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MINORS AND PLANAR EMBEDDINGS OF DIGRAPHS

JAMIE DAVID SNEDDON

A thesis submitted in partial fulfilment of the requirements of the degree of
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Jamie David Sneddon

March 9, 2004
Abstract

Embedding graphs in surfaces is the central concept of topological graph theory. Classifying embeddability of graphs is motivated by Kuratowski’s Theorem and Robertson-Seymour theory, which confirms that the set of obstructions to embeddability in an arbitrary surface is finite.

We consider embedding directed graphs in surfaces, with restrictions on the direction of arcs in the local rotation at each vertex. Clustered planar digraphs have planar embeddings in which, at each vertex, all of the in-arcs occur sequentially in the local rotation. Three different variations of minors are presented, each of which produces a finite set of obstructions to clustered planarity. These variations include new operations on digraphs, and measures which refine the partial ordering.

Tournaments are digraphs with exactly one edge between every distinct pair of vertices. The domination graph of a tournament is a graph with the same vertices, and an edge between two of the vertices if every other vertex is beaten by one of those two vertices. We present two variations of domination graphs, and investigate the relationships between them and their limitations. We investigate those graphs which may be domination graphs of tournaments using excluded minors. Such graphs have a finite set of obstructions under a modification of the minor partial order.
Acknowledgements

*A mathematician is a device for turning coffee into theorems.*

Paul Erdős

I could not have come this far without the support and encouragement of Paul Bonnington, who has been my supervisor from the beginning, in 1999. If Paul Erdős was correct, then I must credit some central Auckland coffee houses (where our research meetings have been conducted for several years) for at least the occasional proposition. Over coffee Paul suggested an unusual word for me to attempt to use as part of a definition in this work; the only exercise for the reader is to find it.

Marston Conder has been my co-supervisor for the last part of my research, and has always had time to hear about my recent work, which I greatly appreciate. He replaced Margaret Morton as one of my supervisors after she passed away in August 2000.

Margaret’s contribution to my research has been deeply missed. She was the lecturer for my first course in combinatorics, and got me hooked. I knew then that discrete mathematics was for me. I like to recall that when I suggested (partly
in jest) that she take me to Texas with her in 2000, she considered it seriously, decided it was possible, and helped me make it happen.

Paul, Marston and Margaret have given me invaluable assistance in improving my skills of mathematical enquiry and, as I have progressed, my mathematical writing. I couldn’t have asked for better supervisors.

The Mathematics Department and the University of Auckland have been supportive of my research — in providing the means for me to present my work to others, and in providing me with constructive employment. I’m grateful that the University of Auckland Graduate Research Fund has contributed to my travel to Denton in Texas and Aveiro in Portugal for research and conferences. Support from a Marsden Fund grant (administered by the Royal Society of New Zealand) UOA-825 has also been of valuable financial assistance.

I am thankful to have been able to work with several co-authors in the course of my degree. Many thanks to Patty McKenna and Margaret Morton for working with me on new domination conditions for tournaments; it was my first published paper and one of Margaret’s last. My work on embedding ss-digraphs and clustered planar digraphs has been encouraged, supported, complemented and co-authored by Paul Bonnington and Marston Conder.

My investigation of obstructions to clustered planarity has been like hunting for something, and the outcomes of the hunt have been reported to family and friends. I am grateful for the enthusiasm they have shown for my announcements that I had found new obstructions and operations, or in some cases, eliminated them. They have shown more interest in my work than anyone could reasonably expect, for which I am forever grateful.

Throughout my studies, my wife Liz has given me her unstinting support and has tried to understand what I am always on about. Her love, understanding and
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Jamie Sneddon
March 2004
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