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Jonathan J. Reeves
Queen's University

Conrad A. Blyth
The University of Auckland

Christopher M. Triggs
The University of Auckland

John P. Small
The University of Auckland

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Jonathan J. Reeves
Department of Economics
Queen's University, Canada

Conrad A. Blyth
Department of Economics
University of Auckland

Christopher M. Triggs
Department of Statistics
University of Auckland

John P. Small
Department of Economics
University of Auckland

Abstract. *This paper proposes a novel derivation of the Hodrick-Prescott (HP) minimization problem which leads to a generalization of the Hodrick-Prescott filter. The main result is the development of a new filter to extract a localized maximum-likelihood estimate of the cycle from a series. This new filter, the multivariate normal cyclical (MNC) filter, makes only a general assumption about the cyclical nature of the series. The output from this filtering procedure is from a nonlinear optimization routine.*

Keywords. business cycle, macroeconomic series decomposition, maximum-likelihood estimation under constraints

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1 Introduction

Hodrick and Prescott (1980) proposed the HP filter to decompose a macroeconomic time series into a nonstationary trend component and a stationary cyclical residual component. The filter has become extremely popular in applied macroeconomics in the last 15 years.

As a general principle, a macroeconomic time series can be decomposed into its seasonal variations, a "business-cycle" component, irregular short-term movements, and its long-term trend component. Kydland and Prescott (1990) emphasized that any definition of the trend and cycle is necessarily statistical, and that "a decomposition is a representation of the data. A representation is useful if, in light of theory, it reveals some interesting patterns in the data."

It is standard practice for the economic series first to be seasonally adjusted. The HP filter is then applied to the seasonally adjusted series. The cyclical residual obtained is an estimate of the combined cyclical and irregular component of the series.

The HP filter has been used by numerous authors to determine the stylized facts of a business cycle; for example, Danthine and Girardin (1989), Kydland and Prescott (1990), Backhus and Kehoe (1992), Brandner and Neusser (1992), and Kim, Buckle, and Hall (1994). King and Rebelo (1993) discussed in detail the HP filter from the perspective of the time and frequency domains, motivating it as a generalization of the exponential smoothing filter. Recently, the HP filter has been criticized for generating spurious business-cycle periodicity when there is no cycle present in the original data (Harvey and Jaeger 1991; Jaeger 1994; Cogley and Nason 1995).

This paper first briefly examines some of the main properties of the HP filter and under what conditions the HP minimization problem is statistically justified. A new derivation of the HP minimization problem is developed using maximum-likelihood estimation, which leads to a complete generalization of the HP minimization problem.

Lucas (1977) defined business cycles as deviations of aggregate real output from trend, which corresponds statistically to estimating a residual, i.e., analyzing a detrended series. However, there are cases where decomposing the cyclical residual into a purely cyclical component and a separate irregular component is useful. In particular, often it is difficult to detect turning points of a cycle for a series with a cyclical residual strongly affected with irregulars. Thus, we have employed constrained maximum-likelihood estimation to develop a new filter, the multivariate normal cyclical (MNC) filter, which extracts an empirical cycle from a series. The Baxter and King (1995) band-pass filter is also attempting to measure a similar unobserved component.

2 The Hodrick-Prescott (1980) Filter

Given an observed series y_t, let $y_t = x_t + c_t$, with $\boldsymbol{y}^T = (y_1, y_2, \ldots, y_N)$, $\boldsymbol{x}^T = (x_1, x_2, \ldots, x_N)$, and $\boldsymbol{c}^T = (c_1, c_2, \ldots, c_N)$, where x_t denotes the unobserved trend component at time t and c_t the unobserved cyclical residual at time t. The HP trend $\widehat{\boldsymbol{x}}$ can be obtained as the solution to the following convex minimization problem:

$$\min_{[x_t]_{t=1}^N} \left[\sum_{t=1}^{N} (y_t - x_t)^2 + \lambda \sum_{t=2}^{N-1} ((x_{t+1} - x_t) - (x_t - x_{t-1}))^2 \right] \qquad \lambda > 0 \tag{1}$$

Here, λ is usually known as the smoothing parameter. As λ becomes larger, the HP estimated trend curve becomes smoother. The term being squared in the second sum of Equation (1), $(x_{t+1} - x_t) - (x_t - x_{t-1})$, or $\Delta^2 x_t$, is an approximation to the second derivative of x at time t. There are two opposing forces in the HP minimization problem. One force is attempting to minimize the sum of squared cyclical residuals. The other force is attempting to minimize the sum of squared $\Delta^2 x_t$. The smoothing parameter, λ, gives relative weight to these two opposing forces.

2.1 First-order conditions of the HP minimization problem

The following HP first-order conditions are derived by setting the gradient vector of Equation (1) equal to zero. The first-order conditions are:

$$\begin{aligned}
c_1 &= \lambda(x_1 - 2x_2 + x_3) \\
c_2 &= \lambda(-2x_1 + 5x_2 - 4x_3 + x_4) \\
c_t &= \lambda(x_{t-2} - 4x_{t-1} + 6x_t - 4x_{t+1} + x_{t+2}), \quad t = 3, 4, 5, \ldots, N-2 \\
c_{N-1} &= \lambda(x_{N-3} - 4x_{N-2} + 5x_{N-1} - 2x_N) \\
c_N &= \lambda(x_{N-2} - 2x_{N-1} + x_N)
\end{aligned}$$

or more compactly,

$$\boldsymbol{c} = \lambda \boldsymbol{F} \boldsymbol{x}$$

$$\text{where} \quad \boldsymbol{F} = \begin{bmatrix}
1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & & & 0 \\
-2 & 5 & -4 & 1 & 0 & \cdots & \cdots & & & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & 0 \\
0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
0 & \cdots & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\
0 & \cdots & & \cdots & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & \cdots & \cdots & & & 0 & 1 & -4 & 5 & -2 \\
0 & \cdots & \cdots & \cdots & & & 0 & 1 & -2 & 1
\end{bmatrix}$$

which implies that

$$\boldsymbol{y} = (\lambda \boldsymbol{F} + \boldsymbol{I})\boldsymbol{x}$$

thus, the HP trend

$$\widehat{\boldsymbol{x}} = (\lambda \boldsymbol{F} + \boldsymbol{I})^{-1} \boldsymbol{y}$$

and

$$\widehat{\boldsymbol{c}} = \boldsymbol{y} - \widehat{\boldsymbol{x}}$$

Except for the four endpoints, the first-order conditions state that

$$c_t = \lambda \Delta^4 x_t, \ \text{for } t = 3, 4, 5, \ldots, N-2$$

This fact was stated by King and Rebelo (1993), “We show that the HP filter—when applied to large samples—contains a centered fourth difference and hence renders stationary time series that are ‘difference-stationary’ and, indeed, integrated of higher order.” However, the undesirable nature of this property appears to have been overlooked. There is no apparent economic reason why the cyclical residual should be proportional to the fourth difference of the trend.

