PRIMITIVE PERMUTATION GROUPS WITH A SUBORBIT OF LENGTH 5 AND VERTEX-PRIMITIVE GRAPHS OF VALENCY 5

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Abstract. We classify finite primitive permutation groups having a suborbit of length 5. As a corollary, we obtain a classification of finite vertex-primitive graphs of valency 5. In the process, we also classify finite almost simple groups that have a maximal subgroup isomorphic to Alt(5) or Sym(5).

1. Introduction

All graphs and groups considered in this paper are finite. Let G be a transitive permutation group on a set V and let $v \in V$. An orbit of the point-stabiliser G_v is called a *suborbit* of G and it is *non-trivial* unless it is $\{v\}$. If G is a group of automorphisms of a digraph Γ with vertex set V and arc set A, and if G is transitive on A, then the set of out-neighbours of v, $\Gamma(v) = \{u|(v,u) \in A\}$, is a suborbit of G, and its length is the (out)-valency of Γ.

Using groups to study graphs and digraphs of small valency has a long history, reaching back at least to Tutte's seminal work [26, 27] on s-arc-transitive cubic graphs. Moreover, this work of Tutte, and that of Sims [25] on primitive groups with a suborbit of length 3, led to conjectures which were not proved for more than 20 years until the classification of finite simple groups could be brought to bear in [33] and [6], respectively. This paper is concerned both with primitive groups having a small suborbit and arc-transitive graphs of small valency, and solving our problems requires application of the finite simple group classification.

It is an easy exercise to show that if a primitive permutation group has a non-trivial suborbit of length one, then it must be regular of prime order, while if it has a suborbit of length two, it must be dihedral of degree an odd prime. Primitive groups with a suborbit of length three have a more complicated structure. Classifying them was accomplished by Wong [39] using the work of Sims [25]. The classification of primitive groups with a suborbit of length four is even more difficult. After some partial results by Sims [25] and Quirin [24], this was finally completed by Wang [29] using the classification of finite simple groups.

Wang then turned his attention to the case of primitive groups with a suborbit of length 5. He proved some strong partial results [31, 32] but was unable to complete this project. This classification is the main result of our paper.

Theorem 1.1. *A primitive permutation group G has a suborbit of length 5 if and only if (G, G_v) appears in Table 1 or 2.*

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Note that each row in Table 1 corresponds to a unique primitive permutation group G with a suborbit of length 5 whereas, in Table 2, there exists one group for each value of the parameter p. (Throughout this paper, p always denotes a prime, while D_n denotes a dihedral group of order $2n$. When reading the tables, it is useful to keep in mind exceptional isomorphisms such as: $\mathrm{PSL}(2,5) \cong \mathrm{Alt}(5)$, $\mathrm{PGL}(2,5) \cong \mathrm{Sym}(5)$, $\mathrm{PSL}(2,9) \cong \mathrm{Alt}(6)$ and $\mathrm{P\Gamma L}(2,9) \cong \mathrm{Aut}(\mathrm{Sym}(6))$).

	G	G_v	$\lvert G : G_v \rvert$
(1)	$\mathrm{Alt}(5)$	D_5	6
(2)	$\mathrm{Sym}(5)$	$\mathrm{AGL}(1,5)$	6
(3)	$\mathrm{PGL}(2,9)$	D_{10}	36
(4)	M_{10}	$\mathrm{AGL}(1,5)$	36
(5)	$\mathrm{P\Gamma L}(2,9)$	$\mathrm{AGL}(1,5) \times \mathbb{Z}_2$	36
(6)	$\mathrm{PGL}(2,11)$	D_{10}	66
(7)	$\mathrm{Alt}(9)$	$(\mathrm{Alt}(4) \times \mathrm{Alt}(5)) \rtimes \mathbb{Z}_2$	126
(8)	$\mathrm{Sym}(9)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	126
(9)	$\mathrm{PSL}(2,19)$	D_{10}	171
(10)	$\mathrm{Suz}(8)$	$\mathrm{AGL}(1,5)$	1 456
(11)	J_3	$\mathrm{AGL}(2,4)$	17 442
(12)	$\mathrm{J}_3 \rtimes \mathbb{Z}_2$	$\mathrm{A\Gamma L}(2,4)$	17 442
(13)	Th	$\mathrm{Sym}(5)$	756 216 199 065 600

TABLE 1. Primitive groups with a suborbit of length 5: sporadic examples.

The classification of primitive groups with suborbits of length three or four was used by Li, Lu and Marušič [20] to obtain a classification of arc-transitive vertex-primitive graphs of valency three or four. Similarly, as an application of Theorem 1.1, we prove the following:

Theorem 1.2. *A 5-valent graph Γ is vertex-primitive if and only if $(\mathrm{Aut}(\Gamma), \mathrm{Aut}(\Gamma)_v)$ appears in Table 3.*

Note that a few well-known graphs appear in Table 3: the Clebsch graph in row (1), the Sylvester graph in row (2), the Odd graph O_5 in row (4) and, when $p = 3$, the complete graph on 6 vertices in row (9) (recall that $\mathrm{P\Sigma L}(2,9) \cong \mathrm{Sym}(6)$).

In the process of proving Theorem 1.1, we are led to classify almost simple groups admitting a maximal subgroup isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$.

Theorem 1.3. *An almost simple group G has a maximal subgroup M isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$ if and only if G appears in Table 4 or 5, respectively. Moreover, the third column in these tables gives the number c of conjugacy classes of such subgroups in G, while the fourth column gives the structure of $\mathrm{N}_G(H)/H$, where H is a subgroup of index 5 in M.*

As a consequence of our results, we are also able to prove the following two corollaries. (A graph is called *half-arc-transitive* if its automorphism group acts transitively on its vertex-set and on its edge-set, but not on its arc-set.)

	G	G_v	$\lvert G:G_v\rvert$	Conditions
(1)	$\mathbb{Z}_p \rtimes \mathbb{Z}_5$	$\mathbb{Z}_5$	p	$p \equiv 1 \pmod 5$
(2)	$\mathbb{Z}_p^2 \rtimes \mathbb{Z}_5$	$\mathbb{Z}_5$	p^2	$p \equiv -1 \pmod 5$
(3)	$\mathbb{Z}_p^4 \rtimes \mathbb{Z}_5$	$\mathbb{Z}_5$	p^4	$p \equiv \pm 2 \pmod 5$
(4)	$\mathbb{Z}_p^2 \rtimes \mathrm{D}_5$	D_5	p^2	$p \equiv \pm 1 \pmod 5$
(5)	$\mathbb{Z}_p^4 \rtimes \mathrm{D}_5$	D_5	p^4	$p \equiv \pm 2 \pmod 5$
(6)	$\mathbb{Z}_p^4 \rtimes \mathrm{AGL}(1,5)$	$\mathrm{AGL}(1,5)$	p^4	$p \neq 5$
(7)	$\mathbb{Z}_p^4 \rtimes \mathrm{Alt}(5)$	$\mathrm{Alt}(5)$	p^4	$p \neq 5$
(8)	$\mathbb{Z}_p^4 \rtimes \mathrm{Sym}(5)$	$\mathrm{Sym}(5)$	p^4	$p \neq 5$
(9)	$\mathrm{PSL}(2,p)$	$\mathrm{Alt}(5)$	$\frac{p^3-p}{120}$	$p \equiv \pm 1, \pm 9 \pmod{40}$
(10)	$\mathrm{PSL}(2,p^2)$	$\mathrm{Alt}(5)$	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
(11)	$\mathrm{P\Sigma L}(2,p^2)$	$\mathrm{Sym}(5)$	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
(12)	$\mathrm{PSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{240}$	$p \equiv \pm 1 \pmod 8$
(13)	$\mathrm{PSp}(6,p)$	$\mathrm{Alt}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{120}$	$p \equiv \pm 3, \pm 13 \pmod{40}$
(14)	$\mathrm{PGSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{120}$	$p \equiv \pm 3 \pmod 8$, $p \geqslant 11$

Table 2. Primitive groups with a suborbit of length 5: infinite families.

	$\mathrm{Aut}(\Gamma)$	$\mathrm{Aut}(\Gamma)_v$	$\lvert \mathrm{V}(\Gamma)\rvert$	Conditions
(1)	$\mathbb{Z}_2^4 \rtimes \mathrm{Sym}(5)$	$\mathrm{Sym}(5)$	16	
(2)	$\mathrm{P\Gamma L}(2,9)$	$\mathrm{AGL}(1,5) \times \mathbb{Z}_2$	36	
(3)	$\mathrm{PGL}(2,11)$	D_{10}	66	
(4)	$\mathrm{Sym}(9)$	$\mathrm{Sym}(4) \times \mathrm{Sym}(5)$	126	
(5)	$\mathrm{Suz}(8)$	$\mathrm{AGL}(1,5)$	1 456	
(6)	$\mathrm{J}_3 \rtimes 2$	$\mathrm{A\Gamma L}(2,4)$	17 442	
(7)	Th	$\mathrm{Sym}(5)$	756 216 199 065 600	
(8)	$\mathrm{PSL}(2,p)$	$\mathrm{Alt}(5)$	$\frac{p^3-p}{120}$	$p \equiv \pm 1, \pm 9 \pmod{40}$
(9)	$\mathrm{P\Sigma L}(2,p^2)$	$\mathrm{Sym}(5)$	$\frac{p^6-p^2}{120}$	$p \equiv \pm 3 \pmod{10}$
(10)	$\mathrm{PSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{240}$	$p \equiv \pm 1 \pmod 8$
(11)	$\mathrm{PGSp}(6,p)$	$\mathrm{Sym}(5)$	$\frac{p^9(p^6-1)(p^4-1)(p^2-1)}{120}$	$p \equiv \pm 3 \pmod 8$, $p \geqslant 11$

Table 3. Vertex-primitive graphs of valency 5.

