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Robust Stabilization of Time Delay Nonlinear Systems With a Triangular Structure

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ABSTRACT

In this paper, we examine the problem of robust stabilization of time-delay nonlinear systems which are in parametric strict feedback (PSF) form. We propose an iterative procedure of stabilizing controller construction similiar to *backstepping procedure* for this class of time-delay nonlinear systems.

1. INTRODUCTION

Robust stabilization for nonlinear structural systems has long been an interesting and challenging problem. Since the inception of the *matching conditions* in the control literature by Leitmann's paper [7], a number of systematic design procedures has been developed to stabilize nonlinear systems satisfying these conditions; see [3, 2, 1, 4, 11, 5, 8] for example. With the recent development of geometric theory for nonlinear feedback systems, there emerges a number of robust controller design techniques for linearizable nonlinear systems with mismatched nonlinearities/uncertainties. In [6], an iterative procedure known as *backstepping* has been developed to design adaptive controllers which globally (locally) stabilize systems which are in so-called parametric-strict-feedback (parametric-pure-feedback) form. In [9], the authors considered systems which are of a much more general form than the PSF. An iterative procedure, similiar to backstepping procedure has been used to construct a globally stabilizing state feedback controllers for those systems. Nonlinear systems with block-traingular structure have been further studied in [10] where the backstepping procedure is also used to provide global stabilization.

Recently, the problem of stabilization of uncertain time-delay systems has been of great interest to many researchers. The motivation of this paper stems from the fact that all the aforementioned results assumed that the nonlinear systems under investigation are free of time-delay. As we know that, in general, the existence of time delay degrades the control performance and sometimes makes the closed-loop stabilization difficult, especially when the systems are nonlinear. What we intend to do in this paper is to propose an iterative procedure for designing a stabilizing controller for uncertain time-delay nonlinear systems which are in PSF form.

The rest of the paper is organised as follows: In Section 2, we give a brief description of nonlinear systems which will be discussed throughout the paper. In Section 3 we propose an iterative robust controller design procedure for the class of nonlinear systems outlined in Section 2. Some conclusions are drawn in Section 4.

2. SYSTEM AND PRELIMINARIES

The class of single input time delay nonlinear systems to be considered in this paper is given by

$$\Sigma \begin{cases} \dot{x}_i(t) &= F_i(w_i(t)) + H_i(w_i(t-\tau)) \\ & \quad + G_1(w_i(t))x_{i+1}; \;\; i = 1, \cdots, n-1 \\ \dot{x}_n(t) &= F_n(w_n(t)) + H_n(w_n(t-\tau)) \\ & \quad + G_n(w_n(t))u \end{cases}$$

where x_i are the state variables, $w_i(t) = [x_1(t), \cdots, x_i(t)]$ are the state vectors, $w_i(t-\tau) = [x_1(t-\tau), \cdots, x_i(t-\tau)]$ are the delay state vector and $u \in \mathbb{R}$ is the control input of the system. The nonlinear functions $F_i(\cdot), H_i(\cdot), G_i(\cdot)$ and $H_i(\cdot)$ with $F_i(0) = 0$ and $H_i(0) = 0$ are assumed to satisfy the following conditions:

C1: The nonlinear function $F_i(w_i(t))$ is assumed to be a Carathéodory function and to satisfy

$$|F_i(w_i(t))| \leq \sum_{j=1}^{i} |x_j| K_{F_{ij}}(w_j(t))$$

where $K_{F_{ij}}(w_j(t))$ are known smooth nonlinear functions.

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C2: The nonlinear function $H_i(w_i(t-\tau))$ is assumed to be a Carathéodory function and to satisfy

$$|H_i(w_i(t-\tau))| \leq \sum_{j=1}^{i} |x_j(t-\tau)| K_{H_{ij}}(w_j(t-\tau))$$

where $K_{H_{ij}}(w_j(t-\tau))$ are known smooth nonlinear functions.

C3: $0 < G_i(w_i) \leq \gamma_i$ where γ_i are known positive constants.

