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FORMULÆ FOR THE EXTENDED LAPLACE INTEGRAL AND THEIR STATISTICAL APPLICATIONS

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Abstract. We propose an extension of the Laplace integral and derive formulæ to evaluate it over finite intervals. This integral is a generalization of the gamma function and modified Bessel function of the third kind. Consequently, our results provide not only formulæ in terms of the complementary error function to evaluate the incomplete gamma functions, but also those for the lower and upper incomplete Bessel functions. Statistically, our formulæ allow for the derivation of the distribution functions of the generalized inverse Gaussian (GIG) and gamma distributions in terms of the complementary error function, which have not been documented in the literature.

1. Introduction

In a classical paper Whittaker [13] expressed several functions: the parabolic cylinder, error, incomplete gamma, logarithmic integral, cosine integral and the modified Bessel function of the third kind (which will be called the Bessel function hereafter), in terms of the Whittaker function. Since then, it appears that the only documented functional relations between the error and the incomplete gamma functions are entries **8.4.1,6** in [9], which give explicit formulæ to calculate special cases of the latter in terms of the former. Moreover, although the evaluation of the incomplete Bessel function has attracted a significant research effort, the concepts of lower and upper incomplete Bessel functions and formulæ to evaluate them have not been documented in the literature. The purpose of our paper is to derive formulæ in terms of the complementary error function to evaluate an extension of an integral due to Laplace over finite intervals. This extended Laplace integral is a generalization of those which appear in the gamma and Bessel functions. Our formulæ establish previously–unknown relations between the complementary error function and incomplete Bessel functions, and they extend known relations between the complementary error function and incomplete gamma functions. Consequently, they allow for the derivation of explicit formulæ for the lower and upper incomplete forms of these functions, which have important applications in many fields of applied sciences and engineering. For instance, we show their applications in statistical science by deriving explicit formulæ for the cumulative distribution function (c.d.f.) and its complementary function (c.c.d.f.) of the family of the generalized inverse Gaussian distributions, which include the gamma distributions.

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In this paper, Section 2 and 3 derive formulæ to evaluate the extended incomplete Laplace integrals, and Section 4 discusses their statistical applications. The final section describes the implementation of these formulæ in the R language. The symbols Erf() and Erfc() denote the error and complementary error functions respectively.

1.1. Generalized incomplete gamma functions. The incomplete gamma function was studied by Legendre and Prym. Chaudhry and Zubair proposed the class of incomplete generalized gamma function [2], defined as

$$\gamma_\nu(x,b) = \int_0^x e^{-t-b/t}\, t^{\nu-1}\, dt = 2\int_0^{\sqrt{x}} e^{-(\xi^2+b/\xi^2)}\, \xi^{2\nu-1}\, d\xi\,, \quad \text{where } t=\xi^2 \quad (1)$$

$$\Gamma_\nu(x,b) = \int_x^\infty e^{-t-b/t}\, t^{\nu-1}\, dt = 2\int_{\sqrt{x}}^\infty e^{-(\xi^2+b/\xi^2)}\, \xi^{2\nu-1}\, d\xi\,. \quad (2)$$

For $\nu = j + \frac{1}{2}$ with $j = 0, 1, 2, 3, \dots$

- if $b = 0$ then the generalized incomplete gamma functions (1) and (2) reduce to the well–known lower and upper incomplete gamma functions respectively. Expressions for $\gamma_{1/2}(x,0)$ and $\Gamma_{1/2}(x,0)$ are given by **8.4.1,6** in [9];
- if $b > 0$ then Chaudhry and Zubair's approaches result in expressions for $\Gamma_\nu(x,b)$ in terms of the Horn hypergeometric series of two variables, See [2, p.57] and references therein.

To evaluate the generalized incomplete gamma function, instead of evaluating (1) and (2) as was attempted in [2], we derive formulæ for (1) and (2). Consequently, we are able to provide explicit formulæ in terms of the complementary error function for $\gamma_\nu(x,b)$ and $\Gamma_\nu(x,b)$ when $\nu > 0$ and $b \geq 0$. Moreover, if $\nu < 0$ then we also obtain expressions for $\gamma_\nu(x,b)$ and $\Gamma_\nu(x,b)$ in cases where $b > 0$, and those for $\Gamma_\nu(x,b)$ when $b = 0$. Note that $\gamma_\nu(x,b)$ is not defined when $\nu < 0$ and $b = 0$.

1.2. Incomplete Bessel functions. One of the integral representations of the modified Bessel function of the third kind with argument z and order λ is given by

$$K_\lambda(z) = \frac{1}{(2z)^\lambda}\int_0^\infty e^{-\left\{z^2\,\xi^2 + 1/(4\,\xi^2)\right\}}\, \xi^{-2\lambda-1}\, d\xi\,, \qquad z > 0\,. \quad (3)$$

See [6, p.50] for the derivation of other common integral representations of $K_\lambda(z)$ from (3). The task of evaluating the incomplete Bessel function has been considered by many authors. However, effort has been focussed on the numerical approach.

In this paper, we deal with the challenge of evaluating the incomplete Bessel function from the analytical approach, to obtain expressions in terms of the complementary error function for the *lower and upper incomplete Bessel functions* defined as

$$\widehat{K}_\lambda(z,x) = \frac{1}{(2z)^\lambda}\int_0^x e^{-\left\{z^2\,\xi^2 + 1/(4\,\xi^2)\right\}}\, \xi^{-2\lambda-1}\, d\xi\,, \quad (4)$$

$$\widetilde{K}_\lambda(z,x) = \frac{1}{(2z)^\lambda}\int_x^\infty e^{-\left\{z^2\,\xi^2 + 1/(4\,\xi^2)\right\}}\, \xi^{-2\lambda-1}\, d\xi\,. \quad (5)$$

Currently, well–known commercial software such as Maple 15 is not able to *evaluate symbolically* these integrals, even in the simplest cases where $\lambda = -1/2$.

