: (i) $\mathrm{SL}(2,2)$, (ii) $\mathrm{SL}(2,3)$, (iii) $\mathrm{SU}(3,2^2)$, (iv) $\Omega(3,3)$, or (v) $\Omega^+(4,p^f)$. (i) If $X = \mathrm{SL}(2,2)$, then $Y = \mathrm{GL}(2,2)$ is soluble and the upper bound of (1) holds strictly if $H < Y$, and exactly when $H = Y = L_0$. (ii) If $X = \mathrm{SL}(2,3)$, then $Y = \mathrm{GL}(2,3)$ is soluble, so $c(H) \leqslant c(\mathrm{GL}(2,3)) = 5$ which is strictly less than the upper bound in (1). (iii) If $X = \mathrm{SU}(3,2^2)$, then $|Y| = |\mathrm{GU}(3,2^2)| = 2^3 \cdot 3^4$ and hence $c(H) \leqslant c(Y) = 7$, and (1) holds strictly as $7 < \frac{23}{3}$. (iv) If $X = \Omega(3,3)$, then $|Y| = |\{\pm 1\} \times \mathrm{SO}(3,3)| = 2^4 \cdot 3$ and hence $c(H) \leqslant c(Y) = 5$, again, (1) holds strictly as $5 < 8\log_2 3 - \frac{13}{3}$. (v) Finally, suppose that $X = \Omega^+(4,p^f)$, and note that X

[2] Some authors use the notation $\mathrm{SU}(d, p^{f/2})$ and $\mathrm{GU}(d, p^{f/2})$ writing the square root of the field size.

modulo scalars is the direct product $\overline{X} := \mathrm{PSL}(2,p^f) \times \mathrm{PSL}(2,p^f)$, and $H \leqslant \mathsf{C}_{p^f-1}.\overline{X}.\mathsf{D}_8$ if p is odd, and $H \leqslant \mathsf{C}_{p^f-1}.\overline{X}.\mathsf{C}_2$ if $p = 2$. If $p^f = 2$ or 3, then $H \leqslant (\mathsf{S}_3 \times \mathsf{S}_3).\mathsf{C}_2$ or $\mathsf{C}_2.(\mathsf{A}_4 \times \mathsf{A}_4).\mathsf{D}_8$, and hence $c(H)$ is at most 5 or 10, respectively. In each case this is strictly less than the upper bound in (1). Suppose now that $p^f \geqslant 4$. Since H contains X, we have $c(H) \leqslant c(\mathsf{C}_{p^f-1}) + c(\overline{X}) + c(\mathsf{D}_8) < f\log_2 p + 5$. This expression is less than $\left(\frac{8}{3}\log_2 p - 1\right)(4f) - \log_2 f - \frac{4}{3}$ for all $p^f \geqslant 4$.

For all the other cases X, modulo scalars, is a nonabelian simple group $\overline{X}$ and using information from [15, Table 2.1.C] about $\mathrm{Out}(\overline{X})$, we see that $c(H) < 2f\log_2 p + \log_2 f + 3$, and this is at most $\left(\frac{8}{3}\log_2 p - 1\right)df - \log_2 f - \frac{4}{3}$ if and only if

$$\frac{2\log_2 f + 13/3}{f} \leqslant \frac{8d-6}{3}\log_2 p - d.$$

The left side is at most 13/3 (taking $f = 1$) while the right side is at least $(5d-6)/3$ (taking $p = 2$), and $13/3 \leqslant (5d-6)/3$ provided $d \geqslant 4$. Similarly considering the value of the right side for $p = 3$, we see that the inequality holds for all $d \geqslant 3$ when $p \geqslant 3$. This leaves the cases $d = 2$ and $(d,p) = (3,2)$. Suppose first that $(d,p) = (3,2)$. The inequality is easily seen to hold for $f \geqslant 3$. Thus $f \leqslant 2$ and $H = X = \mathrm{SL}(3,2^f)$ or $\mathrm{SU}(3,4)$, so $c(H) \leqslant 3$ and the bound (1) holds strictly. Finally let $d = 2$, so $p^f \geqslant 4$. Here $c(H) < 2f\log_2 p + 1$, and this is strictly less than the upper bound in (1) for all $p^f \geqslant 4$. This concludes the proof in Case 8. □

We are now in the case where $X \not\leqslant H \leqslant Y$ where the subgroups X and Y are defined in the preamble to Case 8. We apply the Aschbacher classification [1] of subgroups of Y which do not contain X. Given our analysis above, and by the definition of X, the only remaining possibility is H, modulo scalars, is almost simple; and its quasisimple normal subgroup S is absolutely irreducible and primitive on V. This case is called type $\mathcal{C}_9$.