To simplify our controller design, we introduce the following nonlinear coordinates transformation:

$$\mathcal{T}\begin{cases} x_1 &= z_1 \\ x_i &= z_i - \phi_{i-1}(r_{i-1}); \quad i = 2, \cdots, n. \end{cases} \tag{2.1}$$

where $r_i(t) = [z_1(t), \cdots, z_i(t)]$ and $\phi_{i-1}(r_{i-1})$ with $\phi_{i-1}(0) = 0$ are some smooth functions. The system Σ under transformation $\mathcal{T}$ becomes

$$\begin{aligned} \dot{z}_i &= f_i(r_i(t)) + h_i(r_i(t-\tau)) \\ &\quad + g_i(r_i(t))[z_{i+1} + \phi_i(r_i(t))] + \psi_{i-1}(r_i(t)) \\ &\quad + \sum_{p=1}^{i-1} \frac{\partial \phi_{i-1}(r_{i-1})}{\partial z_p} h_p(r_p(t-\tau)) \qquad (2.2) \\ \dot{z}_n &= f_n(r_n(t)) + h_n(r_n(t-\tau)) \\ &\quad + g_n(r_n(t))u + \psi_{n-1}(r_n(t)) \\ &\quad + \sum_{p=1}^{n-1} \frac{\partial \phi_{n-1}(r_{n-1})}{\partial z_p} h_p(r_p(t-\tau)) \qquad (2.3) \end{aligned}$$

where $f_i(r_i(t)), h_i(r_i(t-\tau))$ and $g_i(r_i(t))$ are functions $F_i(w_i(t)), H_i(w_i(t-\tau))$ and $G_i(w_i(t))$ in new coordinates, respectively. Also

$$\begin{aligned} \psi_{i-1}(r_i(t)) &= \sum_{p=1}^{i-1} \frac{\partial \phi_{i-1}(r_{i-1})}{\partial z_p} \{f_p(r_p) \\ &\quad + g_p(r_p)[z_{p+1} + \phi_p(r_p)]\} \qquad (2.4) \end{aligned}$$

It is readily seen that Conditions $C1-C3$ under transformation $\mathcal{T}$ are given as:

TC1: $|f_i(r_i)| \leq \sum_{j=1}^{i} |z_j| \rho_{ij}(r_j(t))$ where $\rho_{ij}(w_j(t))$ are known smooth nonlinear functions.

TC2: $|h_i(r_i(t-\tau))| \leq \sum_{j=1}^{i} |z_j(t-\tau)| \varrho_{ij}(r_j(t-\tau))$ where $\varrho_{ij}(z_j(t-\tau))$ are known smooth nonlinear functions.

TC3: $0 < g_i(r_i) \leq \gamma_i$ where γ_i are known positive constants.

Remark 1. We stress that conditions given in TC1 and TC2 are very weak. As the matter of fact, for any given smooth function, say $M(x_1, x_2, \cdots, x_i)$ with $M(0) = 0$, there always exist smooth functions $m_i(x_1, \cdots, x_i)$ such that

$$\begin{aligned} M(x_1, x_2, \cdots, x_i) &= x_1 m_1(x_1) + x_2 m_2(x_1, x_2) \\ &\quad + \cdots + x_i m_i(x_1, \cdots, x_i). \end{aligned}$$

3. MAIN RESULT

In this section, we present a design procedure, similiar to the *backstepping* procedure for constructing a nonlinear asymptotically stabilizing controller for the system Σ satisfying Conditions C1-C3. The proof is given by explicitly show how to construct a nonlinear asymptotically stabilizing controller.

Theorem 3.1. *If the system (Σ) satisfies Conditions C1-C3, then there exists a state feedback controller such that the closed-loop system is asymptotically stable.*

Proof. Construction of a Σ's controller: The design procedure adopted here is very similar to *backstepping procedure* [10, 6, 9].