1.3. Evaluation of the extended incomplete Laplace integrals. The key to our approach is to evaluate an integral proposed by Laplace and its extension over finite intervals. The former is the integral to which the Schlömilch transformation was applied to evaluate

$$\int_0^\infty e^{-\left(a\xi^2 + b/\xi^2\right)}\, d\xi = \frac{\sqrt{\pi}}{2\sqrt{a}}\, e^{-2\sqrt{ab}}, \qquad a>0,\ b>0\,, \tag{6}$$

which is entry **3.325** in [5]. Here, we propose an extension of (6) denoted as

$$L_\lambda(a,b) = \int_0^\infty e^{-(a\xi^2+b/\xi^2)}\xi^{-2\lambda-1}\, d\xi, \qquad \lambda\in\mathbb{R}\,,$$

and derive formulæ to evaluate *lower and upper* ***extended*** *incomplete Laplace integrals*:

$$\widehat{L}_\lambda(x,a,b) = \int_0^x e^{-(a\xi^2+b/\xi^2)}\xi^{-2\lambda-1}\, d\xi\,, \tag{7}$$

and

$$\widetilde{L}_\lambda(x,a,b) = \int_x^\infty e^{-(a\xi^2+b/\xi^2)}\xi^{-2\lambda-1}\, d\xi\,, \qquad x\in\mathbb{R}^+ \tag{8}$$

for $\lambda = \pm\left(j+\frac{1}{2}\right)$, $j = 0,1,2,\ldots$. These formulæ can be used to evaluate (1) and (2), as they are special cases of (7) and (8) when $a=1$ and $\nu=-\lambda$. Similarly, (4) and (5) are special cases for $a=z^2$ and $b=\frac{1}{4}$.

2. Analytical Evaluation of the Incomplete Laplace Integral

This section evaluates the integrals $\widehat{L}_\lambda(x,a,b)$ and $\widetilde{L}_\lambda(x,a,b)$ when $\lambda=\pm\frac{1}{2}$ (i.e., $j=0$) using the
Schlömilch transformation, see [11]. The results are then used to derive formulæ for the extended incomplete Laplace integrals (i.e. $j=1,2,\ldots$) in the next section.

Theorem 2.1. *Let $a>0$, $b>0$. Then*

$$\int_0^x e^{-\left(a\xi^2+b/\xi^2\right)}\, d\xi = \frac{1}{4}\sqrt{\frac{\pi}{a}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) - e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right] \tag{9}$$

Proof. In order to evaluate

$$\int_0^x e^{-\left(a\xi^2+b/\xi^2\right)}\, d\xi,$$

firstly apply a change of variable $\zeta=\sqrt{a}\,\xi$ to obtain

$$\int_0^x e^{-\left(a\xi^2+b/\xi^2\right)}\, d\xi = \frac{1}{\sqrt{a}}\int_0^{x\sqrt{a}} e^{-\left(\zeta^2+ab/\zeta^2\right)}\, d\zeta = \frac{1}{\sqrt{a}}\int_0^{x\sqrt{a}} e^{-\left(\psi^2+ab/\psi^2\right)}\, d\psi. \tag{10}$$

Let $c=ab$ and then another change of variable $\psi=\sqrt{c}/\zeta$ in (10), so it can be rewritten as

$$\int_0^x e^{-\left(a\xi^2+b/\xi^2\right)}\, d\xi \;=\; \frac{1}{\sqrt{a}}\int_{\sqrt{b}/x}^\infty e^{-\left(\zeta^2+c/\zeta^2\right)}\frac{\sqrt{c}}{\zeta^2}\, d\zeta. \tag{11}$$

From (10)–(11), it follows that

$$\int_0^x e^{-\left(a\xi^2+b/\xi^2\right)}\,d\xi = \frac{1}{2\sqrt{a}}\left[\int_0^{x\sqrt{a}} e^{-\left(\zeta^2+c/\zeta^2\right)}\,d\zeta + \int_{\sqrt{b}/x}^{\infty} e^{-\left(\psi^2+c/\psi^2\right)}\frac{\sqrt{c}}{\psi^2}\,d\psi\right]. \tag{12}$$

The following relation holds

$$\int_{\sqrt{b}/x}^{\infty} e^{-\left(\psi^2+c/\psi^2\right)}\,d\psi = \int_0^{x\sqrt{a}} e^{-\left(\zeta^2+c/\zeta^2\right)}\frac{\sqrt{c}}{\zeta^2}\,d\zeta, \qquad \text{where} \qquad \psi = \sqrt{c}/\zeta. \tag{13}$$

Plugging both sides of (13) into the RHS of (12) yields

$$\frac{1}{2\sqrt{a}}\left[\int_0^{x\sqrt{a}} e^{-\left(\zeta^2+c/\zeta^2\right)}\left(1+\frac{\sqrt{c}}{\zeta^2}\right)d\zeta - \int_{\sqrt{b}/x}^{\infty} e^{-\left(\psi^2+c/\psi^2\right)}\left(1-\frac{\sqrt{c}}{\psi^2}\right)d\psi\right]. \tag{14}$$