Case 9. $H \in \mathcal{C}_9$. In this final case, H has a quasisimple normal subgroup S which is absolutely irreducible and primitive on V. Thus, if Z is the subgroup of scalars in H and $T := S/(S \cap Z)$, then $T \trianglelefteq H/Z \leqslant \mathrm{Aut}(T)$ and T is a nonabelian simple group. Therefore $c(H) = c(S \cap Z) + c(T) + c(H/S) < f\log_2 p + 1 + c(\mathrm{Out}(T))$.

Suppose first that $c(\mathrm{Out}(T)) \leqslant 2$ and thus $c(H) < f\log_2 p + 3$. It is not hard to show that this is less than the upper bound in (1), unless $(d,p^f) = (3,2)$ or $d = 2$. Since $\mathrm{GL}(3,2)$ contains no $\mathcal{C}_9$-subgroups, we must have $d = 2$. Here $T = \mathsf{A}_5$, $p^f \geqslant 4$, and $\mathrm{Out}(T) = \mathsf{C}_2$. Thus $c(H) < f\log_2 p + 2$, and this is less than the upper bound in (1).

We may therefore assume that $c(\mathrm{Out}(T)) \geqslant 3$. By considering the possibilities for T when $d = 3$, we can then exclude the case $d = 3$ and so we assume that $d \geqslant 4$. The fact that $c(\mathrm{Out}(T)) \geqslant 3$ implies (using the classification of finite simple groups) that T is a simple group of Lie type (keeping in mind the exceptional isomorphism $\mathsf{A}_6 \cong \mathrm{PSL}(2,9)$). It follows from [17, Theorem 4.1] that $|H/Z| \leqslant p^{3fd}$. By Lemma 2.3(b), $|\mathrm{Out}(T)| \leqslant \log_2 |T|$, and so $|\mathrm{Out}(T)| \leqslant 3fd \log_2 p$ which yields

$$c(H) < f \log_2 p + 1 + \log_2(3df) + \log_2 \log_2 p.$$

Suppose that the following inequality holds

$$f \log_2 p + 1 + \log_2 3 + \frac{df}{2} + \log_2 \log_2 p \leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - \frac{f}{2} - \frac{4}{3}. \tag{6}$$

Using (6) and the fact that $\log_2 f \leqslant \frac{f}{2}$ for $f \neq 3$, we see that, for $f \neq 3$,

$$\begin{aligned} c(H) < f \log_2 p + 1 + \log_2 3 + \frac{df}{2} + \log_2 \log_2 p &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - \frac{f}{2} - \frac{4}{3} \\ &\leqslant \left(\frac{8}{3}\log_2 p - 1\right) df - \log_2 f - \frac{4}{3}, \end{aligned}$$

as required. Consider the case when $f = 3$. Since $d \geqslant 4$ and $p \geqslant 2$ we have

$$\frac{7}{3} + 2\log_2 3 \leqslant 11 \leqslant \left(\frac{7d}{2} - 3\right) \log_2 p - \log_2 \log_2 p + \frac{9d}{2}\left(\log_2 p - 1\right).$$

Rearranging gives the desired bound

$$c(H) < 3\log_2 p + 1 + \log_2 3 + \frac{3d}{2} + \log_2 \log_2 p \leqslant \left(\frac{8}{3}\log_2 p - 1\right) 3d - \log_2 3 - \frac{4}{3}.$$

Thus it remains to assume the opposite of (6). This is equivalent to

$$0 > \frac{8}{3} df \log_2 p - f \log_2 p - \frac{3df}{2} - \frac{f}{2} - \frac{7}{3} - \log_2 3 - \log_2 \log_2 p. \tag{7}$$

For fixed d and f, the right side of (7) is an increasing function of p. Similarly, for fixed f and p, the right side of (7) is an increasing function of d. Setting $p = 2$ and $d = 4$ shows

$$\begin{aligned} &\frac{8}{3} df \log_2 p - f \log_2 p - \frac{3df}{2} - \frac{f}{2} - \frac{7}{3} - \log_2 3 - \log_2 \log_2 p \\ &\geqslant \frac{32f}{3} - f - 6f - \frac{f}{2} - \frac{7}{3} - \log_2 3 \\ &= \frac{19f}{6} - \frac{7}{3} - \log_2 3. \end{aligned}$$