Step 1. Under transformation $\mathcal{T}$, the system Σ's first equation becomes

$$\begin{aligned} \dot{z}_1 &= f_1(z_1(t)) + h_1(z_1(t-\tau)) \\ &\quad + g_1(z_1(t))[z_2 + \phi_1(z_1)] \end{aligned} \tag{3.1}$$

First let ignore the term $g_1(z_1(t))z_2$ and choose the Lyapunov function $V_1(z_1) = \frac{1}{2}z_1^2(t) + \int_{t-\tau}^{t} S(z_1(\sigma))\, d\sigma$ where $S(z_1(\sigma))$ is a positive function yet to be determined. The time-derivative of $V_1(z_1)$ along (3.1) reads

$$\begin{aligned} \dot{V}_1(z_1) &= z_1(t)[f_1(z_1(t)) + h_1(z_1(t-\tau)) \\ &\quad + g_1(z_1(t))\phi_1(z_1, t)] \\ &\quad + S(z_1(t)) - S(z_1(t-\tau)) \end{aligned} \tag{3.2}$$

By triangular inequality and Conditions TC1 and TC2, (3.2) becomes

$$\begin{aligned} \dot{V}_1(z_1) &\leq z_1^2(t)\rho_1(z_1(t)) + \tfrac{1}{2}z_1^2(t) \\ &\quad + \tfrac{1}{2}z_1^2(t-\tau)\varrho_{11}^2(z_1(t-\tau)) \\ &\quad + S(z_1(t)) - S(z_1(t-\tau)) \\ &\quad + z_1(t)g_1(z_1(t))\phi_1(z_1). \end{aligned} \tag{3.3}$$

Choose

$$S(z_1(\sigma)) = \frac{1}{2}z_1^2(\sigma)\left[n\varrho_{11}^2(z_1(\sigma)) + \sum_{i=2}^{n} \varrho_{i1}^2(z_1(\sigma))\right]$$

we have

$$\begin{aligned} \dot{V}_1(z_1) &\leq z_1^2(t)\{\rho_1(z_1(t)) + \frac{1}{2} \\ &\quad + \frac{1}{2}\left[n\varrho_{11}^2(z_1(t)) + \sum_{i=2}^{n} \varrho_{i1}^2(z_1(t))\right]\} \end{aligned}$$

$$+ z_1(t)g_1(z_1(t))\phi_1(z_1) - \frac{1}{2}z_1^2(t-\tau)\left[(n-1)\varrho_{11}^2(z_1(t-\tau)) + \sum_{i=2}^{n}\varrho_{i1}^2(z_1(t-\tau))\right] \quad (3.4)$$

In order to have $\dot{V}_1(z_1) \le 0$, we may select $\phi_1(z_1)$ as

$$\phi_1(z_1) = -\frac{1}{\gamma_1}z_1(t)[\frac{n}{2}c_1 + \varphi(z_1(t))] \quad (3.5)$$

where

$$\varphi(z_1(t)) = \frac{1}{2} + \rho_1(z_1(t)) + \frac{1}{2}\left[n\varrho_{11}^2(z_1(t)) + \sum_{i=2}^{n}\varrho_{i1}^2(z_1(t))\right] \quad (3.6)$$

and c_1 is a positive constant. Substitute (3.5) into (3.4) yields

$$\dot{V}_1(z_1) \le -\frac{n}{2}c_1z_1^2(t) - \frac{1}{2}z_1^2(t-\tau) \times [(n-1)\varrho_{11}^2(z_1(t-\tau)) + \sum_{i=2}^{n}\varrho_{i1}^2(z_1(t-\tau))] \le 0. \quad (3.7)$$

Step 2. The second equation of system Σ under transformation $\mathcal{T}$ becomes

$$\begin{aligned}\dot{z}_2 &= f_2(r_2(t)) + h_2(r_2(t-\tau)) \\ &+ g_2(r_2)[z_3 + \phi_2(r_2)] \\ &+ \psi_1(r_2) + \tfrac{\partial\phi_1(z_1)}{\partial z_1}h_1(z_1(t-\tau)\end{aligned} \quad (3.8)$$

where

$$\psi_1(r_2) = \frac{\partial\phi_1(z_1)}{\partial z_1}\{f_1(z_1(t)) + g_1(z_1(t))[z_2 + \phi_1(z_1)]\}. \quad (3.9)$$