To evaluate

$$I_1 = e^{-2\sqrt{c}}\int_0^{x\sqrt{a}} e^{-\left(\zeta-\sqrt{c}/\zeta\right)^2}\left(1+\frac{\sqrt{c}}{\zeta^2}\right)d\zeta$$

let $\rho = \zeta - \sqrt{c}/\zeta$, and then

$$I_1 = e^{-2\sqrt{c}}\int_{\sqrt{b}/x - x\sqrt{a}}^{\infty} e^{-\rho^2}\,d\rho = \frac{\sqrt{\pi}\,e^{-2\sqrt{c}}}{2}\,\mathrm{Erfc}\left(\frac{\sqrt{b}}{x} - x\sqrt{a}\right). \tag{15}$$

To evaluate

$$I_2 = e^{2\sqrt{c}}\int_{\sqrt{b}/x}^{\infty} e^{-\left(\psi+\sqrt{c}/\psi\right)^2}\left(1-\frac{\sqrt{c}}{\psi^2}\right)d\psi$$

let $\rho = \psi + \sqrt{c}/\psi$, and then

$$I_2 = e^{2\sqrt{c}}\int_{\sqrt{b}/x + x\sqrt{a}}^{\infty} e^{-\rho^2}\,d\rho = \frac{\sqrt{\pi}\,e^{2\sqrt{c}}}{2}\,\mathrm{Erfc}\left(\frac{\sqrt{b}}{x} + x\sqrt{a}\right). \tag{16}$$

From (15) and (16), the evaluation result is given by

$$\int_0^x e^{-\left(a\xi^2+b/\xi^2\right)}\,d\xi = \frac{1}{4}\sqrt{\frac{\pi}{a}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) - e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right] \tag{17}$$

as required. □

Corollary 1.

$$\int_x^{\infty} e^{-\left(a\xi^2+b/\xi^2\right)}\,d\xi =$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}}\left[2 - \mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) + e^{4\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right] \tag{18}$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{a}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(x\sqrt{a} - \sqrt{b}/x\right) + e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]. \tag{19}$$

Proof. The sum of the RHSs of (18) and (9) equals the RHS of (6), as required by the sum of the corresponding LHS of these expressions. Use $\mathrm{Erfc}(-z) = 2 - \mathrm{Erfc}(z)$ to obtain (19) from (18). □

Corollary 2.

$$\int_0^\infty e^{-\left(a\xi^2 + b/\xi^2\right)}\,\xi^{-2}\,d\xi = \int_0^\infty e^{-\left(b\phi^2 + a\phi^2\right)}\,d\phi = \frac{\sqrt{\pi}}{2\sqrt{b}}\,e^{-2\sqrt{ab}}\,. \tag{20}$$

Proof. The integral with respect to ϕ is obtained by change of variable $\phi = \xi^{-1}$. It is then evaluated using (6). □

Corollary 3.

$$\int_0^x e^{-\left(a\xi^2 + b/\xi^2\right)}\,\xi^{-2}\,d\xi = \frac{1}{4}\sqrt{\frac{\pi}{b}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) + e^{2\sqrt{ab}}\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]. \tag{21}$$

Proof. The following relation holds

$$\int_0^x e^{-\left(a\xi^2 + b/\xi^2\right)}\,\xi^{-2}\,d\xi = \int_{1/x}^\infty e^{-\left(b\phi^2 + a/\phi^2\right)}\,d\phi \qquad \text{where} \qquad \phi = \xi^{-1}. \tag{22}$$

Apply (17) and let $x = y^{-1}$, and then the RHS of (22) can be calculated using

$$\int_y^\infty e^{-\left(b\phi^2 + a/\phi^2\right)}\,d\phi = \int_0^\infty e^{-\left(b\phi^2 + a/\phi^2\right)}\,d\phi - \int_0^y e^{-\left(b\phi^2 + a/\phi^2\right)}\,d\phi$$

$$= \frac{e^{-2\sqrt{ab}}}{2}\sqrt{\frac{\pi}{b}} - \frac{1}{4}\sqrt{\frac{\pi}{b}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{a}/y - y\sqrt{b}\right) - e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{a}/y + y\sqrt{b}\right)\right],$$

which is rearranged to give

$$= \frac{1}{4}\sqrt{\frac{\pi}{b}}\left\{e^{-2\sqrt{ab}}\left[1 + 1 - \mathrm{Erfc}\left(\sqrt{a}/y - y\sqrt{b}\right)\right] + e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{a}/y + y\sqrt{b}\right)\right\}.$$

Now, $1 - \mathrm{Erfc}\left(\sqrt{a}/y - y\sqrt{b}\right) = \mathrm{Erf}\left(\sqrt{a}/y - y\sqrt{b}\right) = -\mathrm{Erf}\left(y\sqrt{b} - \sqrt{a}/y\right)$, and so

$$\int_y^\infty e^{-\left(b\phi^2 + a/\phi^2\right)}\,d\phi =$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{b}}\left\{e^{-2\sqrt{ab}}\left[1 - \mathrm{Erf}\left(y\sqrt{b} - \sqrt{a}/y\right)\right] + e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{a}/y + y\sqrt{b}\right)\right\}.$$

Since $1 - \mathrm{Erf}\left(y\sqrt{b} - \sqrt{a}/y\right) = \mathrm{Erfc}\left(y\sqrt{b} - \sqrt{a}/y\right)$, $y = x^{-1}$ and using (22) we get

$$\int_0^x e^{-\left(a\xi^2 + b/\xi^2\right)}\,\xi^{-2}\,d\xi =$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{b}}\left[e^{-2\sqrt{ab}}\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) + e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right] \tag{23}$$

as required. □

Corollary 4.

$$\int_x^\infty e^{-\left(a\xi^2 + b/\xi^2\right)}\,\xi^{-2}\,d\xi = \tag{24}$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{b}}\left\{e^{-2\sqrt{ab}}\left[2 - \operatorname{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right)\right] - e^{2\sqrt{ab}}\operatorname{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right\} \tag{25}$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{b}}\left[e^{-2\sqrt{ab}}\operatorname{Erfc}\left(x\sqrt{a} - \sqrt{b}/x\right) - e^{2\sqrt{ab}}\operatorname{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right].$$

Proof. The sum of the RHSs of (21) and (25) equals the RHS of (20), as required by the sum of the corresponding LHS of these expressions. □

3. Evaluation of the Extended Incomplete Laplace Integral

This section details the steps required to derive formulæ for (7) when $\lambda = -j - \frac{1}{2}$, by using the results in Section 2 and the calculation of high–order derivatives of a composite function applying Bell coefficients in [4]. The evaluation of (7) and (8) for $\lambda = j + \frac{1}{2}$ is shown as a corollary.

