

Vertex-transitive Haar graphs that are not Cayley graphs

Marston Conder,^a István Estélyi,^{b,c,d} Tomaž Pisanski^{e,c}

^a Mathematics Department, University of Auckland, Private Bag 92019, Auckland 1142, New Zealand

^b FMF, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

^c IAM, University of Primorska, Muzejski trg 2, 6000 Koper, Slovenia,

^d NTIS, University of West Bohemia, Technická 8, 306 14 Plzeň 3, Czech Republic

^e FAMNIT, University of Primorska, Glagoljaška 8, 6000 Koper, Slovenia

Dedicated to Egon Schulte and Károly Bezdek on the occasion of their 60th birthdays

Abstract

In a recent paper in *Electron. J. Combin.* 23 (2016), Estélyi and Pisanski raised a question whether there exist vertex-transitive Haar graphs that are not Cayley graphs. In this note we construct an infinite family of trivalent Haar graphs that are vertex-transitive but non-Cayley. The smallest example has 40 vertices and is the well-known Kronecker cover over the dodecahedron graph $G(10, 2)$, occurring as the graph ‘40’ in the Foster census of connected symmetric trivalent graphs.

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1 Introduction

Let G be a group, and let S be a subset of G with $1_G \notin S$. Then the *Cayley graph* $\text{Cay}(G, S)$ is the graph with vertex-set G and with edges of the form $\{g, sg\}$ for all $g \in G$ and $s \in S$. Equivalently, since all edges can be written in the form $\{1, s\}g$, this is a covering graph over a single-vertex graph having loops and semi-edges, with voltages taken from S : the order of a voltage over a semi-edge is 2 (corresponding to an

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E-mail addresses: m.conder@auckland.ac.nz (Marston Conder), istvan.estelyi@student.fmf.uni-lj.si (István Estélyi), tomaz.pisanski@upr.si (Tomaž Pisanski).

involution in S), while the order of a voltage over a loop is greater than 2 (corresponding to a non-involution in S). Note that we may assume $S = S^{-1}$.

A natural generalisation of Cayley graphs are the so called *Haar graphs*, introduced in [16] by Hladnik et al, as follows. A *dipole* is a graph with two vertices, say black and white, and parallel edges (each from the *white* vertex to the *black* vertex), but no loops. Given a group G and an arbitrary subset S of G , the *Haar graph* $H(G, S)$ is the regular G -cover of a dipole with $|S|$ parallel edges, labeled by elements of S . In other words, the vertex-set of $H(G, S)$ is $G \times \{0, 1\}$, and the edges are of the form $\{(g, 0), (sg, 1)\}$ for all $g \in G$ and $s \in S$. If it is not ambiguous, we use the notation $(x, 0) \sim (y, 1)$ to indicate an edge $\{(x, 0), (y, 1)\}$ of $H(G, S)$. The name ‘Haar graph’ comes from the fact that when G is an abelian group, the Schur norm of the corresponding adjacency matrix can be easily evaluated via the so-called Haar integral on G (see [15]).

Note that the group G acts on $H(G, S)$ as a group of automorphisms, by right multiplication, and moreover, G acts regularly on each of the two parts of $H(G, S)$, namely $\{(g, 0) : g \in G\}$ and $\{(g, 1) : g \in G\}$. Conversely, if Γ is any bipartite graph and its automorphism group $\text{Aut } \Gamma$ has a subgroup G that acts regularly on each part of Γ , then Γ is a Haar graph — indeed Γ is isomorphic to $H(G, S)$ where S is determined by the edges incident with a given vertex of Γ .