Corollary 1.4. *There is no half-arc-transitive vertex-primitive graph of valency* 10.

Corollary 1.5. *There are infinitely many half-arc-transitive vertex-primitive graphs of valency* 12.

	G	c	$\mathrm{N}_G(H)/H$	Conditions
(1)	$\mathrm{Sym}(5)$	1	$\mathbb{Z}_2$	
(2)	J_2	1	1	
(3)	$\mathrm{PSL}(2,p)$	2	1	$p \equiv \pm 11, \pm 19 \pmod{40}$
(4)	$\mathrm{PSL}(2,p)$	2	$\mathbb{Z}_2$	$p \equiv \pm 1, \pm 9 \pmod{40}$
(5)	$\mathrm{PSL}(2,p^2)$	2	$\mathbb{Z}_2$	$p \equiv \pm 3 \pmod{10}$
(6)	$\mathrm{PSL}(2,2^{2r})$	1	1	r prime
(7)	$\mathrm{PSL}(2,5^r)$	1	1	r odd prime
(8)	$\mathrm{PSp}(6,3)$	1	$\mathbb{Z}_3$	
(9)	$\mathrm{PSp}(6,p)$	1	$\mathbb{Z}_{p-1}$	$p \equiv 13, 37, 43, 67 \pmod{120}$
(10)	$\mathrm{PSp}(6,p)$	1	$\mathbb{Z}_{p+1}$	$p \equiv 53, 77, 83, 107 \pmod{120}$

TABLE 4. Almost simple groups with maximal Alt(5).

	G	c	$\mathrm{N}_G(H)/H$	Conditions
(1)	$\mathrm{Alt}(7)$	1	1	
(2)	M_{11}	1	1	
(3)	$\mathrm{M}_{12} \rtimes \mathbb{Z}_2$	1	1	
(4)	$\mathrm{J}_2 \rtimes \mathbb{Z}_2$	1	1	
(5)	Th	1	$\mathbb{Z}_2$	
(6)	$\mathrm{PSL}(2,5^2)$	2	1	
(7)	$\mathrm{P\Sigma L}(2,p^2)$	2	$\mathbb{Z}_2$	$p \equiv \pm 3 \pmod{10}$
(8)	$\mathrm{PSL}(2,2^{2r}) \rtimes \mathbb{Z}_2$	1	1	r odd prime
(9)	$\mathrm{PGL}(2,5^r)$	1	1	r odd prime
(10)	$\mathrm{PSL}(3,4) \rtimes \langle \sigma \rangle$	1	1	σ a graph-field automorphism
(11)	$\mathrm{PSL}(3,5)$	1	1	
(12)	$\mathrm{PSp}(6,p)$	2	$\mathbb{Z}_2$	$p \equiv \pm 1 \pmod 8$
(13)	$\mathrm{PGSp}(6,3)$	1	1	
(14)	$\mathrm{PGSp}(6,p)$	1	$\mathbb{Z}_2$	$p \equiv \pm 3 \pmod 8$, $p \geqslant 11$

TABLE 5. Almost simple groups with maximal Sym(5).

It is easy to see that a half-arc-transitive graph must have even valency. In [20], it was proved that there is no vertex-primitive example of valency at most 8. The two results above thus imply that 12 is the smallest valency for a half-arc-transitive vertex-primitive graph, solving [20, Problem 1.3].

After some preliminaries in Section 2, we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. These proofs are conditional on the proof of Theorem 1.3 which, being slightly more technical, is delayed until Section 5. We then prove Corollary 1.4 in Section 6 and Corollary 1.5 in Section 7. Before moving on to these proofs, we correct a few mistakes in the literature on this subject that have, as far as we know, gone undetected until now.

Remark 1.6. To increase our confidence in the correctness of our results, we have checked them against databases of known examples whenever possible. More specifically, we have checked Tables 1–3 against the database of primitive groups of degree less than 4 096 [8], and Tables 4 and 5 against the database of almost simple groups of order at most 16 000 000 implemented in MAGMA [2].

While we were at it, we have also rechecked large (but not all) parts of [19], [20], [31] and [32] (our paper does not rely on the first two). In the process, we found the following mistakes.

- In [19, Theorem 1.2], it is mistakenly claimed that there exists an infinite family of 5-valent vertex-primitive 4-arc-transitive graphs. In fact, as can be inferred from Table 3, there is a unique such graph and it has order 17442.
- In [20, Table 2], in the first row, one should have $p \equiv 1 \pmod 4$. In the third row, the condition "$p \equiv \pm 1 \pmod 8$, $p \neq 7$" should be replaced by "$p \equiv \pm 1 \pmod{24}$". In the fifth row, one should have $p \neq 3$. Finally, in the last row, $\mathrm{Aut}(\Gamma)$ should be $\mathrm{PSL}(3,7).2$ and the vertex-stabiliser should be $\mathrm{Sym}(4) \times \mathrm{Sym}(3)$. (See also next item.)
- In [20, Table 3], the case when $G = \mathrm{PSL}(3,7).\langle\sigma\rangle$ where σ is a graph automorphism is missing. (This can be traced back to a typographical error in [29, Theorem 1.4(6)] where it should read $\mathrm{PSL}(3,7).2$ instead of $\mathrm{PSL}(3,7).3$. Note that the correct version appears in Theorem 1.3(2) of the same paper.)
- The case $G = \mathrm{Sym}(5)$ is missing from [31, Theorem 2.3(4)]. It is missed as in compiling [31, Theorem 0.1(3)] from [30] it is overlooked that [30] is only for primitive groups that are not 2-transitive.
- In the main theorem of [32], the case $\mathrm{soc}(G) = \mathrm{J}_3$ is missing. (Indeed, this example was already known to Weiss [34].) The error is in the proof of [32, Proposition 2.2] where the possibility is considered but erroneously discounted as, using the notation of the reference, V is not the unique maximal abelian normal subgroup of $G_{\alpha\beta}$ and so is not a characteristic subgroup of $G_{\alpha\beta}$.

2. Preliminaries

In this section, as well as in Sections 6 and 7, we will need the notion of a digraph. Since this terminology has many usages, we formalise ours here. A *digraph* Γ is a pair (V, A) where V is a set and A is a binary relation on V. The set V is called the *vertex-set* of Γ and its elements are the *vertices*, while A is the *arc-set* and its elements *arcs*. If A is a symmetric relation, then Γ is called a *graph*.

If $(u, v) \in A$, then v is an *out-neighbour* of u and u is an *in-neighbour* of v. The number of out-neighbours of v is its *out-valency*. If this does not depend on the choice of v, then it is the *out-valency* of Γ. An *automorphism* of Γ is a permutation of V that preserves A. We say that Γ is *G-arc-transitive* if G is a group of automorphisms of Γ that acts transitively on A.

The following easy lemma will prove useful. Here, for a (not necessarily normal) subgroup H of a group G, G/H denotes the set of right H-cosets in G.

Lemma 2.1. *Let $d \geqslant 2$, let G be a non-regular primitive permutation group on V and let $v \in V$ such that G_v has a unique conjugacy class of subgroups of index d, and these subgroups are maximal and self-normalising in G_v. Let H be a representative* *of this conjugacy class, and let $N = \mathrm{N}_G(H)$ be the normaliser of H in G. The following hold.*

(1) There is a one-to-one correspondence between G-arc-transitive digraphs with vertex-set V and out-valency d, and elements Hg of $(N/H) \setminus \{H\}$.

(2) Such a digraph is a graph if and only if Hg has order 2 in N/H.

(3) G_v has an orbit of length d if and only if $(N/H) \setminus \{H\} \neq \emptyset$ or, equivalently, $N > H$.

Proof. We prove the three claims in order.

(1) Let Γ be a G-arc-transitive digraph with vertex-set V and out-valency d. Since Γ is G-arc-transitive, Γ also has in-valency d. Also G_v is transitive on the d in-neighbours of v and hence the stabilisers of these in-neighbours form a conjugacy class of subgroups of G_v of index d. As there is a unique such conjugacy class and H is contained in it, we have $H = G_{uv}$ for some in-neighbour u of v. Since H is a self-normalising proper subgroup of G_v, it follows that u is the unique in-neighbour of v fixed by H (this is easily proved, and, for example, is set as an exercise in [11, Exercise 1.6.3]). The same argument on out-neighbours shows that $H = G_{vw}$ for a unique out-neighbour w of v. Since Γ is G-arc-transitive, there exists a unique coset $Hg \in G/H$ such that $(u, v)^g = (v, w)$. Note that $H^g = G_{uv}^g = G_{vw} = H$, and hence $Hg \in N/H$. Also $u \neq v$, and hence $Hg \neq H$. Thus $\varphi : \Gamma \to Hg$ is a well-defined map from G-arc-transitive digraphs of out-valency d to $(N/H) \setminus \{H\}$. We show that φ is a bijection.

To show that φ is onto, let $Hg \in (N/H) \setminus \{H\}$, and let $w = v^g$. Since $g \notin G_v$, $w \neq v$. Let Γ be the G-arc-transitive digraph with arc-set $(v, w)^G$. We have $H = H^g \leqslant (G_v)^g = G_w$ and thus $H \leqslant G_{vw}$. Since G is primitive but not regular, we have $G_{vw} < G_v$. As H is maximal in G_v, it follows that $H = G_{vw}$, and hence Γ has out-valency d. Finally, as in the previous paragraph, $G_{vw} = H = G_{uv}$ for some unique in-neighbour u of v, and we have $(u, v)^g = (v, w)$ so $\varphi(\Gamma) = Hg$.