However, $\frac{19f}{6} - \frac{7}{3} - \log_2 3 \geqslant 0$ for $f \geqslant 2$ contrary to (7). Hence the solutions to (7) have $f = 1$. It is easy to check that (7) is satisfied for $(d, p, f) = (4, 2, 1)$ only. One can then check that, if $H \leqslant \mathrm{GL}(4, 2) \cong \mathsf{A}_8$, then $c(\mathrm{Out}(T)) \leqslant 2$. This case was handled earlier. This final argument completes the proof of both the $\mathcal{C}_9$ case, and Theorem 1.4. □

5. Proofs of Theorems 1.3 and 1.5

Proof of Theorem 1.5. The degree of a non-affine primitive group is at least 5, and if G is such a group of degree 5 then $G = \mathsf{A}_5$ or S_5, $c(G) \leqslant 2$, and the bound $c_{\mathrm{na}} \log_2 n - \frac{4}{3}$ equals $\frac{10}{3\log_2 5}\log_2 5 - \frac{4}{3} = 2$. Thus $c(G) \leqslant c_{\mathrm{na}} \log_2 n - \frac{4}{3}$ holds with equality if and only if $G = \mathsf{S}_5$, so the result holds if $n = 5$.

Assume now that $n > 5$ and that Theorem 1.5 holds for groups of degree less than n. Let G be a non-affine primitive permutation group of degree n. We first treat the cases: almost simple, and simple diagonal (which includes both types HS and SD in the type descriptions in [21, Section 3]). Then we treat all other cases together since in these remaining cases G is contained in a wreath product in product action. (This includes types HC, TW, CD and PA as described in [21, Section 3].)

Case 1. Almost Simple. In this case, $T \trianglelefteqslant G \leqslant \mathrm{Aut}(T)$, where T is a nonabelian simple group. Suppose first that $T = \mathsf{A}_6$. Then $c(G) \leqslant 3$. If $n \geqslant 10$, then it follows that $c_{\mathrm{na}} \log_2 n - \frac{4}{3} = \frac{10}{3\log_2 5}(\log_2 5 + 1) - \frac{4}{3} > 3 \geqslant c(G)$. On the other hand if $n < 10$ then $n = 6$, $c(G) \leqslant 2$ (since then $G \leqslant \mathsf{S}_6$), and $c_{\mathrm{na}} \log_2 n - \frac{4}{3} > 2$. Thus Theorem 1.5 holds with strict inequality if $T = \mathsf{A}_6$. Suppose now that $T \neq \mathsf{A}_6$. Then by Lemma 2.3,

$$c(G) \leqslant 1 + c(\mathrm{Out}(T)) \leqslant 1 + \log_2 |\mathrm{Out}(T)| < 1 + \log_2(n/2) = \log_2 n.$$

If $n \geqslant 11$ then $\log_2 n < 1.4\log_2 n - \frac{4}{3} < c_{\mathrm{na}} \log_2 n - \frac{4}{3}$, as required. So assume that $6 \leqslant n \leqslant 10$. For these degrees the possible almost simple groups are known and in all case $c(G) \leqslant 2$ (since $T \neq \mathsf{A}_6$), which is strictly less than $c_{\mathrm{na}} \log_2 n - \frac{4}{3}$.

Case 2. Simple Diagonal. In this case, the socle N of G (the product of the minimal normal subgroups) has the form $N = T^k$, where T is a nonabelian simple group, $k \geqslant 2$, and $n = |T|^{k-1}$. Further G/N is isomorphic to a subgroup H of $\mathrm{Out}(T) \times \mathsf{S}_k$. Thus H is a subdirect subgroup of $H_1 \times H_2$, for some $H_1 \leqslant \mathrm{Out}(T)$ and $H_2 \leqslant \mathsf{S}_k$. Furthermore, either H_2 is a transitive subgroup of S_k, or $k = 2$ and $H_2 = 1$ (for types HS and SD respectively). In either case, $c(H_2) \leqslant \frac{4}{3}(k-1)$, by Theorem 1.2, and $c(H) \leqslant c(H_1) + c(H_2)$ by Lemma 2.1(b). Moreover, by Lemma 2.3(a), $|\mathrm{Out}(T)| < |T|/16$, (since T has a subgroup of order at least 8 and hence of index at most $|T|/8$). Thus

$$\begin{aligned} c(G) = k + c(H) \leqslant k + c(H_1) + c(H_2) &\overset{1.2,2.3}{\leqslant} k + \log_2\left(\frac{|T|}{16}\right) + \frac{4}{3}(k-1) \\ &= \frac{7}{3}k - 4 + \log_2|T| - \frac{4}{3} = \frac{7k-12}{3} + \log_2|T| - \frac{4}{3}. \end{aligned}$$