Haar graphs form a special subclass of the more general class of *bi-Cayley graphs*, which are graphs that admit a semiregular group of automorphisms with two orbits of equal size. Every bi-Cayley graph can be realised as follows. Let L and R be subsets of a group G such that $L = L^{-1}$, $R = R^{-1}$ and $1 \notin L \cup R$, and let S be any subset of G . Now take a dipole with edges labelled by elements of S , and add $|L|$ loops to the white (or ‘left’) vertex and label these by elements of L , and similarly add $|R|$ loops to the black (or ‘right’) vertex and label these by elements of R . This is a voltage graph, and the bi-Cayley graph $\text{BCay}(G, L, R, S)$ is its regular G -cover. The vertex-set of $\text{BCay}(G, L, R, S)$ is $G \times \{0, 1\}$, and the edges are of three forms: $\{(g, 0), (lg, 0)\}$ for $l \in L$, $\{(g, 1), (rg, 1)\}$ for $r \in R$, and $\{(g, 0), (sg, 1)\}$ for $s \in S$, for all $g \in G$. Note that the Haar graph $H(G, S)$ is exactly the same as the bi-Cayley graph $\text{BCay}(G, \emptyset, \emptyset, S)$.

Recently bi-Cayley graphs (and Haar graphs in particular) have been investigated by several authors — see [8, 9, 10, 16, 17, 18, 19, 20, 21, 23, 24, 25, 28, 30], for example.

It is known that every Haar graph over an abelian group is a Cayley graph (see [23]). More precisely, if A is an abelian group, then a Haar graph over A is a Cayley graph over the corresponding *generalised dihedral group* $D(A)$, which is the group generated by A and the automorphism of A that inverts every element of A (see [26]). The authors of [16] considered only cyclic Haar graphs — that is, Haar graphs $H(G, S)$ where G is a cyclic group. In [9], the second and third authors of this paper extended the study of Haar graphs to those over non-abelian groups, and found some that are not vertex-transitive, and some others that are Cayley graphs. The existence of Haar graphs that are vertex-transitive but non-Cayley remained open, and led to the following question.

Problem 1. *Is there a non-abelian group G and a subset S of G such that the Haar graph $H(G, S)$ is vertex-transitive but non-Cayley?*

In this note we give a positive answer to the above question, by exhibiting an infinite

family of trivalent examples, coming from a family of double covers of generalised Petersen graphs. These graphs, which we denote by $D(n, r)$ for any integers n and r with $n \geq 3$ and $0 < r < n$, are described in Section 2. They have been considered previously by other authors (see later); in particular, by a theorem of Feng and Zhou [30], it is known exactly which of the graphs $D(n, r)$ are vertex-transitive, and which are Cayley. Then in Section 3 we determine necessary and sufficient conditions for $D(n, r)$ to be a Haar graph, and this provides the answer to Problem 1 in Section 4.

2 The graphs $D(n, r)$ and their properties

Let $G(n, r)$ be the generalised Petersen graph on $2n$ vertices with span r . By $D(n, r)$ we denote a double cover of $G(n, r)$, in which the edges get non-trivial voltage if and only if they belong to the ‘inner rim’ (see below). This gives a class of graphs that was introduced by Zhou and Feng [29] under the name of *double generalised Petersen graphs*, and studied recently also by Kutnar and Petecki [22]. In both [29] and [22], the notation $DP(n, r)$ was used for the graph $D(n, r)$.

It is easy to define the vertices and edges of the graph $D(n, r)$ explicitly. There are four types of vertices, called u_i, v_i, w_i and z_i (for $i \in \mathbb{Z}_n$), and three types of edges, given by the sets

$$\begin{aligned} \Omega &= \{\{u_i, u_{i+1}\}, \{z_i, z_{i+1}\} : i \in \mathbb{Z}_n\} && \text{(the ‘outer’ edges),} \\ \Sigma &= \{\{u_i, v_i\}, \{z_i, w_i\} : i \in \mathbb{Z}_n\} && \text{(the ‘spokes’), and} \\ I &= \{\{v_i, w_{i+r}\}, \{v_i, w_{i-r}\} : i \in \mathbb{Z}_n\} && \text{(the ‘inner’ edges).} \end{aligned}$$

This definition makes it easy to see that each $D(n, r)$ is a special tetracirculant [13], which is a cyclic cover $\Sigma_0(n, a, k, b)$ over the voltage graph given in Figure 1. To see this, simply take $a = b = 1$ and $k = 2r$, and then $D(n, r) \cong \Sigma_0(n, 1, 2r, 1)$.