Also, empirically, $\widehat{\Delta^2 x_t}$ is often strongly cyclical over time; i.e., the HP change in the rate of growth is often strongly cyclical, which is mainly an artifact of the HP filter.

Another property, derived directly from the first-order conditions, is that $\sum_{t=1}^{N} \widehat{c_t} = 0$.

Table 1
λ versus $\hat{\lambda}$ for U.S. real GNP

λ	Empirical $\hat{\lambda}$
1	1.194152
16	21.85446
200	366.3405
400	883.3563
1,000	3020.623
1,600	5835.797
6,400	38688.98

2.2 Optimality of the HP filter

Hodrick and Prescott (1980) showed that when both the change in the growth rate (i.e., $\boldsymbol{\Delta}^2\boldsymbol{x}$) and the cyclical component ($\boldsymbol{c}$) are orthogonal white-noise processes, the HP filter is an optimal filter in the sense of minimizing the mean square error. This is proved by using the standard signal-extraction formulae. Further, the parameter $\lambda = \frac{\sigma_c^2}{\sigma_x^2}$, where σ_c^2 denotes the variance of c_t and σ_x^2 denotes the variance of $\Delta^2 x_t$. Hodrick and Prescott (1980) used a "prior view that a five percent cyclical component is moderately large as is a one-eighth of one percent change in the rate of growth in a quarter. This led us to select $\lambda^{\frac{1}{2}} = \frac{5}{\frac{1}{8}}$ or $\lambda = 1{,}600$ as a value for the smoothing parameter."

This justification for choosing $\lambda = 1{,}600$ is weak in practice when the series is cyclical, because the above result has only been shown to be applicable when the cyclical component ($\boldsymbol{c}$) is a white-noise process. Not surprisingly, the empirical

$$\hat{\lambda} = \frac{\sum_{t=1}^{N} \frac{(\widehat{c_t} - \widehat{\bar{c}})^2}{N}}{\sum_{t=2}^{N-1} \frac{(\widehat{\Delta^2 x_t} - \widehat{\overline{\Delta^2 x}})^2}{N-2}}$$

is considerably different from a sensibly, a priori, chosen lambda when the series is cyclical. When the sample estimate of a parameter is considerably different from the prior belief of the actual value of the parameter, this prior belief should be questioned. This disparity between a sensibly chosen λ and its corresponding empirical $\hat{\lambda}$ is shown in Table 1. The series used is logged U.S. real GNP.[1]

A sensible choice of λ must be at least 200 for a cyclical series, as generally for $\lambda < 200$ the HP trend curve still contains a cyclical component. This can be seen by examining plots of the HP trend and plots of the spectral density functions of HP trends for a range of λ values.

When the HP filter is applied to a cyclical series, it is not sensible to look for an optimal value of λ based on the above Hodrick and Prescott (1980) result. Instead, a value of λ must be imposed into the filtering procedure (a point noted by Harvey and Jaeger [1991].) A value of $\lambda = 1{,}600$ has become the standard λ that is used for quarterly data, commonly called HP1600 filtering. It generates a cyclical residual that accords with expectations of what is generally believed to be the "business cycle." However, for λ in the range of 400–6,400, the character of the cyclical residual is very similar.

King and Rebelo (1993) found that if innovations to the growth and cyclical components are uncorrelated, then a necessary condition for the HP filtering procedure to be optimal is that the stochastic growth component have a random-walk growth rate, i.e., that it be second-difference stationary in an extension of the Nelson and Plosser (1982) terminology. However, this condition is not sufficient. For the HP filter to be optimal, they must further require that the cycle consist of uncorrelated events or that there be an identical dynamic mechanism that propagates changes in the growth rate and innovations to the business-cycle component.

[1]Unless otherwise stated, all series used are Organisation for Economic Co-operation and Development official statistics, seasonally adjusted by the national authority. The U.S. real GNP series is U.S. GNP constant prices.

2.3 A new alternative derivation of the HP minimization problem

Making similar assumptions to Hodrick and Prescott's (1980), we present a new motivation for the HP minimization problem.

Assume that

1. $\boldsymbol{\Delta}^2 \boldsymbol{x} \sim N(\boldsymbol{0}, \sigma_x^2 \boldsymbol{I})$,
2. $\boldsymbol{c} \sim N(\boldsymbol{0}, \sigma_c^2 \boldsymbol{I})$,
3. $\boldsymbol{\Delta}^2 \boldsymbol{x}$ is independent of $\boldsymbol{c}$, and
4. σ_x^2 and σ_c^2 are known.

This implies that the joint probability density function of $\boldsymbol{\Delta}^2 \boldsymbol{x}$ and $\boldsymbol{c}$ is

$$\begin{aligned} f(\boldsymbol{\Delta}^2 \boldsymbol{x}, \boldsymbol{c} | \sigma_x^2, \sigma_c^2) &= f(\boldsymbol{\Delta}^2 \boldsymbol{x} | \sigma_x^2) f(\boldsymbol{c} | \sigma_c^2) \\ &= \frac{1}{(2\sigma_x^2 \pi)^{\frac{N-2}{2}}} \exp\left\{-\frac{1}{2} \frac{(\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}}{\sigma_x^2}\right\} \frac{1}{(2\sigma_c^2 \pi)^{\frac{N}{2}}} \exp\left\{-\frac{1}{2} \frac{\boldsymbol{c}^T \boldsymbol{c}}{\sigma_c^2}\right\} \\ &= \frac{1}{(2\pi\sigma_c^2)^{\frac{N}{2}} (2\pi\sigma_x^2)^{\frac{N-2}{2}}} \exp\left\{-\frac{1}{2}\left(\frac{\boldsymbol{c}^T \boldsymbol{c}}{\sigma_c^2} + \frac{(\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}}{\sigma_x^2}\right)\right\} \end{aligned} \tag{2}$$

Given $\boldsymbol{y}$, Equation (2) can now be looked upon as a likelihood function, and we can find a maximum-likelihood estimate of $\boldsymbol{x}$, by choosing $\boldsymbol{x}$ so as to maximize equation (2).