Theorem 3.1. *Let a, b, x be positive real and j be a non–negative integer, then the **lower** incomplete Bessel function when $\lambda = -j - \frac{1}{2}$ is given by:*

$$\widehat{L}_{-1/2}(x,a,b) = \frac{1}{4}\sqrt{\frac{\pi}{a}}\left[e^{-2\sqrt{ab}}\operatorname{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) - e^{2\sqrt{ab}}\operatorname{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right], \quad \text{where} \quad j = 0. \tag{26}$$

If $j \geq 1$, denote

$$\eta = j - k, \quad \kappa = k - s, \quad u = \frac{\sqrt{b}}{x} - x\sqrt{a}, \quad \text{and} \quad p = \frac{\sqrt{b}}{x} + x\sqrt{a}, \tag{27}$$

then we have

$$\widehat{L}_\lambda(x,a,b) = (-1)^j\frac{\sqrt{\pi}}{4}\sum_{k=0}^{j}\binom{j}{k}\left[\frac{\partial^\eta \operatorname{Erfc}(u)}{\partial a^\eta}\frac{\partial^k}{\partial a^k}\left(\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right) - \frac{\partial^\eta \operatorname{Erfc}(p)}{\partial a^\eta}\frac{\partial^k}{\partial a^k}\left(\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right)\right]. \tag{28}$$

With

$$\frac{\partial^\eta \operatorname{Erfc}(u)}{\partial a^\eta} = \frac{-2\,e^{-u^2}}{\sqrt{\pi}}\left[\Delta_{\eta,1}[\widehat{g}(a)] + \sum_{\beta=2}^{\eta}\left(\sum_{n=1}^{\beta-1}(-1)^n\Lambda_{\beta-1,n}[g(a)]\right)\Delta_{\eta,\beta}[\widehat{g}(a)]\right], \tag{29}$$

$$\frac{\partial^k}{\partial a^k}\left(\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right) = e^{-2\sqrt{ab}}\left[\sum_{s=0}^{k-1}\binom{k}{s}\widehat{n}_s(a)\sum_{i=1}^{\kappa}(-1)^i\widehat{\Lambda}_{\kappa,i}\left[\breve{g}(a)\right] + \widehat{n}_k(a)\right], \tag{30}$$

$$\frac{\partial^\eta \operatorname{Erfc}(p)}{\partial a^\eta} = \frac{-2\,e^{-p^2}}{\sqrt{\pi}}\left[\Delta_{\eta,1}[\widehat{p}(a)] + \sum_{\beta=2}^{\eta}\left(\sum_{n=1}^{\beta-1}(-1)^n\Lambda_{\beta-1,n}[p(a)]\right)\Delta_{\eta,\beta}[\widehat{p}(a)]\right], \tag{31}$$

$$\frac{\partial^k}{\partial a^k}\left(\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right) = e^{2\sqrt{ab}}\left[\sum_{s=0}^{k-1}\binom{k}{s}\widehat{n}_s(a)\sum_{i=1}^{\kappa}\widehat{\Lambda}_{\kappa,i}\left[\breve{g}(a)\right] + \widehat{n}_k(a)\right], \tag{32}$$

where

$$\Lambda_{\beta-1,n}[g(a)] = B_{\beta-1,n}\big(g_1(a), g_2(a), \ldots, g_{\beta-n}(a)\big), \tag{33}$$

$$\Lambda_{\beta-1,n}[p(a)] = B_{\beta-1,n}\big(p_1(a), p_2(a), \ldots, p_{\beta-n}(a)\big), \tag{34}$$

$$\Delta_{\eta,\beta}[\widehat{g}(a)] = B_{\eta,\beta}\big(\widehat{g}_1(a), \widehat{g}_2(a), \ldots, \widehat{g}_{\eta-\beta+1}(a)\big)\,, \tag{35}$$

$$\Delta_{\eta,\beta}[\widehat{p}(a)] = B_{\eta,\beta}\big(\widehat{p}_1(a), \widehat{p}_2(a), \ldots, \widehat{p}_{\eta-\beta+1}(a)\big)\,, \tag{36}$$

$$\widehat{\Lambda}_{\kappa,i}[\breve{g}(a)] = B_{\kappa,i}\big(\breve{g}_1(a), \breve{g}_2(a), \ldots, \breve{g}_{\kappa-i+1}(a)\big)\,. \tag{37}$$