To show that φ is one-to-one, suppose that $\Gamma = (V, A)$ and $\Delta = (V, B)$ are G-arc-transitive digraphs with vertex-set V and out-valency d, with images $\varphi(\Gamma) = Hg$ and $\varphi(\Delta) = Hk$ such that $Hg = Hk$. By the first paragraph of the proof, $H = G_{uv}$ where $A = (u, v)^G$, $u^g = v$ and $g \in N \setminus G_v$, and also $H = G_{xv}$, where $B = (x, v)^G$, $x^k = v$ and $k \in N \setminus G_v$. Since $Hg = Hk$, we have $k = hg$ for some $h \in H$, and so $u^g = v = x^k = x^{hg} = x^g$. Hence $u = x$ which implies that $A = B$ and $\Gamma = \Delta$.

(2) Suppose that Γ is a G-arc-transitive graph with vertex-set V and valency d. Adopting the notation from the first paragraph of the proof of (1), we have $u = w$. Hence $(u, v)^g = (v, u)$ and $g^2 \in G_{uv} = H$. In other words, Hg has order 2.

Conversely, if Hg is an element of order 2 in $(N/H) \setminus \{H\}$ then, adopting the notation from the second paragraph of the proof of (1), we have that $g^2 \in H = G_{vw}$. Hence $w^g = v^{g^2} = v$ and $(w, v) = (v, w)^g$, so Γ is a graph.

(3) Suppose that G_v has an orbit of length d. Let w be an element of that orbit, and let Γ be the digraph with vertex-set V and arc-set $(v, w)^G$. Clearly, Γ is G-arc-transitive and has out-valency d. Hence, by (1), $(N/H) \setminus \{H\} \neq \emptyset$.

Conversely, if $(N/H) \setminus \{H\} \neq \emptyset$, then, by (1), there exists a G-arc-transitive digraph of out-valency d and thus G_v has an orbit of length d. □

2.1. **Brauer characters.** We will often use the Brauer character tables of $\mathrm{Sym}(n)$, $\mathrm{Alt}(n)$ and their double covers for $n = 4$ or 5. For an algebraically closed field F of characteristic p and a group G, the Brauer character β of a finite-dimensional F-representation φ of G is a function that maps each p-regular element g of G to the sum of lifts to $\mathbb{C}$ of the eigenvalues of $\varphi(g)$ (see [14] for definitions).

The *degree* of β is $\beta(1)$, which equals the dimension of φ, and the image of β lies in the ring of algebraic integers in $\mathbb{C}$. By [14, Lemma 15.2], there is a ring homomorphism $^{-}$ from the ring of algebraic integers to F with the property that $\overline{\beta(g)} = \chi(g)$ for all p-regular elements g of G, where χ is the character of φ. In particular, if $\beta(g)$ is an integer, then $\chi(g)$ lies in the prime subfield of F. The Brauer character of φ is the sum of the Brauer characters of the irreducible constituents of φ, and two irreducible representations are equivalent precisely when their Brauer characters are equal [14, Theorem 15.5], which occurs exactly when their characters are equal [14, Corollary 9.22]. The Brauer character table describes the irreducible Brauer characters. If $p \nmid |G|$, then the Brauer table is the same as the complex character table [14, Theorem 15.13] and, for the groups above, can be accessed in GAP [12] or Magma [2], as well as the Atlas [7] when $n = 5$. Otherwise, the Brauer table can be accessed in GAP, or the Brauer Atlas [16] when $n = 5$. See [16] for details on how to read the tables.

The following theory will be used in conjunction with Brauer character tables. Let G be a group and F a field. If V is an FG-module and H is a subgroup of G, then V is also an FH-module which we denote by $V \downarrow H$. An irreducible FG-module V is *absolutely irreducible* if the extension of scalars $V \otimes E$ is irreducible for every field extension E of F. Note that V is absolutely irreducible if and only if $\mathrm{End}_{FG}(V) = F$ [13, Lemma VII.2.2], where $\mathrm{End}_{FG}(V)$ denotes the ring of FG-endomorphisms of V. The field F is a *splitting field* for G if every irreducible FG-module is absolutely irreducible. For a character χ of an FG-module V and a subfield K of F, let $K(\chi)$ denote the subfield of F generated by K and the image of χ.

Now suppose that G is one of the groups above. By the Brauer character tables of these groups and [13, Theorem VII.2.6], $F = \mathrm{GF}(q^2)$ is a splitting field for G for any prime power q, so every irreducible representation of G over the algebraic closure of F can be realised over F. Let $K = \mathrm{GF}(q)$. If V is an irreducible FG-module with character χ, then either $K(\chi) = K$ and $V = U \otimes F$ for some absolutely irreducible KG-module U [13, Theorem VII.1.17], or $K(\chi) = F$ and V is an irreducible KG-module of dimension $2\dim_F(V)$ [13, Theorem VII.1.16]. Moreover, every irreducible KG-module arises in this way. Indeed, suppose that W is an irreducible KG-module that is not absolutely irreducible, and let $k := \mathrm{End}_{KG}(W)$. Then k is a finite field by Wedderburn's theorem, and W is an absolutely irreducible kG-module where k-scalar multiplication is defined to be evaluation. Let χ be the character of W as a kG-module. Then $k = K(\chi)$ by [13, Theorem VII.1.16], and $K(\chi) \subseteq F$ by [13, Theorem VII.2.6] (or the Brauer tables). Hence $k = F$ and W is an irreducible FG-module, as desired. Further, $W \otimes F$ (with W as a KG-module) is a direct sum of two non-isomorphic irreducible FG-modules with the same dimension, one of which is W as an FG-module [13, Lemma VII.1.15 and Theorem VII.1.16].

To determine whether $K(\chi) = K$ or $K(\chi) = F$, we can use the Brauer character table of G. To see this, let β be the Brauer character corresponding to the FG-module V. Now $K(\chi) = K$ exactly when $\overline{\beta(g)} \in K$ for all p-regular $g \in G$ (since for each $h \in G$, $\chi(h) = \chi(h')$ for some p-regular $h' \in G$). In particular, if β has no irrational values, then $K(\chi) = K$. Otherwise, we can use [16, Appendix 1] (or [3, Section 4.2]) to determine whether $^-$ maps the irrational values of β into K.

3. Proof of Theorem 1.1

For the rest of this section, let G be a primitive group, let G_v be one of its point-stabilisers, let Δ be an orbit of G_v of length 5 and let G_v^Δ be the permutation group induced by the action of G_v on Δ.

We first report the following result of Wang [31]. (As noted earlier, the case corresponding to Table 1 (2) was mistakenly omitted in [31].)

Theorem 3.1. *G_v^Δ is soluble if and only if (G, G_v) appears in Table 1 (1–6,9,10) or Table 2 (1–6).*

It thus remains to consider the case when G_v^Δ is not soluble. Since it is a transitive permutation group of degree 5, it must be isomorphic to either $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. We first consider the case when G_v does not act faithfully on Δ.

Theorem 3.2. *$G_v^\Delta \in \{\mathrm{Alt}(5), \mathrm{Sym}(5)\}$ and G_v does not act faithfully on Δ if and only if (G, G_v) appears in Table 1 (7,8,11,12).*

Proof. This is essentially a result of Wang [32], except that the author left open the case when G is isomorphic to one of the Monster or Baby Monster sporadic groups and G_v is a maximal 2-local subgroup of G. Also, as noted earlier, the case where $\mathrm{soc}(G) = J_3$ is missed in [32]. By [18, Theorem 5.2] or [32], the order of G_v divides $2^{14} \cdot 3^2 \cdot 5$. This is impossible for the Monster by [4], and for the Baby Monster by [36]. Moreover, while running some computations, we noticed that Wang mistakenly excluded the cases corresponding to Table 1 (11,12). □

By Theorems 3.1 and 3.2, it suffices to consider the case when $G_v \cong G_v^\Delta \in \{\mathrm{Alt}(5), \mathrm{Sym}(5)\}$. Since $\mathrm{Alt}(5)$ and $\mathrm{Sym}(5)$ are 2-transitive, it follows by [23, Theorem A] that either G is almost simple, or it has a unique minimal normal subgroup which is regular. We deal with the latter case in the next two results. (Recall that a primitive group is *affine* if it has an elementary abelian regular normal subgroup.)

Lemma 3.3. *If $G_v \in \{\mathrm{Alt}(5), \mathrm{Sym}(5)\}$ and G has a unique minimal normal subgroup which is regular, then G is affine.*

Proof. Let N be the unique minimal normal subgroup of G. If N is abelian, then G is affine. We thus assume that N is non-abelian and hence $N = T^m$ for some non-abelian simple group T. Write $N = T_1 \times \cdots \times T_m$ and let $X = \mathrm{N}_{G_v}(T_1)$.

By [11, Theorem 4.7B], $m \geqslant 6$, the action by conjugation of G_v on $\{T_1, \ldots, T_m\}$ is faithful and transitive, and X has a composition factor isomorphic to T. The only non-abelian composition factor of G_v is $\mathrm{Alt}(5)$ and thus $m = |G_v : X| \leqslant 2$, which is a contradiction. □

Lemma 3.4. *G is affine and $G_v \in \{\mathrm{Alt}(5), \mathrm{Sym}(5)\}$ if and only if (G, G_v) appears in Table 2 (7,8).*

Proof. We assume that G is affine and $G_v \in \{\mathrm{Alt}(5), \mathrm{Sym}(5)\}$. By definition, G has an elementary abelian regular normal subgroup V, with $V \cong \mathbb{Z}_p^d$. Note that $G = V \rtimes G_v$. We view V as a faithful irreducible $\mathrm{GF}(p)G_v$-module.

Let H be a subgroup of index 5 in G_v and let $\mathrm{C}_V(H)$ be the centraliser of H in V. Since H is self-normalising in G_v, it follows that $\mathrm{N}_G(H) = \mathrm{C}_V(H) \rtimes H$. Lemma 2.1(3) implies that $\mathrm{C}_V(H) \neq 0$, so the trivial $\mathrm{GF}(p)H$-module is a submodule of V.