Maximizing Equation (2) is equivalent to minimizing

$$\begin{aligned} &\frac{\boldsymbol{c}^T \boldsymbol{c}}{\sigma_c^2} + \frac{(\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}}{\sigma_x^2} \\ \Longleftrightarrow &\min_{[x_t]_{t=1}^N} \left[\boldsymbol{c}^T \boldsymbol{c} + \frac{\sigma_c^2}{\sigma_x^2} (\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}\right] \\ \Longleftrightarrow &\min_{[x_t]_{t=1}^N} [\boldsymbol{c}^T \boldsymbol{c} + \lambda (\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}], \qquad \text{where } \lambda = \frac{\sigma_c^2}{\sigma_x^2} \end{aligned}$$

Thus, under the above conditions, a maximum-likelihood estimate of the trend is being extracted when the HP filter is applied.

2.4 Improved HP filtering

If we assume that σ_x^2 and σ_c^2 are not known, we obtain maximum-likelihood estimates of σ_x^2 and σ_c^2, along with $\boldsymbol{x}$, i.e.,

$$\max_{\sigma_x^2, \sigma_c^2, [x_t]_{t=1}^N} \left[\frac{1}{(2\pi\sigma_c^2)^{\frac{N}{2}} (2\pi\sigma_x^2)^{\frac{N-2}{2}}} \exp\left\{-\frac{1}{2}\left(\frac{\boldsymbol{c}^T \boldsymbol{c}}{\sigma_c^2} + \frac{(\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}}{\sigma_x^2}\right)\right\}\right] \tag{3}$$

The maximum-likelihood estimate of σ_x^2 is

$$\widehat{\sigma_x^2} = \sum_{t=2}^{N-1} \frac{(\widehat{\Delta^2 x_t})^2}{N-2} = \frac{\widehat{(\boldsymbol{\Delta}^2 \boldsymbol{x})}^T \widehat{\boldsymbol{\Delta}^2 \boldsymbol{x}}}{N-2} \tag{4}$$

The maximum-likelihood estimate of σ_c^2 is

$$\widehat{\sigma_c^2} = \sum_{t=1}^{N} \frac{(\widehat{c_t})^2}{N} = \frac{\widehat{\boldsymbol{c}}^T \widehat{\boldsymbol{c}}}{N} \tag{5}$$

And the maximum-likelihood estimate of $\boldsymbol{x}$ must satisfy the standard first-order conditions of the HP minimization problem.

The first-order conditions of the improved HP optimization problem are

$$\widehat{\boldsymbol{c}} = \frac{\widehat{\sigma_c^2}}{\widehat{\sigma_x^2}} \boldsymbol{F}\widehat{\boldsymbol{x}} = \frac{\widehat{\boldsymbol{c}}^T \widehat{\boldsymbol{c}}(N-2)}{(\widehat{\boldsymbol{\Delta}^2 \boldsymbol{x}})^T \widehat{\boldsymbol{\Delta}^2 \boldsymbol{x}} N} \boldsymbol{F}\widehat{\boldsymbol{x}}$$

The improved HP optimization problem, Equation (3), can be simplified to

$$\min_{[x_t]_{t=1}^N} \left[\log(\boldsymbol{c}^T \boldsymbol{c}) + \frac{N-2}{N} \log((\boldsymbol{\Delta}^2 \boldsymbol{x})^T \boldsymbol{\Delta}^2 \boldsymbol{x}) \right] \tag{6}$$

This can be seen by substituting Equations (4) and (5) into Equation (3).

The fundamental problem with these optimization problems is that they have only been given strong statistical justification when $\boldsymbol{c}$ is a white-noise random vector, i.e., there is no cycle present in the series. The HP filtering has only been justified when the series consists of a slowly changing trend and white noise. The filter had been popular in actuarial science, where many series are not cyclical, before it was applied to economics. The improved HP filter is not suitable for application to a series that does not have a smoothly changing trend.

3 Generalized HP Filtering

The assumptions made in the motivation of the HP minimization problem are altered to allow for general covariance matrices that can model cyclical behavior.

Assume that

1. $\boldsymbol{\Delta}^2 \boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{\Psi})$,
2. $\boldsymbol{c} \sim N(\boldsymbol{0}, \boldsymbol{\Omega})$, and
3. $\boldsymbol{\Delta}^2 \boldsymbol{x}$ is independent of $\boldsymbol{c}$.

This implies that

$$\begin{aligned} f(\boldsymbol{\Delta}^2 \boldsymbol{x}, \boldsymbol{c}) &= f(\boldsymbol{\Delta}^2 \boldsymbol{x}) f(\boldsymbol{c}) \\ &= \frac{1}{(\det(\boldsymbol{\Psi})(2\pi)^{N-2})^{1/2}} \exp\left\{-\frac{1}{2} \boldsymbol{\Delta}^2 \boldsymbol{x}^T \boldsymbol{\Psi}^{-1} \boldsymbol{\Delta}^2 \boldsymbol{x}\right\} \frac{1}{(\det(\boldsymbol{\Omega})(2\pi)^N)^{1/2}} \exp\left\{-\frac{1}{2} \boldsymbol{c}^T \boldsymbol{\Omega}^{-1} \boldsymbol{c}\right\} \end{aligned} \tag{7}$$

Given $\boldsymbol{y}$, we now want to choose $\boldsymbol{x}$ so as to maximize Equation (7). This is the generalized HP optimization problem. However, in most situations, $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$ are unknown. One possible way to implement generalized HP filtering is to make the additional assumptions that $\boldsymbol{\Delta}^2 \boldsymbol{x}$ and $\boldsymbol{c}$ are stationary processes, and that the theoretical population parameters, $\boldsymbol{\Psi}$ and $\boldsymbol{\Omega}$, equal their corresponding sample values. That is, maximize Equation (7) subject to the following constraints:[2]