For integer γ, the following relations hold

$$g_\gamma(a) = \frac{d^\gamma\,(u^2)}{du^\gamma} = \begin{cases} 2\,u, & \textit{for} \quad \gamma = 1, \\ 2, & \textit{for} \quad \gamma = 2, \\ 0, & \textit{for} \quad \gamma > 2, \end{cases} \qquad p_\gamma(a) = \frac{d^\gamma\,(p^2)}{dp^\gamma}\,, \tag{38}$$

$$\widehat{g}_\gamma(a) = \frac{d^\gamma(-x\,\sqrt{a})}{d^\gamma a} = \frac{(-1)^{\gamma+1}\,x}{a^{\gamma-1/2}} \prod_{m=1}^{\gamma}\left(m - \frac{3}{2}\right), \qquad \widehat{p}_\gamma(a) = \frac{d^\gamma(x\,\sqrt{a})}{d^\gamma a} = -\widehat{g}_\gamma(a)\,, \tag{39}$$

$$\breve{g}_\gamma(a) = \frac{d^\gamma(2\,\sqrt{ab})}{d^\gamma a} = \frac{(-1)^\gamma\,2\,b^\gamma}{(ab)^{\gamma-1/2}} \prod_{m=1}^{\gamma}\left(m - \frac{3}{2}\right)\,,$$

$$\widehat{n}_\gamma(a) = \frac{d^\gamma}{da^\gamma}\left(\frac{1}{\sqrt{a}}\right) = \frac{(-1)^\gamma}{a^{\gamma+1/2}} \prod_{m=1}^{\gamma}\left(m - \frac{1}{2}\right)\,. \tag{40}$$

Proof. If $j = 0$ then (7) is (9), and thus

$$\widehat{L}_{-1/2}(x, a, b) = \frac{1}{4}\sqrt{\frac{\pi}{a}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) - e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right] \tag{41}$$

as required. If $j = 1, 2, 3, \ldots$ then substituting $\lambda = -j - \frac{1}{2}$ into (7) and a change of variable $\theta = \xi^2$, say, yields

$$\widehat{L}_{-j-1/2}(x, a, b) = \frac{1}{2}\int_0^{x^2} e^{-(a\xi + b/\xi)}\,\xi^{\,j-1/2}\,d\xi\,.$$

When parametric differentiation wrt a is applied to the integral

$$M_j(a) = \int_0^{x^2} e^{-(a\xi+b/\xi)}\,\xi^{\,j-1/2}\,d\xi\,,$$

the relation

$$\frac{\partial^j M_j(a)}{\partial a^j} = -M_{j+1}(a) \tag{42}$$

holds. Thus, we use (41), (42) and induction to obtain

$$\widehat{L}_{-j-1/2}(x, a, b) \;=\; (-1)^j\,\frac{\sqrt{\pi}}{4}\,\frac{\partial^j}{\partial a^j}\left\{\frac{1}{\sqrt{a}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) - e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]\right\}\,. \tag{43}$$

This can be rewritten as

$$\widehat{L}_{-j-1/2}(x,a,b) = (-1)^j \, \frac{\sqrt{\pi}}{4} \, [N(a) - M(a)] \,, \quad j = 1, 2, \ldots, \qquad \text{where}$$

$$N(a) = \frac{\partial^j}{\partial a^j}\left[\operatorname{Erfc}\left(\frac{\sqrt{b}}{x} - x\sqrt{a}\right)\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right] = \sum_{k=0}^{j}\binom{j}{k}\left[\frac{\partial^{\eta}\operatorname{Erfc}(u)}{\partial a^{\eta}}\frac{\partial^k}{\partial a^k}\left(\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right)\right], \tag{44}$$

$$M(a) = \frac{\partial^j}{\partial a^j}\left[\operatorname{Erfc}\left(\frac{\sqrt{b}}{x} + x\sqrt{a}\right)\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right] = \sum_{k=0}^{j}\binom{j}{k}\left[\frac{\partial^{\eta}\operatorname{Erfc}(p)}{\partial a^{\eta}}\frac{\partial^k}{\partial a^k}\left(\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right)\right]. \tag{45}$$

□

Note that (44) and (45) are obtainable by applying the Leibniz rule for the j^{th} derivative of a product of two factors. The objective now becomes to calculate $N(a)$ and $M(a)$ by evaluating their components one by one.

3.1. Higher derivatives of $\operatorname{Erfc}(\mathbf{u})$ **wrt** a**.** This task amounts to calculating higher derivatives of composite functions. Applying the Faà di Bruno theorem, see [4, p.139], we have

$$\frac{\partial^{\eta}\operatorname{Erfc}(u)}{\partial a^{\eta}} = \sum_{\beta=1}^{\eta}\frac{d^{\beta}\operatorname{Erfc}(u)}{du^{\beta}}\,\Delta_{\eta,\beta}[\widehat{g}(a)], \qquad \eta = 1, 2, \ldots, \tag{46}$$

where the quantity $\Delta_{\eta,\beta}[\widehat{g}(a)]$ is evaluated using (35) and (39). Applying the Faà di Bruno theorem again and using Appendix B, we get

$$\frac{d^{\beta}\operatorname{Erfc}(u)}{du^{\beta}} = \frac{-2\,e^{-u^2}}{\sqrt{\pi}}\left(1 + \sum_{n=1}^{\beta-1}(-1)^n\,\Lambda_{\beta-1,n}[g(a)]\right), \qquad \beta = 1, 2, 3, \ldots\,, \tag{47}$$

where $\Lambda_{\beta-1,n}[g(a)]$ is calculated using (33) and (38). Plugging (47) into (46) yields

$$\frac{\partial^{\eta}\operatorname{Erfc}(u)}{\partial a^{\eta}} = \frac{-2\,e^{-u^2}}{\sqrt{\pi}}\left[\Delta_{\eta,1}[\widehat{g}(a)] + \sum_{\beta=2}^{\eta}\left(\sum_{n=1}^{\beta-1}(-1)^n\Lambda_{\beta-1,n}[g(a)]\right)\Delta_{\eta,\beta}[\widehat{g}(a)]\right] \tag{48}$$

as given in (29).