Suppose that $p > 5$ and $G_v = \mathrm{Sym}(5)$. In this case, V is isomorphic to a Specht module S^μ for some partition μ of 5. Since the trivial $\mathrm{GF}(p)H$-module is a submodule of V, [15, Theorem 9.3] implies that we can remove an element from one of the parts of μ and obtain the partition (4). If $\mu = (5)$, then V is the trivial module, a contradiction. Hence $\mu = (4, 1)$, in which case $d = 4$ and (G, G_v) appears in Table 2 (8), and conversely, the pair (G, G_v) has the required properties.

Suppose that $p > 5$ and $G_v = \mathrm{Alt}(5)$. In particular, $H = \mathrm{Alt}(4)$. Using the Brauer character tables of Alt(4) and Alt(5), we determine that $d = 4$. Hence (G, G_v) appears in Table 2 (7), and conversely, the pair (G, G_v) has the required properties.

Finally, suppose that $p \leqslant 5$. Using MAGMA, we determine that $p \leqslant 3$ and V is the deleted permutation module. Hence $d = 4$ and (G, G_v) appears in Table 2 (7,8), and conversely, the pairs (G, G_v) are examples. □

By Lemmas 3.3 and 3.4 and the remark preceding them, we may now assume that G is an almost simple group. In particular, by Theorem 1.3, the possible groups G appear in Table 4 or 5. In view of Lemma 2.1 (3), Theorem 1.1 now follows by going through these tables and ignoring the rows with $\mathrm{N}_G(H)/H = 1$. (Row (1) of Table 4 must also be ignored as M is not core-free in G in this case.)

4. Proof of Theorem 1.2

Let Γ be a 5-valent vertex-primitive graph and let $G = \mathrm{Aut}(\Gamma)$. We first show that Γ is G-arc-transitive. Suppose, on the contrary, that Γ is not G-arc-transitive and thus $G_v^{\Gamma(v)}$ is intransitive. If $G_v^{\Gamma(v)}$ has a fixed point then, since G is primitive, it is regular and cyclic of prime order at least 7. However, a non-trivial regular abelian group G of odd order cannot be the full automorphism group of a graph since the permutation sending each element to its inverse is a nontrivial automorphism with a fixed point. Thus $G_v^{\Gamma(v)}$ has two orbits, one of length 2 and one of length 3. Having an orbit of length 2 implies that G_v is a 2-group, contradicting the fact that G_v has an orbit of length 3. This concludes the proof that Γ is G-arc-transitive. In particular, G_v has an orbit of length 5, and hence, by Theorem 1.1, (G, G_v) appears in Table 1 or 2. It follows that G is either affine or almost simple.

If G is of affine type, it has a regular elementary abelian subgroup R and Γ is a Cayley graph on R, with connection set S, say. Recall that S generates R and that $|S| = 5$. Since S is inverse-closed, this implies that $R \cong \mathbb{Z}_2^a$ for some $a \leqslant 5$ and thus $|R| \leqslant 32$. It is then easy to check that Γ appears in Table 3 (1) and, conversely, that the graph in Table 3 (1) does exist and has the required properties.

We may now assume that G is almost simple. If G_v is not isomorphic to Alt(5) or Sym(5) then, by Tables 1 and 2, there are only finitely many possibilities for Γ (in fact, it has order at most 17442) and we can deal with them on a case-by-case basis, by computer if necessary. We obtain the graphs in Table 3 rows (2-4) and (6).

We may therefore assume that G_v is isomorphic to Alt(5) or Sym(5). In particular, G appears in Table 4 or 5. Note that, in these tables, $\mathrm{N}_G(H)/H$ always has at most one element of order 2. By Lemma 2.1 (2), it follows that $|\mathrm{N}_G(H)/H|$ is even and that Γ is uniquely determined by G and G_v. Note that the number of choices for G_v for a given G corresponds to the number of conjugacy classes of maximal Alt(5) or Sym(5) in G, which is listed in the third column of Tables 4 and 5, respectively. It can be checked that, in the cases where there are multiple conjugacy classes, the classes are fused by an outer automorphism of G and hence the different conjugacy classes give rise to isomorphic graphs. Finally, note that the groups appearing in rows (5), (9) and (10) of Table 4 are subgroups of the ones appearing in rows (7) and (14) of Table 5. In particular, the former can be ignored as G will not be the (full) automorphism group of Γ in these cases. Finally, the groups in row (4) of Table 4 and rows (5, 7, 12, 14) of Table 5 lead to the graphs in rows (8, 7, 9, 10, 11) of Table 3. (Row (1) of Table 4 must also be ignored for the same reason as in the last section.)

5. Proof of Theorem 1.3

Throughout this section, let G be an almost simple group with socle T, let M be a maximal subgroup of G isomorphic to Alt(5) or Sym(5), and let H be a subgroup of index 5 in M. We prove Theorem 1.3 via a sequence of lemmas.

Lemma 5.1. *Theorem 1.3 holds if T is an alternating group.*

Proof. Suppose that $T \cong \mathrm{Alt}(n)$ for some $n \geqslant 5$. The case $n \leqslant 10$ can be handled in various ways, including by computer, and we find that G appears in Table 4 rows (1,5) or Table 5 rows (1,7). (Recall that $\mathrm{Alt}(6) \cong \mathrm{PSL}(2,9)$ and $\mathrm{Sym}(6) \cong \mathrm{P\Sigma L}(2,9)$.) We thus assume that $n \geqslant 11$. Note that $\mathrm{Alt}(n) \leqslant G \leqslant \mathrm{Sym}(n)$ and we may view G as a permutation group of degree n in the natural way. If M is an intransitive subgroup of G, then $(\mathrm{Sym}(k)\times\mathrm{Sym}(m))\cap\mathrm{Alt}(n) \leqslant M$ where $n = k+m$, a contradiction since $n \geqslant 8$. If M is imprimitive, then $M = (\mathrm{Sym}(k)\wr\mathrm{Sym}(m))\cap G$ where $n = km$ and $k, m \geqslant 2$, so $\mathrm{Sym}(k)^m \cap \mathrm{Alt}(n)$ is a normal subgroup of M, a contradiction. (Here we need no restrictions on n.) Finally, Alt(5) and Sym(5) have no primitive actions of degree greater than 10. □

Lemma 5.2. *Theorem 1.3 holds if T is a sporadic simple group.*

Proof. Suppose that T is a sporadic simple group. By [4, 22], we may assume that T is not the Monster. The maximal subgroups of the remaining sporadic groups can be found in a variety of places, including [38] or the Atlas [7] (whose lists are not always complete). Most of the cases can be handled in a straightforward manner using the GAP package AtlasRep [37], and we find that G appears in Table 4 (2) or Table 5 (2–5).

The only case which presents some difficulty is when $G = \mathrm{Th}$, the Thompson sporadic group and $M \cong \mathrm{Sym}(5)$. A computation yields that $|\mathrm{N}_G(H) : H|$ is non-trivial and we give a few details. The difficulty arises because the minimum degree of a permutation representation of Th is 143 127 000. Combined with the order of Th, this makes it computationally very hard to do any non-trivial calculations directly. To overcome this problem, we perform most calculations in one of the maximal subgroups of Th, only "pulling back" to the full group when computations in two different maximal subgroups have to be reconciled. Even using these tricks, the task is computationally non-trivial. We used MAGMA as it seems to perform better with very high degree permutation representations than GAP.

It follows from the Atlas [7] that there is a unique choice for the conjugacy class of M and, clearly, there is a unique choice for the conjugacy class of H in M. Note that $\mathrm{N}_{\mathrm{Th}}(H)$ is a 2-local subgroup (as it normalises the Klein 4-subgroup of H) and therefore, by [35, Theorem 2.2] it must lie in either M_2 or M_3, which are maximal subgroups of Th isomorphic to $2^5.L_5(2)$ and $2^{1+8}.A_9$ (in Atlas notation), respectively.

We then use information from the Atlas [7] to find a permutation representation of degree 143 127 000 for M_2 and M_3. Despite the very high degree, the fact that the order of M_i is known means that it is possible to construct a base and strong generating set for M_i using randomised algorithms. It is then easy to determine the orbits of M_i, and by taking the action of M_i on one of these orbits, obtain a faithful representation of M_i of a more reasonable degree. With a representation of relatively low degree (less than 10^6), it is possible to compute all the subgroups of M_i isomorphic to Sym(4) and determine their normalisers (in M_i).

Carrying out this process, we find that M_2 has a single conjugacy class of subgroups isomorphic to Sym(4), while M_3 has four such classes. To identify which of these classes contain H, we pull them back into the degree 143 127 000 representation of Th. Due to the extremely high degree, it is impossible to test directly the conjugacy of these groups in Th, but we can compute simple invariants of them. In particular, we can determine the number of points fixed by a representative of each class. It turns out that only one conjugacy class matches the number of fixed points of H, thereby identifying H as conjugate to a particular subgroup of M_3. We can then compute the normalizer in M_3 of H to find that it has order 48, completing the verification of Table 5 (5). □

We may now assume that T is a group of Lie type. By [9], it is not an exceptional group, so it must be a classical group. Let V be the natural module for the covering group of T, let n be the dimension of V, let q be the order of the underlying field and p its characteristic.

Lemma 5.3. *If T is a classical group, then either G is as in Table 5 (10,11) or T is isomorphic to one of* $\mathrm{PSL}(2,q)$ *or* $\mathrm{PSp}(6,p)$.

Proof. For a subgroup K of $\mathrm{P\Gamma L}(V)$, we denote the preimage of K in $\Gamma\mathrm{L}(V)$ by $\widehat{K}$. That is, K is the image of $\widehat{K}$ under the homomorphism $\phi : \Gamma\mathrm{L}(V) \to \Gamma\mathrm{L}(V)/\mathrm{Z}(\mathrm{GL}(V))$.

