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ASSESSMENT OF SEISMIC DAMAGE TO CIVIL STRUCTURES USING STATISTICAL PATTERN RECOGNITION TECHNIQUES AND TIME SERIES ANALYSIS

By

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Abstract

The ability to estimate seismic induced damage to civil infrastructure is undoubtedly one of the most important challenges faced by structural engineers. In this research two complementary methods of damage estimation using either knowledge of the structure and earthquake or recorded structural responses were investigated. These methods gave different natured estimates, either prediction or detection, which are suitable for different applications. Firstly, damage to a structure was predicted based on analysis of structural and ground motion properties. Secondly, damage to a structure was detected and assessed by analysing the structural response under dynamic excitation.

In the first approach, basic structural and ground motion properties were used to characterise a broad group of structures and earthquakes. These properties were used as inputs into a Back-Propagation (BP) Artificial Neural Network (ANN) and related to a damage index that quantified the extent of damage to the structure. A set of prior structural analyses was required to train the ANN before useful predictions could be made. Applied to 2D Reinforced Concrete (RC) frames, the method was capable of predicting with good accuracy damage to frames of varying stiffness, strength and topology whilst subjected to a range of ground motion severities.

In the second approach, Autoregressive (AR) models were used to fit the acceleration time histories obtained when the structure was in both undamaged and damaged states. The AR coefficients were selected as damage sensitive features and statistical pattern recognition techniques were investigated for interpreting changes in the values of these features caused by damage. Initially, an offline damage detection method was developed in which BP ANNs were used for both classification and quantification tasks where the percentage remaining stiffness at a specific location was estimated. The method was applied to three experimental structures; a 3-storey bookshelf structure, the ASCE Phase II Experimental SHM Benchmark Structure and a RC column. In addition, for damage classification tasks only, the supervised classification techniques of Nearest Neighbour and Learning Vector Quantisation were found to be effective while Self-Organising Maps, an unsupervised classification method, showed promising results. Finally, an online damage detection method was developed based on recursive identification of the AR models using the forgetting factor and Kalman filter approaches. A linear 3-DOF model with time varying stiffness was investigated and the results showed that damage could be detected and quantified as it occurred. Nonlinear damage detection was addressed with the investigation of a 1-DOF bilinear oscillator and a 3-DOF Bouc-Wen hysteretic system. In both cases the on-set of nonlinearity was detected using Outlier analysis.

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Table of Contents

Abstract...i
Acknowledgements...iii
Table of Contents...v
List of Figures...ix
List of Tables...xiii
List of Abbreviations...xvi
Notation...xvii

1. Introduction

1.1 Background and motivation for research...1-1
  1.1.1 Seismic damage prediction...1-1
  1.1.2 Seismic damage detection...1-2
1.2 Objective, contribution and scope of research...1-3
1.3 Outcomes of thesis...1-5
1.4 Layout of thesis...1-5
1.5 References...1-6

2. Literature review

2.1 Estimation of seismic-induced damage...2-1
  2.1.1 Previous New Zealand studies on seismic damage estimation...2-2
  2.1.2 Seismic vulnerability functions...2-2
  2.1.2 ANN seismic damage prediction...2-4
2.2 Structural Health Monitoring...2-6
  2.2.1 Frequency domain...2-6
  2.2.2 Time domain...2-8
  2.2.3 Environment effects and operating conditions...2-11
  2.2.4 Time series methods...2-12
2.3 Artificial Neural Networks...2-14
  2.3.1 Historical development...2-15
  2.3.2 Applications in structural engineering...2-17
2.4 Conclusions...2-18

2.4 References..2-19

3. Theory of Back-Propagation Artificial Neural Networks

3.1 Back-Propagation algorithm..3-1
3.2 References..3-8

4. Prediction of seismic-induced damage to 2D reinforced concrete frames

4.1 Application to regular 2D RC frames...4-2
  4.1.1 Structural and ground motion parameters...4-2
  4.1.1 Nonlinear FEM simulations for damage data generation..........................4-4
  4.1.2 Damage quantification...4-7
4.2 Training and testing of ANNs for damage prediction....................................4-8
4.3 Prediction of seismic damage...4-10
4.4 Conclusions..4-13
4.5 References..4-13