3.2. Higher derivatives of $\operatorname{Erfc}(p)$ **wrt** a**.** Following the above steps we get (31)

$$\frac{\partial^{\eta}\operatorname{Erfc}(p)}{\partial a^{\eta}} = \frac{-2\,e^{-p^2}}{\sqrt{\pi}}\left[\Delta_{\eta,1}[\widehat{p}(a)] + \sum_{\beta=2}^{\eta}\left(\sum_{n=1}^{\beta-1}(-1)^n\Lambda_{\beta-1,n}[p(a)]\right)\Delta_{\eta,\beta}[\widehat{p}(a)]\right].$$

where $\Lambda_{\beta-1,n}[p(a)]$ is evaluated using (34) and (38), while $\Delta_{\eta,\beta}[\widehat{p}(a)]$ is calculated using (36) and (39).

3.3. Higher derivatives of $e^{-2\sqrt{ab}}a^{-1/2}$ and $e^{2\sqrt{ab}}a^{-1/2}$ wrt a. Applying the Leibniz rule for the k^{th} derivative of a product of two factors $e^{-2\sqrt{ab}}$ and $a^{-1/2}$ and using the value of κ given by (27, we get

$$\frac{\partial^k}{\partial a^k}\left(\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right) = \sum_{s=0}^{k}\binom{k}{s}\frac{\partial^\kappa}{\partial a^\kappa}e^{-2\sqrt{ab}}\frac{d^s}{da^s}\frac{1}{\sqrt{a}}, \tag{49}$$

$$\frac{\partial^k}{\partial a^k}\left(\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right) = \sum_{s=0}^{k}\binom{k}{s}\frac{\partial^\kappa}{\partial a^\kappa}e^{2\sqrt{ab}}\frac{d^s}{da^s}\frac{1}{\sqrt{a}}. \tag{50}$$

The expression $\widehat{n}_s(a)$ to evaluate the higher derivatives of $a^{-1/2}$ is given in (40). However, those of the composite functions are obtained by applying the Faà di Bruno formula

$$\frac{\partial^\kappa}{\partial a^\kappa}e^{-2\sqrt{ab}} = e^{-2\sqrt{ab}}\sum_{i=1}^{\kappa}(-1)^i B_{\kappa,i}\left(\breve{g}_1(a), \breve{g}_2(a), \ldots, \breve{g}_{\kappa-i+1}(a)\right),$$

$$\frac{\partial^\kappa}{\partial a^\kappa}e^{2\sqrt{ab}} = e^{2\sqrt{ab}}\sum_{i=1}^{\kappa}B_{\kappa,i}\left(\breve{g}_1(a), \breve{g}_2(a), \ldots, \breve{g}_{\kappa-i+1}(a)\right),$$

which are then substituted back into (49) and (50) to give

$$\frac{\partial^k}{\partial a^k}\left(\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right) = e^{-2\sqrt{ab}}\left[\sum_{s=0}^{k-1}\binom{k}{s}\widehat{n}_s(a)\sum_{i=1}^{\kappa}(-1)^i\widehat{\Lambda}_{\kappa,i}\left[\breve{g}(a)\right] + \widehat{n}_k(a)\right],$$

$$\frac{\partial^k}{\partial a^k}\left(\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right) = e^{2\sqrt{ab}}\left[\sum_{s=0}^{k-1}\binom{k}{s}\widehat{n}_s(a)\sum_{i=1}^{\kappa}\widehat{\Lambda}_{\kappa,i}\left[\breve{g}(a)\right] + \widehat{n}_k(a)\right],$$

which are (30) and (32) respectively. The evaluation of $\widehat{\Lambda}_{\kappa,i}[\breve{g}(a)]$ is given by (37).

Corollary 5. *If $\lambda = -j - \frac{1}{2}$ then the **upper** incomplete Laplace integral (8) is given by*

$$\widetilde{L}_{-j-1/2}(x,a,b) = \\ = (-1)^j\frac{\sqrt{\pi}}{4}\frac{\partial^j}{\partial a^j}\left\{\frac{1}{\sqrt{a}}\left[e^{-2\sqrt{ab}}\operatorname{Erfc}\left(x\sqrt{a} - \sqrt{b}/x\right) + e^{2\sqrt{ab}}\operatorname{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]\right\}.$$

Proof. To evaluate (8) when $\lambda = -j - \frac{1}{2}$, from Corollary 1 we have

$$\widetilde{L}_{-j-1/2}(x,a,b) = \frac{1}{4}\sqrt{\frac{\pi}{a}}\left[e^{-2\sqrt{ab}}\operatorname{Erfc}\left(x\sqrt{a} - \sqrt{b}/x\right) + e^{2\sqrt{ab}}\operatorname{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right], \quad \text{for} \quad j = 0.$$

If $j \geq 1$ and let $v = x\sqrt{a} - \frac{\sqrt{b}}{x}$ then

$$\widetilde{L}_{-j-1/2}(x,a,b) = \\ = (-1)^j\frac{\sqrt{\pi}}{4}\sum_{k=0}^{j}\binom{j}{k}\left[\frac{\partial^\eta\operatorname{Erfc}(v)}{\partial a^\eta}\frac{\partial^k}{\partial a^k}\left(\frac{e^{-2\sqrt{ab}}}{\sqrt{a}}\right) + \frac{\partial^\eta\operatorname{Erfc}(p)}{\partial a^\eta}\frac{\partial^k}{\partial a^k}\left(\frac{e^{2\sqrt{ab}}}{\sqrt{a}}\right)\right],$$

which can be evaluated straightforwardly by applying the results in Theorem 3.1. Here