Choose a new Lyapunov function as

$$V_2(r_2) = V_1(z_1) + \frac{1}{2}z_2^2 + \int_{t-\tau}^{t} S_2(r_2(\sigma))\, d\sigma \quad (3.10)$$

where $S_2(r_2(\sigma))$ is a positive function yet to be computed. Using (3.7), it is easy to show that the time derivative of (3.10) along (3.1) and (3.8) is given by

$$\begin{aligned}\dot{V}_2(r_2) &\le -\frac{n}{2}c_1z_1^2 - \frac{1}{2}z_1^2(t-\tau) \times \\ &\left[\sum_{i=2}^{n}\varrho_{i1}^2(z_1(t-\tau)) + (n-1)\varrho_{11}^2(z_1(t-\tau))\right] \\ &+ z_1g_1(z_1(t))z_2 + z_2\{f_2(r_2(t)) + h_2(z_1(r_2(t-\tau)) \\ &+ g_2(r_2(t))[z_3 + \phi_2(r_2)] + \psi_1(r_2(t)) \\ &+ \frac{\partial\phi_1(z_1)}{\partial z_1}h_1(z_1(t-\tau))\} \\ &+ S_2(r_2(t)) - S_2(r_2(t-\tau))\end{aligned} \quad (3.11)$$

Again using triangular inequality and Conditions TC1 and TC2, (3.11) becomes

$$\begin{aligned}\dot{V}_2(r_2) &\le -\frac{n}{2}c_1z_1^2 - \frac{1}{2}z_1^2(t-\tau) \times \\ &\left[\sum_{i=3}^{n}\varrho_{i1}^2(z_1(t-\tau)) + (n-2)\varrho_{11}^2(z_1(t-\tau))\right] \\ &+ z_2^2[\rho_2(r_2(t)) + 1] + \frac{1}{2}z_2^2(t-\tau)\varrho_{22}^2(r_2(t-\tau)) + \\ &+ z_2\psi_1(r_2) + +z_1g_1(z_1)z_2 + \frac{1}{2}z_2^2\left(\frac{\partial\phi_1(z_1)}{\partial z_1}\right)^2 \\ &+ z_2(t)g_2(r_2(t))[z_3 + \phi_2(r_2(t))] \\ &+ S_2(r_2(t)) - S_2(r_2(t-\tau))\end{aligned} \quad (3.12)$$

Let us derive an upper bound for the term $z_2\psi_1(z_1) + z_1g_1(z_1)z_2$. Using triangular inequality, we have

$$\begin{aligned}&|z_2[\psi_1(z_1) + z_1g_1(z_1)]| \le \\ &\frac{1}{2}z_2^2\left\{\left(\frac{\partial\phi_1(z_1)}{\partial z_1}[\rho_1(z_1(t)) + (\frac{n}{2}c_1 + \varphi(z_1))] + g_1(z_1)\right)^2\right. \\ &\left.+ 2\left|\frac{\partial\phi_1(z_1)}{\partial z_1}\right|\gamma_1\right\} + \frac{1}{2}z_1^2 \le \frac{1}{2}z_2^2\Phi_1(z_1) + \frac{1}{2}z_1^2\end{aligned} \quad (3.13)$$

where $\Phi_1(z_1)$ is a known smooth function (smoothness of $\Phi_1(z_1)$ is assured by Conditions TC1 and TC2). Choose

$$S(r_2(\sigma)) = \frac{1}{2}z_2^2(\sigma)[(n-1)\varrho_{22}^2(r_2(\sigma)) + \sum_{i=3}^{n}\varrho_{i2}^2(r_2(\sigma))] \quad (3.14)$$

and with the upper bound derived in (3.13) and the choice of $S(r_2(\sigma))$ given in (3.14), (3.12) becomes