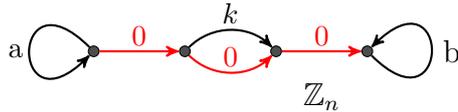


Figure 1: Voltage graph defining the tetracirculant $\Sigma_0(n, a, k, b)$

We now describe some other properties of the graphs $D(n, r)$ which are helpful. Many of these properties are already known, but we explain them here in detail for completeness.

Proposition 1. *Every $D(n, r)$ is connected.*

Proof. Clearly all of the u_i lie in the same component as each other, as do all the z_j . Next, all the v_i lie in the same component as the u_i , and similarly, all the w_j lie in the same component as the z_j . Finally, there are edges between the vertices v_i and some of the w_j , and this makes the whole graph connected. \square

Proposition 2. *The graph $D(n, r)$ is bipartite if and only if n is even.*

Proof. If n is odd, then the vertices u_i lie in a cycle of odd length, and so the graph is not bipartite. On the other hand, if n is even, then the graph is bipartite, with one part containing the vertices u_i and $w_{i\pm r}$ for even i and the vertices v_j and $z_{j\pm r}$ for odd j . \square

We now consider automorphisms of the graphs $D(n, r)$. Some automorphisms are apparent from the definition, such as these, which were noted in [22]:

$$\begin{aligned} \alpha : \quad & u_i \mapsto u_{i+1}, \quad v_i \mapsto v_{i+1}, \quad w_i \mapsto w_{i+1}, \quad z_i \mapsto z_{i+1} && \text{(rotation),} \\ \beta : \quad & u_i \mapsto z_i, \quad v_i \mapsto w_i, \quad w_i \mapsto v_i, \quad z_i \mapsto u_i && \text{(flip symmetry),} \\ \gamma : \quad & u_i \mapsto u_{-i}, \quad v_i \mapsto v_{-i}, \quad w_i \mapsto w_{-i}, \quad z_i \mapsto z_{-i} && \text{(reflection).} \end{aligned}$$

Immediately we obtain the following:

Proposition 3. *The automorphism group of the graph $D(n, r)$ has at most two orbits on vertices, namely the set of all u_i and all z_j , and the set of all v_i and all w_j .*

Note also that α and β commute with each other. In fact, Zhou and Feng [29] proved that $D(n, r)$ is isomorphic to the bi-Cayley graph $\text{BCay}(G, R, L, \{1\})$ over the abelian group $G = \langle \alpha, \beta \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_2$, and $R = \{\alpha, \alpha^{-1}\}$ and $L = \{\alpha^r \beta, \alpha^{-r} \beta\}$.

Next, we consider isomorphisms among the graphs $G(n, r)$ and $D(n, r)$.

Proposition 4. *For every n and r , the graph $D(n, r)$ is isomorphic to $D(n, n-r)$, and $D(2n, r)$ is isomorphic to $D(2n, n-r)$.*

Proof. First, $D(n, r) \cong D(n, n-r)$ because $G(n, r)$ is identical to $G(n, n-r)$. For the second part, consider a 180 degree rotation of the two ‘inner’ layers, namely $w_i \mapsto w_{i+n}$ and $z_i \mapsto z_{i+n}$ for all i . This shows that $D(2n, r)$ is isomorphic to $D(2n, n+r)$, and then applying the first part gives $D(2n, r) \cong D(2n, 2n - (n+r)) = D(2n, n-r)$. \square

Here we note that it can happen that the graphs $D(n, r)$ and $D(n, s)$ are different when $G(n, r)$ is isomorphic to $G(n, s)$. For instance, $G(7, 2)$ is isomorphic to $G(7, 3)$ but $D(7, 2)$ is not isomorphic to $D(7, 3)$, since $D(7, 3)$ is planar but $D(7, 2)$ is not.