Let the i, jth position of the $\boldsymbol{\Psi}$ matrix be $\psi_{i,j}$, and

$$\begin{aligned} \psi_{1,j} &= \textstyle\sum_{t=2}^{N-j} \frac{(\Delta^2 x_t - \overline{\Delta^2 x})(\Delta^2 x_{t+j-1} - \overline{\Delta^2 x})}{N-2} && \text{for } j = 1, 2, \ldots, N-2 \\ \psi_{i,j} &= \psi_{1,j-i+1} && \text{for } i = 2, 3, \ldots, N-2,\ j = i, i+1, \ldots, N-2 \\ \psi_{j,i} &= \psi_{i,j} && \text{for } i = 1, 2, \ldots, N-2,\ j = 1, 2, \ldots, N-2 \end{aligned}$$

Let the i, jth position of the $\boldsymbol{\Omega}$ matrix be $\omega_{i,j}$, and

$$\begin{aligned} \omega_{1,j} &= \textstyle\sum_{t=1}^{N-j+1} \frac{(c_t - \bar{c})(c_{t+j-1} - \bar{c})}{N} && \text{for } j = 1, 2, \ldots, N \\ \omega_{i,j} &= \omega_{1,j-i+1} && \text{for } i = 2, 3, \ldots, N,\ j = i, i+1, \ldots, N \\ \omega_{j,i} &= \omega_{i,j} && \text{for } i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, N \end{aligned}$$

[2]The following equations for sample autocovariances ensure that the corresponding sample autocovariance matrix is positive definite.

This version of generalized HP filtering is a difficult numerical problem to solve. If this type of generalized HP filtering were to be implemented, it may be more practical to let the $\boldsymbol{\Psi}$ covariance matrix be proportional to the identity matrix, with all the corresponding constraints removed. This would reduce the complexity of the numerical problem to be solved.

These versions of generalized HP filtering give an extremely smooth estimate of the trend curve. This arises from the assumption that $\boldsymbol{\Delta}^2\boldsymbol{x}$ is distributed with mean vector $\mathbf{0}$.

There are a number of possible statistical and/or economic modeling assumptions that can be incorporated into the generalized HP optimization problem. Another example is provided in Appendix A. In Appendix B we provide another modification to the HP filter in an application to financial series.

The generalized HP procedure is different from the approaches of Harvey (1985) and Watson (1986). These types of approaches make parametric assumptions about the trend and cyclical component, whereas our approach allows for a more general stochastic process to be generating the unobserved components of the data.

4 The Multivariate Normal Cyclical Filter

Since it is practically impossible to know the true nature of the trend component of a macroeconomic series, we propose the following procedure to extract the dominant cyclical component from a series.

Let $c_t = y_t - x_t$ = unobserved cyclical component at time t, and x_t = unobserved trend and irregular component at time t.

Assume that

$$\boldsymbol{c} \sim N\left(\mathbf{0}, \sigma_c^2 \boldsymbol{\Omega}\right)$$

This implies that

$$f(\boldsymbol{c}) = \frac{1}{(\det(\boldsymbol{\Omega})(2\sigma_c^2\pi)^N)^{1/2}} \exp\left\{-\frac{1}{2}\frac{\boldsymbol{c}^T\boldsymbol{\Omega}^{-1}\boldsymbol{c}}{\sigma_c^2}\right\} \tag{8}$$

Given $\boldsymbol{y}$, we want to choose $\boldsymbol{x}$ so as to find a maximum of Equation (8), in an N-dimensional region where we have a prior belief that the "actual cycle" is contained. Again, σ_c^2 and $\boldsymbol{\Omega}$ are unknown; thus, again assume that $\boldsymbol{c}$ is a stationary process with its theoretical population parameters, σ_c^2 and $\boldsymbol{\Omega}$, equal to their corresponding sample values.

The MNC estimate of the cycle, $\widehat{\boldsymbol{c}}$, is the value of $\boldsymbol{c}$ which maximizes Equation (8) over the given region, subject to the following constraints:

$$\sigma_c^2 = \sum_{t=1}^{N} \frac{(c_t - \bar{c})^2}{N}$$

Let the i, jth position of the $\boldsymbol{\Omega}$ matrix be $\omega_{i,j}$, and

$$\begin{aligned}
\omega_{1,j} &= \frac{\sum_{t=1}^{N-j+1} \frac{(c_t-\bar{c})(c_{t+j-1}-\bar{c})}{N}}{\sum_{t=1}^{N} \frac{(c_t-\bar{c})^2}{N}} && \text{for } j = 1, 2, \ldots, N \\
\omega_{i,j} &= \omega_{1,j-i+1} && \text{for } i = 2, 3, \ldots, N, \ j = i, i+1, \ldots, N \\
\omega_{j,i} &= \omega_{i,j} && \text{for } i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, N
\end{aligned}$$

The strength of the MNC filter comes from the general assumptions made about the cyclical component of the series. Also, no explicit assumption is made about the nature of the trend component. However, the procedure is dependent on the region over which the maximum is found.

The assumption behind the MNC filter of the cycle having a multivariate normal distribution is not a restrictive assumption, because the multivariate normal distribution is a flexible distribution. The assumption of the sample parameters being equal to their corresponding population parameters is a desirable property.

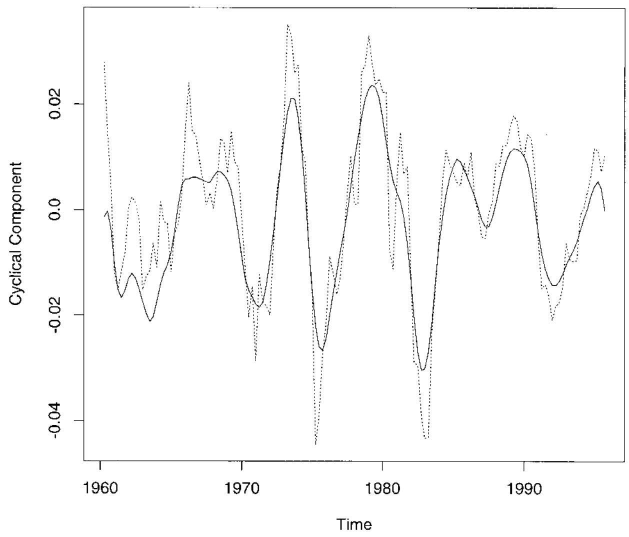

Figure 1
The cyclical residual and cyclical component of logged U.S. real GNP, using the HP1600 and MNC filters. The HP1600 cyclical residual is represented by the dashed line, and the MNC1600 cyclical component is represented by the solid line.