$$\begin{aligned}\dot{V}_2(r_2) &\le -\frac{(n-1)}{2}c_1z_1^2 - \frac{1}{2}z_1^2(t-\tau) \times \\ &\left[\sum_{i=3}^{n}\varrho_{i1}^2(z_1(t-\tau)) + (n-2)\varrho_{11}^2(z_1(t-\tau))\right] \\ &+ z_2^2\left[\rho_2(r_2) + 1 + \left(\frac{\partial\phi_1(z_1)}{\partial z_1}\right)^2 + \frac{1}{2}\Phi_1(z_1)\right] \\ &+ \frac{1}{2}z_2^2(t)\left[(n-1)\varrho_{22}^2(r_2(t)) + \sum_{i=3}^{n}\varrho_{i2}^2(r_2(t))\right] \\ &+ z_2(t)g_2(r_2)[z_3 + \phi_2(r_2)] - \frac{1}{2}z_2^2(t-\tau) \times\end{aligned}$$

$$\left[(n-2)\varrho_{22}^2(r_2(t-\tau))+\sum_{i=3}^{n}\varrho_{i2}^2(r_2(t-\tau))\right] \quad (3.15)$$

Similiar to **Step 1**, we first forget about the term containing z_3. Choose

$$\phi_2(r_2)=-\frac{1}{\gamma_2}z_2\left[\frac{(n-1)}{2}c_1z_2+\varphi_2(r_2)\right] \quad (3.16)$$

where

$$\varphi_2(r_2)=\rho_2(r_2)+1+\left(\frac{\partial\phi_1(z_1)}{\partial z_1}\right)^2+\frac{1}{2}\Phi_1(z_1)$$
$$+\frac{1}{2}\left[(n-1)\varrho_{22}^2(r_2(t))+\sum_{i=3}^{n}\varrho_{i2}^2(r_2(t))\right] \quad (3.17)$$

and with the choice of $\phi_2(r_2)$ given in (3.16), (3.15) becomes

$$\dot V_2(r_2)\le-\frac{(n-1)c_1}{2}\|r_2\|^2-\frac{1}{2}z_1^2(t-\tau)\times$$
$$\left[(n-2)\varrho_{11}^2(z_1(t-\tau))\sum_{i=3}^{n}\varrho_{i1}^2(z_1(t-\tau))\right]$$
$$-\frac{1}{2}z_2^2(t-\tau)[\sum_{i=3}^{n}\varrho_{i2}^2(r_2(t-\tau))$$
$$+(n-2)\varrho_{22}^2(r_2(t-\tau))]. \quad (3.18)$$

Step k ($3\le k\le n-1$). Under transformation $\mathcal{T}$, the system Σ's kth equation becomes

$$\begin{aligned}\dot z_k &= f_k(r_k)+h_k(r_k(t-\tau))+\psi_{k-1}(r_k(t))\\ &\quad +g_k(r_k(t))[z_{k+1}+\phi_k(r_k(t))]\\ &\quad +\textstyle\sum_{p=1}^{k-1}\frac{\partial\phi_{k-1}(r_{k-1})}{\partial z_p}h_p(r_r(t-\tau))\end{aligned} \quad (3.19)$$

where

$$\psi_{k-1}(r_k)=\sum_{p=1}^{k-1}\frac{\partial\phi_{k-1}(r_{k-1})}{\partial z_p}\{f_p(r_p)$$
$$+g_p(r_p)[z_{p+1}+\phi_p(r_p)]\}. \quad (3.20)$$

Choose a new Lyapunov function as

$$V_k(r_k)=V_{k-1}(r_{k-1})+\frac{1}{2}z_k^2+\int_{t-\tau}^{t}S_k(r_k(\sigma))\,d\sigma \quad (3.21)$$

where $S_k(r_i(\sigma))$ is a function yet to be found. Follow from **Steps 1** and **2**, it can be shown that the time-derivative of $V_k(r_k)$ reads

$$\dot V_k(r_k)\le-\frac{(n+2-k)}{2}c_1\sum_{p=1}^{k-1}z_k^2$$
$$+z_{k-1}^tg_k(r_k)z_k-\frac{1}{2}\sum_{j=1}^{k-1}z_j^2(t-\tau)\times$$
$$\left\{(n-k+1)\varrho_{jj}^2(r_k(t-\tau)+\sum_{i=k}^{n}\varrho_{ij}^2(r_k(t-\tau))\right\}$$
$$+z_k\{f_k(r_k)+g_k(r_k)[z_{k+1}+\phi_k(r_k)]$$
$$+\psi_{k-1}(r_{k-1})+\sum_{p=1}^{i-1}\frac{\partial\phi_{k-1}(r_k)}{\partial z_p}h_p(z_p(t-\tau))\right\}$$
$$+S_k(r_k(t))-S_k(r_k(t-\tau)). \quad (3.22)$$