Also we have the following:

Proposition 5. *For every r , the graph $D(2r+1, r)$ is planar, and isomorphic to the generalised Petersen graph $G(4r+2, 2)$.*

Proof. To see that $D(2r+1, r)$ is planar, first note that since r is coprime to $2r+1$, the edges between the vertices v_i and w_j give a cycle of length $2(2r+1)$, namely $(v_0, w_{-r}, v_1, w_{1-r}, v_2, w_{2-r}, \dots, v_{-2}, w_{r-1}, v_{-1}, w_r)$. Now draw three concentric circles, with the middle one for this $2(2r+1)$ -cycle, the inside one for the $(2r+1)$ -cycle $(u_0, u_1, \dots, u_{2r})$, and the outside one for the $(2r+1)$ -cycle $(z_0, z_1, \dots, z_{2r})$, in a consistent order, and then add the spoke edges $\{u_i, v_i\}$ and $\{w_i, z_i\}$ in the natural way. In the resulting planar drawing of $D(2r+1, r)$, there is an inner face of length $2r+1$ (with the u_i as vertices), then two layers of pentagonal faces (bounded by cycles of the

form $(u_i, v_i, w_{i-r}, v_{i+1}, u_{i+1})$ and $(v_j, w_{j+r}, z_{j+r}, z_{j-r}, w_{j-r})$, and an outer face of length $2r+1$ (with the z_j as vertices). After doing this, it is also easy to see that $D(2r+1, r)$ is isomorphic to the generalised Petersen graph $G(4r+2, 2)$, with the spoke edges joining vertices of the large $2(2r+1)$ -cycle (on the vertices v_i and w_j) to the two $(2r+1)$ -cycles (on the vertices u_i and vertices z_j respectively). \square

In particular, the graph $D(5, 2)$ is isomorphic to the dodecahedral graph $G(10, 2)$, and hence $D(5, 2)$ is vertex-transitive. But as we will see, it is not a Haar graph.

Finally in this section, we consider the questions of which of the graphs $D(n, r)$ are vertex-transitive, and which are Cayley (or equivalently, which have the property that $\text{Aut}(D(n, r))$ has a subgroup that acts regularly on vertices). Recall that $\text{Aut}(D(n, r))$ has at most two orbits on vertices, and just one when $(n, r) = (5, 2)$. The complete picture was determined by Feng and Zhou in [30, Theorem 1.3], as follows:

Theorem 6. *The graph $D(n, r)$ is vertex-transitive if and only if $n = 5$ and $r = \pm 2$, or n is even and $r^2 \equiv \pm 1 \pmod{\frac{n}{2}}$. In the first case, $D(n, r)$ is isomorphic to the dodecahedral graph $G(10, 2)$, which is non-Cayley, and in the second case, if $r^2 \equiv 1 \pmod{\frac{n}{2}}$ then $D(n, r)$ is a Cayley graph, while if $r^2 \equiv -1 \pmod{\frac{n}{2}}$ then $D(n, r)$ is non-Cayley.*

3 The graphs $D(n, r)$ as Haar graphs

Recall that a Haar graph is a regular cover of a dipole, and also a bi-Cayley graph. Also we have the following, proved in a different way in [9, Proposition 5]:

Proposition 7. *A Cayley graph is a Haar graph if and only if it is bipartite.*

Proof. Let Γ be a Cayley graph, say for a group K . Then K acts on Γ as a group of automorphisms, and acts regularly on the vertices of Γ . Now if Γ is a Haar graph, then by definition Γ is bipartite. Conversely, suppose Γ is bipartite. Then the subgroup G of K preserving each of the two parts of Γ has index 2 in K , and acts regularly on each part, so Γ is a Haar graph (by the argument given in the third paragraph of the Introduction). \square

We can now prove our main theorem:

Theorem 8. *$D(n, r)$ is a Haar graph if and only if it is vertex-transitive and n is even.*

Proof. First, we note that $D(n, r)$ is bipartite if and only if n is even, by Proposition 2, and hence we may suppose that n is even, and then show that under that assumption, $D(n, r)$ is a Haar graph if and only if it is vertex-transitive.