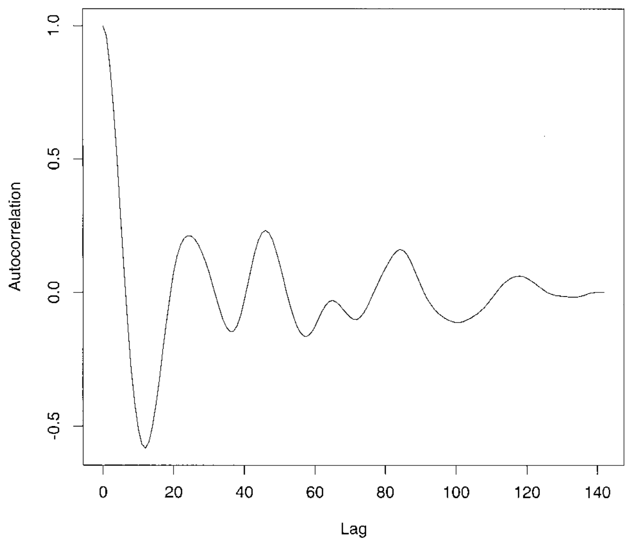

Figure 2
Autocorrelation function of the MNC1600 cyclical component of logged U.S. real GNP.

4.1 Application of the MNC filter

In the applications of the MNC filter throughout this paper, the region over which the maximum is found is centered around the HP1600 $\widehat{\boldsymbol{x}}$ vector.[3] Figures 1 and 3 are typical results from applying this filtering procedure. The HP1600 $\widehat{\boldsymbol{x}}$ vector is chosen as the starting value for a localized numerical optimization search, because we believe that this is a reasonable first estimate of the trend and irregular vector. The numerical algorithm used to maximize equation (8) throughout this paper, subject to the constraints, is the Nelder and Mead (1965) simplex method for function minimization. Thus, in these applications, the region over which the

[3]C programs that run these procedures are available from Jonathan Reeves, P.O. Box 48111, Auckland, New Zealand; fax (+ 64 - 9) 8288734; e-mail reevesj@qed.econ.queensu.ca.

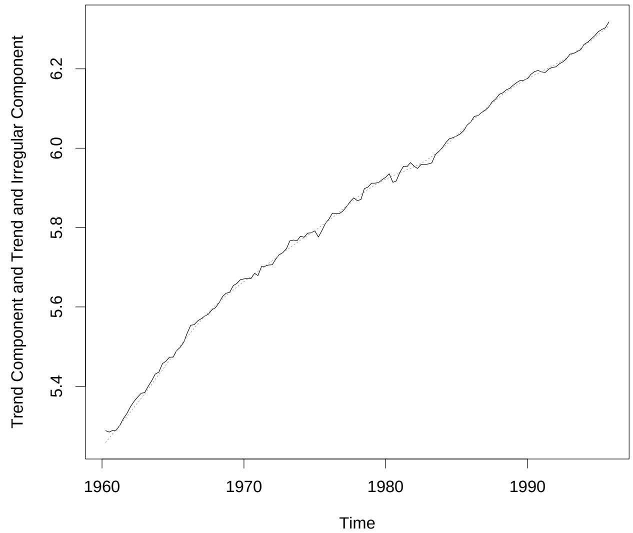

Figure 3
Trend component and trend and irregular component of logged U.S. real GNP, using the HP1600 and MNC filters. The HP1600 trend is represented by the dashed line, and the MNC1600 trend and irregular component is represented by the solid line.

maximum is found is partially determined by the Nelder and Mead algorithm. The covariance constraints have been substituted directly into the objective function, Equation (8).

The first point to notice on comparing the HP1600 cyclical residual with the corresponding MNC estimate of the cycle, i.e., the MNC estimate of the cycle from starting the minimization algorithm from the HP1600 cyclical residual (MNC1600), is that the MNC procedure has smoothed the HP1600 cyclical residual. The degree of smoothing that occurs is primarily determined by two generally accepted principles of statistics: first, the principle of maximum-likelihood estimation under constraints; and second, the principle of setting population parameters equal to their corresponding sample parameters, a principle most often applied in method of moments estimation.

We propose an approach that involves studying the cyclical residual and the purely cyclical component. The HP1600 cyclical residual appears to be a reasonable estimate of the unobserved cyclical residual, and the corresponding MNC estimate of the cycle is a reasonable estimate of the dominant cyclical component for a cyclical series of a reasonable length.

In practical terms, the only strong assumption made in applying the MNC1600 filter is that of assuming the “actual cycle” is similar to the dominant features of the HP1600 cyclical residual.

Figure 1 gives a clear estimate of the cyclical nature of the U.S. real GNP. The estimated cyclical turning points are easily determined from the MNC estimates. When the HP1600 cyclical residual is examined by visual inspection, it is sometimes difficult to determine precise turning points. However, the MNC estimate of the cycle in Figure 1 is quite similar to the dominant component of the HP1600 cyclical residual.

By the nature of a cyclical component, individual peaks and troughs are not isolated events; they are part of an ongoing cyclical pattern that corresponds to a particular correlation structure in the cyclical component over time. The MNC procedure is designed to pick up a “likely” correlation structure in the cyclical component and hence a likely estimate of the cyclical component of the series.

Thus, the short trough and peak episode in the cyclical residual of the U.S. real GNP in the early 1980s, which does not occur in the corresponding part of the MNC estimate of the cycle, indicates that possibly this short trough and peak episode was not part of the cyclical nature of the U.S. real GNP, and perhaps was more likely just an irregular movement in the series.