Again by triangular inequality and Conditions TC1 and TC2, Eq. (3.22) becomes

$$\dot V_k(r_k)\le-\frac{(n+2-k)}{2}c_1\sum_{r=1}^{k-1}z_k^2-\frac{1}{2}\sum_{j=1}^{k-1}z_j^2(t-\tau)$$
$$\left\{(n-k)\varrho_{jj}^2(r_k(t-\tau)+\sum_{i=k+1}^{n}\varrho_{ij}^2(r_k(t-\tau))\right\}$$
$$+z_{k-1}g_{k-1}(r_{k-1})z_k+z_k^2\{\rho_k(r_k)+\frac{k}{2}\}$$
$$+\frac{1}{2}z_k^2(t-\tau)\varrho_{kk}(r_k(t-\tau))$$
$$+z_kg_k(r_k)[z_{k+1}+\phi_k(r_k)]$$
$$+z_k\psi_{k-1}(r_k)+\frac{1}{2}z_k^2\sum_{p=1}^{k-1}\left(\frac{\partial\phi_{k-1}(r_{k-1})}{\partial z_p}\right)^2$$
$$+S_k(r_k(t))-S_k(r_k(t-\tau)) \quad (3.23)$$

Let us derive an upper bound for the term $z_k\psi_{k-1}(r_{k-1})+z_{k-1}g_{k-1}(r_{k-1})z_k$. By using triangular inequality, we obtain

$$|z_k[\psi_{k-1}(r_{k-1})+z_{k-1}g_{k-1}(r_{k-1})]|\le\frac{1}{2}\sum_{p=1}^{k-1}z_p^2$$
$$+\frac{1}{2}z_k^2\sum_{p=1}^{k-1}\left\{2\sum_{p=1}^{k-1}\left|\frac{\partial\phi_{k-1}(r_{k-1})}{\partial z_p}\right|\gamma_p\right.$$
$$+\left(\frac{\partial\phi_{k-1}(r_{k-1})}{\partial z_p}[\rho_p(r_p)+\varphi_p(r_p)\right.$$
$$\left.\left.\frac{(n-1-p)}{2}c_1]+g_{k-1}(r_{k-1})\right)^2\right\}$$
$$\le\frac{1}{2}z_k^2\Phi_{k-1}(r_{k-1})+\frac{1}{2}\sum_{p=1}^{k-1}z_p^2 \quad (3.24)$$

where $\Phi_{k-1}(r_{k-1})$ is some smooth nonlinear function. Once again, forget about the term containing z_{k+1}. Choose

$$S_k(r_k(\sigma))=\frac{1}{2}z_k^2(\sigma)[(n-k+1)\varrho_{kk}^2(r_k(\sigma))$$
$$+\sum_{i=k+1}^{n}\varrho_{ik}^2(r_k(\sigma))] \quad (3.25)$$

and with the choice of $S_k(r_k(\sigma))$ given in (3.25) and the upper obtained in (3.24), we have

$$\begin{aligned} &\dot{V}_k(r_k) \le -\tfrac{(n+1-k)}{2}c_1\textstyle\sum_{r=1}^{k} z_{k-1}^2 \\ &-\tfrac{1}{2}\textstyle\sum_{j=1}^{k} z_j^2(t-\tau)\left\{(n-k)\varrho_{jj}^2(r_k(t-\tau)) + \sum_{i=k+1}^{n}\varrho_{ij}^2(r_k(t-\tau))\right\} \\ &+z_k^2\{\rho_k(r_k)+\tfrac{k}{2}+\tfrac{1}{2}\Phi_{k-1}(r_{k-1})\} \\ &+ z_k g_k(r_k)\phi_k(r_k) + \tfrac{1}{2}z_k^2\textstyle\sum_{p=1}^{k-1}\left(\frac{\partial\phi_{k-1}(r_{k-1})}{\partial z_p}\right)^2 \\ &+\tfrac{1}{2}z_k^2(t)[(n-k+1)\varrho_{kk}^2(r_k(t)) + \textstyle\sum_{i=k+1}^{n}\varrho_{ik}^2(r_k(t))] \end{aligned} \tag{3.26}$$