5. Experimental structures, modal testing and system identification

5.1 3-storey bookshelf structure..5-1
  5.1.1 Modal analysis..5-4
    5.1.1.1 Frequency Response Function via broadband excitation.......................5-4
    5.1.1.2 Frequency Response Functions via frequency sweep tests....................5-6
    5.1.1.3 State-space system identification..5-9
    5.1.1.4 Free vibration decay method...5-11
    5.1.1.5 Half-power bandwidth method..5-12
  5.1.2 Discussion and summary..5-13
5.2 RC column...5-15
  5.2.1 Modal analysis..5-15
5.3 Conclusions..5-19
5.4 References..5-20

6. Damage detection and quantification using time series analysis and Artificial Neural Networks

6.1 Time series analysis...6-2
  6.1.1 Autoregressive Models...6-3

6.2 Application to 3-storey bookshelf structure....6-4
    6.2.1 Model updating....6-6
    6.2.2 AR model order selection and diagnostic checking....6-9
    6.2.3 Damage classification using output-only model....6-17
    6.2.4 Damage detection, localisation and quantification using output-only model....6-18
    6.2.5 Damage classification with a known input....6-19
    6.2.6 Damage detection, localisation and quantification with a known input....6-20
6.3 Application to ACSE Phase II Experimental SHM Benchmark Structure....6-21
    6.3.1 AR model selection and diagnostic checking....6-23
    6.3.2 Damage classification....6-25
6.4 Application to a RC column....6-27
    6.4.1 AR model selection and diagnostic checking....6-30
    6.4.2 Damage classification....6-31
    6.4.3 Damage quantification....6-33
6.5 Conclusions....6-35
6.6 References....6-36

7. Nearest Neighbour, Learning Vector Quantisation and Self-Organising Maps for damage classification using time series analysis

7.1 Theory....7-2
    7.1.1 Nearest Neighbour classification....7-2
    7.1.2 Learning Vector Quantisation....7-3
    7.1.3 Self-Organising Maps....7-4
    7.1.4 Principal Component Analysis....7-4
    7.1.5 Sammon mapping....7-5
7.2 Application to 3-storey bookshelf structure....7-6
7.3 Application to ACSE Phase II Experimental SHM Benchmark Structure....7-9
7.4 Application to RC column....7-15
    7.4.1 Damage classification....7-16
    7.4.2 Damage quantification....7-18
7.5 Conclusions....7-19
7.6 References....7-21

8. Online damage detection using recursive identification of time series and Artificial Neural Networks

8.1 Recursive identification of AR time series 8-2
  8.1.1 Forgetting factor method 8-2
  8.1.2 Kalman filter 8-3
8.2 Intervention analysis and outlier detection 8-4
8.3 Application to a linear 3-DOF model 8-6
  8.3.1 Forgetting factor approach 8-7
  8.3.2 Kalman filter approach 8-8
  8.3.3 Effect of measurement noise 8-9
8.4 Application to a 1-DOF bilinear oscillator 8-11
8.5 Application to a 3-DOF Bouc-Wen hysteretic system 8-12
8.6 Conclusions 8-18
8.7 References 8-19

9. Conclusions

9.1 Seismic damage prediction 9-1
9.2 Seismic damage detection 9-2

List of Figures

Figure 3.1. A single hidden layer BP ANN..3-2

Figure 3.2. Function of the k^{th} neuron in the hidden layer 3-3

Figure 4.1. Giberson one-component beam..4-6

Figure 4.2. Modified Takeda hysteretic rule. 4-7

Figure 4.3. Predicted vs. actual damage comparison for BP ANN.....................................4-9

Figure 4.4. Damage to 5-storey by 3-bay frame with scaled Imperial Valley record and varying beam reinforcement ratio (a) p=0.008, (b) p=0.012, (c) p=0.016 and (d) p=0.20. 4-11