Suppose first that $n \leqslant 6$. The maximal subgroups of the classical groups of dimension at most 6 are given in [3]. The tables at the end of this book are especially useful. Care must be taken due to the fact that the tables give the structure of the pre-images in the matrix group instead of the projective group. One must also have in mind the many exceptional isomorphisms involving Alt(5) and Sym(5) (and other isomorphisms, such as $\mathrm{PSp}(4,2) \cong \mathrm{Sym}(6)$). With this in mind, one finds that, apart from the two examples which appear in Table 5 (10,11), all examples have T isomorphic to either $\mathrm{PSL}(2,q)$ or $\mathrm{PSp}(6,p)$.

From now on, we assume that $n \geqslant 7$. In particular, $T \nleqslant M$ and, since M is maximal in G, $TM = G$ and $G/T \cong M/(T \cap M)$. By the Schreier conjecture, G/T is soluble, and hence $T \cap M \neq 1$. Let $X = \mathrm{soc}(M) \cong \mathrm{Alt}(5)$. Then $X \leqslant T$, $M = \mathrm{N}_G(X)$ and $|G : T| = |M/(T \cap M)| \leqslant |M/X| \leqslant 2$. In particular, if

$T = \mathrm{P}\Omega^+(8,q)$, then G does not contain a triality automorphism. Our argument is aided by Aschbacher's theorem for maximal subgroups of classical groups as developed in [17]. Since $n \geqslant 7$, either $G \leqslant \mathrm{P\Gamma L}(V)$ or $T = \mathrm{PSL}(V)$ and G contains a graph automorphism. In both cases, G acts on the set of subspaces of V.

Suppose that M is the stabiliser in G of a nontrivial decomposition $V = U \oplus W$. Let $m = \dim(W)$. Without loss of generality, we may assume that $m \geqslant \lceil n/2 \rceil \geqslant 4$. Let $\widehat{Y}$ be the subgroup of $\Gamma\mathrm{L}(W)$ induced on W by $\widehat{M_W}$. In the case where $T \neq \mathrm{PSL}(V)$, the maximality of M implies that either U and W are both nondegenerate, or U and W are both totally singular of dimension $n/2$. If either $T = \mathrm{PSL}(V)$ or both U and W are totally singular of dimension $n/2$, then $\widehat{Y}$ contains $\mathrm{SL}(W)$ as a normal subgroup. However, $m \geqslant 4$, contradicting the fact that M is isomorphic to one of $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. Thus $T \neq \mathrm{PSL}(V)$ and both U and W are nondegenerate. In particular, $\widehat{Y}$ contains one of $\mathrm{SU}(W)$, $\mathrm{Sp}(W)$ or $\Omega^\epsilon(W)$ as a normal subgroup. Since none of $\mathrm{PSU}(m,q)$ for $m \geqslant 4$, or $\mathrm{PSp}(m,q)$ for $m \geqslant 4$, or $\mathrm{P}\Omega^\epsilon(m,q)$ for $m \geqslant 5$, have $\mathrm{Alt}(5)$ as a composition factor, it follows that G is an orthogonal group and $m = 4$. Since $m \geqslant n/2$ it follows that $n = 7$ or 8. Thus $\widehat{M}$ contains either $\Omega(3,q) \times \Omega^{\epsilon_2}(4,q)$ or $\Omega^{\epsilon_1}(4,q) \times \Omega^{\epsilon_2}(4,q)$ as a normal subgroup. Note that, if $n = 7$, then q is odd. Also $\Omega(3,q) \cong \mathrm{PSL}(2,q)$ for q odd, $\Omega^-(4,q) \cong \mathrm{PSL}(2,q^2)$ and $\Omega^+(4,q) \cong \mathrm{SL}(2,q) \circ \mathrm{SL}(2,q)$. Since M is insoluble and has $\mathrm{Alt}(5)$ as a unique non-abelian composition factor it follows that $n \neq 7$. Moreover, when $n = 8$ we must have that $\epsilon_1 = +$, $\epsilon_2 = -$ and $q = 2$. In this case, the stabiliser of a decomposition in G will be 3-local (as $\mathrm{PSL}(2,2) \cong \mathrm{Sym}(3)$) which M is not. This contradiction completes the proof that M is not the stabiliser in G of a decomposition $V = U \oplus W$.

Suppose now that M fixes some nontrivial subspace U. As M is maximal in G, it is the stabiliser of U in G. Since M is not p-local, M is not a parabolic subgroup. In particular, $T \neq \mathrm{PSL}(V)$ and U is not a totally singular subspace. Thus, according to Aschbacher's theorem (see [17, Table 4.1A]), U is either nondegenerate or $p = 2$, G is an orthogonal group and U is a nonsingular 1-space. The latter is not possible as the stabiliser of such a 1-space in $\mathrm{P}\Omega^\pm(n,q)$ is isomorphic to $\mathrm{Sp}_{n-2}(q)$, which is not contained in $\mathrm{Sym}(5)$. It follows that U is nondegenerate and M also fixes $U^\perp$ and the decomposition $V = U \oplus U^\perp$. This contradicts the previous paragraph.

We may now assume that M does not fix any nontrivial subspace of V. Suppose that, on the other hand, X does fix a nontrivial subspace U. Since $M = \mathrm{N}_G(X)$, there is another subspace W fixed by X such that M fixes the set $\{U, W\}$. Moreover, as M is maximal in G, it is the stabiliser in G of $\{U, W\}$ and either $U < W$ or $V = U \oplus W$. The latter case contradicts an earlier statement. In the former case, since M does not fix W, we must have that $T = \mathrm{PSL}(V)$ and G contains a graph automorphism (recall that $n \geqslant 7$ so $T \ncong \mathrm{PSp}(4,q)$). However, this contradicts M not being p-local.

We have shown that X does not fix any nontrivial subspace of V and hence $\widehat{X}$ is irreducible. By [1, 31.1], we have $\widehat{X} = \widehat{X}' \circ \mathrm{Z}(\widehat{X})$. Since $\mathrm{Z}(\widehat{X})$ consists of scalars, it follows that $\widehat{X}'$ is irreducible on V. Moreover, since $\widehat{X}'$ is a perfect central extension of $\mathrm{Alt}(5)$, it is isomorphic to $\mathrm{Alt}(5)$ or $2 \cdot \mathrm{Alt}(5)$. In the Brauer character table of $\widehat{X}'$, we see that the Brauer characters with no irrational values have degree at most 6, while those with some irrational value have degree at most 3. Thus the (not necessarily absolutely) irreducible representations of $\widehat{X}'$ have dimension at most 6 (see Section 2.1), contradicting our assumption that $n \geqslant 7$. □

The next lemma follows from Dickson's classification of the subgroups of $\mathrm{PGL}(2,q)$ [10].

Lemma 5.4. *A subgroup of* $\mathrm{PSL}(2,q)$ *isomorphic to* $\mathrm{Alt}(4)$ *is self-normalising if and only if* q *is even or* $q \equiv \pm 3 \pmod 8$. *For* q *odd,* $\mathrm{Sym}(4)$ *is a self-normalising subgroup of* $\mathrm{PGL}(2,q)$ *and it is the normaliser of an* $\mathrm{Alt}(4)$.

Lemma 5.5. *Theorem 1.3 holds when* $T \cong \mathrm{PSL}(2,q)$.

Proof. Since $\mathrm{PSL}(2,5) \cong \mathrm{Alt}(5)$, we see from [3, Table 8.1] that $\mathrm{PSL}(2,5^2)$ has two classes of maximal subgroups isomorphic to $\mathrm{Sym}(5)$. This gives row (6) of Table 5. The same isomorphism also yields that there is a unique conjugacy class of maximal $\mathrm{Alt}(5)$ subgroups in $\mathrm{PSL}(2,5^r)$ for r an odd prime and a unique conjugacy class of maximal $\mathrm{Sym}(5)$ subgroups in $\mathrm{PGL}(2,5^r)$ (and no such maximal subgroups when $q = p^r$ with r not prime). Since r is odd, $5^r \equiv -3 \pmod 8$ and Lemma 5.4 implies row (7) of Table 4 and row (9) of Table 5.

Since $\mathrm{PSL}(2,4) \cong \mathrm{Alt}(5)$, we see from [3, Table 8.1] that $\mathrm{Alt}(5)$ is a maximal subgroup of $\mathrm{PSL}(2,2^{2r})$ for r an odd prime and there is a unique conjugacy class of such subgroups. Such a subgroup is normalised by a field automorphism of T of order $2r$. When $r = 2$, such an $\mathrm{Alt}(5)$ is the centraliser of the field automorphism of order two but when r is odd the centraliser of a field automorphism of order two is $\mathrm{PSL}(2,2^r)$, which does not contain an $\mathrm{Alt}(5)$. Thus when r is odd, the normaliser of $\mathrm{Alt}(5)$ in $\mathrm{PSL}(2,2^{2r}).2$ is $\mathrm{Sym}(5)$ and is a maximal subgroup. Again there is a unique conjugacy class of such subgroups. Lemma 5.4 then yields row (6) of Table 4 and row (8) of Table 5.