$$\frac{\partial^\eta \mathrm{Erfc}(v)}{\partial a^\eta} = \frac{-2\,e^{-v^2}}{\sqrt{\pi}}\left[\Delta_{\eta,1}[\widehat{p}(a)] + \sum_{\beta=2}^{\eta}\left(\sum_{n=1}^{\beta-1}(-1)^n\Lambda_{\beta-1,n}[v(a)]\right)\Delta_{\eta,\beta}[\widehat{p}(a)]\right],$$

and

$$v_\gamma(a) = \frac{d^\gamma\,(v^2)}{dp^\gamma}.$$

□

Corollary 6. *Let a, b, x be positive real, j be a non–negative integer and $\lambda = j+\frac{1}{2}$. Then the **lower** incomplete Bessel function is given by*

$$\int_0^x e^{-(a\xi^2+b/\xi^2)}\xi^{-2\lambda-1}\,d\xi = \frac{1}{2}\int_0^{x^2} e^{-(a\xi+b/\xi)}\xi^{\,-j-3/2} = \,d\xi \tag{51}$$

$$= (-1)^j\,\frac{\sqrt{\pi}}{4}\,\frac{\partial^j}{\partial b^j}\left\{\frac{1}{\sqrt{b}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) + e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]\right\} \tag{52}$$

Proof. Equation (51) is formed by substituting $\lambda = j+\frac{1}{2}$ into the LHS and a change of variable $\kappa = \xi^2$. It is shown that (52) holds by letting

$$M_j(b) = \int_0^{\widehat{m}} e^{-(a\xi+b/\xi)}\xi^{\,-j-3/2}\,d\xi \qquad \forall\;\widehat{m} > 0,$$

which satisfies

$$\frac{\partial M_j(b)}{\partial b} = -M_{j+1}(b). \tag{53}$$

If $j=0$ then the LHS of (51) is the integral in (21), and

$$\widehat{L}_{1/2}(x,a,b) = \frac{1}{4}\sqrt{\frac{\pi}{b}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x - x\sqrt{a}\right) + e^{2\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]. \tag{54}$$

For $j = 1, 2, \ldots$, Equation (52) is established by using (53) and (54) and induction. Higher-order derivatives wrt b are calculated by the steps in Theorem 3.1. □

Corollary 7. *If $a > 0$, $b > 0$ and $\lambda = j+\frac{1}{2}$, then the evaluation of the **upper** extended incomplete Laplace function* (8) *is given by*

$$\int_x^\infty e^{-(a\xi^2+b/\xi^2)}\xi^{-2\lambda-1}\,d\xi \;=$$

$$= (-1)^j\,\frac{\sqrt{\pi}}{4}\,\frac{\partial^j}{\partial b^j}\left\{\frac{1}{\sqrt{b}}\left[e^{-2\sqrt{ab}}\,\mathrm{Erfc}\left(x\,\sqrt{a} - \sqrt{b}/x\right) - e^{2\,\sqrt{ab}}\,\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]\right\}. \tag{55}$$

Proof. If $j=0$ then $\widetilde{L}_{j+1/2}(x,a,b)$ reduces to the integral in (25). Thus, we have

$$\widetilde{L}_{1/2}(x,a,b) = \frac{1}{4}\sqrt{\frac{\pi}{b}}\left[e^{-2\sqrt{ab}}\mathrm{Erfc}\left(x\,\sqrt{a} - \sqrt{b}/x\right) - e^{2\,\sqrt{ab}}\mathrm{Erfc}\left(\sqrt{b}/x + x\sqrt{a}\right)\right]. \tag{56}$$

If $j = 1, 2, 3, \ldots$ then using (53) and (56) we obtain (55). A higher derivative of order j wrt b is evaluated using the steps in Theorem 3.1. □

3.4. Evaluation of the incomplete Laplace integral in cases where $\lambda = -j - \frac{1}{2}$ and $b = 0$. The evaluation of the incomplete Laplace integral in these cases is required to evaluate the standard lower and upper incomplete $\gamma(\nu, b = 0)$ and $\Gamma(\nu, b = 0)$, respectively. In these cases, the **lower** and **upper** extended incomplete Laplace integral can be straightforwardly obtained by substituting $b = 0$ into the formulæ in Theorem 3.1 and Corollary 5 respectively.

4. Statistical Applications

Statistically, the formulæ in Section 3 have important applications, because they allow for the derivation of explicit analytical formulæ for the distribution functions of the gamma and generalized inverse Gaussian distributions. Firstly, we establish the relationship between the incomplete gamma and incomplete Bessel functions with the obtained formulæ. From (1)–(2) and (7)–(8), we have

$$\gamma_\nu(x, b) = 2\,\widehat{L}_{-\lambda}(\sqrt{x}, 1, b)\,, \tag{57}$$

$$\Gamma_\nu(x, b) = 2\,\widetilde{L}_{-\lambda}(\sqrt{x}, 1, b)\,, \tag{58}$$

which allow for the derivation of explicit formulæ for the lower and upper generalized incomplete gamma functions $\gamma_\nu(x, b)$ and $\Gamma_\nu(x, b)$, when λ equals half of an odd integer. Hence, those for the standard lower and upper incomplete gamma functions $\gamma_\nu(x)$ and $\Gamma_\nu(x)$ (i.e., $a = 1, b = 0$) are also obtainable as special cases. Currently, only the formulæ for $\nu = \frac{1}{2}$ are available in the literature. Note that (1) is not defined when $\nu < 0$ and $b = 0$.

Similarly, expressions for the lower and upper incomplete Bessel functions are

$$\widehat{K}_\lambda(z, x) = \frac{\widehat{L}_\lambda(x, z^2, 1/4)}{(2z)^\lambda}\,, \tag{59}$$

$$\widetilde{K}_\lambda(z, x) = \frac{\widetilde{L}_\lambda(x, z^2, 1/4)}{(2z)^\lambda}\,, \tag{60}$$

respectively for $\lambda = \pm(j + \frac{1}{2})$.