Figure 4.5. Comparison of damage predicted by FEM and ANN to a 4-storey by 3-bay frame. 4-12

Figure 4.6. Comparison of damage predicted by FEM and ANN for the Colinga earthquake. 4-13

Figure 5.1. 3-storey bookshelf structure (a) general view (b) diagram of accelerometer locations and external dimensions (c) detail of column-plate joint..............................5-2

Figure 5.2. Shake table equipment (a) pump and (b) PID controller (below) and data acquisition box (above)..5-3

Figure 5.3. FRF for the 3rd storey of 3-storey bookshelf structure. 5-6

Figure 5.4. Acceleration response curve for 3-storey bookshelf structure at the 3rd storey showing the 1st natural frequency..5-7

Figure 5.5. Acceleration response curve for 3-storey bookshelf structure at the 3rd storey showing the 2nd natural frequency..5-8

Figure 5.6. Acceleration response curve for 3-storey bookshelf structure at the 3rd storey showing the 3rd natural frequency. 5-8

Figure 5.7. Actual and identified state-space response of 3-storey bookshelf structure (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 5-9

Figure 5.8. Free vibration decay of the 3rd storey of 3-storey bookshelf structure excited in the 1st mode..5-12

Figure 5.9. Half-power bandwidth method. 5-13

Figure 5.10. Mode shapes for 3-storey bookshelf structure obtained from the state-space model and acceleration data (a) 1st mode, (b) 2nd mode and (c) 3rd mode...................5-14

Figure 5.11. RC column (a) general arrangement, (b) shaker and (c) drawings. 5-16
Figure 5.12. FRF in *x*-direction for RC column. 5-17

Figure 6.1. Mode shapes of 3-storey bookshelf structure in each damage state (a) 1st mode, (b) 2nd mode and (c) 3rd mode. 6-6
Figure 6.2. 3-DOF lumped mass-spring model used for modelling the 3-storey bookshelf structure. 6-7
Figure 6.3. Sample ACF for 3-storey bookshelf structure (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-11
Figure 6.4. Sample PACF for 3-storey bookshelf structure (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-11
Figure 6.5. Variation of AIC with AR model order for 3-storey bookshelf structure. 6-12
Figure 6.6. Actual response and one-step ahead prediction for 3-storey bookshelf structure using AR(24) models (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-13
Figure 6.7. Residual errors from AR(24) models for 3-storey bookshelf structure (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-14
Figure 6.8. Histogram of residual time series for AR(24) models (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-14
Figure 6.9. Sample ACFs of residual time series using AR(24) models (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-16
Figure 6.10. Sample PACF of residual time series using AR(24) models (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-16
Figure 6.11. 1st storey AR coefficient distribution for 3-storey bookshelf structure in damage states (a) D0 and (b) D1...6-17
Figure 6.12. Damage states in 3-storey bookshelf structure showing percentage of lateral stiffness at each storey...6-18
Figure 6.13. Detected damage for 3-storey bookshelf structure using AR models in states (a) D0, (b) D1, (c) D2 and (d) D3. 6-19
Figure 6.14. Detected vs. actual damage for 3-storey bookshelf structure using ARX models (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 6-21
Figure 6.15. ASCE Phase II Experimental SHM Benchmark Structure (a) general view and (b) beam-column joint and bracing. 6-22
Figure 6.16. Model identification of ASCE structure (a) sample ACF of responses, (b) sample PACF or responses, (c) sample ACF of residuals and (d) sample PACF of residuals. 6-24

Figure 6.17. Variation of AIC with AR model order for ASCE Phase II Experimental Benchmark Structure. 6-24
Figure 6.18. Results of damage classification from ASCE structure using subset AR and accelerometer measurements. 6-26
Figure 6.19. Photographs of RC column in damaged states D1-D5. 6-28
Figure 6.20. FRF curves for RC column in states D0, D1, D4 and D6. 6-29
Figure 6.21. Model identification of RC column (a) sample ACF of response, (b) sample PACF of response, (c) sample ACF of residuals and (d) sample PACF of residuals. 6-30
Figure 6.22. Variation of AIC with AR model order for RC column. 6-31
Figure 6.23. Histograms of the 1st AR coefficient from RC column in damage state (a) D0, (b) D1, (c) D4 and (d) D6. 6-32
Figure 6.24. Detected vs. actual damage for RC column Case I using AR models. 6-34
Figure 6.25. Detected vs. actual damage for RC column Case II using AR models. 6-34