Using [3, Table 8.1], we see that $\mathrm{Alt}(5)$ is a maximal subgroup of $\mathrm{PSL}(2,p)$ for $p \equiv \pm 1 \pmod{10}$. There are two classes of such maximals and they are self-normalising in $\mathrm{PSL}(2,p)$. This gives rows (3) and (4) of Table 4 with the normaliser of an $\mathrm{Alt}(4)$ given by Lemma 5.4. We also see that there are two classes of maximal $\mathrm{Alt}(5)$ subgroups in $\mathrm{PSL}(2,p^2)$ when $p \equiv \pm 3 \pmod{10}$. Since $p^2 \equiv 1 \pmod 8$, by Lemma 5.4 the normaliser in T of an $\mathrm{Alt}(4)$ is $\mathrm{Sym}(4)$ and we get row (5) of Table 4. Finally, each of these $\mathrm{Alt}(5)$ subgroups is normalised but not centralised by a field automorphism. Hence we obtain two conjugacy classes of maximal $\mathrm{Sym}(5)$ subgroups in $\mathrm{P\Sigma L}(2,p^2)$. The normaliser of an $\mathrm{Alt}(4)$ in $\mathrm{P\Sigma L}(2,p^2)$ is then $\mathrm{Sym}(4) \times \mathbb{Z}_2$ and hence the $\mathrm{Sym}(4)$ in $\mathrm{P\Sigma L}(2,p^2)$ has normaliser twice as large. Hence we have row (7) of Table 5. □

Before dealing with the case where $T \cong \mathrm{PSp}(6,p)$ we need a couple of lemmas. For a permutation group X fixing a set U, we denote by X^U the permutation group induced by X on U.

Lemma 5.6. [28, p.36] *Let* p *be an odd prime. A semisimple element* A *of* $\mathrm{GL}(d,p)$ *is conjugate to an element of* $\mathrm{Sp}(d,p)$ *if and only if* A *is conjugate to* $(A^{-1})^T$.

Lemma 5.7. [17, Lemmas 4.1.1 and 4.1.12] *Let* $X \leqslant \mathrm{Sp}(d,p)$ *and suppose that* X *fixes an* m*-dimensional subspace* U *of the natural module. If* X^U *is irreducible, then* U *is either nondegenerate or totally isotropic. Moreover*

(i) *if* U *is nondegenerate then* $\mathrm{Sp}(d,p)_U \cong \mathrm{Sp}(m,p) \times \mathrm{Sp}(d-m,p)$;

(ii) *if* $(|X|,p) = 1$ *and* U *is totally isotropic, then* X *fixes another totally isotropic subspace* U^* *such that* U *and* U^* *are disjoint and* $\dim U = \dim U^*$. *Moreover,*

$$(\mathrm{Sp}(d,p)_{U \oplus U^*})^{U \oplus U^*} = \left\{ \begin{bmatrix} A & 0 \\ 0 & (A^{-1})^T \end{bmatrix} \mid A \in \mathrm{GL}(m,p) \right\}.$$

Lemma 5.8. *Theorem 1.3 holds when* $T \cong \mathrm{PSp}(6,p)$.

Proof. The cases when $p \leqslant 5$ can be verified by a MAGMA calculation. (In this case, we obtain row (8) of Table 4 and row (13) of Table 5.) We assume now that $p \geqslant 7$.

Suppose first that $p \equiv \pm 1 \pmod 8$. By [3, Table 8.29], $M \cong \mathrm{Sym}(5)$, $G = T$ and there are two possibilities for the conjugacy class containing M. Let $\widehat{M} \cong 2 \cdot \mathrm{Sym}(5)^-$ be the preimage of M in $\mathrm{Sp}(6,p)$ and let $\widehat{H} \cong 2 \cdot \mathrm{Sym}(4)^-$ be the index five subgroup of $\widehat{M}$ corresponding to H. Note that $V \downarrow \widehat{M}$ is absolutely irreducible. By considering the Brauer character tables for $2 \cdot \mathrm{Sym}(5)^-$ and $2 \cdot \mathrm{Sym}(4)^-$, we deduce that $V \downarrow \widehat{H} = U \oplus W$ where U and W are absolutely irreducible representations of $\widehat{H}$ over $\mathrm{GF}(p)$ with degree two and four respectively. Since $p \geqslant 7$, Lemma 5.7 implies that U and W are nondegenerate and hence the stabiliser in $\mathrm{Sp}(6,p)$ of this decomposition is $\mathrm{Sp}(2,p) \times \mathrm{Sp}(4,p)$. Since $\widehat{H}$ is absolutely irreducible on U and W, it follows from Schur's Lemma that the centraliser of $\widehat{H}$ in $\mathrm{Sp}(6,p)$ is $\mathrm{Z}(\mathrm{Sp}(2,p)) \times \mathrm{Z}(\mathrm{Sp}(4,p))$. By Lemma 5.4, $\mathrm{Sym}(4)$ is self-normalising in $\mathrm{PSL}(2,p)$ and hence $\widehat{H}$ is self-normalising in $\mathrm{SL}(2,p) = \mathrm{Sp}(2,p)$. Thus $\mathrm{N}_{\mathrm{Sp}(6,p)}(\widehat{H}) = \mathrm{C}_{\mathrm{Sp}(6,p)}(\widehat{H})\widehat{H}$ and so $|\,\mathrm{N}_G(H) : H| = 2$. This verifies Row (12) of Table 5.

Next suppose that $p \equiv \pm 3 \pmod 8$. It follows from [3, Table 8.29] that $M \cong \mathrm{Alt}(5) \leqslant \mathrm{PSp}(6,p)$. Let $\widehat{M} \cong 2 \cdot \mathrm{Alt}(5)$ be the preimage of M in $\mathrm{Sp}(6,p)$. When $p \equiv \pm 3, \pm 13 \pmod{40}$, [3, Table 8.29] asserts that M is maximal in $\mathrm{PSp}(6,p)$ and, moreover, $X := \mathrm{N}_{\mathrm{PGSp}(6,p)}(M) \cong \mathrm{Sym}(5)$ is a maximal subgroup of $\mathrm{PGSp}(6,p)$. When $p \equiv \pm 11, \pm 19 \pmod{40}$, M is not maximal in $\mathrm{PSp}(6,p)$ but $X := \mathrm{N}_{\mathrm{PGSp}(6,p)}(M) \cong \mathrm{Sym}(5)$ is maximal in $\mathrm{PGSp}(6,p)$. As usual, we denote the preimage of X in $\mathrm{GSp}(6,p)$ by $\widehat{X}$. Let $\widehat{H} = 2 \cdot \mathrm{Alt}(4)$ be the subgroup of $\widehat{M}$ corresponding to H. Note that $V \downarrow \widehat{M}$ is absolutely irreducible. Let χ be the character for $V \downarrow \widehat{M}$ and let F be a splitting field for $2 \cdot \mathrm{Alt}(4)$. By the Brauer character table of $2 \cdot \mathrm{Alt}(4)$, we conclude that $\chi = \chi_1 + \chi_2 + \chi_3$ over F, where the χ_i are the three irreducible representations of $2 \cdot \mathrm{Alt}(4)$ of degree two. Moreover, when $p \equiv 1 \pmod 3$, we may take $F = \mathrm{GF}(p)$, and when $p \equiv 2 \pmod 3$, we may take $F = \mathrm{GF}(p^2)$. We divide our analysis into these two cases.

Suppose first that $p \equiv 1 \pmod 3$ and $F = \mathrm{GF}(p)$. In this case V splits as the sum of three irreducible spaces U, W_1 and W_2 for $\widehat{H}$ of dimension two. By looking at the character tables and using Lemmas 5.6 and 5.7, it follows that U is nondegenerate while W_1 and W_2 are complementary totally isotropic subspaces. By Lemma 5.7, the partwise stabiliser in $\mathrm{Sp}(6,p)$ of the decomposition of V preserved by $\widehat{H}$ is $\mathrm{Sp}(2,p) \times \mathrm{GL}(2,p)$. Since the actions of $\widehat{H}$ on W_1 and W_2 are dual, the centraliser in $\mathrm{Sp}(6,p)$ of $\widehat{H} = 2 \cdot \mathrm{Alt}(4)$ is $Z_1 \times \mathrm{Z}(\mathrm{GL}(2,p))$ where $Z_1 = \mathrm{Z}(\mathrm{Sp}(2,p))$. By Lemma 5.4, $\mathrm{Alt}(4)$ is self-normalising in $\mathrm{PSp}(2,p) \cong \mathrm{PSL}(2,p)$ when $p \equiv \pm 3 \pmod 8$ and hence $\mathrm{N}_{\mathrm{Sp}(6,p)}(\widehat{H}) = \widehat{H}\,\mathrm{C}_{\mathrm{Sp}(6,p)}(\widehat{H})$. Thus $\mathrm{N}_{\mathrm{PSp}(6,p)}(H)/H$ is a cyclic group of order $p-1$. This verifies row (9) of Table 4.

Now consider $\widehat{X}$. It has an index five subgroup $\widehat{R}$ containing $Z = \mathrm{Z}(\mathrm{GSp}(6,p)) \cong \mathbb{Z}_{p-1}$ such that $R = \widehat{R}/Z \cong \mathrm{Sym}(4)$ and $\widehat{R}$ normalises $\widehat{H}$. Now $\widehat{R}$ must preserve the decomposition $V = U \perp (W_1 \oplus W_2)$ fixed by $\widehat{H}$. The partwise stabiliser of this partition in $\mathrm{GSp}(6,p)$ is $(\mathrm{Sp}(2,p) \times \mathrm{GL}(2,p)) \rtimes \langle \delta \rangle$ where δ is an element of order $p-1$ that centralises the $\mathrm{GL}(2,p)$ and generates $\mathrm{GSp}(2,p)$ with $\mathrm{Sp}(2,p)$. Since $\widehat{R}$ contains an element that does not centralise $\widehat{H}$, it follows that $\widehat{R}$ must interchange

W_1 and W_2. In particular, $|\,\mathrm{C}_Z(\widehat{R})| = 2$. It follows that $\mathrm{C}_{\mathrm{GSp}(6,p)}(\widehat{R}) = Z_1 Z$ and hence $|\,\mathrm{N}_{\mathrm{PGSp}(6,p)}(R) : R| = 2$. This verifies row (14) of Table 5 when $p \equiv 1 \pmod 3$.