Choose $\phi_k(r_k)$ as

$$\phi_k(r_k) = -\frac{1}{\gamma_k}z_k\left[\frac{(n+1-k)}{2}c_1+\varphi_k(r_k)\right] \tag{3.27}$$

where

$$\varphi_k(r_k) = \rho_k(r_k)+\frac{k}{2}+\frac{1}{2}\Phi_{k-1}(r_{k-1})+\frac{1}{2}z_k^2(t)\times \left[(n-k+1)\varrho_{kk}^2(r_k(t))+\sum_{i=k+1}^{n}\varrho_{ik}^2(r_k(t))\right] \tag{3.28}$$

Follow the same analysis as in $k=2$ case, we have

$$\begin{aligned} &\dot{V}_k(r_k) \le -\tfrac{(n-k)}{2}c_1\textstyle\sum_{p=1}^{k} z_p^2 \\ &-\tfrac{1}{2}\textstyle\sum_{j=1}^{k} z_j^2(t-\tau)\left\{(n-k)\varrho_{jj}^2(r_k(t-\tau)) + \sum_{i=k+1}^{n}\varrho_{ij}^2(r_k(t-\tau))\right\} \end{aligned} \tag{3.29}$$

Step n. Under transformation $\mathcal{T}$, the last equation becomes

$$\begin{aligned} \dot{z}_n &= f_i(r_n)+h_n(r_n(t-\tau))+g_n(r_n)u \\ &+\psi_{n-1}(r_n)+\sum_{p=1}^{n-1}\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}h_p(r_p(t-\tau)). \end{aligned} \tag{3.30}$$

where

$$\psi_{n-1}(r_n) = \textstyle\sum_{p=1}^{n-1}\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}\{f_p(r_p) + g_p(r_p)[z_{p+1}+\phi_p(r_p)]\} \tag{3.31}$$

Choose a new Lyapunov function as

$$V_n(r_n) = V_{n-1}(r_{n-1})+\frac{1}{2}z_n^2+\int_{t-\tau}^{t}S_n(r_n(\sigma))\,d\sigma \tag{3.32}$$

where $S_n(r_n(\sigma))$ is a function yet to be determined. It can be shown that the time derivative of (3.32) reads

$$\begin{aligned} &\dot{V}_n(r_n) \le -c_1\textstyle\sum_{p=1}^{n-1} z_p^2 - \tfrac{1}{2}\sum_{j=1}^{n-1} z_j^2(t-\tau)\times \\ &\{\varrho_{jj}^2(r_k(t-\tau))+\varrho_{nj}^2(r_n(t-\tau))\} \\ &+z_{n-1}g_n(r_n)z_n + z_n\{f_n(r_n)+g_n(r_n)u \\ &+\textstyle\sum_{p=1}^{n-1}\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}h_p(z_p(t-\tau))\} \\ &+\psi_{n-1}(r_n)+S_n(r_n(t))-S_n(r_n(t-\tau)). \end{aligned} \tag{3.33}$$

With triangular inequality and Conditions TC1 and TC2, (3.22) becomes

$$\begin{aligned} &\dot{V}_k(r_k) \le -c_1\textstyle\sum_{p=1}^{k-1} z_p^2 + z_{n-1}g_n(r_n)z_n \\ &+z_n^2\{\rho_n(r_n)+\tfrac{n}{2}\} \\ &+\tfrac{1}{2}z_n^2(t-\tau)\varrho_{nn}^2(r_n(t-\tau)) \\ &+z_n g_n(r_n)u + z_n\psi_{n-1}(r_n) \\ &+\tfrac{1}{2}z_n^2\textstyle\sum_{p=1}^{n-1}\left(\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}\right)^2 \\ &+S_n(r_n(t))-S_n(r_n(t-\tau)). \end{aligned} \tag{3.34}$$