Figure 7.1. Projection of 3-storey bookshelf structure data onto the first two principal components. 7-7
Figure 7.2. Projection of 3-storey bookshelf structure data via Sammon mapping. 7-7
Figure 7.3. Projection of ASCE structure data on the first two principal components. 7-10
Figure 7.4. Projection of ASCE structure data via Sammon mapping. 7-11
Figure 7.5. Number of misclassifications using subsets of AR coefficient and/or accelerometers for ASCE structure. 7-12
Figure 7.6. Clustering of data from ASCE structure using SOM on PCA data. 7-14
Figure 7.7. Clustering of data from ASCE structure using SOM on Sammon projection data. 7-15
Figure 7.8. Scatter plot of the 1st vs. 2nd AR coefficients for RC column in all Case I damage states. 7-16
Figure 7.9. Damage quantification results for RC column. 7-19

Figure 8.1. Linear 3-DOF lumped-mass model. 8-7
Figure 8.2 Damage detection in 3-DOF model using forgetting factor approach (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 8-8
Figure 8.3 Damage detection in 3-DOF model using Kalman filter approach (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 8-9

Figure 8.4 Damage detection in 3-DOF model with 2% RMS noise using forgetting factor approach (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 8-10

Figure 8.5 Damage detection in 3-DOF model with 5% RMS noise using forgetting factor approach (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 8-10

Figure 8.6. Damage detection in a 1-DOF elastoplastic oscillator (a) 1st AR coefficient, (b) actual yielding and (c) detected yielding (d) magnitude of outlier. 8-12

Figure 8.7. Hysteresis loops for 3-DOF Bouc-Wen structure (a) 1st storey, (b) 2nd storey and (c) 3rd storey. 8-14

Figure 8.8. 1st AR coefficient for the 3-DOF Bouc-Wen structure (a) 1st storey, (b) 2nd storey and (c) 3rd storey.. 8-15

Figure 8.9. Detection of yielding in a 3-DOF Bouc-Wen model (a) actual 1st storey yield, (b) actual 2nd storey yield, (c) detected yields from 1st storey, (d) detected yields from 2nd storey and (e) detected yields from 3rd storey. 8-16

Figure 8.10. Magnitude of outliers in a 3-DOF Bouc-Wen model a) 1st storey, b) 2nd storey and c) 3rd storey. 8-17

Figure 8.11. Detection of yielding in a 3-DOF Bouc-Wen model with 1% RMS noise (a) actual 1st storey yield, (b) actual 2nd storey yield, (c) detected yields from 1st storey, (d) detected yields from 2nd storey and (e) detected yields from 3rd storey. 8-18

List of Tables

Table 4.1. Frame topologies, beam and column reinforcement ratios, concrete strengths and damping ratios used in numerical simulations. 4-3
Table 4.2. Member dimensions for different frame topologies. 4-3
Table 4.3. Properties of earthquakes used in numerical simulations. 4-4
Table 4.4. Classification of structural damage based on the Park and Ang damage index.... 4-8
Table 4.5. Standard deviation of errors produced by ANNs with different numbers of hidden layer neurons. 4-8
Table 4.5. Properties of Coalinga earthquake.. 4-12

Table 5.1. Frequencies and viscous damping ratios for 3-storey bookshelf structure obtained from state-space model. 5-11
Table 5.2. Normalised mode shapes for 3-storey bookshelf structure obtained from state-space model. 5-11
Table 5.3. Mode shapes for 3-storey bookshelf structure extracted from acceleration data. 5-13
Table 5.4. Summary of natural frequencies for 3-storey bookshelf structure. 5-14
Table 5.5. Summary of viscous damping ratios for 3-storey bookshelf structure. 5-14
Table 5.6. Summary of RC column modal analysis results. 5-19