One direction is easy. Suppose $\Gamma = D(n, r)$ is a Haar graph, say $H(G, S)$. Then by the definition of a Haar graph given in the Introduction, the subgroup G_R of $\text{Aut } \Gamma$ induced by G has two orbits on vertices, namely the two parts of the bipartition of Γ . On the other hand, by Proposition 3, all the vertices u_i lie in the same orbit of $\text{Aut } \Gamma$; and then since these vertices lie in both parts of Γ , it follows that $\text{Aut } \Gamma$ has a single orbit on vertices. Thus Γ is vertex-transitive.

For the converse, suppose that $\Gamma = D(n, r)$ is vertex-transitive, and let $m = \frac{n}{2}$. Then by Theorem 6, we know that $r^2 \equiv \pm 1 \pmod{m}$. Also by Proposition 4 we may suppose that $0 < r < m$, and further, we may suppose that r is odd, because if r is even then m is odd, and then by Proposition 4 we can replace r by $m - r$. We now proceed by considering separately the two cases $r^2 \equiv \pm 1 \pmod{m}$.

Case (a): Suppose that $r^2 \equiv 1 \pmod{m}$. Then by Theorem 6, we know that $D(n, r)$ is a Cayley graph, and also since it is bipartite, it follows from Proposition 7 that it is a Haar graph as well.

Case (b): Suppose that $r^2 \equiv -1 \pmod{m}$. In this case we construct a group of automorphisms of $D(n, r)$ that acts regularly on each part of $D(n, r)$. To do this, we take the automorphism α from the previous section, given by

$$\alpha : u_i \mapsto u_{i+1}, \quad v_i \mapsto v_{i+1}, \quad w_i \mapsto w_{i+1}, \quad z_i \mapsto z_{i+1},$$

and then take an additional automorphism δ , given by

$$\begin{aligned} \delta : u_i &\mapsto v_{ri+1}, \quad v_i \mapsto u_{ri+1}, \quad w_i \mapsto z_{ri+1}, \quad z_i \mapsto w_{ri+1} && \text{if } m \text{ is odd and } i \text{ is even,} \\ \delta : u_i &\mapsto w_{ri+1}, \quad v_i \mapsto z_{ri+1}, \quad w_i \mapsto u_{ri+1}, \quad z_i \mapsto v_{ri+1} && \text{if } m \text{ is odd and } i \text{ is odd,} \end{aligned}$$

or

$$\begin{aligned} \delta : u_i &\mapsto v_{ri+1}, \quad v_i \mapsto u_{ri+1}, \quad w_i \mapsto z_{ri+m+1}, \quad z_i \mapsto w_{ri+m+1} && \text{if } m \text{ is even and } i \text{ is even,} \\ \delta : u_i &\mapsto w_{ri+1}, \quad v_i \mapsto z_{ri+1}, \quad w_i \mapsto u_{ri+m+1}, \quad z_i \mapsto v_{ri+m+1} && \text{if } m \text{ is even and } i \text{ is odd.} \end{aligned}$$

It is a straightforward exercise to verify that δ preserves the edge-set $\Omega \cup \Sigma \cup I$ of $D(n, r)$, and also preserves the two parts of $D(n, r)$, given in the proof of Proposition 2. To do the former, it is important to note that $r^2 \equiv 1 \pmod{4}$ (because r is odd), and hence that $r^2 \equiv -1 \pmod{n}$ when m is odd, while $r^2 \equiv m-1 \pmod{n}$ when m is even. For example, if m and i are even then $\{v_i, w_{i+r}\}^\delta = \{u_{ri+1}, u_{r(i+r)+m+1}\} = \{u_{ri+1}, u_{ri}\}$.