Table 2
U.S. business cycle turning points[4]

Method	Troughs	Peaks
NBER	61.1; 64.4; 67.4; 70.4; 75.1; 82.4	60.1; 62.2; 66.2; 69.1; 73.1; 78.4
DOC	60.4; 63.1; 67.2; 70.4; 75.2; 82.1; 86.2	62.2; 65.4; 67.4; 73.1; 78.2; 84.1
MNC1600	61.3; 63.3; 67.1; 70.4; 75.4; 82.3; 87.1; 91.4	60.3; 62.1; 66.3; 68.2; 73.2; 79.1; 85.1; 89.2; 95.1

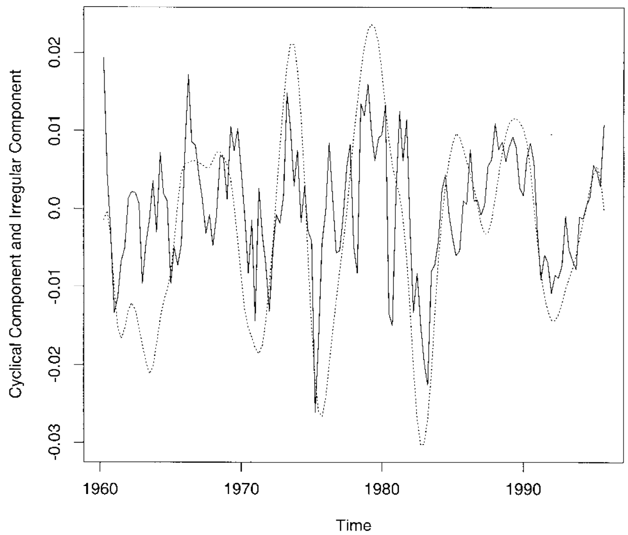

Figure 4
Cyclical component and irregular component of logged U.S. real GNP, using the MNC and HP1600 filters. The MNC1600 cyclical component is represented by the dashed line, and the estimate of the irregular component is represented by the solid line.

Table 2 shows that there is a reasonably close correspondence with previously believed cyclical turning points and the MNC1600 cyclical turning points for the U.S. real GNP.

Another filter could be applied to the estimated trend and irregular component from the MNC filter to extract an estimate of the trend component. One possibility is to apply a filter, based on maximum-likelihood estimation, which assumes that the trend component is a random walk with drift. Another possibility is to apply the HP1600 filter, which would generate an estimated trend component that would be almost identical to the estimated trend component from applying the HP1600 filter to the original series. The residual then is the estimated irregular component, which could be interpreted as random shocks.

In Figure 4, the bold curve is the residual from applying the HP1600 filter to the estimated trend and irregular component from the MNC filter. Thus, we have an estimate of the “business cycle” and an estimate of the irregular component, or random shocks, of the U.S. real GNP.

5 Conclusion

The HP1600 filter is probably best described as simply a standardized way of drawing a smooth trend curve through a cyclical series. The end result of this procedure is an estimate of the cyclical residual of the series. After over 15 years of use in economics, it has become established that the HP1600 cyclical residual is a reasonable estimate of the unobserved cyclical and irregular component of a series. However, from a purely

[4]NBER refers to the NBER chronology reported by the Center for International Business Cycle Research at Columbia University. DOC refers to the Higgings and Poole chronology compiled from the DOC composite index of leading indicators. Both are taken from Canova (1994), and have been checked against those reported by Niemira (1991) and Simkins (1994). The MNC1600 turning points come from the MNC1600 cyclical component of the U.S. real GNP, seasonally adjusted.

statistical point of view, the HP filter's applicability to a cyclical series has not been justified. The HP filter has been shown to be statistically justified, through minimizing the mean square error or through maximum-likelihood estimation, only when the detrended series is a white-noise process. When the HP1600 filter is applied to a cyclical series, a value for the smoothing parameter that generates what is believed to be a likely estimate of the cyclical residual of the series is simply being imposed.

One fundamental property of the HP filter that appears to have been overlooked is that the HP cyclical residual is proportional to the fourth difference of the trend. This property raises serious questions about the validity of the HP filter.

It is suggested here that if the HP1600 estimate of the trend is taken as the starting value to apply the MNC filter, it is likely that a reasonable estimate of the cyclical component of the series is being extracted. This approach to extracting a cyclical component from a series is statistically justified, through maximum-likelihood estimation under constraints.

The most useful insights could come from an integrated approach that jointly analyzes the HP1600 cyclical residual and the corresponding MNC estimate of the dominant cyclical component of the series. An immediate use of the MNC filter is that of dating business-cycle turning points.

Appendix A: An Application of Generalized HP Filtering

Razzak and Dennis (1995) proposed the HP filter with a nonconstant (time-varying) λ:

$$\min_{[x_t]_{t=1}^{N}} \left[\sum_{t=1}^{N} (y_t - x_t)^2 + \sum_{t=2}^{N-1} \lambda_t ((x_{t+1} - x_t) - (x_t - x_{t-1}))^2 \right] \qquad \lambda_t > 0 \tag{9}$$

Using the generalized HP optimization framework, one can show that Razzak and Dennis (1995) are in fact extracting a maximum-likelihood estimate of the trend under the conditions that:

1. $\boldsymbol{\Delta}^2 \boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{\Psi})$,
2. $\boldsymbol{c} \sim N(\boldsymbol{0}, \sigma_c^2 \boldsymbol{I})$,
3. $\boldsymbol{\Delta}^2 \boldsymbol{x}$ is independent of $\boldsymbol{c}$,
4. $\sigma_{x_t}^2$ and σ_c^2 are known, $\lambda_t = \frac{\sigma_c^2}{\sigma_{x_t}^2}$, $t = 2, 3, \ldots, N-1$, and

$$\boldsymbol{\Psi} = \begin{bmatrix}
\sigma_{x_2}^2 & 0 & 0 & 0 & \cdots & \cdots & \cdots & & \cdots & 0 \\
0 & \sigma_{x_3}^2 & 0 & 0 & 0 & \cdots & \cdots & & & 0 \\
0 & 0 & \sigma_{x_4}^2 & 0 & 0 & 0 & \cdots & & \cdots & 0 \\
0 & 0 & 0 & \sigma_{x_5}^2 & 0 & 0 & 0 & \cdots & & 0 \\
0 & 0 & 0 & 0 & \sigma_{x_6}^2 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
\vdots & & & & & & & & & \vdots \\
0 & \cdots & & \cdots & 0 & 0 & 0 & \sigma_{x_{N-3}}^2 & 0 & 0 \\
0 & \cdots & \cdots & & & 0 & 0 & 0 & \sigma_{x_{N-2}}^2 & 0 \\
0 & \cdots & \cdots & \cdots & & & 0 & 0 & 0 & \sigma_{x_{N-1}}^2
\end{bmatrix}$$