Table 6.1. Natural frequencies and percentage changes at different damage states for 3-storey bookshelf structure. 6-5
Table 6.2. Mode shapes at different damage states for 3-storey bookshelf structure. 6-5
Table 6.3. Earthquake records used to excite 3-storey bookshelf structure. 6-6
Table 6.4. Analytical frequencies and MAC values after updating for 3-storey bookshelf structure. 6-8
Table 6.5. Updated stiffnesses from analytical models for 3-storey bookshelf structure. 6-8
Table 6.6. Final stiffness for damage states for 3-storey bookshelf structure. 6-9
Table 6.7. Final analytical frequencies and MAC values for 3-storey bookshelf structure. .. 6-9
Table 6.8. Classification results from 3-storey bookshelf structure using AR models........ 6-17
Table 6.9. Damage at each storey as a ratio of initial stiffness for 3-storey bookshelf structure. 6-18

Table 6.10. ANN identified damage as a ratio of initial stiffness in 3-storey bookshelf structure using AR models. 6-19
Table 6.11. Classification results from 3-storey bookshelf structure using ARX models. .. 6-20
Table 6.12. ANN identified damage as a ratio of initial stiffness in 3-storey bookshelf structure using ARX model. 6-21
Table 6.13. Damage configurations for ASCE Phase II Experimental SHM Benchmark Structure. 6-22
Table 6.14. Number of misclassifications using PCA reduced data from ASCE structure. 6-27
Table 6.15. Description of damage states for RC column. 6-28
Table 6.16. Modal analysis results for RC column. 6-29
Table 6.17. Case I classification results for RC column. 6-32
Table 6.18. Case II classification results for RC column. 6-32
Table 6.19. Stiffness and damage as a ratio of initial stiffness at each state for RC column...... 6-33
Table 6.20. ANN identified damage as a ratio of remaining stiffness to initial stiffness for RC column. 6-35

Table 7.1. Number and percentage of misclassifications using NN classification for 3-storey structure using PCA. 7-8
Table 7.2. Number and percentage of misclassifications using NN classification for 3-storey structure using subset AR coefficient selection. 7-9
Table 7.3. Number and percentage of misclassifications using LVQ classification for 3-storey structure. 7-9
Table 7.4. Number and percentage of misclassifications using NN classification for ASCE structure. 7-11
Table 7.5. Number and percentage of misclassifications using LVQ classification for ASCE structure. 7-13
Table 7.6. Comparison of true classification and clustering from SOM on PCA reduced data for ASCE structure. 7-14
Table 7.7. Comparison of true classification and clustering from SOM on Sammon projection data for ASCE structure. 7-15
Table 7.8. Number and percentage of misclassifications using NN classification for RC column Case I. 7-17

Table 7.9. Number and percentage of misclassifications using LVQ classification for RC column Case I. 7-17
Table 7.10. Number and percentage of misclassifications using NN and LVQ classification for RC column Case II.......... 7-17
Table 7.11. Mean detected damage quantification for RC column. 7-19

Table 8.1. Detected yielding in a 3-DOF Bouc-Wen model. 8-17

List of Abbreviations

ACF	Autocorrelation Function
ANN	Artificial Neural Network
AR	Autoregressive
ARMAX	Autoregressive-Moving Average with eXogenous input
ARX	Autoregressive with eXogenous input
ASCE	American Society of Civil Engineers
BP	Back-Propagation
CSD	Cross Spectral Density
DFT	Discrete Fourier Transform
DOF	Degree Of Freedom
ERA	Eigenvalue Realisation Algorithm
FEM	Finite Element Method
FRF	Frequency Response Function
IFR	Impulse Response Function
LVQ	Learning Vector Quantisation
MA	Moving Average
MAC	Modal Assurance Criterion
NN	Nearest Neighbour
PCA	Principal Component Analysis
PCF	Partial Autocorrelation Function
PEM	Prediction Error Method
PGA	Peak Ground Acceleration
PGD	Peak Ground Displacement
PGV	Peak Ground Velocity
PSD	Power Spectral Density
RC	Reinforced Concrete
SHM	Structural Health Monitoring
SI	Spectrum Intensity
SOM	Self-Organising Maps