It is also easy to see that conjugation by δ takes α^2 to α^{2r} , and so the subgroup G of $\text{Aut}(D(n, r))$ generated by α^2 and δ is isomorphic to the semi-direct product $\mathbb{Z}_m \rtimes_r \mathbb{Z}_4$. In particular, G has order $4m = 2n$. Also G acts transitively and hence regularly on each of the two parts of $D(n, r)$, and therefore $D(n, r)$ is a Haar graph. \square

4 Vertex-transitive Haar graphs that are not Cayley graphs

Combining Theorems 6 and 8, we have the following, in answer to Problem 1:

Theorem 9.

- (a) *If n is odd, or if n is even and $r^2 \not\equiv \pm 1 \pmod{\frac{n}{2}}$, then $D(n, r)$ is not a Haar graph, and is vertex-transitive only when $(n, r) = (5, \pm 2)$;*
- (b) *If n is even and $r^2 \equiv 1 \pmod{\frac{n}{2}}$, then $D(n, r)$ is a Haar graph and a Cayley graph;*
- (c) *If n is even and $r^2 \equiv -1 \pmod{\frac{n}{2}}$, then $D(n, r)$ is a Haar graph and is vertex-transitive but not a Cayley graph.*

Corollary 10. *If $m > 2$ and $r^2 \equiv -1 \pmod{m}$, then $D(2m, r)$ is a Haar graph that is vertex-transitive but non-Cayley. In particular, there are infinitely many such graphs.*

Proof. The first part follows immediately from Theorem 9, and the second part follows from a well known fact in number theory, namely that -1 is a square mod m if and only if m or $m/2$ is a product of primes $p \equiv 1 \pmod{4}$ (see [14, Chapter 6]), or simply by taking $m = r^2 + 1$ for each integer $r \geq 2$. \square

We discovered the first few of these examples during the week of the conference *Geometry and Symmetry*, held in 2015 at Veszprém, Hungary, to celebrate the 60th birthdays of Károly Bezdek and Egon Schulte.

The smallest of our examples is $D(10, 2)$, of order 40, occurring when $m = 5$ and $r \equiv \pm 2$ or $\pm 3 \pmod{10}$ (noting that $m - r = 3$ when $(m, r) = (5, 2)$). We will show that this is also the smallest Haar graph that is vertex-transitive and non-Cayley. It is a Kronecker cover over the dodecahedral graph $G(10, 2)$, and is also a double cover over the Desargues graph $G(10, 3)$. These graphs are illustrated in Figure 2.