Under these conditions, the likelihood function is proportional to

$$\exp\left\{ -\frac{1}{2} \left(\frac{\sum_{t=1}^{N} (y_t - x_t)^2}{\sigma_c^2} + \sum_{t=2}^{N-1} \frac{((x_{t+1} - x_t) - (x_t - x_{t-1}))^2}{\sigma_{x_t}^2} \right) \right\} \tag{10}$$

Maximizing the above function with respect to $\boldsymbol{x}$ is equivalent to minimizing

$$
\begin{aligned}
&\frac{\sum_{t=1}^{N}(y_t - x_t)^2}{\sigma_c^2} + \sum_{t=2}^{N-1}\frac{((x_{t+1}-x_t)-(x_t-x_{t-1}))^2}{\sigma_{x_t}^2} \\
\iff &\min_{[x_t]_{t=1}^{N}}\left[\sum_{t=1}^{N}(y_t-x_t)^2 + \sum_{t=2}^{N-1}\frac{\sigma_c^2}{\sigma_{x_t}^2}((x_{t+1}-x_t)-(x_t-x_{t-1}))^2\right] \\
\iff &\min_{[x_t]_{t=1}^{N}}\left[\sum_{t=1}^{N}(y_t-x_t)^2 + \sum_{t=2}^{N-1}\lambda_t((x_{t+1}-x_t)-(x_t-x_{t-1}))^2\right], \qquad \text{where } \lambda_t = \frac{\sigma_c^2}{\sigma_{x_t}^2}
\end{aligned}
$$

Thus, under the above conditions, a maximum-likelihood estimate of the trend is being extracted when the HP filter with a nonconstant λ is applied.

The HP filter with a nonconstant λ is an example of how researchers can incorporate prior information about the structure of the economy into the generalized HP optimization problem.

Appendix B: Extracting the Signal from the Foreign Exchange Rate

The HP filter can be modified to make it more applicable to financial series. Many financial series such as foreign exchange rates do not have the increasing trend component that is evident in national income series. The trend (signal) or smoothly changing component of a financial series may be more appropriately modeled as requiring three differencing operations to have approximately zero mean for each time period, i.e., modeling the signal as locally parabolic (as opposed to only the two differencing operations that are assumed for the HP filtering of national income series). The residual from this filtering procedure will be an irregular component.

The HP3rd filter

Derivation of the HP3rd minimization problem Assume that

1. $\boldsymbol{\Delta}^3\boldsymbol{x} \sim N(\boldsymbol{0}, \sigma_x^2\boldsymbol{I})$,
2. $\boldsymbol{i} \sim N(\boldsymbol{0}, \sigma_i^2\boldsymbol{I})$, where $i_t = y_t - x_t$,
3. $\boldsymbol{\Delta}^3\boldsymbol{x}$ is independent of $\boldsymbol{i}$, and
4. σ_x^2 and σ_i^2 are known.

This implies that the joint probability-density function of $\boldsymbol{\Delta}^3\boldsymbol{x}$ and $\boldsymbol{i}$ is

$$
\begin{aligned}
f(\boldsymbol{\Delta}^3\boldsymbol{x}, \boldsymbol{i}|\sigma_x^2, \sigma_i^2) &= f(\boldsymbol{\Delta}^3\boldsymbol{x}|\sigma_x^2)f(\boldsymbol{i}|\sigma_i^2) \\
&= \frac{1}{(2\sigma_x^2\pi)^{\frac{N-3}{2}}}\exp\left\{-\frac{1}{2}\frac{(\boldsymbol{\Delta}^3\boldsymbol{x})^T\boldsymbol{\Delta}^3\boldsymbol{x}}{\sigma_x^2}\right\}\frac{1}{(2\sigma_i^2\pi)^{\frac{N}{2}}}\exp\left\{-\frac{1}{2}\frac{\boldsymbol{i}^T\boldsymbol{i}}{\sigma_i^2}\right\} \\
&= \frac{1}{(2\pi\sigma_i^2)^{\frac{N}{2}}(2\pi\sigma_x^2)^{\frac{N-3}{2}}}\exp\left\{-\frac{1}{2}\left(\frac{\boldsymbol{i}^T\boldsymbol{i}}{\sigma_i^2}+\frac{(\boldsymbol{\Delta}^3\boldsymbol{x})^T\boldsymbol{\Delta}^3\boldsymbol{x}}{\sigma_x^2}\right)\right\} \qquad (11)
\end{aligned}
$$

Given this, Equation (11) can now be looked upon as a likelihood function, and we can find a maximum-likelihood estimate of $\boldsymbol{x}$ by choosing $\boldsymbol{x}$ so as to maximize Equation (11).

Maximizing Equation (11) is equivalent to minimizing

$$\frac{\boldsymbol{i}^T\boldsymbol{i}}{\sigma_i^2}+\frac{(\boldsymbol{\Delta}^3\boldsymbol{x})^T\boldsymbol{\Delta}^3\boldsymbol{x}}{\sigma_x^2}$$

$$\Longleftrightarrow \min_{[x_t]_{t=1}^N}\left[\boldsymbol{i}^T\boldsymbol{i}+\frac{\sigma_i^2}{\sigma_x^2}(\boldsymbol{\Delta}^3\boldsymbol{x})^T\boldsymbol{\Delta}^3\boldsymbol{x}\right]$$

$$\Longleftrightarrow \min_{[x_t]_{t=1}^N}[\boldsymbol{i}^T\boldsymbol{i}+\lambda(\boldsymbol{\Delta}^3\boldsymbol{x})^T\boldsymbol{\Delta}^3\boldsymbol{x}],\qquad \text{where } \lambda=\frac{\sigma_i^2}{\sigma_x^2} \tag{12}$$

Thus, under the above conditions, a maximum-likelihood estimate of the signal is being extracted when the HP3rd filter is applied.