Again, let us derive an upper bound for the term $z_n\psi_{n-1}(r_n)+z_{n-1}g_n(r_n)z_n$. By using triangular inequality, we have

$$\begin{aligned} &|z_n[\psi_{n-1}(r_{n-1})+z_{n-1}g_k(r_n)]| \le \tfrac{1}{2}\textstyle\sum_{p=1}^{n-1} z_p^2 \\ &+\tfrac{1}{2}z_n^2\textstyle\sum_{p=1}^{n-1}\Big\{\Big(\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}\times \\ &[\rho_p(r_p)+\tfrac{(n-1-p)}{2}c_1+\varphi_p(r_p)]+g_n(r_n)\Big)^2 \\ &+\textstyle\sum_{p=1}^{n-1}\left|\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}\right|\gamma_p+\tfrac{1}{2}\gamma_{n-1}^2\Big\} \\ &\le \tfrac{1}{2}z_n^2\Phi_{n-1}(r_{n-1})+\tfrac{1}{2}\textstyle\sum_{p=1}^{n-1} z_p^2 \end{aligned} \tag{3.35}$$

where $\Phi_{n-1}(r_{n-1})$ is a smooth function. Choose

$$S_n(r_n(\sigma)) = \frac{1}{2}z_n^2(\sigma)\left[\varrho_{nn}^2(r_n(\sigma))\right] \tag{3.36}$$

and with the choice of $S_n(r_n(\sigma))$ given in (3.36) and the upper bound derived in (3.35), we have

$$\begin{aligned} &\dot{V}_n(r_n) \le -\tfrac{1}{2}c_1\textstyle\sum_{p=1}^{n-1} z_p^2 + z_n^2\{\rho_n(r_n) \\ &+\tfrac{n}{2}+\tfrac{1}{2}\Phi_{n-1}(r_{n-1})\} + z_n g_n(r_n)u \\ &+\tfrac{1}{2}z_n^2\textstyle\sum_{p=1}^{n-1}\left(\frac{\partial\phi_{n-1}(r_{n-1})}{\partial z_p}\right)^2 \\ &+\tfrac{1}{2}z_n^2(t)\varrho_{nn}^2(r_n(t)) \end{aligned} \tag{3.37}$$

Now choose $u(t)$ as

$$u(t) = -\frac{1}{\gamma_n}z_n\left[\frac{1}{2}c_1+\varphi_n^2(r_n)\right] \tag{3.38}$$

where

$$\begin{aligned} \varphi_n(r_n) &= \rho_n(r_n)+\frac{n}{2}+\frac{1}{2}\Phi_{n-1}(r_{n-1}) \\ &+\frac{1}{2}z_n^2(t)\varrho_{nn}^2(r_n(t)) \end{aligned} \tag{3.39}$$

Follow the same analysis as in $k=2$ case, we obtain

$$\dot{V}_n(r_n) \le -\frac{1}{2}c_1\sum_{p=1}^{n} z_p^2. \tag{3.40}$$

Clearly, (3.40) implies that the variables z_i are exponentially stable. Now all we need to do is to show that system Σ is asymptotically stable or x_i are asymptotically stable. From the transformation defined in (2.1), we have the following relationship:

$$|x_1| = |z_1| \quad (3.41)$$

$$\begin{aligned} |x_i| &= |z_i + \phi_i(r_i)| \\ &\leq |z_i| + |\phi_i(r_i)|, \quad \text{for } i = 2, 3, \cdots, n \end{aligned} \quad (3.42)$$

Note that $\phi_i(0)$ are zero because of $\rho_i(0) = 0$ and $\varphi_i(0) = 0$. Clearly from (3.42), we deduce that x_i are asymptotically stable. ∇∇∇

4. CONCLUSIONS

In this paper, the problem of robust stabilization of time delay nonlinear system with a triangular structure has been addressed. An iterative procedure with similiar to *backstepping procedure* has been proposed to construct a robust control which stabilizes this class of time-delay nonliner systems.

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