Since $c_{\mathrm{na}} > 1.4$, it is sufficient to prove that this expression is at most

$$1.4\log_2 n - \frac{4}{3} = 1.4(k-1)\log_2|T| - \frac{4}{3} = \frac{7k-12}{5}\log_2|T| + \log_2|T| - \frac{4}{3}.$$

This is true since $\log_2|T| \geqslant 5/3$. Hence the bound holds strictly.

CASE 3. PRODUCT ACTION. In this final case $n = a^b$ with $b \geqslant 2$, and G is a subwreath subgroup of $A \wr B$, where B is a transitive permutation group of degree b and A is a primitive permutation group of degree a. Moreover, either A is almost simple (for G of type PA), or of simple diagonal type (namely of type SD if G has type CD, and of type HS if G has type HC or TW). In either of these cases, it follows by induction and Cases 1 and 2 that $c(A) \leqslant c_{\mathrm{na}}\log_2 a - \frac{4}{3}$. Also, by Lemma 2.1(c), $c(G) \leqslant b \cdot c(A) + c(B)$. Thus by Theorem 1.2,

$$c(G) \leqslant b\cdot c(A) + c(B) \leqslant b\left(c_{\mathrm{na}}\log_2 a - \frac{4}{3}\right) + \frac{4}{3}(b-1) = bc_{\mathrm{na}}\log_2 a - \frac{4}{3} = c_{\mathrm{na}}\log_2 n - \frac{4}{3},$$

and equality holds if and only if all of the following hold:

$$c(A) = c_{\mathrm{na}}\log_2 a - \frac{4}{3}, \quad c(B) = \frac{4}{3}(b-1) \quad \text{and} \quad c(G) = b\cdot c(A) + c(B).$$

By induction, Theorem 1.2, and Lemma 2.1(c), it follows that $G = A \wr B$, $b = 4^k$ and $B = T_k$ for some $k \geqslant 1$ (since $b > 1$), and A is one of the groups listed in Theorem 1.5. Since A is almost simple or of simple diagonal type it follows that $a = 5$ and $A = \mathsf{S}_5$. Thus $n = 5^{4^k}$ and $G = \mathsf{S}_5 \wr T_k$ in product action. This completes the proof of Theorem 1.5. □

PROOF OF THEOREM 1.3. For $n \leqslant 4$, the result can be checked by inspection. Note that, for $n = 4$, the bound is met by $G = P_1 = \mathsf{S}_4$. Henceforth assume that $n \geqslant 5$, that G is a primitive permutation group of degree n, and inductively that Theorem 1.3 holds for groups of degree less than n. If G is non-affine then, by Theorem 1.5, $c(G) \leqslant c_{\mathrm{na}}\log_2 n - \frac{4}{3}$. Since $c_{\mathrm{na}} < \frac{8}{3}$, Theorem 1.3 holds with a strict inequality in this case.

Thus we may assume that G is of affine type, so $n = p^d$ for some prime p and integer $d \geqslant 1$, and $G = (\mathsf{C}_p)^d \rtimes H$ where H is an irreducible subgroup of $\mathrm{GL}(d,p)$. Thus by

Theorem 1.4, $c(H) \leqslant (\frac{8}{3}\log_2 p - 1)d - \frac{4}{3}$, and therefore

$$c(G) = d + c(H) \overset{1.4}{\leqslant} d + \left(\frac{8}{3}\log_2 p - 1\right) d - \frac{4}{3} = \frac{8d}{3}\log_2 p - \frac{4}{3} = \frac{8}{3}\log_2 n - \frac{4}{3}.$$