Since (57)–(58) and (59)–(60) appear in the distribution functions of the gamma and GIG distributions, they allow for the derivation of explicit formulæ for the cumulative and complementary cumulative distribution functions of these distributions. Here we show the derivation of the c.d.f. and c.c.d.f. of the GIG which has probability density distribution

$$\mathrm{GIG}(w|\lambda, \chi, \psi) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})}\, e^{-\left(\chi w^{-1} + \psi w\right)/2}\, w^{\lambda-1}, \qquad w > 0 \tag{61}$$

where $\chi > 0$, $\psi > 0$, $\lambda \in \mathbb{R}$. This gives c.c.d.f. as

$$F(w > r|\lambda, \chi, \psi) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\chi\psi})} \int_r^\infty e^{-\left(\chi w^{-1} + \psi w\right)/2}\, w^{\lambda-1}\, dw\,, \tag{62}$$

where r is a quantile. Although a family of the GIG distributions are obtainable by varying the value of λ, currently only the c.d.f. of the special case with $\lambda = -1/2$ is available. That was obtained in [3], [10] and [14] by applying different methods.

None of these approaches, however, involves analytical evaluation of the incomplete Bessel function. Here, for $\lambda = \pm(j + \frac{1}{2})$, from Appendix A we obtain the c.c.d.f.

$$F(w > r|\lambda, \chi, \psi) = \frac{\widehat{L}_\lambda(x, z^2, 1/4)}{(2z)^\lambda K_\lambda(z)} \tag{63}$$

where $x = 1/\sqrt{2\psi r}$. Similarly, the c.d.f. is given by

$$F(w \le r|\lambda, \chi, \psi) = \frac{\widetilde{L}_\lambda(x, z^2, 1/4)}{(2z)^\lambda K_\lambda(z)}\,. \tag{64}$$

5. Numerical Computation in the R Language

We implemented (57), (58) as the routine gamma_inc_err; (59), (60) as the routine besselK_inc_err and (61) as the routine pgig in the first Author's R package **frmqa**, see [12] and [11]. There are two important computational issues which are worth highlighting. The first issue concerns computational speed of our algorithm. We used the package **partitions**, see [7], to calculate the Bell coefficients, which are obtainable by calculating the number of unrestricted partitions of integer j by solving the relation $\lambda = \pm\left(j + \frac{1}{2}\right)$. Because the computation intensifies significantly as λ increases, it has significant effect on the computational speed. In our experience, it may take up to 3 minutes to complete a computation if $|\lambda| > \frac{15}{2}$.

Secondly, because our formulæ involve addition and subtraction operations, if we use floating–point numbers which are the default accuracy setting in R, then loss of accuracy occurs when these algebraic calculations involve a number less than the smallest positive floating–point number ε (i.e. number such that $1 + \varepsilon \neq 1$, normally $\varepsilon = 2.220446 \times 10^{-16}$). To address this accuracy issue, we use the arbitrarily precise numbers instead of R *double* precision numbers. This is achieved by calling functions in the R package **Rmpfr**, see [8], which converts and performs all calculation using the Multiple Precision Floating–Point Reliably, where all arithmetic and mathematical functions work via the (GNU) C library **MPFR**. Consequently, our computation becomes much more accurate, even when involving numbers that are much smaller than ε.

Appendix A. Relations Between the Incomplete Bessel and Extended Incomplete Laplace Integrals

$$\begin{aligned}\widehat{K}_\lambda(z, x) &= \frac{1}{(2z)^\lambda}\int_0^x e^{-\left\{z^2\,\xi^2 + 1/(4\,\xi^2)\right\}}\,\xi^{-2\lambda-1}\,d\xi \\ &= \frac{1}{2}\int_{\widehat{u}}^\infty e^{-\,z/2\,(\xi + 1/\xi)}\,\xi^{\lambda-1}\,d\xi \qquad (65) \\ &= \frac{1}{2}\left(\frac{\psi}{\chi}\right)^{\lambda/2}\int_r^\infty e^{-1/2\,(\chi/w + \psi\,w)}w^{\lambda-1}\,dw\,. \qquad (66)\end{aligned}$$

To obtain (65), replace ξ^2 by $(2z\xi)^{-1}$ with $\widehat{u} = (2zx^2)^{-1}$. By substituting $z = (\chi\psi)^{1/2}$ into (65) and letting $\xi = w(\psi/\chi)^{1/2}$, (66) was obtained with $r = (2\psi x^2)^{-1}$.

Appendix B. Derivative of the Complementary Error Function

Entry **7.1.19** in [1] gives

$$\frac{d^{k+1}}{dt^{k+1}}\,\mathrm{Erf}(t) = (-1)^k\,\frac{2}{\sqrt{\pi}}\,H_k(t)\,e^{-t^2},$$

for $k = 0, 1, 2, 3, \ldots$. The Hermite polynomial is defined as

$$H_k(t) = (-1)^k\,e^{t^2}\,\frac{d^k}{dt^k}e^{-t^2},$$

and so we get

$$\begin{aligned}\frac{d^k}{dt^k}\,\mathrm{Erfc}(t) &= (-1)^{2k-1}\frac{2}{\sqrt{\pi}}\,\frac{d^{k-1}}{dt^{k-1}}\,e^{-t^2}\\ &= -\frac{2}{\sqrt{\pi}}\,\frac{d^{k-1}}{dt^{k-1}}\,e^{-t^2},\end{aligned}$$

for $k = 1, 2, 3, \ldots$

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