We now assume that $p \equiv 2 \pmod 3$ and $F = \mathrm{GF}(p^2)$. It follows from the Brauer character table of $2 \cdot \mathrm{Alt}(4)$ that χ_1 can be realised over $\mathrm{GF}(p)$ while χ_2 and χ_3 cannot, hence the restriction of V to $\widehat{H}$ must decompose as $V = U \oplus W$ with $\dim(U) = 2$ and $\dim(W) = 4$. Since $\dim(U) \neq \dim(W)$ it follows from Lemma 5.7 that U and W are both nondegenerate and hence the stabiliser of this decomposition in $\mathrm{Sp}(6,p)$ is $\mathrm{Sp}(2,p) \times \mathrm{Sp}(4,p)$. Moreover, the image of $2 \cdot \mathrm{Alt}(4)$ in the group induced on W is contained in the subgroup $\mathbb{Z}_{p+1} \circ \mathrm{Sp}(2,p^2)$. Thus the centraliser of $\widehat{H}$ in $\mathrm{Sp}(6,p)$ is equal to $Z_1 \times Z_2$ where $Z_1 = \mathrm{Z}(\mathrm{Sp}(2,q))$ and Z_2 has order $p+1$. Since $p \equiv \pm 3 \pmod 8$, we again have that $\mathrm{N}_{\mathrm{Sp}(6,p)}(\widehat{H}) = \widehat{H}\,\mathrm{C}_{\mathrm{Sp}(6,p)}(\widehat{H})$ and hence $\mathrm{N}_{\mathrm{PSp}(6,p)}(H)/H$ is cyclic of order $p+1$. This verifies row (10) of Table 4.

Now consider $\widehat{X}$ and again let $Z = \mathrm{Z}(\mathrm{GSp}(6,p))$. Again it has an index five subgroup $\widehat{R}$ containing Z such that $R = \widehat{R}/Z \cong \mathrm{Sym}(4)$ and $\widehat{R}$ normalises $\widehat{H}$. Also $\widehat{R}$ must preserve the decomposition of $V = U \oplus W$ preserved by $\widehat{H}$. The stabiliser in $\mathrm{GSp}(6,p)$ of this decomposition is $(\mathrm{Sp}(2,p) \times \mathrm{Sp}(4,p)) \rtimes \langle \delta \rangle$ where δ has order $p-1$ and acts as an outer automorphism of order $p-1$ on both $\mathrm{Sp}(2,p)$ and $\mathrm{Sp}(4,p)$. Consider $\widehat{H}$ acting on $V' = V \otimes \mathrm{GF}(p^2)$ as a 6-dimensional space over $\mathrm{GF}(p^2)$. Since $\mathrm{GF}(p^2)$ is a splitting field for $\widehat{H}$, we have that $\widehat{H}$ decomposes V' as a nondegenerate 2-space and two totally isotropic 2-spaces. The partwise stabiliser in $\mathrm{GSp}(6,p^2)$ of this decomposition is $(\mathrm{Sp}(2,p^2) \times \mathrm{GL}(2,p^2)) \rtimes \langle \delta \rangle$ where δ is an element of order p^2-1 that centralises the $\mathrm{GL}(2,p^2)$ and, together with $\mathrm{Sp}(2,p^2)$, generates $\mathrm{GSp}(2,p^2)$. Since $\widehat{R} \backslash \widehat{H}$ contains an element that does not centralise $\widehat{H}$, it follows that $\widehat{R}$ (when viewed as acting on V') must interchange the two totally isotropic 2-spaces. Thus $\widehat{R}$ is absolutely irreducible on W. Hence $\mathrm{C}_{GSp(6,p)}(\widehat{R}) = Z_1 Z$ and $|\,\mathrm{N}_{\mathrm{PGSp}(6,p)}(R) : R| = 2$. This completes the verification of row (14) of Table 5. □

6. Proof of Corollary 1.4

Suppose, to the contrary, that Γ is a half-arc-transitive vertex-primitive graph of valency 10, let G be its automorphism group, and let (u,v) be an arc of Γ. Let $\vec{\Gamma}$ be the digraph with the same vertex-set $\mathrm{V}\,\Gamma$ as Γ and with arc-set $(u,v)^G$. Note that $\vec{\Gamma}$ is an asymmetric G-arc-transitive digraph of out-valency 5. In particular, G_v has an orbit of length 5 and (G, G_v) appears in Table 1 or 2. It follows that G is either affine or almost simple.

If G is of affine type, it has a regular elementary abelian subgroup R and Γ is a Cayley graph on R, with connection set S, say. Since R is abelian, the permutation sending every element of R to its inverse is an automorphism of Γ. On the other hand, if $s \in S$, then the composition of the inversion map with multiplication by s is an automorphism of Γ that reverses the arc $(1,s)$, contradicting the fact that Γ is half-arc-transitive.

We may now assume that G is almost simple. If G_v is not isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$ then, as in the proof of Theorem 1.2, there are only finitely many possibilities which can be handled on a case-by-case basis. These yield no examples. We may therefore assume that G_v is isomorphic to $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$. In particular, by Theorem 1.3, G appears in Table 4 or 5. By Lemma 2.1 (1-2), we may restrict our attention to rows where $\mathrm{N}_G(H)/H$ contains an element of order at least 3.

In particular, $G \cong \mathrm{PSp}(6,p)$ for some prime p with $p \equiv \pm 3, \pm 13 \pmod{40}$ and $G_v \cong \mathrm{Alt}(5)$.

Let $H = G_{uv}$ and note that $H \cong \mathrm{Alt}(4)$. Let $G^* = \mathrm{PGSp}(6,p)$. By [3, Table 8.29], $\mathrm{N}_{G^*}(G_v) \cong \mathrm{Sym}(5)$ and hence $G^* \leqslant \mathrm{N}_{\mathrm{Sym}(\mathrm{V}\,\Gamma)}(G)$ and $G_v^* = \mathrm{N}_{G^*}(G_v)$. Let Δ be the orbit of G_v^* containing u. If Δ is also an orbit of G_v, then $(u,v)^{G^*} = (u,v)^G$ and G^* is contained in the automorphism group G of Γ, a contradiction. Since $|G_v^* : G_v| = 2$, the only other possibility is that Δ is a union of two orbits of G_v of the same size, namely 5. In particular $|\Delta| = 10$. It follows that $G_{uv}^* = H$. Let $H^* = \mathrm{N}_{G_v^*}(H)$. Note that $H^* \cong \mathrm{Sym}(4)$. Since H is a characteristic subgroup of H^*, we have that $\mathrm{N}_{G^*}(H) = \mathrm{N}_{G^*}(H^*)$. If $p = 3$, then Table 5 implies that $\mathrm{N}_{G^*}(H) = H^*$. If $p \neq 3$, then it follows by Table 5 (and the fact that $\mathrm{Sym}(4)$ is a complete group) that $\mathrm{N}_{G^*}(H) = H^* \times Z$ for some $Z \cong \mathbb{Z}_2$. In both cases, we have that $\mathrm{N}_{G^*}(H)/H$ is an elementary abelian 2-group.

Let Γ^* be the digraph with vertex-set $\mathrm{V}\,\Gamma$ and with arc-set $(u,v)^{G^*}$. Since $|\Delta| = 10$, Γ^* has out-valency 10. Let w' be an out-neighbour of v. As Γ^* is G^*-arc-transitive, H and $G_{vw'}^*$ are conjugate in G and, in particular, isomorphic. On the other hand, G_v^* has a unique conjugacy class of subgroups isomorphic to $\mathrm{Alt}(4)$, and hence H and $G_{vw'}^*$ are conjugate in G_v^*. It follows that $H = G_{vw}^*$ for some out-neighbour w of v in Γ^*. Since Γ^* is G^*-arc-transitive, there exists $g \in G^*$ such that $(u,v)^g = (v,w)$. Note that g normalises H. By the previous paragraph, this implies that $g^2 \in H$. However $u^{g^2} = v^g = w$ and so $u = w$ and Γ^* is actually a graph. Since $G < G^*$, $\vec{\Gamma}$ is a sub-digraph of Γ^* and hence $\Gamma^* = \Gamma$. This implies that G^* is contained in the automorphism group of Γ which is a contradiction.

7. Proof of Corollary 1.5

We first need the following lemma.

Lemma 7.1. *Let p be a prime with $p \equiv 7, 23 \pmod{40}$, let $G = \mathrm{PSp}(6,p)$ and let M be a maximal subgroup of G isomorphic to* $\mathrm{Sym}(5)$*. If H is a subgroup of index 6 in M, then* $\mathrm{N}_G(H)/H \cong \mathbb{Z}_{p+1}$.