Moreover, by Theorem 1.4 (and since H is an irreducible linear group over the field $\mathbb{F}_p$), equality occurs if and only if $p = 2$, $d = 2^{2k+1} = 2 \cdot 4^k$ for some $k \geqslant 0$, and H is linearly isomorphic to L_k (recall $n = p^d \neq 4$). Thus $n = 2^d = 4^{4^k}$ with $k \geqslant 1$, and

$$G = \mathsf{C}_2^{2\cdot 4^k} \rtimes L_k = (\mathsf{C}_2^2)^{4^k} \rtimes (\mathrm{GL}(2,2) \wr T_k) = (\mathsf{C}_2^2 \rtimes \mathrm{GL}(2,2)) \wr T_k \cong \mathsf{S}_4 \wr T_k = P_k. \qquad \square$$

6. Proof and examples for Theorem 1.7

Proof. Let $G \leqslant \mathrm{Sym}(\Omega)$ with $n = |\Omega|$.

(a) Suppose first that G is quasiprimitive but not primitive. Let Δ be a system of maximal (proper) blocks of imprimitivity for G in Ω, and let $d = |\Delta|$. Then $d \leqslant n/2$ and $d \mid n$ since G is imprimitive. Also, G^Δ is primitive as Δ is maximal. Since G is quasiprimitive, the kernel of the action of G on Δ is trivial, and so $G \cong G^\Delta$. Thus G and G^Δ have isomorphic socles. If G were of affine type, then $\mathrm{soc}(G)$ would be abelian and regular. As $\mathrm{soc}(G^\Delta)$ is abelian, it is regular on Δ. This proves that $n = |\mathrm{soc}(G)| = |\mathrm{soc}(G^\Delta)| = d$, a contradiction. Thus G is non-affine, and so, by Theorem 1.5, we have the required bound

$$c(G) = c(G^\Delta) \leqslant c_{\mathrm{na}} \log_2 d - \frac{4}{3} \leqslant c_{\mathrm{na}} \log_2 \frac{n}{2} - \frac{4}{3} = c_{\mathrm{na}}(\log_2 n - 1) - \frac{4}{3}.$$

(b) Let $G \leqslant \mathrm{Sym}(\Omega)$ be semiprimitive but not quasiprimitive. As G is not quasiprimitive, G must have a nontrivial intransitive normal subgroup. Let M be a maximal such normal subgroup of G, let Σ be the set of M-orbits, and let $m = |M|$. Since G is semiprimitive and M is intransitive, we have $G^\Sigma \cong G/M$ by [5, Lemma 2.4]. We now show that G^Σ is quasiprimitive. Suppose $N^\Sigma \trianglelefteqslant G^\Sigma$ where $N \trianglelefteqslant G$. If $N \leqslant M$, then N^Σ is trivial. If $N \not\leqslant M$, then $M < NM \trianglelefteqslant G$, and by the maximality of M, NM is transitive on Ω, and hence N^Σ is transitive on Σ. Therefore G^Σ is quasiprimitive. Using the argument in the previous paragraph, G^Σ is isomorphic to a primitive permutation group of degree dividing $|\Sigma|$.

Since M is an intransitive normal subgroup of the semiprimitive group G, M is semiregular, and hence $|\Sigma| = |\Omega|/|M| = n/m$. By the previous paragraph, G^Σ is isomorphic to a primitive permutation group of degree r dividing n/m. By Theorem 1.3, $c(G^\Sigma) \leqslant \frac{8}{3}\log_2 r - \frac{4}{3}$. As $|M| = m \geqslant 2$, the desired bound is proved as follows

$$\begin{aligned} c(G) = c(G/M) + c(M) = c(G^\Sigma) + c(M) &\leqslant \frac{8}{3}\log_2 r - \frac{4}{3} + c(M) \\ &\leqslant \frac{8}{3}\log_2\left(\frac{n}{m}\right) - \frac{4}{3} + \log_2 m = \frac{8}{3}\log_2 n - \frac{4}{3} - \frac{5}{3}\log_2 m \leqslant \frac{8}{3}\log_2 n - \frac{9}{3}. \end{aligned}$$ □