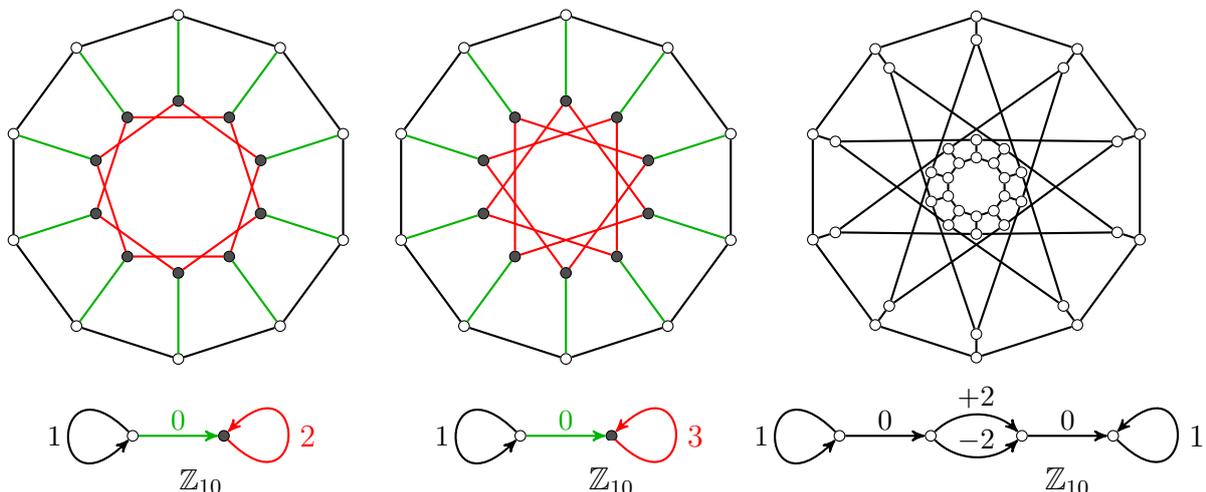


Figure 2: The dodecahedral graph $G(10, 2)$, the Desargues graph $G(10, 3)$, and the Haar graph $D(10, 2) \cong D(10, 3) \cong \mathbf{F40}$

The graph $D(10, 2)$ was known by R.M. Foster as early as the late 1930s, and appears as the graph ‘40’ (alternatively known as ‘**F40**’) in the *Foster Census* of connected symmetric trivalent graphs [7]. It was also studied in [27] by Asia Ivić Weiss (the chair of the Veszprém conference), and by Betten, Brinkmann and Pisanski in [1], and Boben, Grünbaum, Pisanski and Žitnik in [2]. It has girth 8, and automorphism group of order 480, and it is not just vertex-transitive, but is also arc-transitive. Moreover, by a very recent theorem of Kutnar and Petecki [22], the graph $D(n, r)$ is arc-transitive only when $(n, r) = (5, 2)$ or $(10, 2)$ or $(10, 3)$. This implies that **F40** is the only example from the family of graphs $D(n, r)$ that is arc-transitive but non-Cayley.

In fact, **F40** is the smallest vertex-transitive non-Cayley Haar graph, in terms of both the graph order and the number of edges. We found this by running a MAGMA [3]

computation to construct all Haar graphs with at most 40 vertices or at most 60 edges, with a check for which of the graphs are vertex-transitive but non-Cayley. Incidentally, this computation shows that there are 60 different examples of order 40, with valencies running between 3 and 17, but just one of valency 3, namely **F40**.

Finally, there are many other examples of vertex-transitive non-Cayley Haar graphs that are not of the form $D(n, r)$, including 3-valent examples of orders 80, 112, 120 and 128, and higher-valent examples of orders 48, 64, 72, 78 and 80. Among the 3-valent examples, many are arc-transitive, including the graphs **F80** and **F640** in the Foster census [7] and its extended version in [5, 6], and others in the first author's complete set of all arc-transitive trivalent graphs of order up to 10000 described on his website [4]. Most of these 'small' examples are abelian regular covers of **F40**, of orders 1280, 2560, 3240, 5000, 5120, 6480, 6720, 9720 and 10000, and are 3-arc-regular, but two others are 2-arc-regular of type 2^2 , with orders 6174 and 8064, and these are abelian regular covers of the Pappus graph (**F18**) and the Coxeter graph (**F28**) respectively.

In particular, the graph of order 6174 is a member of an infinite family of 2-arc-regular covers of the Pappus graph, investigated in [12, 11]. Each graph in this family is a 2-arc-regular 3-valent graph of type 2^2 and order $18n^3$ for some $n \geq 7$ for which \mathbb{Z}_n contains a root of the polynomial $x^2 + x + 1$. Also each member of this family is Haar but not Cayley, since the 2-arc-regular subgroup of type 2^2 in the automorphism group of the Pappus graph contains a subgroup of order 9 acting regularly on each of the two parts of the graph, but contains no subgroup of order 18 acting transitively on the vertices. Hence we have another infinite family of examples of vertex-transitive non-Cayley Haar graphs, but of considerably larger orders.

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