Proof. First, note that M actually exists by Theorem 1.3. Note also that $\mathrm{Sym}(5)$ has a unique conjugacy class of subgroups of index 6. These subgroups are maximal and isomorphic to $\mathrm{AGL}(1,5)$. Let $\widehat{M}$ and $\widehat{H}$ be the preimage of M and H in $\mathrm{Sp}(6,p)$, respectively. Note that $\widehat{M} \cong 2\cdot\mathrm{Sym}(5)^-$ and $\widehat{H} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_8$. Let V be the natural 6-dimensional module for $\mathrm{Sp}(6,p)$ over $\mathrm{GF}(p)$. Since $p \equiv 7 \pmod 8$, it follows from the Brauer character tables of $\widehat{M}$ and $\widehat{H}$ (available in Magma, for example) that $V \downarrow \widehat{H}$ splits as a sum of an absolutely irreducible 4-dimensional subspace W and an irreducible but not absolutely irreducible subspace U of dimension 2. Moreover, $\widehat{H}$ is faithful on W while elements of order 5 in $\widehat{H}$ act trivially on U. Since $p \geqslant 7$, Lemma 5.7 implies that U and W are nondegenerate and hence the stabiliser in $\mathrm{Sp}(6,p)$ of this decomposition is $\mathrm{Sp}(2,p)\times\mathrm{Sp}(4,p)$. By Schur's Lemma, $\mathrm{C}_{\mathrm{Sp}(6,p)}(\widehat{H}) = Z_1 \times Z_2$ where $Z_1 \cong \mathbb{Z}_{p+1}$ and $Z_2 = \mathrm{Z}(\mathrm{Sp}(4,p)) \cong \mathbb{Z}_2$. Since elements of order 5 in $\widehat{H}$ act trivially on U, any element of $\mathrm{N}_{\mathrm{Sp}(6,p)}(\widehat{H}) \setminus \mathrm{C}_{\mathrm{Sp}(6,p)}(\widehat{H})\widehat{H}$, must centralise the elements of order 5 in $\widehat{H}$. Moreover, for $p \equiv -1 \pmod 8$, the normaliser in $\mathrm{Sp}(2,p)$ of a cyclic group of order 8 is $Q_{2(p+1)}$. Now 5 divides p^2+1, and so the centraliser C of an element of order 5 in $\mathrm{Sp}(4,p)$ is cyclic of order p^2+1 (see [5, Proposition 3.4.3 and Remark 3.4.4]). However, 4 does not divide p^2+1 and so the Sylow 2-subgroup of C is equal to $\mathrm{Z}(\mathrm{Sp}(4,p)) = Z_2$. Thus $\mathrm{N}_{\mathrm{Sp}(6,p)}(\widehat{H}) = C_{\mathrm{Sp}(6,p)}(\widehat{H})\widehat{H}$ and so

$$\mathrm{N}_G(H)/H \cong \mathrm{N}_{\mathrm{Sp}(6,p)}(\widehat{H})/\widehat{H} \cong C_{\mathrm{Sp}(6,p)}(\widehat{H})/\,\mathrm{Z}(\widehat{H}) = (Z_1 \times Z_2)/Z_2 \cong \mathbb{Z}_{p+1}.$$

□

We are now ready to prove Corollary 1.5. Let p be a prime with $p \equiv 7, 23 \pmod{40}$ and let $G = \mathrm{PSp}(6,p)$. By Theorem 1.3, there exists a maximal subgroup M of G isomorphic to $\mathrm{Sym}(5)$. Note that $\mathrm{Sym}(5)$ has a unique conjugacy class of subgroups of index 6, and these subgroups are maximal and not normal. Let H be a subgroup of index 6 in M. By Lemma 7.1, $\mathrm{N}_G(H)/H \cong \mathbb{Z}_{p+1}$. By Lemma 2.1, there exists Γ' a G-arc-transitive digraph of out-valency 6 that is not a graph. Let Γ be the underlying graph of Γ' and let A be the automorphism group of Γ. Note that Γ has valency 12 and that $G \leqslant A \leqslant \mathrm{Sym}(V\Gamma)$. Since $\mathrm{Alt}(V\Gamma) \nleqslant A$, it follows from [21] that $\mathrm{soc}(A) = G$ and thus $A = G$ or $A = \mathrm{PGSp}(6,p)$. However, by [3, Table 8.29], M is self-normalising in $\mathrm{PGSp}(6,p)$ and thus $\mathrm{PGSp}(6,p) \nleqslant \mathrm{Sym}(V\Gamma)$. It follows that $A = G$ and hence Γ is half-arc-transitive.

As there are infinitely many primes p with $p \equiv 7, 23 \pmod{40}$, this proves that there are infinitely many vertex-primitive half-arc-transitive graphs of valency 12, as required.

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References

[1] Aschbacher, M. Finite group theory. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 1986.

[2] Bosma, W., Cannon, J., Playoust, C. The Magma algebra system. I. The user language. *J. Symbolic Comput.* **24** (1997), 235–265.

[3] Bray, J. N., Holt, D. F., Roney-Dougal, C. M. The maximal subgroups of the low-dimensional finite classical groups. With a foreword by Martin Liebeck. London Mathematical Society Lecture Note Series, 407. Cambridge University Press, Cambridge, 2013.

[4] Bray, J. N., Wilson, R. A. Explicit representations of maximal subgroups of the Monster. *J. Algebra* **300** (2006), 834–857.

[5] Burness, T. C, Giudici, M. Classical Groups, Derangements and Primes, Cambridge University Press, Australian Mathematical Society Lecture Series, 25, 2016.

[6] Cameron, P. J., Praeger, C. E., Saxl, J., Seitz, G. M. On the Sims conjecture and distance transitive graphs. *Bull. London Math. Soc.* **15** (1983), 499–506.

[7] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A., Wilson, R. A. Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With computational assistance from J. G. Thackray. Oxford University Press, Eynsham, 1985.

[8] Coutts, H. J., Quick, M., Roney-Dougal, C. M. The primitive permutation groups of degree less than 4096. *Comm. Algebra* **39** (2011), 3526–3546.

[9] Craven, D. A. Alternating subgroups of exceptional groups of Lie type, *Proc. London Math. Soc.* **115** (2017), 449–501.

[10] Dickson, L. E. Linear groups: With an exposition of the Galois field theory. Dover Publications Inc., New York, 1958.

[11] Dixon, J. D., Mortimer, B. Permutation groups. Graduate Texts in Mathematics, 163. Springer-Verlag, New York, 1996.

[12] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.8.3*; 2016, (http://www.gap-system.org).

[13] Huppert, B., Blackburn, N. Finite groups. II. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 242. AMD, 44. Springer-Verlag, Berlin-New York, 1982.

[14] Isaacs, I.M. Character Theory of Finite Groups. Dover Publications Inc., New York, 1994.

[15] James, G. D. The representation theory of the symmetric groups, Lecture Notes in Mathematics, 682. Springer, Berlin, 1978.

[16] Jansen, C., Lux, K., Parker, R., Wilson, R. An atlas of Brauer characters. Appendix 2 by T. Breuer and S. Norton. London Mathematical Society Monographs. New Series, 11. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[17] Kleidman, P., Liebeck, M. The subgroup structure of the finite classical groups. London Mathematical Society Lecture Note Series, 129. Cambridge University Press, Cambridge, 1990.

[18] Knapp, W. Primitive Permutationsgruppen mit einem zweifach primitiven Subkonstituenten. *J. Algebra* **38** (1976), 146–162.

[19] Li, C. H. The finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geqslant 4$. *Trans. Amer. Math. Soc.* **353** (2001), 3511–3529.

[20] Li, C. H., Lu, Z. P., Marušič, D. On primitive permutation groups with small suborbits and their orbital graphs. *J. Algebra* **279** (2004), 749–770.

[21] Liebeck, M. W., Praeger, C. E., Saxl, J. A classification of the maximal subgroups of the finite alternating and symmetric groups. *J. Algebra* **111** (1987), 365–383.

[22] Norton, S. P., Wilson, R. A. A correction to the 41-structure of the Monster, a construction of a new maximal subgroup $L_2(41)$ and a new Moonshine phenomenon. *J. London Math. Soc.* **87** (2013), 943–962.

[23] Praeger, C. E. Primitive permutation groups with a doubly transitive subconstituent. *J. Austral. Math. Soc. Ser. A* **45** (1988), 66–77.

[24] Quirin, W. L. Primitive permutation groups with small orbitals. *Math. Z.* **122** (1971), 267–274.

[25] Sims, C. C. Graphs and finite permutation groups. *Math. Z.* **95** (1967), 76–86.

[26] Tutte, W. T. A family of cubical graphs. *Proc. Camb. Philos. Soc.* **43** (1947), 459–474.

[27] Tutte, W. T. On the symmetry of cubic graphs. *Can. J. Math.* **11** (1959), 621–624.

[28] Wall, G. E. On the conjugacy classes in the unitary, symplectic and orthogonal groups. *J. Austral. Math. Soc.*, **3** (1963), 1–62.

[29] Wang, J. The primitive permutation groups with an orbital of length 4. *Comm. Algebra* **20** (1992), 889–921.

[30] Wang, J. Primitive permutation groups with a sharply 2-transitive subconstituent of prime degree *Comm. Algebra*, **20** (1992), 923–941.

[31] Wang, J. Primitive permutation groups with a solvable subconstituent of degree 5. *Beijing Daxue Xuebao Ziran Kexue Ban* **31** (1995), 520–526.

[32] Wang, J. Primitive permutation groups with an unfaithful subconstituent containing A_5. *Algebra Colloq.* **3** (1996), 11–18.

[33] Weiss, R. The nonexistence of 8-transitive graphs. *Combinatorica* **1** (1981), 309–311.

[34] Weiss, R. A characterization and another construction of Janko's group J_3. *Trans. Amer. Math. Soc.* **298** (1986), 621–633.

[35] Wilson, R. A. Some subgroups of the Thompson group. *J. Austral. Math. Soc. Ser. A* **44** (1988), 17–32.

[36] Wilson, R. A. The maximal subgroups of the Baby Monster, I. *J. Algebra* **211** (1999), 1–14.

[37] Wilson, R. A., Parker, R. A., Nickerson, S., Bray, J. N., Breuer, T. AtlasRep, A GAP Interface to the Atlas of Group Representations, Version 1.5, 2011. (http://www.gap-system.org/Packages/atlasrep.html).

[38] Wilson, R. A., Walsh, P., Tripp, J., Suleiman, I., Parker, R. A., Norton, S. P., Nickerson, S., Linton, S., Bray, J. N., Abbot, R. ATLAS of Finite Group Representations. (http://brauer.maths.qmul.ac.uk/Atlas/).

[39] Wong, W. J. Determination of a class of primitive permutation groups. *Math. Z.* **99** (1967), 235–246.

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