Remark 6.1. We claim that, if equality holds in Theorem 1.7(b), then n is a power of 2, G is a $\{2,3\}$-group (and hence soluble), and each plinth of G is a 2-group (see the definition after Theorem 1.7 and [13, Corollary 3.11]). Suppose that equality holds in Theorem 1.7(b) and hence in the displayed equation above. To start with, this means that $r = n/m$ and hence that G^Σ is a primitive permutation group of degree n/m. Moreover equality must hold in Theorem 1.3 for G^Σ. Thus $G^\Sigma = P_k$ for some k and G^Σ is of affine type where $n/m = 4^{4^k}$. Furthermore equality holding implies that $c(M) = \log_2 m$, so M is a 2-group and m is a 2-power. Thus n is a 2-power, and G is a (soluble) $\{2,3\}$-group. Let K be an arbitrary plinth of G, and let L be a normal subgroup of G, properly contained in K, and maximal respect to these properties. By the definition of a plinth, L is intransitive and hence semiregular. Hence L is a 2-group since n is a 2-power. Also it follows from the maximality of L that K/L is a transitive minimal normal subgroup of G/L, and acts faithfully on the set of, say r, L-orbits in Ω. Now $r = n/|L|$, and so r is a 2-power. Then since G/L is soluble, its transitive minimal normal subgroup K/L must be an elementary abelian group of 2-power order r. Hence K is a 2-group, proving the claim. △

Example 6.2. We will construct infinitely many groups $H_0, H_1, \ldots,$ for which the bound in Theorem 1.7(b) is attained.

Consider $\mathrm{GL}(2,3)$ as a permutation group of degree 8 on the set Δ of non-zero vectors of $\mathbb{F}_3^2$. Let $k \geqslant 0$ and let $H_k = \mathrm{GL}(2,3) \wr T_k$ act in its product action on Δ^{4^k}. Let B_k be the base group of H_k (so that $H_k = B_k \rtimes T_k$), and let Z_k be the center of B_k. As T_k has degree 4^k, we have $B_k \cong \mathrm{GL}(2,3)^{4^k}$ and $Z_k \cong \mathsf{C}_2^{4^k}$. View Z_k as a vector space over the field $\mathbb{F}_2$ with basis consisting of the generators of the 4^k copies of $Z(\mathrm{GL}(2,3))$. Let N_k be the codimension 1 subspace of Z_k comprising vectors with coordinates summing to zero in $\mathbb{F}_2$. Note that N_k is an intransitive normal subgroup of H_k.

Since $\mathrm{GL}(2,3)$ is semiprimitive on Δ, H_k is semiprimitive on Δ^{4^k}, by [13, Theorem 9.7]. Hence N_k is semiregular on Δ^{4^k}. It follows by [13, Lemma 3.1] that H_k/N_k acts faithfully and semiprimitively on the set Ω of N_k-orbits in Δ^{4^k}. Here H_k/N_k has degree

$$|\Omega| = \frac{8^{4^k}}{|N_k|} = \frac{(2\cdot 4)^{4^k}}{2^{4^k-1}} = 2\cdot 4^{4^k}$$

while

$$c(H_k/N_k) = c(H_k) - c(N_k) = 5\cdot 4^k + \frac{4}{3}(4^k-1) - (4^k-1) = \frac{16}{3}4^k - \frac{1}{3} = \frac{8}{3}\log_2(2\cdot 4^{4^k}) - 3,$$

as in Theorem 1.7(b). Note also that H_k/N_k is not quasiprimitive on Ω since it has a normal subgroup Z_k/N_k of order 2 and, as $|\Omega| > 2$, Z_k/N_k is intransitive on Ω.

Remark 6.3. We show that the bound in Theorem 1.7(a), is never attained. Suppose to the contrary that G is quasiprimitive of degree n, but not primitive, and that $c(G) = c_{\mathrm{na}}(\log_2 n - 1) - \frac{4}{3}$. It follows from the proof of Theorem 1.7(a) that G acts primitively on a set Δ of $n/2$ blocks of imprimitivity each of size 2, and that the induced primitive group G^Δ is not affine, and $c(G^\Delta)$ achieves the upper bound of Theorem 1.5. Thus $G \cong G^\Delta = \mathsf{S}_5 \wr T_k$ in product action and the stabiliser of a block $\delta\in\Delta$ is $G_\delta \cong \mathsf{S}_4\wr T_k$. Since G is quasiprimitive on n points, the stabiliser in $N = (\mathsf{A}_5)^{4^k}$ of a point $\alpha\in\delta$ is a subgroup of index 2 in $N_\delta \cong (\mathsf{A}_4)^{4^k}$. However, no such subgroup exists. △

In Example 6.4 we provide an infinite family of quasiprimitive groups G which are not primitive and are such that the composition lengths $c(G)$ grow logarithmically with the degree. The competing requirements for such a construction are (a) to use a simple group such as A_5 for the direct factors of the socle, and a group T_k permuting the factors of the socle to make $c(G)$ large relative to the degree; and (b) to define the point stabiliser to ensure that the socle is transitive.