Notation

The following notation is used throughout this thesis:

$\mathbf{0}$	null matrix
$\mathbf{A}$	state matrix
A	cross-sectional area
a	AR coefficient
$\mathbf{B}$	input matrix
b	exogenous coefficient
$\mathbf{C}$	damping matrix, output matrix
Cov	covariance operator
c	MA coefficient, viscous damping coefficient
$\mathbf{D}$	feedthrough matrix
D	Park and Ang damage index, distance between vectors
$\mathbf{d}$	vector of desired ANN outputs
$\mathbf{E}$	selection matrix
E	error, expectation operator
$\mathbf{e}$	error vector
e	error
F	discrete Fourier transform of force, frequency, force
f	frequency
$\mathbf{H}$	Hankel matrix, covariance matrix of noise
H	frequency response function
$\mathbf{I}$	identity matrix
I	second moment of area
Im	imaginary part
$\mathbf{J}$	Jacobian matrix
$\mathbf{K}$	stiffness matrix
k	stiffness
$\mathbf{L}$	gain matrix
$\mathbf{M}$	mass matrix
$\mathbf{m}$	codebook vector
m	mass

na	AR order
nb	exogenous input order
nc	MA order
$\mathbf{o}$	vector of ANN outputs
$\mathbf{P}$	estimated covariance matrix
P	Partial Autocorrelation Function
$\mathbf{Q}$	covariance matrix of noise
q	backshift operator
$\mathbf{R}$	matrix of singular vectors
R	autocorrelation function, cross-correlation function
Re	real part
r	bilinear factor
$\mathbf{S}$	matrix of singular vectors, sensitivity matrix
S	power spectral density, cross-spectral density, spectrum response
$\mathbf{T}$	transition matrix
T	natural period
t	time
$\mathbf{u}$	state-space input vector
u	weighted sum of inputs, displacement, input
$\mathbf{V}$	matrix of singular vectors
Var	variance operator
$\mathbf{w}$	ANN weights vector
X	discrete Fourier transform of response
$\mathbf{x}$	state vector, feature vector
x	input, excitation
$\mathbf{Y}$	matrix of previous time series output
$\mathbf{y}$	output vector, vector of current time series outputs
y	output, time series value
$\mathbf{Z}$	measurement matrix
$\mathbf{z}$	principal component
α	plastic angle, model parameter for Takeda hysteresis, Rayleigh damping, LVQ, SOM, Bouc-Wen hysteresis

β	model parameter for Takeda hysteresis, Rayleigh damping, Bouc-Wen hysteresis
$\mathbf{\Phi}$	matrix of mode shapes
ϕ	mode shape
$\boldsymbol{\varphi}$	vector of previous time series values
γ	covariance
$\mathbf{\Lambda}$	matrix of eigenvalues, matrix of singular values
λ	iteration parameter, eigenvalue, forgetting factor
$\boldsymbol{\theta}$	parameter vector of time series model coefficients, vector of updating parameters
ρ	Autocorrelation Function
$\mathbf{\Sigma}$	covariance matrix, matrix of singular values for ERA
σ	standard deviation
τ	time
ω	natural radial frequency
ξ	damping ratio

$\{\}$	time series
$\lvert\,\rvert$	absolute value

Subscripts:

a	analytical
$accel$	accelerometers
c	continuous time system, closest codebook vector, complex number
$comp$	complex number
$crit$	crticial
E	Euclidean metric
e	experimental
f	input
g	gross area, gross second moment of area
i	index, iteration step
j	index

k	time step, index, iteration step
M	Mahalanobis metric
m	measurement
r	input
$real$	real number
s	sampling frequency
V	spectral velocity
x	output

Superscripts:

T	matrix transpose
^	estimated value
-	mean value
+	pseudoinverse