First-order conditions of the HP3rd minimization problem The following HP3rd first-order conditions are derived by setting the gradient vector of Equation (12) equal to zero. The first-order conditions are

$$\boldsymbol{i}=\lambda\boldsymbol{F}\boldsymbol{x}$$

where

$$\boldsymbol{F}=\begin{bmatrix}
1 & -3 & 3 & -1 & 0 & \cdots & \cdots & \cdots & & & & 0\\
-3 & 10 & -12 & 6 & -1 & 0 & \cdots & \cdots & & & & 0\\
3 & -12 & 19 & -15 & 6 & -1 & 0 & \cdots & & & & 0\\
-1 & 6 & -15 & 20 & -15 & 6 & -1 & 0 & \cdots & & & 0\\
0 & -1 & 6 & -15 & 20 & -15 & 6 & -1 & 0 & \cdots & & 0\\
0 & 0 & -1 & 6 & -15 & 20 & -15 & 6 & -1 & 0 & \cdots & 0\\
\vdots & & & & & & & & & & & \vdots\\
\vdots & & & & & & & & & & & \vdots\\
\vdots & & & & & & & & & & & \vdots\\
0 & \cdots & \cdots & 0 & -1 & 6 & -15 & 20 & -15 & 6 & -1 & 0\\
0 & & \cdots & \cdots & 0 & -1 & 6 & -15 & 20 & -15 & 6 & -1\\
0 & & \cdots & & \cdots & 0 & -1 & 6 & -15 & 19 & -12 & 3\\
0 & & \cdots & \cdots & & & 0 & -1 & 6 & -12 & 10 & -3\\
0 & & \cdots & \cdots & \cdots & & & 0 & -1 & 3 & -3 & 1
\end{bmatrix}$$

which implies that

$$\boldsymbol{y}=(\lambda\boldsymbol{F}+\boldsymbol{I})\boldsymbol{x}$$

thus, the HP trend

$$\widehat{\boldsymbol{x}}=(\lambda\boldsymbol{F}+\boldsymbol{I})^{-1}\boldsymbol{y}$$

and

$$\widehat{\boldsymbol{i}}=\boldsymbol{y}-\widehat{\boldsymbol{x}}$$

This filter is a symmetric moving-average filter, as is the original HP filter.

Application of the HP3rd filter We chose $\lambda = 16$ as an appropriate λ for extracting the signal from a monthly foreign exchange rate. The HP3rd filter with $\lambda = 16$ generates what we believe is a likely estimate of the signal and irregular component of the monthly foreign exchange rate series. The results of applying this filtering procedure to the monthly U.S. trade weighted index are presented in Figures 5, 6, and 7.

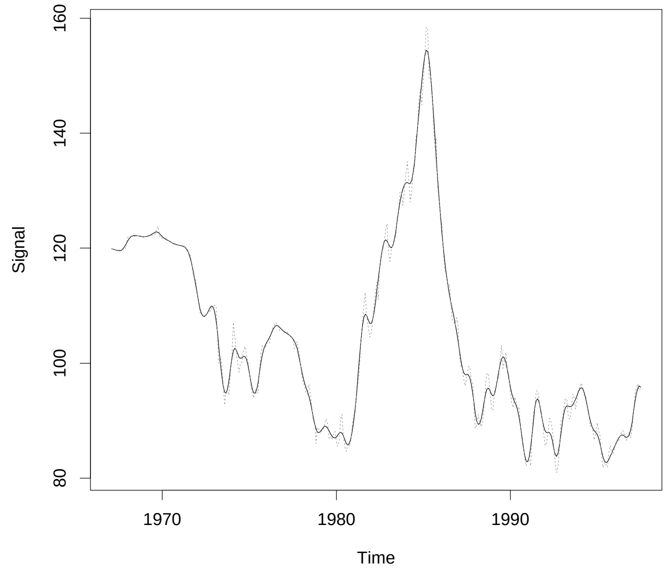

Figure 5
Signal of monthly U.S. trade weighted index using the HP3rd-differenced filter, with $\lambda = 16$. The monthly exchange rate at time t is represented by the dashed line, and the signal at time t is represented by the solid line. The raw data is from the Federal Reserve Bank of St. Louis.

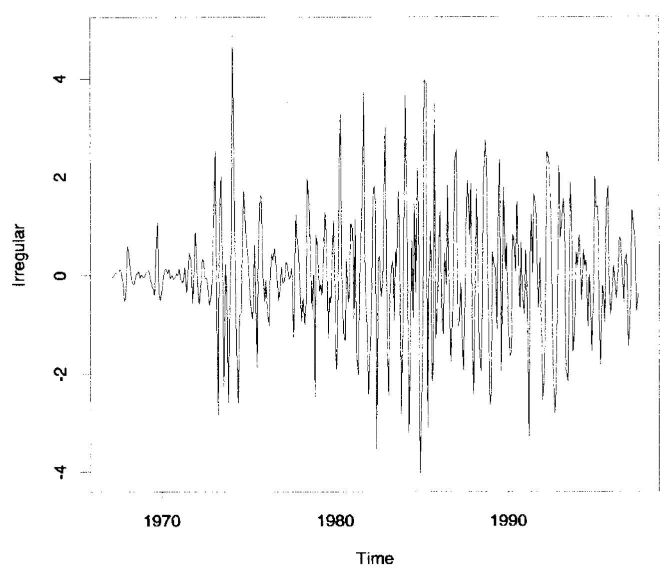

Figure 6
Irregular component of monthly U.S. trade weighted index using the HP3rd-differenced filter with $\lambda = 16$.

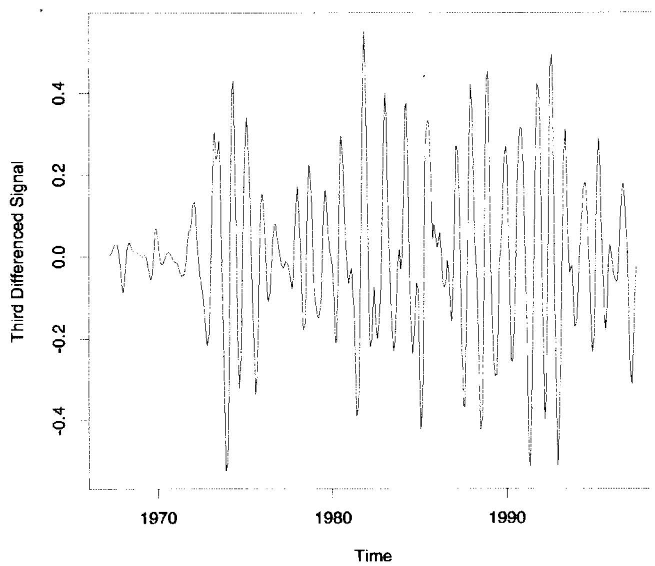

Figure 7
Third-differenced signal of monthly U.S. trade weighted index using the HP3rd-differenced filter with $\lambda = 16$.

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