Example 6.4. Let k be a positive integer, and consider the group $X = \mathsf{S}_5 \wr T_k$, which has a primitive action of degree $d = 5^{4^k}$ on a set Δ, and satisfies $c(X) = c_{\mathrm{na}}\log_2 d - \frac{4}{3}$, by Theorem 1.5. There is an element $\delta\in\Delta$ such that $X_\delta = \mathsf{S}_4\wr T_k$ where each factor S_4 of the base group of X_δ is the stabiliser in S_5 of the point 5.

Let $N = \mathsf{A}_5^{4^k}$, the unique minimal normal subgroup of X, and $B = \mathsf{S}_5^{4^k}$, the base group of X. Let $B_0 = \mathsf{S}_2^{4^k}$ denote the subgroup of B which projects to $\langle(1,2)\rangle$ on each factor S_5 of B, so $B = N \rtimes B_0$. Also $B_0 \leqslant G_\delta$ and B_0 normalises $N_\delta = \mathsf{A}_4^{4^k}$.

The transitive conjugation-action of the top group T_k on the 4^k factors S_5 of B preserves a system of imprimitivity with 4^{k-1} blocks of size 4. Let $D = \mathrm{Diag}(\mathsf{S}_2^4)$ and let $M = D^{4^{k-1}}$ be the subgroup of B_0 such that the image of M under projection to S_2^4 is

D, for each of the 4^{k-1} blocks of size 4. Then M is T_k-invariant, being constant on each minimal block for T_k (of size 4).

Define G to be the subgroup $G = N.M.T_k$ of X. Since G contains the top group T_k, it follows that N is a minimal normal subgroup of G, and in fact it is the unique minimal normal subgroup since $C_G(N) = C_X(N) = 1$. It is not difficult to see that $G_\delta = N_\delta.M.T_k$ is maximal and core-free in G, so G acts faithfully and primitively on Δ of degree $d = 5^{4^k}$.

We define a subgroup H of G_δ such that G acts quasiprimitively on the coset space $\Omega = [G : H]$. Let $O_2(N_\delta) \cong (\mathsf{C}_2^2)^{4^k}$ be the largest normal 2-subgroup of N_δ, and let $D_1 = \mathrm{Diag}(\mathsf{S}_3^4)$ (with S_3 fixing points $4,5$) and $M_1 = D_1^{4^{k-1}}$, so M_1 contains M and M_1 is T_k-invariant. Let $H = O_2(N_\delta).M_1.T_k$. Then H is a subgroup of G_δ of index

$$|G_\delta : H| = |N_\delta : H \cap N_\delta| = 3^{4^k - 4^{k-1}} = 3^{3.4^{k-1}}.$$

Since G_δ is a core-free subgroup of G, so is H and hence G acts transitively and faithfully on Ω. Moreover, the displayed equation implies that N is transitive on Ω. Since N is the unique minimal normal subgroup of G, G is quasiprimitive (but not primitive) on Ω.

The degree is $n = |\Omega| = |\Delta||G_\delta : H| = 5^{4^k} \cdot 3^{3.4^{k-1}} = x^{4^k}$, where $x = 5 \cdot 3^{3/4}$. Thus $\log_2 n = 4^k \log_2 x$. Also (using Theorem 1.2)

$$c(G) = c(N) + c(M) + c(T_k) = 4^k + 4^{k-1} + \frac{4}{3}(4^k - 1) = 4^k\left(1 + \frac{1}{4} + \frac{4}{3}\right) - \frac{4}{3} = \frac{31}{12}\,4^k - \frac{4}{3}.$$

It follows that $c(G) = c\log_2 n - \frac{4}{3}$, where $c = \frac{31}{12\log_2 x} = \frac{31}{12\log_2 5 + 9\log_2 3} > 0.7358$.

Acknowledgments

We would like to thank the Centre for the Mathematics of Symmetry and Computation for its support, as this project grew from a problem posed by the first author at the annual CMSC retreat in 2017. The first two authors gratefully acknowledge the support of the Australian Research Council Discovery Grant DP160102323. The third author thanks the Australian Government for the support of a Research Training Program grant.

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